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# An extended Fatou–Shishikura inequality and wandering branch continua for polynomials

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Let $P$ be a polynomial of degree $d$ with Julia set $J_P$. Let $\widetilde{N}$ be the number of non-repelling cycles of $P$. By the famous Fatou–Shishikura inequality $\widetilde{N} \leq d-1$. The goal of the paper is to improve this bound. The new count includes *wandering collections of non-(pre)critical branch continua*, i.e., collections of continua or points $Q_i \subset J_P$ *all* of whose images are pairwise disjoint, contain no critical points, and contain the limit sets of $\mathrm{eval}(Q_i) \geq 3$ external rays. Also, we relate individual cycles, which are either non-repelling or repelling with no periodic rays landing, to individual critical points that are recurrent in a weak sense.
A weak version of the inequality reads

$$\widetilde{N} + N_{irr} + \chi + \sum_i (\mathrm{eval}(Q_i) - 2) \leq d - 1$$

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Julia set
Wandering continuum

where $N_{irr}$ counts repelling cycles with no periodic rays landing at points in the cycle, $\{Q_i\}$ form a wandering collection $\mathcal{B}_{\mathbb{C}}$ of non-(pre)critical branch continua, $\chi = 1$ if $\mathcal{B}_{\mathbb{C}}$ is non-empty, and $\chi = 0$ otherwise.



## 1. Introduction

In the dynamics of iterated rational maps, it is a frequent observation that many interesting dynamical features are largely determined by the dynamics of critical points. The classical Fatou–Shishikura inequality states in the polynomial case that a complex polynomial of degree $d \geq 2$ has at most $d-1$ non-repelling periodic orbits in $\mathbb{C}$. We extend this in several ways.

- *Wandering (eventual) branch continua*, defined below, are included in the count (such continua are either proper subsets of periodic components of the Julia set or wandering components of the Julia set); note that we allow continua to be points. In the simplest case, such a continuum corresponds to a point $z$ in the Julia set that is the landing point of 3 or more external rays so that no point in the forward orbit of $z$ is critical or periodic.
- Together with non-repelling periodic orbits, we also count orbits of repelling periodic points that are not landing points of *periodic* external rays (such points may exist if the Julia set is not connected and then must be components of the Julia set).
- Specific critical points are associated to the aforementioned periodic orbits and wandering branch continua: (a) every non-repelling periodic orbit and every repelling periodic orbit without periodic rays has at least one associated critical point, so that different orbits are associated to different critical points, and (b) wandering branch continua require other critical points not associated to any periodic orbits.
- The inequality is sharpened by counting not all critical points, but certain "weak equivalence classes of weakly recurrent critical points" (other restrictions on critical points apply as well).
- The key idea is that various phenomena counted on the left hand side of the inequality can be associated with critical points counted on the right. In the case of wandering eventual branch continua the association is not as direct as in the case of specific periodic points, but sufficient for our purpose.

Let $P$ be a polynomial of degree $d \geq 2$ with Julia set $J_P$. A *rational ray pair* $\mathcal{R}$ is a pair of (pre)periodic external rays that land at a common point, together with their common landing point; $\mathcal{R}$ *weakly separates* two points $z, w \in \mathbb{C}$ if $z$ and $w$ are in two different components of $\mathbb{C} \setminus \mathcal{R}$. A critical point $c$ is *weakly recurrent* if it belongs to the filled-in Julia set, never maps to a repelling or parabolic point, and for every finite collection $\mathcal{R}_1, \dots, \mathcal{R}_k$ of rational ray pairs there is an $n \geq 0$ such that $c$ and $P^{\circ n}(c)$ are not weakly separated by any ray pair $\mathcal{R}_i$, $1 \leq i \leq k$. Clearly, a *recurrent* critical point is weakly recurrent.

If $J_P$ is not connected, then some external rays of the polynomial $P$ are non-smooth, namely those that contain preimages of escaping critical points or escaping critical points themselves (see Section 6 for details).

In this text, a *continuum* is a non-empty compact connected metric space (we allow it to be a point and call a continuum that is not a point *non-degenerate*). The *valence* $\mathrm{val}_{J_P}(Q)$ *of a continuum* $Q \subset J_P$ is the number of external rays with limit sets in $Q$ (in case $J_P$ is not connected we allow for the possibility of non-smooth external rays, see Section 6). Call $Q$ a *branch continuum* if its valence is 3 or greater. A continuum $Q \subset J_P$ is *wandering* if $P^k(Q) \cap P^m(Q) = \emptyset$ for all $m > k \geq 0$. We show that if $Q$ is wandering then $\mathrm{val}_{J_P}(Q)$ is finite, and show that there exists $m$ such that $\mathrm{val}_{J_P}(P^n(Q)) = m$ for all sufficiently big $n$. If $m > 1$ and $Q$ is contained in a (pre)periodic component $E$ of the Julia set, then $m$ equals the number of components of $P^n(E) \setminus P^n(Q)$, see Corollary 3.6 and Corollary 6.11. Set $\mathrm{eval}_{J_P}(Q) = m$ and call it the *eventual valence* of $Q$. We call a wandering continuum $Q$ an *eventual branch continuum* if $\mathrm{eval}_{J_P}(Q) > 2$. A collection of eventual branch continua is called a *wandering collection* if all their forward images are pairwise disjoint.

Some of our main results are stated in Theorem 1.1. The actual results proven in the body of the paper are significantly stronger than Theorem 1.1, however their statements require additional notions that will be introduced later in the paper. Observe, that if $J_P$ is connected, the results can be stated in topological terms because in this case by Corollary 3.6 the valence of a wandering continuum $Q$ equals the number of components of $J_P \setminus Q$ (i.e., can be defined without invoking external rays); similarly, non-repelling cycles can be defined in a purely topological way. Consequently, the main results also hold for polynomial-like mappings with connected Julia set.

**Theorem 1.1.** *The following facts hold for the polynomial $P$.*

(1) *Every non-repelling periodic orbit has an associated weakly recurrent critical orbit (recurrent in the case of irrationally indifferent orbits), so that distinct non-repelling orbits have distinct associated critical orbits.*
(2) *Every repelling periodic orbit $L$ consisting of points at which no periodic external ray lands, has an associated escaping critical orbit $H$ (such that $H$ is not weakly separated from $L$) so that distinct repelling periodic orbits have different associated critical orbits, see [28].*
(3) *If $P$ has a wandering collection of $m \geq 1$ eventual branch continua $Q_1, \dots, Q_m$, then*

$$1 + \sum_{i=1}^{m} (\mathrm{eval}_{J_P}(Q_i) - 2)$$

*is bounded from above by the number of weakly recurrent critical points, weakly separated from all non-repelling periodic points.*

The relation between special dynamical features and associated critical orbits of a polynomial $P$ with Julia set $J_P$ can be reduced to a count; this will yield an extension of the classical Fatou–Shishikura inequality. More precisely, let us use the following notation.

- $N_{FC}$ is number of different orbits of bounded Fatou domains plus the number of Cremer cycles;
- $N_{irr}$ is the number of repelling cycles without periodic external rays (the subscript *irr* stands for rays with *irrational* arguments);
- $C_{wr}$ is the set of weakly recurrent critical points in periodic components of $J_P$;
- $C'_{wr}$ is the set of weakly recurrent critical points in wandering components of $J_P$;
- $C_{esc}$ is the set of escaping critical points;
- $m$ is the number of eventual branch continua $Q_i$ in a wandering collection $\{Q_1, \dots, Q_m\}$ such that each $Q_i$ is contained in a (pre)periodic component of $J_P$;
- $m'$ is the number of eventual branch continua $Q'_j$ in a wandering collection $\{Q'_1, \dots, Q'_m\}$ such that each $Q'_j$ is a component of $J_P$;
- $N_{co}$ is the number of cycles of components of $J_P$ that contain wandering eventual branch continua;
- $\chi(l)$ is 1 if $l > 0$ and 0 otherwise.

Given a finite (perhaps empty) set of numbers $\{a_i\}_{i=1}^k$, set $\sum_{i=1}^k a_i = 0$ if $k = 0$. Also, let $|A|$ denote the cardinality of a set $A$.

**Theorem 1.2** *(The extended Fatou–Shishikura inequality). For the polynomial $P$ the following inequalities hold:*

$$N_{FC} + N_{co} + \sum_{i=1}^{m} (\mathrm{eval}_{J_P}(Q_i) - 2) \le |C_{wr}|$$

*and*

$$N_{irr} + \chi(m') + \sum_{j=1}^{m'} (\mathrm{eval}_{J_P}(Q'_j) - 2) \le \chi(m')|C'_{wr}| + |C_{esc}|$$

*Summing up, we have*

$$\begin{aligned} &N_{FC} + N_{irr} + N_{co} + \sum_{i=1}^{m} (\mathrm{eval}_{J_P}(Q_i) - 2) + \chi(m') + \sum_{j=1}^{m'} (\mathrm{eval}_{J_P}(Q'_j) - 2) \\ &\quad \le |C_{wr}| + \chi(m')|C'_{wr}| + |C_{esc}| \le d - 1 \end{aligned}$$

We would like to make a few remarks concerning the above results.

(1) An attracting or rationally indifferent cycle is the limit of at least one critical orbit as follows from Fatou [17]; this is the best known case in all results. It is also well-known that every Cremer point and every boundary point of a Siegel disk is a limit point of at least one recurrent critical orbit (see Mañé [31]). The idea to use rational ray pairs to associate different indifferent cycles to different critical points is due to Kiwi [21]. Combining this with a version of Mañé [31] (see [25] or [8]) we show that different Cremer or Siegel cycles can be associated to different individual *recurrent* critical points. This implies that $N_{FC} \leq |C_{wr}|$ which is a version of the first inequality of Theorem 1.2 implying the classical Fatou–Shishikura-inequality for polynomials, i.e. $N_{FC} \leq d - 1$.

(2) Using a recent topological result on fixed points in non-invariant continua (Chapter 7 of [6]), we show that the recurrent critical points associated to Cremer or Siegel cycles *cannot* be associated to wandering eventual branch continua (should the latter exist). Together with combinatorial results of [7,11] this yields the first inequality of Theorem 1.2. The tools similar to those developed in [6] are used in a recent paper [36], devoted to extending isotopies of plane continua onto the entire plane.

(3) If there are no wandering eventual branch continua, the inequalities reduce to $N_{FC} \leq |C_{wr}|$ and $N_{irr} \leq |C_{esc}|$; if there are wandering eventual branch continua, then $N_{co} + \chi(m') \geq 1$, so at least one weakly recurrent critical point is used for the existence of wandering eventual branch continua (more if, e.g., the latter are contained in different cycles of components of the Julia set), in addition to the individual count in the sum $\sum(\mathrm{eval}_{J_P}(Q_i) - 2)$.

(4) The initial version of the Fatou–Shishikura inequality is due to Fatou [17] who proved that any rational map of degree $d$ has at most $4d - 4$ non-repelling periodic cycles (he proved that any pair of indifferent cycles can be perturbed into one attracting and one repelling cycle, and every attracting cycle attracts one of the $2d - 2$ critical points).

Shishikura [48] improved the Fatou bound by proving that there can be at most $2d - 2$ non-repelling cycles: using quasiconformal surgery, he showed that *every* indifferent cycle can be perturbed so as to become attracting. His method allows to show that this bound is sharp. Rationally indifferent periodic orbits may attract more than one critical orbit; this refines the counts above. For rational maps, this inequality also includes Herman rings: each periodic cycle of Herman ring counts for two non-repelling periodic cycles.

For a conceptually different proof of the Fatou–Shishikura inequality, see Epstein's preprint [15]. There push-forwards of quadratic differentials are used and, in certain cases, the count of rationally indifferent orbits is refined (Herman rings are not discussed in the preprint [15]).

(5) For polynomials, we have that every polynomial of degree $d \geq 2$ has at most $d - 1$ non-repelling periodic orbits in $\mathbb{C}$ (this is because $\infty$ is a critical point of multiplicity

$d - 1$, and there are no Herman rings). A simple proof of this inequality in the polynomial case is due to Douady and Hubbard [14]; it is based on perturbations of polynomial-like maps. Conceptually, our approach is close to that of Kiwi [21], yet we use some additional tools and push the inequality further.

(6) The estimates concerning wandering branch points in the locally connected case are obtained in [22,7,11]. For connections between wandering continua and topology of the Julia set, see [33] (Douady–Hubbard examples), [27,45].

(7) We do not use perturbations and directly allocate to each "piece of dynamics" distinct critical points (more precisely, their grand orbits). It allows us to include in the count the wandering eventual branch continua as well. By [9,10] the count of wandering branch continua in degree 3 is sharp: there exist uncountably many cubic polynomials with locally connected non-separating Julia sets which contain a wandering non-(pre)critical branchpoint $z$ of valence 3 so that the inequality in Theorem 1.2 becomes equality (we believe it is sharp in general too).

The relation between non-repelling periodic orbits and critical points is well-known. To briefly motivate the relation between wandering continua and critical points, suppose that $J_P$ is connected and locally connected. For each $y \in J_P$ let $A(y)$ be the set of all angles $\alpha$ such that the external ray $R_\alpha$ lands at $y$. Now, consider the collection of all hyperbolic geodesics in the boundaries of the convex hulls (in the hyperbolic metric on $\mathbb{D}$) of all the sets $A(y), y \in J_P$ taken in the closed unit disk $\overline{\mathbb{D}}$. The set of all such line segments in $\overline{\mathbb{D}}$ forms an *invariant (geometric) lamination* in the sense of Thurston [51].

Consider a non-(pre)periodic non-(pre)critical point $z$ that is the landing point of at least three external rays (the number of such external rays is finite by [22,7]). Then the arguments $\mathcal{A}(z)$ of the external rays landing at $z$ determine a polygon $Q_0 \subset \overline{\mathbb{D}}$. The image point $P(z)$ determines the polygon say, $Q_1$, with vertices $\mathcal{A}(P(z))$ of external rays landing on $P(z)$. Note that if we define the map $\sigma_d$ so that $\sigma_d(w) = w^d$ for $w \in \mathbb{S}^1$, then $\sigma_d(\mathcal{A}(z)) = \mathcal{A}(P(z))$; also, $|\mathcal{A}(z)| = |\mathcal{A}(P(z))|$ if $z$ is not critical.

This yields a sequence of polygons $Q_0, Q_1, \dots \subset \overline{\mathbb{D}}$ with disjoint interiors and hence Euclidean areas converging to 0. If $Q_i$ has a small area, then either *all* its sides are short, or *two* of its sides have almost equal length and the *remaining* sides are short. Under $z^d|_{\mathbb{S}^1}$ lengths of short sides of $Q_i$ increase. A side $s$ of $Q_i$ can have a short image *only* if the endpoints of $s$ have angles that differ by *nearly* $k/d, k = 1, 2, \dots, d - 1$. So, the sequence $Q_i$ must have sides that (a) converge subsequentially to a chord $\ell \in \overline{\mathbb{D}}$ such that (b) the endpoints of $\ell$ in $\mathbb{S}^1$ have angles that differ by *exactly* $k/d$. By (a) $\ell$ corresponds to two different external rays that land at a common point $c$, and by (b) these rays have equal images. This implies that $c$ is a critical point of $P$ and motivates why wandering eventual branch continua are related to critical points.

## 2. Preliminaries

### 2.1. Laminations and locally connected models

#### 2.1.1. Introductory information

Let $\mathbb{D}$ be the open unit disk and $\widehat{\mathbb{C}}$ be the complex sphere. For a compactum $X \subset \mathbb{C}$, let $U^\infty(X)$ be the unbounded component of $\mathbb{C} \setminus X$ and let $T(X) = \mathbb{C} \setminus U^\infty(X)$ be the *topological hull* of $X$. Sometimes we use $U^\infty(X)$ for $\widehat{\mathbb{C}} \setminus T(X)$ (including the point at $\infty$). We say that $X$ is *unshielded* if $X = \mathrm{Bd}(U^\infty(X))$. If $X$ is a continuum then $T(X)$ is a non-separating continuum and there exists a Riemann map $\Psi_X : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to U^\infty(X)$; we always normalize it so that $\Psi_X(\infty) = \infty$ and $\Psi'_X(z)$ tends to a positive real limit as $z \to \infty$.

Now consider a polynomial $P$ of degree $d \geq 2$ with Julia set $J_P$ and filled-in Julia set $K_P = T(J_P)$. Clearly, $J_P$ is unshielded. Extend $z^d : \mathbb{C} \to \mathbb{C}$ to a map $\theta_d$ on $\widehat{\mathbb{C}}$. If $J_P$ is connected then $\Psi_{K_P} = \Psi : \mathbb{C} \setminus \overline{\mathbb{D}} \to U^\infty(K_P)$ is such that $\Psi \circ \theta_d = P \circ \Psi$ [13,34].

#### 2.1.2. Laminations in the locally connected case

Let us suppose for now that $J_P$ is locally connected. Then $\Psi$ extends to a continuous function $\overline{\Psi} : \widehat{\mathbb{C}} \setminus \mathbb{D} \to \overline{\widehat{\mathbb{C}} \setminus K_P}$ and $\overline{\Psi} \circ \theta_d = P \circ \overline{\Psi}$; in particular, we obtain a continuous surjection $\overline{\Psi}\colon \mathrm{Bd}(\mathbb{D}) \to J_P$ (the Carathéodory loop). Identify $S^1 = \mathrm{Bd}(\mathbb{D})$ with $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.

Let $\sigma_d = \sigma = \theta_d|_{\mathbb{S}^1}$, $\psi = \overline{\Psi}|_{\mathbb{S}^1}$. Define an equivalence relation $\sim_P$ on $\mathbb{S}^1$ by $x \sim_P y$ if and only if $\psi(x) = \psi(y)$, and call it the *(d-invariant) lamination (of P)* [7]. Clearly, equivalence classes of $\sim_P$ are pairwise *unlinked* (i.e., their Euclidean convex hulls are disjoint). The quotient space $\mathbb{S}^1/\sim_P = J_{\sim_P}$ is homeomorphic to $J_P$ and the map $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ *induced* by $\sigma$ is topologically conjugate to $P|_{J_P}$. The set $J_{\sim_P}$ is a topological (combinatorial) model of $J_P$ and is called the *topological Julia set*. The induced map $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ serves as a model for $P|_{J_P}$ and is often called a *topological polynomial*. Moreover, one can extend the conjugacy between $P|_{J_P}$ and $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ to a conjugacy on the entire plane. Fig. 1 shows the Julia set called “the Douady rabbit” and the corresponding lamination.

#### 2.1.3. Laminations in the connected case

In his fundamental paper [23] Kiwi extended these ideas to the case of a polynomial $P$ of degree $d \geq 2$ without irrationally indifferent periodic points, *not requiring* that $J_P$ be locally connected. In the case when $J_P$ is connected, he constructed a $d$-invariant lamination $\sim_P$ on $\mathbb{S}^1$ such that $P|_{J_P}$ is semiconjugate to the induced map $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ by a monotone map $m : J_P \to J_{\sim_P}$ (a map is *monotone* if all points have connected preimages). Kiwi's results were extended to *all* polynomials with connected Julia sets in [5]. Equivalences $\sim$ similar to $\sim_P$ can be defined abstractly, without any polynomials. Then they are called *(d-invariant) laminations* and still give rise to similarly constructed *topological Julia sets* $J_{\sim_P}$ and *topological polynomials* $f_{\sim_P}$.

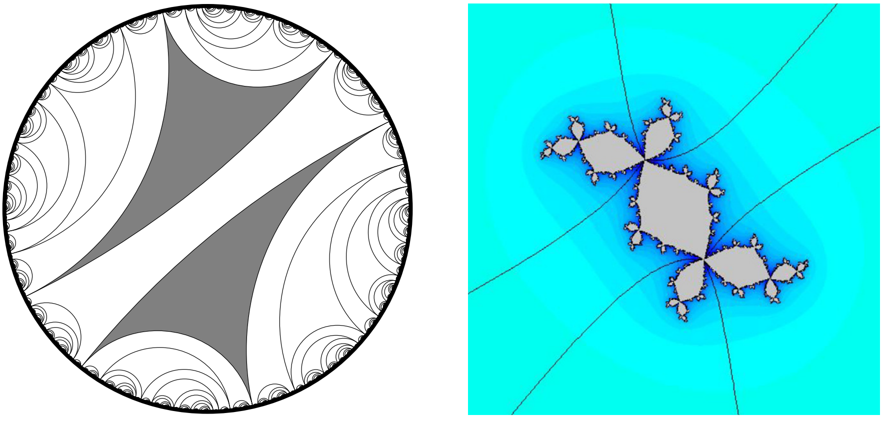

**Fig. 1.** The Douady rabbit and its lamination.

**Theorem 2.1.** *(See [5].) Let $P$ be a polynomial with connected Julia set $J_P$. Then there exists an essentially unique monotone map $\varphi$ of $J_P$ onto a locally connected continuum which is finest in the sense that for* any *monotone map $\psi : J_P \to J'$ onto a locally connected continuum there exists a monotone map $h$ with $\psi = h \circ \varphi$. Moreover, there exists an invariant lamination $\sim_P$ such that $\varphi(J_P) = J_{\sim_P}$ and the map $\varphi$ semiconjugates $P|_{J_P}$ and the topological polynomial $f_{\sim_P}|_{J_{\sim_P}}$.*

In this construction, big pieces of $J_P$ may collapse under $\varphi$. In fact, [5] contains a criterion for the finest map $\varphi$ from Theorem 2.1 to not collapse all of $J_P$ to a point as well as examples of polynomials for which $\varphi(J_P)$ is a point. This shows that the notion of an invariant lamination cannot be applied to all polynomials, even with connected Julia set.

*2.1.4. Geometric prelaminations: Thurston's approach*

The above shows the limitations of the approach based upon laminations as equivalences on $\mathbb{S}^1$. Therefore, in the present paper, we use Thurston's original approach [51] which was different. Instead of equivalences on $\mathbb{S}^1$, Thurston considered closed families of chords in $\overline{\mathbb{D}}$ with certain invariance properties. More precisely, for $A \subset \mathbb{S}^1 \subset \mathbb{C}$, let $\mathrm{Ch}(A)$ be the hyperbolic convex hull of $A$. If $A$ is a $\sim$-class, then call a chord $ab$ (with endpoints $a$ and $b$) on the boundary of $\mathrm{Ch}(A)$ a *leaf*; we allow for $a = b$ and then call the leaf *degenerate* (cf. [51]). Using equivalence classes $A$ of an equivalence relation $\sim$ we get in this way a collection of leaves generated by $\sim$. Thurston's idea was to study collections of leaves abstractly, i.e., without assuming that they are generated by an equivalence relation with specific properties defined on the circle.

**Definition 2.2.** (Cf. [51].) A *geometric prelamination* $\mathcal{L}$ is a set of chords in the closed unit disk $\overline{\mathbb{D}}$ such that any two distinct chords from $\mathcal{L}$ meet at most in an endpoint of both of them. Also, $\mathcal{L}$ is called a *geometric lamination* (*geo-lamination*) if $\bigcup \mathcal{L}$ is closed.

Chords in a geometric prelamination are called *leaves*. If $\mathcal{L}$ is a geo-lamination then $\mathcal{L}^+ = \bigcup \mathcal{L} \cup \mathbb{S}^1$ is a continuum. A geo-lamination can be obtained if we construct a geometric prelamination $\mathcal{L}$ and then add all chords that are limits of sequences of chords from $\mathcal{L}$. Denote the new family of chords by $\overline{\mathcal{L}}$; it is easy to see that $\overline{\mathcal{L}}$ is a geo-lamination.

A *gap* of a geometric prelamination $\mathcal{L}$ is the closure (in $\mathbb{C}$) of a component of $\mathbb{D} \setminus \bigcup \mathcal{L}$ that has interior points. The boundary of a gap consists of leaves in $\overline{\mathcal{L}}$ and points in $\mathbb{S}^1$. The *basis* of a gap or leaf $G$ is $G' = G \cap \mathbb{S}^1$. A gap is *finite* if its basis is finite (i.e., if the gap is a polygon), and *infinite* otherwise. For a closed subset of $\mathbb{S}^1$, we call its convex hull a *(degenerate) leaf or gap* even if it is not coming from any lamination. Slightly abusing the language, we often identify a gap and its basis, or a gap and its boundary. Note that gaps and leaves of an *invariant* lamination have additional properties specified in Definition 2.3.

*2.1.5. Geometric prelaminations and dynamics*

We extend $\sigma$ to $\sigma^* : \overline{\mathcal{L}}^+ \to \overline{\mathbb{D}}$ by mapping each leaf $\ell = ab \in \overline{\mathcal{L}}$ linearly onto the chord $\sigma(a)\sigma(b)$. For a (degenerate) leaf $\ell$, we define $\sigma(\ell)$ as $\mathrm{Ch}(\sigma(\ell'))$.

**Definition 2.3.** (Cf. [51].) A geometric prelamination $\mathcal{L}$ of degree $d$ is said to be *invariant* if the following conditions are satisfied:

(1) (Leaf invariance) For each leaf $\ell \in \mathcal{L}$, $\sigma(\ell)$ is a (degenerate) leaf in $\mathcal{L}$ and, if $\ell$ is non-degenerate, there exist $d$ pairwise disjoint leaves $\ell_1, \dots, \ell_d$ in $\mathcal{L}$ such that for each $i$, $\sigma(\ell_i) = \ell$.
(2) (Gap invariance) For a gap $G$ of $\mathcal{L}$, $\mathrm{Ch}(\sigma(G'))$ is (1) a (degenerate) leaf, or (2) the boundary of a gap $H = \mathrm{Ch}(\sigma(G'))$ of $\mathcal{L}$ and $\sigma^*|_{\mathrm{Bd}(G)} : \mathrm{Bd}(G) \to \mathrm{Bd}(H)$ is a *positively oriented composition of a monotone map and a covering map*. We consider $\sigma(G) = \mathrm{Ch}(\sigma(G'))$ as *defined* only if (1) or (2) is satisfied.

If a geometric prelamination $\mathcal{L}$ satisfies conditions (1)–(2) except for the last part of (1), it is called *forward invariant*. By Thurston [51] if $\mathcal{L}$ is invariant or forward invariant, then $\overline{\mathcal{L}}$ is an invariant or forward invariant geo-lamination. A leaf or gap $G$ is *critical* if $\sigma(G)$ is defined and the map $\sigma^*$ on $\mathrm{Bd}(G)$ (equivalently, if $\sigma|_{G'}$) is not one-to-one.

**Definition 2.4.** Let $\mathcal{C}$ be a collection of pairwise disjoint leaves and gaps such that for every element $G \in \mathcal{C}$, $\sigma(G) \in \mathcal{C}$. Let $\mathcal{L}$ be the set of all leaves in $\mathcal{C}$, all boundary leaves of gaps in $\mathcal{C}$, and of all points in $\mathbb{S}^1$. Then $\mathcal{L}$ is a forward invariant geometric prelamination, $\overline{\mathcal{L}}$ is a forward invariant geometric lamination, and $\mathcal{C}$ is called a *generating family of* $\mathcal{L}$ (or of $\overline{\mathcal{L}}$).

For an element $G$ of $\mathcal{L}$ or $\overline{\mathcal{L}}$ we can talk about its image as either $\sigma(G)$ or $\sigma^*(G)$, and we will use these notations interchangeably. A gap is *periodic* if some iterate of $\sigma$ maps the basis of the gap *into* itself. If $G \in \mathcal{C}$ and $\sigma^n(G) \subset G$ then it follows from the

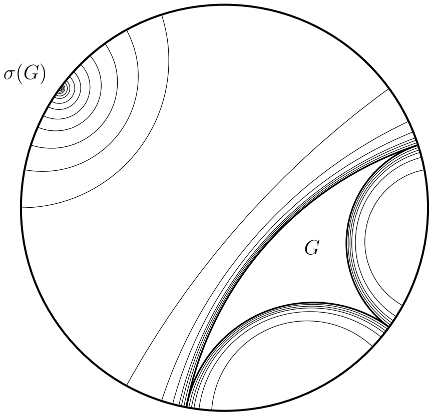


**Fig. 2.** An all-critical triangle.

definition that $\sigma^n(G) = G$. A leaf of $\overline{\mathcal{L}}$ which is the limit of other leaves of $\mathcal{L}$ from one or both sides is called a (one-sided or two-sided) *limit leaf*. A leaf that is not a limit leaf on either side is called *isolated*. If a leaf is not a two-sided limit leaf, then it is a boundary leaf of a gap. We use the term *gap-leaf* for a gap, or a two-sided limit leaf, or a degenerate leaf that is the limit of non-degenerate leaves which separate it from the rest of $\mathbb{S}^1$. Call a gap-leaf $G$ *all-critical* if $\sigma(G)$ is a point. Fig. 2 shows an all-critical triangle with the edges which are one-sided limit leaves.

**Lemma 2.5.** *Let $\mathcal{C}$ be a generating family of a geometric prelamination $\mathcal{L}$ with no critical leaves in $\mathcal{L}$. Then the following claims hold.*

(1) *Let $\ell$ be a critical leaf of $\overline{\mathcal{L}}$. Then $\ell$ is a boundary leaf of an all-critical gap-leaf $G$ of $\overline{\mathcal{L}}$ all boundary leaves of which are limit leaves, $\sigma(G)$ is a point not belonging to any gap or non-degenerate leaf of $\overline{\mathcal{L}}$ and separated from the rest of $\mathbb{S}^1$ by a sequence of leaves of $\mathcal{L}$, and so $\sigma^n(G) \cap G = \emptyset$ for every $n > 0$.*
(2) *If $(\sigma^*)^n(H) \subset H$ for a leaf or gap $H$ of $\overline{\mathcal{L}}$, then $(\sigma^*)^n(H) = H$.*
(3) *If $G$ is a (pre)periodic gap-leaf of $\overline{\mathcal{L}}$ that is not all-critical for $\sigma^n$ for any $n$, then all leaves in* $\mathrm{Bd}(G)$ *are non-(pre)critical and (pre)periodic (in particular, this holds if $G$ is infinite). Moreover, there are at most finitely many periodic leaves in* $\mathrm{Bd}(G)$.

**Proof.** (1) Since $\mathcal{L}$ contains no critical leaves, $\ell$ is the limit leaf of a sequence of leaves $\ell_i$ disjoint from $\ell$. Clearly, $\ell \in \overline{\mathcal{L}}$ lies on the boundary of a gap-leaf $G$ of $\overline{\mathcal{L}}$ and $\ell_i \cap G = \emptyset$. If $\sigma(G)$ is not a point, then $\sigma(\ell_i)$ either cross a leaf in the boundary of $\sigma(G)$, or intersect the interior of $\sigma(G)$, a contradiction (this is where the invariantness of the lamination is used). Hence $G$ is all-critical, and $\sigma(G)$ is separated from the rest of the circle by a sequence of leaves of $\mathcal{L}$. Since the same argument applies to all leaves in $\mathrm{Bd}(G)$, they are all limit leaves. This implies the rest of the lemma (e.g., if the point $\sigma^n(G)$ belongs to a gap or leaf $Q$ of $\overline{\mathcal{L}}$, the leaves $\sigma^n(\ell_i)$ will cross $Q$, a contradiction).

(2) Suppose that $\sigma(H) \subsetneqq H$ (the arguments for $n > 1$ are similar). If there are critical leaves (of any power of $\sigma$) in $\mathrm{Bd}(H)$ then by (1) $H$ is all-critical and $\sigma^n(H) \cap H = \emptyset$ for all $n$. Hence we may assume that $H$ *is a gap without critical leaves in its boundary*. Since $\sigma(H) \subsetneqq H$, $\sigma(H) = \alpha\beta = \sigma(\alpha\beta)$ is an invariant leaf in $\mathrm{Bd}(H)$ and so $H$ is finite. We may assume that $\sigma(\alpha) = \alpha, \sigma(\beta) = \beta$. If there are no limit leaves in $\mathrm{Bd}(H)$, then $H$ is an invariant gap from $\mathcal{C}$, hence $\sigma(H) = H$. So, there are limit leaves in $\mathrm{Bd}(H)$ and $\alpha\beta$, which is their image, is also a limit leaf. If the leaf $\beta\gamma \in \mathrm{Bd}(H)$, adjacent to $\alpha\beta$, is a limit leaf, then images of leaves, approaching $\beta\gamma$, will cross $H$, a contradiction. Hence $\beta\gamma$ is isolated in $\overline{\mathcal{L}}$ and so $\beta\gamma \in \mathcal{L}$. Since there are no critical leaves in $\mathrm{Bd}(H)$, $\sigma(\beta\gamma) = \alpha\beta \in \mathcal{L}$. By Definition 2.4 we conclude that there is an element of $\mathcal{C}$ which contains vertices $\alpha, \beta, \gamma$ and has to coincide with $H$. This implies that $\sigma^*(H) = \sigma(H) = H$, a contradiction.

(3) If there is a (pre)critical leaf in $\mathrm{Bd}(G)$ then by (1) $G$ is all-critical for a power of $\sigma$, a contradiction. Consider $n$ with $(\sigma^*)^n(G)$ periodic of period $m$. Clearly, it is enough to prove the rest of lemma for $(\sigma^*)^n(G) = H$. By (2) and above all leaves in $\mathrm{Bd}(H)$ stay in $\mathrm{Bd}(H)$ under $\sigma^m$ and are all (pre)periodic. We show that they belong to the backward orbits of finitely many periodic leaves. Indeed, any leaf from $\mathrm{Bd}(H)$ of length less than some $\varepsilon(m) = \varepsilon > 0$ increases its length under $\sigma^m$. Since for geometric reasons there are finitely many leaves in $\mathrm{Bd}(H)$ of length greater than $\varepsilon$ and no leaf ever collapses, then for any leaf $\ell$ in $\mathrm{Bd}(H)$ there is a moment right before the length of the leaf drops, and by the above at this moment the image of $\ell$ is a leaf $\ell' \in \mathrm{Bd}(H)$ of length greater than $\varepsilon$. Thus, all leaves in $\mathrm{Bd}(H)$ pass through a finite collection of leaves and are therefore (pre)periodic; moreover, there are at most finitely many periodic leaves in $\mathrm{Bd}(H)$ as desired. The claim about infinite gaps follows from a Theorem of Kiwi [22] by which all infinite gaps are (pre)periodic. $\square$

### *2.2. Hedgehogs*

The contents of the first two paragraphs of this subsection are due to Perez-Marco [37,38]. Consider an irrationally indifferent periodic point $q$ of period 1 and let $\Delta$ be $q$ (in the Cremer case) or the maximal open Siegel disk (in the Siegel case). Suppose that $U$ is a simply connected neighborhood of $\overline{\Delta}$ such that $\overline{U}$ contains no critical point. The *hedgehog* $H(U)$ is defined as the component containing $\Delta$ of the set of all points for which the whole orbit stays in $\overline{U}$ [37,38]; it has the property that $H(U) \cap \mathrm{Bd}(U) \neq \emptyset$. If $\Delta$ is a Siegel disk with a critical point on the boundary, then there are no hedgehogs. In the rest of this subsection $H$ denotes a hedgehog.

It is known that $\mathrm{Bd}(H) \subset J_P$. A hedgehog contains no periodic points other than $q$. Hence if an invariant non-separating continuum contains an irrationally indifferent periodic point and another periodic point, it contains a critical point. Also, $P|_H$ is *recurrent*: there is a sequence $m_n \to \infty$ with $P^{m_n}|_H$ converging uniformly to the identity on $H$. Moreover, the map $P|_H$ is transitive, i.e. there is a dense $G_\delta$-subset of $H$ consisting of points with dense orbits in $H$.

Two hedgehogs intersect only if they are generated by the same $\Delta$; in this case their union is another hedgehog of the same $\Delta$. The *mother hedgehog* $M_q$ [12] is the union of $\overline{\Delta}$ and the closure of the union of all hedgehogs containing $\Delta$. Thus, $M_q$ is *always* non-empty – if there are no true hedgehogs, $M_q = \overline{\Delta}$ (this occurs for a Siegel disk $\Delta$ containing critical points in its boundary). In the Cremer case, $\mathrm{Bd}(M_q) = M_q$.

If the period of $q$ is greater than 1, everything is analogous. Thus, for each point $y$ of $Q = \mathrm{orb}\, q$ its mother hedgehog $M_y$ is defined and invariant under the appropriate power of $P$. The union $M_Q = \cup_{y\in Q} M_y$ is called the *mother hedgehog* of $Q$; clearly, $P(M_Q) = M_Q$.

### 2.3. Continuum theory preliminaries

Here we introduce a few basic notions of Carathéodory's prime end theory (see [34, 39]) and state a continuum theory result from [6]. Let $X$ be an unshielded continuum. A *crosscut* of $X$ (or of $T(X)$) is the image $C \subset U^\infty(X)$ of $(0,1)$ under an embedding $\psi : [0,1] \to \mathbb{C}$ with $\psi(0) \neq \psi(1) \in X$ and $\psi((0,1)) \subset U^\infty(X)$. Let $\mathrm{Sh}(C)$ (the *shadow of* $C$) be the bounded component of $U^\infty(X) \setminus C$.

As above, let $\Psi_X : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to U^\infty(X)$ be a conformal isomorphism with $\Psi_X(\infty) = \infty$ and such that $\Psi'_X(z)$ has a positive real limit as $z \to \infty$. To each angle $\alpha \in \mathbb{S}^1$ we associate the *(conformal) external ray* $R_\alpha$ as the $\Psi_X$-image of the infinite radial segment $\{(1,\infty)e^{2\pi i\alpha}\}$. The *principal set* (or *limit set*) of the ray $R_\alpha$ is the set $\mathrm{Pr}(\alpha) := \overline{R_\alpha} \setminus R_\alpha$. If $\mathrm{Pr}(\alpha) = \{z\}$ is a singleton, then we say that the ray $R_\alpha$ *lands* at $z$. If $X = K_P$ is the filled-in Julia set of a polynomial $P$ of degree $d \geq 2$ and $K_P$ is connected, then $P(R_\alpha) = R_{\sigma(\alpha)}$. In this case, every periodic ray lands at a periodic point of $J_P$, and every repelling or parabolic periodic point in $J_P$ is the landing point of a positive finite number of rays, all of them with the same period [13,34].

For any $\alpha \in \mathbb{S}^1$ there exist two sequences $\beta_1 < \beta_2 < \cdots < \cdots < \gamma_2 < \gamma_1$ of angles-arguments of landing rays with $\lim \beta_i = \lim \gamma_i = \alpha$ such that the landing points of $R_{\alpha_i}$ and $R_{\gamma_i}$ can be joined by a crosscut $Q_i$ with $\mathrm{diam}(Q_i) \to 0$ [34, Lemma 17.9]. The *impression of the ray* $R_\alpha$ (or of the angle $\alpha$) is defined as the set $\mathrm{Imp}(\alpha) = \bigcap \overline{\mathrm{Sh}(Q_i)}$; it does not depend on the sequences $\beta_i$ and $\gamma_i$. Alternatively, the impression $\mathrm{Imp}(\alpha)$ is the set of all limit points of sequences $z_i = \Psi_X(y_i) \in U^\infty(X)$ where $y_i \in \mathbb{C} \setminus \overline{\mathbb{D}}$ are points with $y_i \to \alpha \in \mathbb{S}^1$.

A point $z \in \mathrm{Bd}(X)$ is *accessible* if there exists an injective curve $l\colon [0,1] \to \mathbb{C}$ with $l([0,1)) \subset U^\infty(X)$ and $l(1) = z$. For any injective curve $l\colon [0,1) \to U^\infty(X)$ with $l(t) \to X$ as $t \to 1$, one can define the principal set $\mathrm{Pr}(l) = \overline{l} \setminus l$ as above.

Fig. 3 illustrates the above introduced notions. The ray $R_\alpha$ lands at the point with coordinates $(0,1)$, so $\mathrm{Pr}(\alpha) = \{(0,1)\}$ and the point $(0,1)$ is accessible. However it is easy to see that the impression $\mathrm{Imp}(\alpha)$ of $\alpha$ is the segment connecting $(0,0)$ and $(0,1)$.

By a *CS-point* we mean either a periodic Cremer point or a periodic Siegel point.

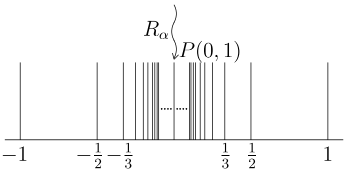


**Fig. 3.** The principal $\mathrm{Pr}(\alpha)$ and the impression $\mathrm{Imp}(\alpha)$ are not the same.

**Theorem 2.6.** *(See [6, a short version of Theorem 4.2].) Let $X \subset J_P$ be a non-separating invariant continuum. If all fixed points in $X$ are repelling or parabolic and* all rays landing at them are fixed *then $X$ is a fixed point. In particular, if all periodic points in $X$ are repelling or parabolic and the number of periodic points in $X$, at which at least two external rays land, is finite, then $X$ is a point. Also, if $X$ is non-degenerate then either $X$ contains a fixed CS-point, or $X$ contains a repelling or parabolic fixed point at which non-fixed rays land.*

### *2.4. Wandering gaps*

Suppose that $A \subset \mathbb{S}^1$ is a finite set with $|A| > 2$ such that (1) all sets $A$, $\sigma(A), \dots$ have pairwise disjoint convex hulls, (2) $\sigma^n \colon A \to \sigma^n(A)$ is injective for all $n \geq 1$, and (3) the sets $\mathrm{Ch}(\sigma^n(A))$ satisfy gap invariance so that we can define images of $\mathrm{Ch}(A)$ under powers of $\sigma$ (see Definition 2.3(2)); then the set $\mathrm{Ch}(A)$ is called a *wandering gap* (here we talk about gaps *in the absence* of a lamination). Thus, in the definition we already assume that $A$ is *non-(pre)critical*. A collection of finite gaps is *wandering* if all *all* images of *all* gaps have disjoint convex hulls. In particular, if $x$ is a wandering non-(pre)critical branch point of a locally connected Julia set, then the external angles of the rays that land at $x$ form a wandering gap.

By the No Wandering Triangle Theorem of Thurston [51], in the quadratic case there are no wandering gaps; Thurston posed the problem of extending this to the higher degree case and emphasized its importance. The theorem was instrumental in the construction of a combinatorial model of the *Mandelbrot set* $\mathcal{M}$ [51]. The next result is due to Kiwi [22]; it says that in an invariant lamination of degree $d$ a *wandering gap* intersects the circle over *at most $d$ angles*. Then in [7] it was proven that for a non-empty wandering collection $\mathcal{B}_{\mathbb{D}}$ of gaps $G_i$ we have $\sum_{\mathcal{B}_{\mathbb{D}}}(|G'_i| - 2) + N' \leq d - 2$ where $N'$ is the number of cycles of infinite gaps in the lamination.

In [4] the role of recurrent critical points in the dynamics of wandering gaps was studied in the cubic case. In [11] the results of [4] were generalized. We need a few definitions. Given a wandering gap $B$, a *limit leaf* of $B$ is a leaf which is a limit of a sequence of convex hulls of images of $B$. Let $L^B_{\mathrm{lim}}$ be the family of such limit leaves of $B$. Clearly, $L^B_{\mathrm{lim}}$ is a forward invariant geo-lamination. Also, a chord $ab$ is called *recurrent* if at least one of its endpoints is recurrent, and *critical* if $\sigma(a) = \sigma(b)$.

**Theorem 2.7.** *(See [11].) Consider a non-empty wandering collection of gaps $G_1, \dots, G_s$. Then the following holds.*

(1) *For each $G_i$ there exist at most $|G'_i| - 1$ recurrent critical chords $t^i_j \in L^{G_i}_{\lim}, 1 \le j \le |G'_i| - 1$ with pairwise disjoint infinite orbits and the same limit set $\omega_i$.*
(2) *For each leaf $\ell \in L^{G_i}_{\lim}$ we have $\ell \cap \omega_i \neq \emptyset$.*
(3) *Let $k'$ be the maximal number of recurrent critical chords from $\bigcup_{i=1}^s L^{G_i}_{\lim}$ with pairwise disjoint orbits. Let $l$ be the number of their distinct $\omega$-limit sets. Then*

$$\sum_{i=1}^{s}(|G'_i| - 2) \le k' - l \le d - 1 - l \le d - 2.$$

## 3. The tools, or disk to plane and back again

In Sections 3, 4 and 5, unless explicitly stated otherwise, *we consider a polynomial $P$ of degree $d$ with **connected** Julia set $J_P$*. We use the following terminology and notation. Recall that irrationally indifferent periodic points are called *CS-points* (i.e., *Cremer* points or *Siegel* points). Also, let $\mathfrak{R}$ be the set of all repelling or parabolic periodic bi-accessible points and their iterated preimages. Let $Y \subset Z$ be two continua (not necessarily subsets of any Julia set). Define $\mathrm{val}'_Z(Y)$ as the number of components of $Z \setminus Y$, and call $Y$ a *cut-continuum of $Z$* if $\mathrm{val}'_Z(Y) > 1$ (i.e., $Z \setminus Y$ is not connected).

Section 3 prepares tools for the rest of the paper. In Subsection 3.1 we show that wandering cut-continua in $J_P$ contain the principal sets of finitely many rays. This creates cuts of the plane. In Subsection 3.2 we consider these cuts, and cuts created by rays landing at points in $\mathfrak{R}$. We associate to them convex hulls of sets of arguments of rays with principal sets in a wandering cut-continua *or* in a point of $\mathfrak{R}$; the boundary leaves of these convex hulls form a geometric prelamination. Cuts of the plane allow us to define *fibers*, i.e. intersections of *closed wedges* created by cuts. This generalizes the notion of fibers as in [46]: in the latter reference, fibers were defined using pairs of dynamic rays that land at common points, and intersecting subsets of the filled Julia set that are not separated by such ray pairs. On the other hand, the parallel construction in the disk allows us to define subsets of the disk corresponding to such fibers. This correspondence plays an important role in what follows.

### *3.1. Wandering continua and their rays*

For a continuum $Z \subset J_P$, let $A(Z)$ be the set of all angles whose rays have principal sets in $Z$. Let $\mathrm{Tail}(Z)$ be the union of $Z$ and all rays with arguments in $A(Z)$ (thus, if there are no rays with principal sets in $Z$, then $\mathrm{Tail}(Z) = Z$). Clearly, in the case when $A(Z) \neq \emptyset$ the set $\mathrm{Tail}(Z)$ is an unbounded connected set which is closed if $A(Z)$ is finite. Also, by $\mathrm{Tail}'(Z)$ we denote the union of $Z$ and long bounded segments of rays with arguments in $A(Z)$ (to get $\mathrm{Tail}'(Z)$, on each ray we choose a point and remove the unbounded segment of this ray to infinity). Note that $|A(Z)| = \mathrm{val}_{J_P}(Z)$ (recall that $\mathrm{val}_{J_P}(Z)$ is defined as the number of external rays to $J_P$ with limit sets in $J_P$).

**Lemma 3.1.** *Let $X$ be an unshielded continuum and $K \subset X$ be a cut-continuum of $X$ which does not separate the plane. If $\mathrm{val}'_X(K) \geq n > 1$, then there are $n$ distinct external rays to $X$ with principal sets in $K$. If $\mathrm{val}_X(K) < \infty$, then $\mathrm{val}'_X(K) = \mathrm{val}_X(K) = m$. If $A(K) = \{\alpha_1 < \alpha_2 < \cdots < \alpha_m < \alpha_{m+1} = \alpha_1\}$ in the sense of the cyclic order, then components $C_j$ of $X \setminus K$ can be numbered so that $C_j$ corresponds to $I_j = (\alpha_j, \alpha_{j+1})$ in the sense that for any $\beta \in I_j$ we have $\mathrm{Imp}(\beta) \subset \overline{C_j} \subset C_j \cup K$, and $\mathrm{Pr}(\beta) \cap C_j \neq \emptyset$.*

**Proof.** First *we show that if $\mathrm{val}'_X(K) \geq n$ then there are at least $n$ external rays with principal sets in $K$*. Collapse $K$ to a point and denote the corresponding collapsing map $\psi$. By the Moore Theorem [35], the resulting topological space is still the plane on which $k = \psi(K)$ is a cutpoint of $\psi(X)$. By a nice result of McMullen (see Theorem 6.6 of [32]), if there are $n > 1$ components of $\psi(X) \setminus k$, then there are at least $n$ external rays to $\psi(X)$ landing at $k$ (if $n = 1$ then there might exist *no* rays with principal sets in $K$). Their $\psi$-preimages are curves non-homotopic outside $X$ with principal sets in $K$. By Lindelöf's theorem (see, e.g., [40]) this implies that there exist at least $n$ external rays with principal sets in $K$.

*Let us now prove that if there are finitely many rays with principal sets in $K$ then their number equals* $\mathrm{val}'_X(K)$. Indeed, in this situation by the previous paragraph $\mathrm{val}'_X(K) = m < \infty$, and there are *at least* $m$ external rays with principal sets in $K$. Let us show that there are *exactly* $m$ such rays. Suppose otherwise. Then there must exist two external rays $R_1$ and $R_2$ with principal sets in $K$ such that one of the wedges formed by $R_1, R_2$ and $K$ contains no points of $X$ while the other wedge contains $X \setminus K$. This implies that *all* external rays contained in the first wedge will have their principal sets in $K$. Since there are infinitely many of them, we get a contradiction with the assumption.

Let us introduce the notation which we need to complete the proof. Namely, let the set of arguments of the rays with principal sets in $K$ be $A(K) = \{\alpha_1 < \alpha_2 < \cdots < \alpha_m < \alpha_{m+1} = \alpha_1\}$ and set $I_j = (\alpha_j, \alpha_{j+1})$.

*Now we show that there is a unique component $C = C_j$ of $X \setminus K$ such that for any angle $\beta \in I_j$ we have $\mathrm{Pr}(\beta) \cap C \neq \emptyset$ and $\mathrm{Imp}(\beta) \subset \overline{C_j} \subset C_j \cup K$*. Denote by $E_j$ the open wedge formed by the rays $R_{\alpha_j}, R_{\alpha_{j+1}}$ and the continuum $K$, such that $E_j$ contains rays of angles from $I_j$. Then there is at least one component of $X \setminus K$ in $E_j$ (otherwise, as in the second paragraph of the proof, infinitely many angles from $I_j$ will have principal sets in $K$, a contradiction). Since $\mathrm{val}'(K) = m$, there is a *unique* component $C_j$ of $X \setminus K$ in $E_j$. Since none of the angles $\beta \in I_j$ can have the principal set inside $K$, $\mathrm{Pr}(\beta) \cap C_j \neq \emptyset$. To see that $\mathrm{Imp}(\beta) \subset \overline{C_j} \subset C_j \cup K$, choose two sequences of angles $\theta_i < \beta < \gamma_i$ such that the rays $R_{\theta_i}, R_{\gamma_i}$ land and connect their landing points $x_i, y_i \in C_j$ with crosscuts $T_i$ forming a fundamental chain of crosscuts. It follows that $\mathrm{Imp}(\beta) = \cap \overline{\mathrm{Sh}(T_i)} \cap X \subset \overline{C_j} \subset C_j \cup K$. $\square$

Observe that Theorem 6.6 of [32] cannot be extended to show that the number of components of $X \setminus \{x\}$ for a cutpoint $x$ always equals the cardinality of the number of external rays to $X$ landing on $x$. E.g., a cone over a Cantor set has a vertex of uncountable valence at which only countably many external rays land. Also, easy examples show that $\mathrm{val}'_X(K)$ can be finite while $|A(K)|$ is uncountable (for example consider an arc $I$ containing a non-degenerate subarc $K$ not containing an endpoint of $I$).

**Lemma 3.2.** *Suppose that $X \subset J_P, Y \subset J_P$ are disjoint continua and there are closed sets $Q \subset A(X), T \subset A(Y)$. Then $Q$ and $T$ are unlinked. Thus, if $A(X)$ and $A(Y)$ are finite, then they are unlinked.*

**Proof.** Clearly, $Q \cap T = \emptyset$. Hence if $Q, T$ are not unlinked, there must exist angles $\alpha, \beta \in Q$ and $\alpha', \beta' \in T$ which are pairwise distinct and such that the chord $\alpha\beta$ intersects the chord $\alpha'\beta'$. For geometric reasons this implies that $X$ and $Y$ intersect, a contradiction. □

Let us now go back to dynamics. If $Z \subset J_P$ is a point of $\mathfrak{R}$, then, by [13,16], $|A(Z)| = \mathrm{val}(Z)$ is finite. We show that wandering cut-continua are, as far as providing a tool for separating the plane and the Julia set, analogous to points of $\mathfrak{R}$. So, assume that $W$ is a wandering cut-continuum and study its dynamics.

**Lemma 3.3.** *If $P$ is a polynomial with arbitrary (perhaps, not connected) Julia set and $W \subset J_P$ is a wandering continuum, then $W$ does not separate the plane.*

**Proof.** If $W$ is separating, the set $T(W)$ contains a Fatou domain which must be (pre)periodic, contradicting the fact that $W$ is wandering. □

Let us now define the grand orbit of a wandering continuum $W$. Take a forward image $W'$ of $W$ so that $P^n(W'), n \geq 0$, contain no critical points. The pullbacks (i.e. components of $P^{-m}(P^k(W'))$) of sets from the forward orbit of $W'$ form the *grand orbit* $\Gamma(W)$ of $W$. The construction is necessary because of the following. Imagine that a forward image $P^m(W)$ of $W$ contains a critical point $c$, but is smaller than the one-step pullback of $P^{n+1}(W)$ containing $P^n(W)$ (i.e. $P^n(W)$ is not "symmetric" with respect to the naturally defined "symmetry" around $c$). Then there is an ambiguity in defining the element of the grand orbit of $W$ containing $P^n(W)$. Our definition allows us to avoid this ambiguity and is consistent because it does not depend on the choice of $W'$ (as long as it satisfies the conditions above).

**Lemma 3.4.** *Suppose that $W \subset J_P$ is a cut-continuum from the grand orbit of a wandering continuum. Then the map $P^n|_{\mathrm{Tail}(W)}$ is not one-to-one if and only if $W$ contains a critical point of $P^n$ (in this case there are two rays in* $\mathrm{Tail}(W)$ *mapped to one ray).*

**Proof.** By Lemma 3.1 the set $\mathrm{Tail}(W)$ includes some rays and is, therefore, non-degenerate. Suppose that $P^n|_{\mathrm{Tail}(W)}$ is not one-to-one. Note that $P^n(\mathrm{Tail}(W)) = \mathrm{Tail}(P^n(W))$. By Lemma 3.3, $P^n(\mathrm{Tail}'(W))$ is a non-degenerate continuum which does not separate the plane, and has no interior in the plane. Then by [19] there is a critical point $c$ of $P^n$ in $\mathrm{Tail}(W)$. Since $J_P$ is connected this implies that in fact $c \in W$.

Now, suppose that there is a critical point $c$ of $P^n$ in $W$. Collapse $W$ and $P^n(W)$ by a map $\psi$ of the plane to points $a$ and $b$. Consider the induced map $g$ from a neighborhood of $a$ to a neighborhood of $b$. Since $c \in W$ is a critical point of $P^n$, the map $g$ is $k$-to-1 with $k > 1$. Take a ray $R$ from $\mathrm{Tail}(W)$, map it forward by $P^n$, and then take all rays which are preimages (pullbacks) of $P^n(R)$. Then $\psi(P^n(R)) = g(\psi(R))$ has $k$ preimage-rays which land at $a$. Hence there are $k$ rays with principal sets in $W$ and the $P^n$-image of these $k$ rays is a single ray. □

Lemma 3.4 allows us to introduce the following notion.

**Definition 3.5.** A wandering continuum $K \subset J_P$ is said to be *non-(pre)critical* if $\mathrm{Tail}(K)$ has the following property: for every $n$ the map $P^n|_{\mathrm{Tail}(K)}$ is one-to-one. By Lemma 3.4, $K$ is non-(pre)critical if and only if $\mathrm{Tail}(K)$ contains no (pre)critical points.

By Lemma 3.4, $\mathrm{eval}_{J_P}(W)$ for a wandering continuum $W$ is well-defined and equals $\mathrm{val}_{J_P}(P^N(W))$ where $N$ is big enough to guarantee that $P^N(W)$ is non-(pre)critical. Also, the claim as in Lemma 3.4 holds for disconnected Julia sets too, and so literally the same definition as Definition 3.5 can be given in that case. However to prove Lemma 3.4 in the disconnected case we need to study in detail the family of external rays in that case, thus we postpone it until Section 6 (see Lemma 6.10).

**Corollary 3.6.** *Let $W \subset J_P$ be a wandering cut-continuum. Then $1 < m = \mathrm{val}_{J_P}(W) \le 2^d$, and there are exactly $m$ positively ordered angles $A(W) = \{\alpha_1 < \alpha_2 < \cdots < \alpha_m < \alpha_{m+1} = \alpha_1\}$ with principal sets in $W$. Also, if $W$ is non-(pre)critical, then $m \le d$, $\mathrm{Ch}(A(W))$ is wandering non-(pre)critical, and $|\sigma^k(A(W))| = m$ for any $k$.*

*In particular, if $Q$ is a wandering cut-continuum or a point of $\mathfrak{R}$, then there are finitely many rays with principal sets in $Q$ and $\mathrm{val}_{J_P}(Q) = \mathrm{val}'_{J_P}(Q)$.*

Recall that $\mathrm{val}'_{J_P}(Q)$ is the number of components of $J_P \setminus Q$.

**Proof.** First *let us show that there are at most $2^d$ external rays of $P$ with principal sets in $W$*. Indeed, otherwise there is a set $Q$ of $2^d + 1$ distinct external rays of $J_P$ whose principal sets are contained in $W$. Then the angles of $\sigma^m(Q)$ will have principal sets in $P^m(W)$ for every $m \ge 0$. Since $W$ is wandering, Lemma 3.2 now implies that all sets $\sigma^m(Q)$ are unlinked. However, by [22] this is impossible.

By Lemma 3.1 the existence and the desired properties of the set of angles $A(W) = \{\alpha_1 < \alpha_2 < \cdots < \alpha_m < \alpha_{m+1} = \alpha_1\}$ follow. Suppose that $W$ is non-(pre)critical; then by definition $\sigma^N|_{A(W)}$ is one-to-one for any $N$, $\mathrm{Ch}(A(W))$ is non-(pre)critical, and

$|\sigma^k(A(W))| = m$ is constant. By [22] this implies that $m \le d$. Finally, the last claim of the corollary follows from Lemma 3.1. $\square$

So, wandering cut-continua in $J_P$ contain the principal sets of finitely many rays and are in this respect analogous to repelling periodic points.

### *3.2. The correspondence between the plane and the disk*

In this subsection we consider cuts of the plane, generated by wandering cut-continua and/or by rays landing at points of $\mathfrak{R}$.

#### *3.2.1. Grand orbits of wandering collections*

We call a collection $\mathcal{B}_\mathbb{C} = \{B^1_\mathbb{C}, \dots, B^k_\mathbb{C}\}$ a *wandering collection of non-(pre)critical cut-continua* if $P^k(B^i_\mathbb{C}) \cap P^l(B^j_\mathbb{C}) = \emptyset$ unless $k = l$ and $i = j$. Take grand orbits $\Gamma(B^i_\mathbb{C})$, as defined right after Lemma 3.3, of the sets $B^i_\mathbb{C}$ and then the union $\Gamma(\mathcal{B}_\mathbb{C}) = \bigcup \Gamma(B^i_\mathbb{C})$, called the *grand orbit* of $\mathcal{B}_\mathbb{C}$. Observe that since the $B^i_\mathbb{C}$'s are non-(pre)critical, the construction of the grand orbit of $\Gamma(\mathcal{B}^i_\mathbb{C})$ is simplified in this case. Let $\Gamma^*(\mathcal{B}_\mathbb{C})$ be the union of all sets from $\Gamma(\mathcal{B}_\mathbb{C})$.

In the case of points of $\mathfrak{R}$ the construction of their grand orbits is easier than for wandering non-(pre)critical cut-continua; in fact, by definition the set $\mathfrak{R}$ is fully invariant, hence we can write $\mathfrak{R} = \Gamma(\mathfrak{R}) = \Gamma^*(\mathfrak{R})$. Let the collection of sets $\Gamma(\mathcal{B}_\mathbb{C}) \cup \mathfrak{R}$ be $\Gamma(\mathcal{B}_\mathbb{C}, \mathfrak{R})$ and the union of all points of these sets be $\Gamma^*(\mathcal{B}_\mathbb{C}, \mathfrak{R})$. For $Q \in \Gamma(\mathcal{B}_\mathbb{C}, \mathfrak{R})$, set $G(Q) = \mathrm{Ch}(A(Q))$.

#### *3.2.2. Some important prelaminations*

By Lemma 3.2, the sets $A(Q)$ with $Q \in \Gamma(\mathcal{B}_\mathbb{C}, \mathfrak{R})$ are pairwise unlinked, hence boundary chords of the sets $G(Q)$ with $Q \in \mathfrak{R}$ $(\Gamma(\mathcal{B}_\mathbb{C}, \mathfrak{R}), \Gamma(\mathcal{B}_\mathbb{C}))$ form a geometric prelamination $\mathcal{L}^\mathfrak{R}$ $(\mathcal{L}^{\mathcal{B}_\mathbb{C},\mathfrak{R}}, \mathcal{L}^{\mathcal{B}_\mathbb{C}})$. Say that the sets $G(Q)$ are *elements of the corresponding prelamination* (even though formally leaves in the boundaries of the sets $G(Q)$, and not the sets $G(Q)$ themselves, are elements of the prelaminations). The closures of these prelaminations are the geo-laminations $\overline{\mathcal{L}^\mathfrak{R}}$, $\overline{\mathcal{L}^{\mathcal{B}_\mathbb{C},\mathfrak{R}}}, \overline{\mathcal{L}^{\mathcal{B}_\mathbb{C}}}$. Observe that by construction all elements $Q$ of the grand orbit $\Gamma(\mathcal{B}_\mathbb{C}, \mathfrak{R})$ have valences greater than 1.

**Definition 3.7.** If we make a statement about *all* geometric prelaminations $\mathcal{L}^\mathfrak{R}$, $\mathcal{L}^{\mathcal{B}_\mathbb{C}}$, $\mathcal{L}^{\mathcal{B}_\mathbb{C},\mathfrak{R}}$, $\overline{\mathcal{L}^\mathfrak{R}}$, $\overline{\mathcal{L}^{\mathcal{B}_\mathbb{C}}}$, $\overline{\mathcal{L}^{\mathcal{B}_\mathbb{C},\mathfrak{R}}}$, we may jointly denote them by $\mathcal{L}$ or $\overline{\mathcal{L}}$. The collections $\Gamma(\mathfrak{R}) = \mathfrak{R}$, $\Gamma(\mathcal{B}_\mathbb{C})$, $\Gamma(\mathcal{B}_\mathbb{C}, \mathfrak{R})$ are sometimes jointly denoted by $\Gamma$ while sets $\mathfrak{R}, \Gamma^*(\mathcal{B}_\mathbb{C}), \Gamma^*(\mathcal{B}_\mathbb{C}, \mathfrak{R})$ are sometimes jointly denoted by $\Gamma^*$. If $\mathfrak{R} = \emptyset$, we take $\overline{\mathcal{L}^\mathfrak{R}}$ as the empty lamination with all leaves degenerate and a unique infinite gap coinciding with $\overline{\mathbb{D}}$.

Recall that a gap-leaf is *all-critical* if its $\sigma$-image is a singleton.

**Lemma 3.8.** *The following claims hold.*

(1) *There are no critical leaves in $\mathcal{L}$; in particular, there are no all-critical gap-leaves in $\mathcal{L}$.*
(2) *The only critical leaves of $\overline{\mathcal{L}}$ must belong to all-critical gap-leaves with all boundary leaves being limit leaves.*
(3) *Boundary leaves of any (pre)periodic gap-leaf are (pre)periodic.*

**Proof.** (1) Let us prove that there are no critical leaves in $\mathcal{L}$. Suppose that $\ell \in \mathcal{L}$ is a critical leaf. Then there is a set $Q \in \Gamma$ with $\ell = \alpha\beta \subset \mathrm{Bd}(G(Q))$. If $Q$ is a periodic point then it cannot be a critical point, hence $\sigma$ is one-to-one on $A(Q)$ and so a critical leaf cannot belong to the boundary of $G(Q)$. Similarly we deal with non-critical preperiodic points.

Let now $Q$ be a wandering continuum or a (pre)periodic critical point. Then by Lemma 3.4, there is a critical point in $Q$. On the other hand, by our assumption $|A(P(Q))| \geq 2$. Hence there is an angle $\gamma \neq \sigma(\alpha)$ whose ray has a principal set in $P(Q)$. Then by pulling back we can see that preimages of $\gamma$ separate preimages of $\sigma(\alpha)$ in $\mathbb{R}/\mathbb{Z}$. This shows that $\alpha$ cannot be adjacent to $\beta$ in $A(Q)$, a contradiction. The rest of the lemma is easy if the gap-leaf is finite and follows from Lemma 2.5 otherwise. □

*3.2.3. Disk to plane and back again*

In this subsection we establish a correspondence between certain subsets on the plane and of the disk. It is generated by the above introduced sets $Q$ and $G(Q)$, $Q \in \Gamma$. If need be, we use the superscript $\Gamma$ in our notation to indicate which family generates the introduced objects, yet mostly $\Gamma$ will be assumed to be fixed, so if it does not cause confusion we will not use $\Gamma$ in the notation. First we introduce a family of *planar cuts*.

**Definition 3.9** *(Planar cuts).* Let $\ell = \alpha\beta \in \mathcal{L}$ and $\alpha \neq \beta \in G'(Q)$ be adjacent angles from $G'(Q)$ where $Q \in \Gamma$. Denote the set $R_\alpha \cup R_\beta \cup Q$ by $\mathrm{Cut}^\ell$ and call it a *planar cut (centered at $Q$ and generated by $\ell$).*

Next we define *planar wedges.*

**Definition 3.10** *(Planar wedges).* Consider the set $W = \mathbb{C} \setminus \mathrm{Cut}^\ell$. Clearly, $W$ is an open set with two components each of which is called a *planar wedge (centered at $Q$ and generated by $\ell$)* and is denoted $W^\ell_{\mathbb{C}}$. By a *closed planar wedge (centered at $Q$ and generated by $\ell$)* $\widehat{W}^\ell_{\mathbb{C}}$ we mean the closure of $W^\ell_{\mathbb{C}} \cup Q$. *Hence, a closed planar wedge is* **not** *the closure of the corresponding (open) planar wedge.* All planar wedges described above are said to *border* on the cut $\mathrm{Cut}^\ell$ and to have $Q$ as their *center.* If $z \in \mathbb{C} \setminus \mathrm{Cut}^\ell$, the closed and open planar wedges defined by $\ell$ and containing $z$ are unique and are denoted by $W^\ell_{\mathbb{C}}(z)$ and $\widehat{W}^\ell_{\mathbb{C}}(z)$.

Fix $Q \in \Gamma$. Then for $z \in \mathbb{C} \setminus \mathrm{Tail}(Q)$ the component of $\mathbb{C} \setminus \mathrm{Tail}(Q)$ containing $z$ is denoted by $W^Q_{\mathbb{C}}(z)$ and is called an *open planar wedge centered at $Q$, containing $z$* (clearly, this is an open planar wedge centered at $Q$). Similarly, $\widehat{W}^Q_{\mathbb{C}}(z)$ is the closure

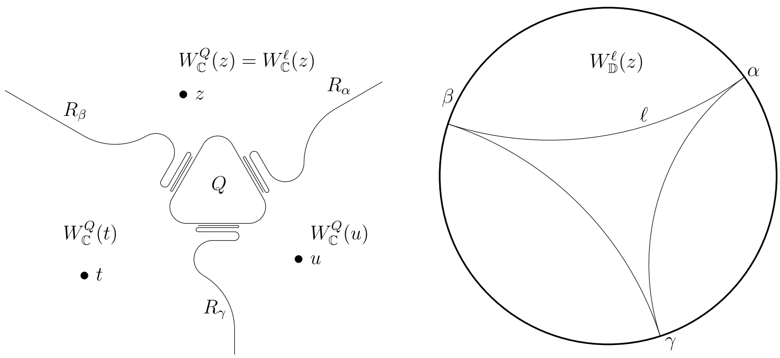


**Fig. 4.** A continuum $Q$ and its planar and disk wedges.

of $W_{\mathbb{C}}^{Q}(z) \cup Q$ and is called a *closed planar wedge centered at $Q$, containing $z$*. Thus, if $z \in K_P \setminus \Gamma^*$ then these wedges are well-defined for any $Q \in \Gamma$.

Fig. 4 shows planar wedges centered at a continuum $Q$ (we assume that $A(Q) = \{\alpha, \beta, \gamma\}$ is the set of all angles whose rays accumulate inside $Q$); it also shows the appropriate leaf $\ell$ on the boundary of the triangle in the unit disk corresponding to $Q$ and the appropriate disk wedge.

The definition of a *disk wedge* is similar to that of a planar wedge.

**Definition 3.11** *(Disk wedges).* Let $\ell = \alpha\beta \in \mathcal{L}$ and $\alpha, \beta \in G'(Q)$ where $Q \in \Gamma$. Let $W_{\mathbb{D}}^{\ell}$ be a component of $\mathbb{D} \setminus \ell$, called a *disk wedge*. Also, let $\widehat{W}_{\mathbb{D}}^{\ell}$ be the closure of $W_{\mathbb{D}}^{\ell}$ called a *closed disk wedge*. These disk wedges are said to be *centered* at $G(Q)$ (or at $Q$), and to *border* on $\ell$. If $z \in \mathbb{D} \setminus \ell$, then closed and open disk wedges defined by $\ell$ and containing $z$ are unique and are denoted by $W_{\mathbb{D}}^{\ell}(z)$ and $\widehat{W}_{\mathbb{D}}^{\ell}(z)$. If $z \in \mathbb{D} \setminus G(Q)$, then there exists a unique leaf $m$ in the boundary of $G(Q)$ which separates $G(Q) \setminus m$ from $z$. Then we define $W_{\mathbb{D}}^{Q}(z)$ as $W_{\mathbb{D}}^{m}(z)$ and define $\widehat{W}_{\mathbb{D}}^{Q}(z)$ as $\widehat{W}_{\mathbb{D}}^{m}(z)$.

Depending on what is known about a wedge, a superscript $Q$ or a superscript $\ell$ is used. Clearly, not only points $z$ but also sets $Y \subset \mathbb{C}$ can define wedges containing $Y$ which are denoted similarly to the above.

The correspondence between planar wedges and disk wedges is as follows: a planar wedge $W_{\mathbb{C}}^{\ell}$ and a disk wedge $W_{\mathbb{D}}^{\ell}$ are *associated (to each other)* if $W_{\mathbb{C}}^{\ell}$ contains rays with arguments coming from the boundary *circle* arc of $W_{\mathbb{D}}^{\ell}$. Associated planar and disk wedges will be denoted the same way except for the subscripts $\mathbb{C}$ and $\mathbb{D}$ respectively. Clearly, there are countably many planar wedges and countably many disk wedges (recall that all these wedges are associated to sets of $\Gamma$). Let us now define *disk blocks and fibers.*

**Definition 3.12** *(Disk blocks and fibers).* A non-empty intersection of finitely many closed disk wedges is said to be a *disk block*. A disk block is said to *border* on its *boundary leaves* which are defined in a natural way.

*Any* intersection $F_{\mathbb{D}}$ of closed disk wedges is called a *disk fiber* if it is minimal in the following sense: for any set $Q \in \Gamma$, either $G(Q)$ is disjoint from $F_{\mathbb{D}}$, or there are two adjacent angles $\alpha, \beta \in A(Q)$ such that the leaf $\alpha\beta$ is contained in $\mathrm{Bd}(F_{\mathbb{D}})$. For a disk fiber $F_{\mathbb{D}}$ we define its *basis* $F'_{\mathbb{D}} = F_{\mathbb{D}} \cap \mathbb{S}^1$ whose points are said to be *vertices* of $F_{\mathbb{D}}$.

Disk fibers are not necessarily disjoint, yet by Lemma 3.13 it is easy to see that if two non-degenerate disk fibers meet, than their intersection is a leaf from $\mathcal{L}$.

**Lemma 3.13.** *Non-degenerate disk fibers are exactly gap-leaves of $\overline{\mathcal{L}}$ and leaves of $\mathcal{L}$. Also, if $G$ is a disk fiber, then $G = \bigcap\{\widehat{W}^{\ell}_{\mathbb{D}}(G) \mid \ell \in \mathcal{L}\}$. Moreover, the $\sigma^*$-image of a disk fiber is a disk fiber.*

**Proof.** A leaf $\ell \in \mathcal{L}$ is a disk fiber because it is the intersection of the two closed wedges generated by $\ell$. Let $G$ be a gap-leaf of $\overline{\mathcal{L}}$ which is not an element of $\mathcal{L}$. Then, if $G$ is a leaf approximated from both sides by leaves of $\mathcal{L}$, the appropriate disk wedges generated by these leaves will have $G$ as their intersection. Suppose that $G$ is a gap of $\overline{\mathcal{L}}$. For each leaf $\ell \subset \mathrm{Bd}(G)$ which belongs to $\mathcal{L}$ choose $W^{\ell}_{\mathbb{D}}(G)$. For each $\ell \subset \mathrm{Bd}(G)$ which does not belong to $\mathcal{L}$ we can choose a sequence of leaves of $\mathcal{L}$ converging to $\ell$ from outside of $G$ and then the sequence of closed disk wedges generated by these leaves, all containing $G$. The intersection of the just constructed family of closed disk wedges is $G$, and clearly $G$ satisfies all the necessary properties, hence $G$ is a disk fiber.

On the other hand, let $G$ be a disk fiber which is neither a leaf of $\mathcal{L}$ nor a gap-leaf of $\overline{\mathcal{L}}$. Suppose that $G$ is a leaf. Since $G$ is not a gap-leaf of $\overline{\mathcal{L}}$, $G$ is a boundary leaf of a gap $H$ of $\overline{\mathcal{L}}$. Since $G$ is not a leaf of $\mathcal{L}$, it is the limit leaf of a sequence of leaves from outside of $H$. Again, since $G$ is not a leaf of $\mathcal{L}$, it follows that $H \subset \widehat{W}^{\ell}_{\mathbb{D}}(G)$ for every $\ell \in \mathcal{L}$, a contradiction with the assumption that $G$ is a fiber. Finally, assume that $G$ is not a leaf. Since by definition $G$ cannot contain any leaves of $\overline{\mathcal{L}}$ in its interior, $G$ must be a gap of $\overline{\mathcal{L}}$. The proof of the remaining two statements of the lemma is left to the reader. □

Now, to define *the planar fiber* of a point, we first define *planar blocks*.

**Definition 3.14** *(Planar blocks).* A non-empty intersection of finitely many closed planar wedges is said to be a *planar block*. In particular, a planar wedge is a planar block. A planar cut whose rays are contained in the boundary of a planar block, is called a *boundary cut (of the block)*, and the block is then said to *border* on its planar cuts.

**Definition 3.15** *(Planar fibers).* If $G$ is a disk fiber, then by Lemma 3.13 $G = \bigcap\{\widehat{W}^{\ell}_{\mathbb{D}}(G) \mid \ell \in \mathcal{L}\}$. If $\{\widehat{W}^{\ell}_{\mathbb{C}}(G)\}$ is the sequence of associated closed planar wedges, then we say that $F_{\mathbb{C}}(G) = \bigcap \widehat{W}^{\ell}_{\mathbb{C}}(G)$ is the *planar fiber of $G$ (or associated to $G$)*. Observe that if $G$ is a leaf $\ell \in \mathcal{L}$, then $F_{\mathbb{C}}(\ell) = \mathrm{Cut}^{\ell}$ and if $Q \in \Gamma$ and $G = G(Q)$, then $F_{\mathbb{C}}(G) = \mathrm{Tail}(Q)$.

Given a point $z \in \mathbb{C} \setminus \bigcup_{E \in \Gamma} \mathrm{Tail}(E)$ and $Q \in \Gamma$, there exists a unique planar wedge $\widehat{W}^{Q}_{\mathbb{C}}(z)$ which contains $z$. For such $z$ we denote by $F_{\mathbb{C}}(z)$ the *planar fiber of $z$*, the

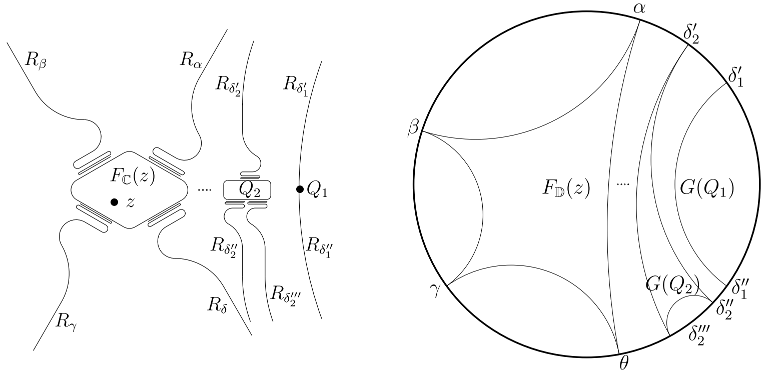


**Fig. 5.** A planar fiber and its associated disk fiber.

intersection of *all* the wedges $\widehat{W}_{\mathbb{C}}^{Q}(z)$. Moreover, for every planar wedge $\widehat{W}_{\mathbb{C}}^{Q}(z)$ let $\widehat{\widehat{W}}_{\mathbb{D}}^{Q}(z)$ be the associated disk wedge. Then it is easy to see that $F_{\mathbb{D}}(z) = \bigcap \widehat{\widehat{W}}_{\mathbb{D}}^{Q}(z)$ is a disk fiber and we call it the *disk fiber of* $z$. We will also say that the fibers $F_{\mathbb{C}}(z)$ and $F_{\mathbb{D}}(z)$ are *associated to each other.*

Fig. 5 shows a planar fiber $F_{\mathbb{C}}$ and its associated disk fiber $F_{\mathbb{D}}(z)$ together with some sets $\mathrm{Tail}(Q), Q \in \Gamma$ and corresponding sets $G(Q)$.

A planar fiber can be represented as a countable intersection of a nested sequence of planar blocks. Clearly, $z \in F_{\mathbb{C}}(z)$. Also, by definition the fiber $F_{\mathbb{C}}^{\mathfrak{R}}(z) \cap K_P$ consists exactly of all points of $K_P$ which are weakly non-separated from $z$. The relation between other types of fibers may be more complicated.

**Lemma 3.16.** *For a point $z \in \mathbb{C} \setminus \bigcup_{E\in\Gamma} \mathrm{Tail}(E)$ let $F_{\mathbb{C}}(z)$ be the planar fiber of $z$, and let $G = F_{\mathbb{D}}(z)$ be the associated disk fiber of $z$. Then $F_{\mathbb{C}}(G) \subset F_{\mathbb{C}}(z)$.*

**Proof.** Consider a planar wedge $\widehat{W}_{\mathbb{C}}^{Q}(z)$ and its associate disk wedge $\widehat{\widehat{W}}_{\mathbb{D}}^{Q}(z)$. Then there exists $\ell \in \mathcal{L}$ such that $\widehat{\widehat{W}}_{\mathbb{D}}^{Q}(z) = \widehat{W}_{\mathbb{D}}^{\ell}(G)$. If now $F_{\mathbb{C}}(G)$ is the associated planar fiber, then $\widehat{W}_{\mathbb{C}}^{\ell}(G) = \widehat{W}_{\mathbb{C}}^{Q}(z)$. Hence $F_{\mathbb{C}}(G) \subset F_{\mathbb{C}}(z)$ as desired. $\square$

Lemma 3.17 is a simple corollary of the definitions.

**Lemma 3.17.** *A planar fiber $F_{\mathbb{C}}$ is the union of the non-separating in the plane continuum $F_{\mathbb{C}} \cap K_P$ and rays with angles in the associated disk fiber. Let $G$ be a disk fiber such that there exists a point $z \in F_{\mathbb{C}}(G) \setminus \bigcup_{E\in\Gamma} \mathrm{Tail}(E)$. Then $G \neq G(Q)$ for any $Q \in \Gamma$, and $G$ is not a leaf of $\mathcal{L}$. Moreover, $F_{\mathbb{C}}(G) = F_{\mathbb{C}}(z)$ and $G = F_{\mathbb{D}}(z) = \mathrm{Ch}\{\alpha \in \mathbb{S}^1 \mid R_\alpha \subset F_{\mathbb{C}}(z)\}$.*

**Proof.** Note that $F_{\mathbb{C}} \cap K_P$ is the intersection of planar continua which do not separate the plane (which are the intersections of the appropriate closed planar wedges and $K_P$).

Hence $F_{\mathbb{C}} \cap K_P$ is a continuum which does not separate the plane. By definition, rays of angles from the associated disk fiber are contained in $F_{\mathbb{C}}$ while all other says are disjoint from $F_{\mathbb{C}}$. This proves the first part of the lemma.

To prove the rest of the lemma, observe first that it easily follows if $G$ is degenerate. Now, let $G$ be a non-degenerate disk fiber such that there exists a point $z \in F_{\mathbb{C}} \setminus \bigcup_{E\in\Gamma} \mathrm{Tail}(E)$. By definition this implies that $G \neq G(Q)$ for any $Q \in \Gamma$, and $G$ is not a leaf of $\mathcal{L}$. Since $G$ is a disk fiber, it now follows from Lemma 3.13 that $G$ is either a double sided limit leaf in $\overline{\mathcal{L}} \setminus \mathcal{L}$ or a gap of $\overline{\mathcal{L}}$ such that $G \neq G(Q)$ for all $Q \in \Gamma$. The required equality $F_{\mathbb{C}}(G) = F_{\mathbb{C}}(z)$ follows since the two families of closed planar wedges whose intersections are, respectively, $F_{\mathbb{C}}(z)$ and $F_{\mathbb{C}}(G)$, are identical. The last claim of the lemma is left to the reader. □

*3.2.4. Dynamics and correspondence between sets*

Notice, that by a Theorem of Kiwi [22] all infinite gaps of $\overline{\mathcal{L}}$ are (pre)periodic. Mark a point in each periodic parabolic Fatou domain, and let AN ("attracting and neutral" points) be the set of all attracting, Siegel, Cremer, or marked points; given $p \in \mathrm{AN}$, let $c(p)$ be the period of $p$ or (for a marked point) the period of its parabolic domain. The next lemma is an application of the tools developed so far. Recall that the linear extension $\sigma^*$ was defined in the beginning of Subsection 2.1.5. Note that if $p \in \mathrm{AN}$, then $p \in K_P \setminus \Gamma^*$ and both $F_{\mathbb{C}}(p)$ and $F_{\mathbb{D}}(p)$ are defined. We now show that the correspondence between disk fibers and planar fibers is dynamical.

We will need the following definition. Let $X$ be a connected topological space. Then $X$ is *unicoherent* provided that for any closed connected subsets $A$ and $B$ of $X$, if $X = A \cup B$, then $A \cap B$ is connected. Thus, an interval is unicoherent while the circle is not.

**Lemma 3.18.** *Let $F_{\mathbb{D}}(z)$ and $F_{\mathbb{C}}(z)$ be the disk fiber and the planar fiber of a point $z \in \mathbb{C} \setminus \Gamma^*$. Then $P(F_{\mathbb{C}}(z)) = F_{\mathbb{C}}(P(z))$ and $\sigma^*(F_{\mathbb{D}}(z)) = F_{\mathbb{D}}(P(z))$ are the planar fiber and the disk fiber of the point $P(z)$. Moreover, if $G$ is a disk fiber and $H = \mathrm{Ch}(\sigma(G'))$, then $H$ is a disk fiber and $P(F_{\mathbb{C}}(G)) = F_{\mathbb{C}}(H)$.*

**Proof.** Suppose that $F_{\mathbb{C}}(z)$ is the planar fiber of a point $z \in \mathbb{C} \setminus \Gamma^*$. Clearly $w = P(z) \in \mathbb{C} \setminus \bigcup_{E\in\Gamma} \mathrm{Tail}(E)$ and the fiber $F_{\mathbb{C}}(w)$ is well-defined.

*We will show first that* $P(F_{\mathbb{C}}(z)) \subset F_{\mathbb{C}}(w)$. Suppose that $x \in F_{\mathbb{C}}(z)$ and $P(x) \notin F_{\mathbb{C}}(w)$. Then there exists $Q \in \Gamma$ such that $w$ and $P(x)$ are in distinct components of $\mathbb{C} \setminus \mathrm{Tail}(Q)$. If $C = P^{-1}(\mathrm{Tail}(Q))$ does not separate $z$ and $x$, there exists an arc $A \subset \mathbb{C} \setminus C$ joining $x$ and $z$. But then $P(A)$ is a continuum in $\mathbb{C} \setminus \mathrm{Tail}(Q)$ joining $w$ and $P(x)$, a contradiction. Hence $C$ separates $x$ and $z$ and, since $\mathbb{C}$ is unicoherent and locally connected, a component $C'$ of $C$ must separate $x$ and $z$. Since $C' = \mathrm{Tail}(Q')$ for some component $Q'$ of $P^{-1}(Q)$, we get a contradiction with the fact that $x \in F_{\mathbb{C}}(z)$. Hence we have shown that $P(F_{\mathbb{C}}(z)) \subset F_{\mathbb{C}}(w)$.

*We show next that* $P(F_{\mathbb{C}}(z)) = F_{\mathbb{C}}(w)$. Suppose that $v \in F_{\mathbb{C}}(w)$ and $P^{-1}(v) \cap F_{\mathbb{C}}(z) = \emptyset$. Since $P^{-1}(v) = \{u_1, \dots, u_d\}$ is finite, there exists for each $j$ a set $Q_j \in \Gamma$ such that $u_j$ and $z$ are in distinct components of $\mathbb{C} \setminus \mathrm{Tail}(Q_j)$. Since $v \in F_{\mathbb{C}}(w)$, there exists an arc $A \subset \mathbb{C} \setminus \bigcup \mathrm{Tail}(P(Q_j))$ joining $w$ and $v$. Since $P$ is an open map, the component $A'$ of $P^{-1}(A)$ containing the point $z$ contains some point $u_j$. Since $A' \cap \mathrm{Tail}(Q_j) = \emptyset$, $u_j$ and $z$ are in the same component of $\mathbb{C} \setminus \mathrm{Tail}(Q_j)$, a contradiction $P^{-1}(v) \cap F_{\mathbb{C}}(z) = \emptyset$. Hence $P(F_{\mathbb{C}}(z)) = F_{\mathbb{C}}(w)$ as desired.

*We show next that $\sigma^*$ maps disk fibers to disk fibers.* Let $G$ be a disk fiber. If $G$ is degenerate, the lemma easily follows. So we can assume that $G$ is a non-degenerate disk fiber. By Lemma 3.13, $G$ is a gap-leaf of $\overline{\mathcal{L}}$ or a leaf of $\mathcal{L}$. If $\sigma^*(G)$ is a leaf of $\mathcal{L}$ or a gap-leaf of $\overline{\mathcal{L}}$, we are done. Otherwise $\sigma^*(G)$ is either a leaf of $\overline{\mathcal{L}} \setminus \mathcal{L}$ or a point. Clearly, this implies that $G$ is a finite gap or a single leaf. If $\mathrm{Bd}(G)$ contains a leaf of $\mathcal{L}$, then $\sigma^*(G)$ will be a leaf of $\mathcal{L}$, a contradiction. Hence $G$ is a finite gap all of whose boundary leaves are one-sided limit leaves. Therefore $\sigma^*(G)$ is a gap-leaf, a contradiction. Note that by the above, $P(F_{\mathbb{C}}(z)) = F_{\mathbb{C}}(w)$. It now follows easily that $\sigma^*(F_{\mathbb{D}}(z)) = F_{\mathbb{D}}(w)$.

Let $G$ be a disk fiber. By the above, $\sigma^*(G) = H$ is also a disk fiber and $H = \mathrm{Ch}(\sigma(G'))$. Let $F_{\mathbb{C}}(G)$ and $F_{\mathbb{C}}(H)$ be the associated planar fibers. *We will show that $P(F_{\mathbb{C}}(G)) = F_{\mathbb{C}}(H)$.* If $G = G(Q)$ for some $Q \in \Gamma$ or $G = \ell$ for some $\ell \in \mathcal{L}$, then it follows easily that $P(F_{\mathbb{C}}(G)) = F_{\mathbb{C}}(H)$ and we are done. Hence we may assume that there exists a point $z \in F_{\mathbb{C}}(G) \setminus \bigcup_{E \in \Gamma} \mathrm{Tail}(E)$. Now, by Lemma 3.17 $F_{\mathbb{C}}(z) = F_{\mathbb{C}}(G)$. By the first part of this lemma, $P(F_{\mathbb{C}}(z)) = F_{\mathbb{C}}(P(z))$.

*Let us show that $F_{\mathbb{C}}(H) = F_{\mathbb{C}}(P(z))$.* To this end, let us show that the arguments of external rays contained in both sets, are the same. Indeed, by the first, already proven, claim of this lemma, $P(F_{\mathbb{C}}(z)) = F_{\mathbb{C}}(P(z))$. Hence the arguments of the rays inside the set $F_{\mathbb{C}}(P(z))$ form a set $\sigma(G)' = H'$ and $F_{\mathbb{D}}(P(z)) = H$. Since $G \neq G(Q)$ for any $Q \in \Gamma$ and $G$ is not a leaf of $\mathcal{L}$, the same holds for $H$. Hence by Lemma 3.17, $F_{\mathbb{C}}(P(z)) = F_{\mathbb{C}}(H)$ and $P(F_{\mathbb{C}}(G)) = F_{\mathbb{C}}(H)$ as desired. □

**Lemma 3.19.** *If $F$ is a planar fiber and $G$ is its associated disk fiber, then the following holds.*

(1) *Let $\alpha\beta$ be a leaf in* $\mathrm{Bd}(G)$ *such that the circular arc $(\alpha, \beta)$ is disjoint from $G'$, and $\gamma \in (\alpha, \beta)$. If $\alpha\beta \notin \mathcal{L}$ then* $\mathrm{Imp}(\gamma) \cap F = \emptyset$. *On the other hand, if $\alpha\beta \in$* $\mathrm{Bd}(G(Q))$ *for some $Q \in \Gamma$ then* $\mathrm{Imp}(\gamma) \cap F \subset Q$. *Moreover, there are at most finitely many angles $\gamma \in (\alpha, \beta)$ with* $\mathrm{Pr}(\gamma) \subset F$.
(2) *If $G$ is finite, there are finitely many angles $\gamma$ with* $\mathrm{Pr}(\gamma) \subset F$.
(3) *There are at most finitely many repelling or parabolic periodic points in $F$ at which two or more external rays land.*
(4) *If $x \in F$ is a preimage of a repelling or a parabolic point, then there exists $\alpha \in G'$ with $R_\alpha \subset F$ landing at $x$.*

**Proof.** If $G = G(Q)$ for some $Q \in \Gamma$, or a leaf of $\mathcal{L}$, or a two-sided limit leaf, the lemma follows easily. Thus, by Lemma 3.13 we may assume that $G$ is a gap of $\overline{\mathcal{L}}$ that is not an element of $\mathcal{L}$.

(1) If $\alpha\beta$ is a limit of leaves in $\mathcal{L}$ with endpoints in $(\alpha, \beta)$ then for all $\gamma \in (\alpha, \beta)$, $\mathrm{Imp}(\gamma) \cap F = \emptyset$. Otherwise $\alpha\beta$ is a boundary leaf of an element $H$ of $\mathcal{L}$ corresponding to $Q \in \Gamma$. Clearly, then $\mathrm{Imp}(\gamma) \cap F \subset Q$ as desired. This proves the first part of (1). Now, by the above if $\alpha\beta$ is a limit of leaves in $\mathcal{L}$ with endpoints in $(\alpha, \beta)$ then there are no angles $\gamma$ with $\mathrm{Pr}(\gamma) \subset F$. If $\alpha\beta$ is a boundary leaf of an element $H$ of $\mathcal{L}$ corresponding to $Q \in \Gamma$ then, again by the above, the fact that $\mathrm{Pr}(\gamma) \subset F$ would imply that $\mathrm{Pr}(\gamma) \subset Q$. Since there are finitely many angles $\gamma'$ with $\mathrm{Pr}(\gamma') \subset Q$, then there are at most finitely many angles $\gamma \in (\alpha, \beta)$ with $\mathrm{Pr}(\gamma) \subset F$.

(2) follows from (1).

(3), (4) *Let us show that if $x \in F$ is a repelling or parabolic point or a preimage of it, then there is at least one (if $x \notin \mathfrak{R}$) and at least two (if $x \in \mathfrak{R}$) rays landing at $x$ and contained in $F$.* Indeed, if $x \notin \mathfrak{R}$ let $C = R \cup \{x\}$ where $R$ is a ray landing at $x$. If $x \in \mathfrak{R}$ let $\widehat{W}_x(F)$ be the closed wedge at $x$ containing $F$, and let $C$ be the union of two boundary rays of $\widehat{W}_x(F)$ and $\{x\}$. Then in either of these cases by definition $C \subset F$.

Now, the previous paragraph immediately implies (4). To prove (3), observe that by the previous paragraph each point of $\mathfrak{R}$ in $F$ corresponds to a boundary leaf of $G$. However by Lemma 2.5 there are at most finitely many periodic leaves in $\mathrm{Bd}(G)$. Hence there are finitely many points of $\mathfrak{R}$ in $F$ which implies (3). □

**Definition 3.20.** We will call an attracting or parabolic Fatou domain a *parattracting domain.*

Lemma 3.21 relates periodic planar fibers and disk fibers.

**Lemma 3.21.** *Let $G$ be a disk fiber which maps into itself by $(\sigma^*)^n$ and let $F = F_{\mathbb{C}}(G)$ be the associated planar fiber. Then in fact $(\sigma^*)^n(G) = G$ and the following claims hold.*

(1) *$P^n(F) = F$.*
(2) *If $G'$ is finite then $F$ is a periodic point. If in addition $|G'| > 1$ then there exists $x \in \mathfrak{R}$ such that $G = G(x)$.*
(3) *If $G'$ is infinite, then there exists $p \in \mathrm{AN}$ such that $F_{\mathbb{D}}(p) = G$. Conversely, for each $p \in \mathrm{AN}$, $F_{\mathbb{D}}(p)$ is an infinite periodic gap. If $\mathfrak{R} \subset \Gamma$, this correspondence between $\mathrm{AN}$ and all infinite periodic gaps of $\overline{\mathcal{L}}$ is one-to-one, and $p$ is a unique point of $\mathrm{AN}$ in $F_{\mathbb{C}}(p)$.*

**Proof.** Assume that $n = 1$. By Lemma 3.13 $G$ is either a gap-leaf, or a leaf of $\mathcal{L}$; by Lemma 3.8, there are no critical leaves in $\mathcal{L}$. Thus, if $G$ is a gap-leaf, then Lemma 2.5(2) implies $\sigma^*(G) = G$ and $\sigma(G') = G'$, and if $G$ is a leaf of $\mathcal{L}$ then $\sigma^*(G) = G$ and $\sigma(G') = G'$ too.

(1) follows immediately from Lemma 3.18.

(2) We consider only the case when $G$ is non-degenerate and $|G'| > 1$; if $G$ is a degenerate gap-leaf (i.e., a point in $\mathbb{S}^1$ which is separated from the rest of $\mathbb{S}^1$ by a

sequence of leaves converging to it), the arguments are almost literally the same and are left to the reader.

*We first prove that $F$ cannot contain a parattracting Fatou domain.* Indeed, otherwise by [43] there are infinitely many repelling periodic points in $\mathrm{Bd}(U)$ which contradicts Lemma 3.19.

*Let us show that a CS-point $p$ cannot belong to $F$.* Indeed, since $G$ is periodic under $\sigma^*$, all the angles in $G'$ are periodic. Hence, since $F$ is closed, their principal sets (i.e., in this case landing points) are contained in $F$. Now, suppose that a CS-point $p$ belongs to $F$. Then by [37,38] (see Subsection 2.2) there is a critical point in $F$ (if not, $F$ is a subset of a *hedgehog* and cannot contain periodic landing points of angles of $G'$). Consider two cases.

(i) Suppose that there exists a critical point $c \in F \cap J_P$. Choose $\alpha \in \mathbb{S}^1$ so that $c \in \mathrm{Imp}(\alpha)$ and $\alpha$ is not periodic (this is possible because, due to the symmetry of the map $P$ around $c$, the set of all angles whose impressions contain $c$ must contain pairs of angles mapping to the same angle). Then $\alpha \notin G'$ (because all angles in $G'$ are periodic) and there exists a boundary leaf $\ell = \beta\gamma$ of $G$ with $\alpha \in (\beta, \gamma)$ and $G' \cap (\beta, \gamma) = \emptyset$.

By Lemma 3.19 the fact that $c \in \mathrm{Imp}(\alpha) \cap F$ implies that $\beta\gamma$ is a boundary leaf of some element $H$ of $\mathcal{L}$ corresponding to $Q \in \Gamma$, and $c \in Q$. Since $\beta\gamma$ is periodic, this implies that $Q$ is a point of $\mathfrak{R}$, a contradiction with $c \in Q$.

(ii) Suppose that $F \cap J_P$ contains no critical points. Let $E$ be the component of $P^{-1}(P(F))$ containing $F$. *We claim that in this case $E = F$.* Indeed, suppose that $F$ is a proper subset of $E$. Then there exists a sequence $z_i \in E \setminus F$ converging to a point $z \in F$. We may assume that one of the following two possibilities holds.

(a) There exists a leaf $\ell = \beta\gamma$ of $\mathcal{L}$ such that the cut $\mathrm{Cut}^\ell$ separates points $z_i$ from $F \setminus \mathrm{Cut}^\ell$. Then $z$ is a periodic point from $\mathfrak{R}$. Therefore $P$ is one-to-one in a small neighborhood of $z$. Choose points $x_i \in F$ such that $P(x_i) = P(z_i)$ for all $i$'s. Then all these points must be positively distant from $z$. Assuming that $x_i \to x \in F$ we see that $x \neq z$. By continuity $P(x) = P(z)$ and so $x$ is a preperiodic point from $\mathfrak{R}$. However, by Lemma 3.19(4) there exists a periodic angle $\theta \in G'$ whose ray lands at $x$, a contradiction.

(b) There exists a sequence of sets $Q_i \in \Gamma$ and boundary leaves $\ell \in \mathrm{Bd}(G(Q_i))$ such that $z_1$ is separated from $z$ by cuts $\mathrm{Cut}^{\ell_i}$. Then it follows that each $Q_i$ intersects $E$ and hence $P(Q_i)$ intersects $P(F)$ for every $i$. However, by Lemma 3.18 $P(F)$ coincides with the planar fiber $F_{\mathbb{C}}(\sigma(G))$ associated with the *finite* disk fiber $\sigma(G)$, a contradiction.

So, we have proved that $E = F$. On the other hand, by the above there is a critical point in $F$. Hence $P|_F$ is a non-trivial branched covering map onto $P(F)$. Choose an angle $\alpha \in G'$ and let $y$ be the landing point of $R_\alpha$. Then there exists a point $y' \neq y$ in $F$ such that $P(y) = P(y')$. Hence again by Lemma 3.19(4) we have a contradiction. This shows that there are no CS-points in $F$.

By the above, $F$ contains no parattracting Fatou domains and no CS-points. By Lemma 3.17, $F \cap K_P$ is a continuum which does not separate the plane. By (1), $P(F \cap K_P) = F \cap K_P$. By Lemma 3.19(3) and Theorem 2.6, $F \cap K_P$ is a periodic point $x$. If $|G'| > 1$ this implies that $x \in \mathfrak{R}$.

(3) *We claim that if $G'$ is infinite and periodic, then* $\mathrm{AN} \cap F \neq \emptyset$. Indeed, otherwise $G'$ is infinite and periodic with neither a CS-point nor a Fatou domain in $F$. As above, by (1), Theorem 2.6 and Lemma 3.19(3), $F \cap K_P$ is a point $x \in \mathfrak{R}$ which is impossible. Notice that by Lemma 3.17 $G = F_{\mathbb{D}}(p)$ as desired.

Now, *let $p \in \mathrm{AN}$ and $P(p) = p$ (otherwise the proof is similar), and prove the second statement of (3).* By Lemma 3.18, $P(F_{\mathbb{C}}(p)) = F_{\mathbb{C}}(p)$ (because $n = 1$ by the assumption). By Lemma 3.18, $\sigma^*(F_{\mathbb{D}}(p)) = F_{\mathbb{D}}(p)$. By (2), $F'_{\mathbb{D}}(p)$ is infinite.

Finally, let $\mathfrak{R} \subset \Gamma$. By [18,22] sets $\mathrm{Tail}(x), x \in \mathfrak{R}$ separate CS-points, attracting points and marked points from each other. Hence the just defined association between infinite periodic gaps of $\overline{\mathcal{L}}$ and points of AN is one-to-one. □

Observe that even if $p$ is an attracting or marked point, the corresponding set $F_{\mathbb{C}}(p)$ is not necessarily the closure of the corresponding Fatou domain. Indeed, suppose that $p$ is a fixed attracting point, $E$ is its Fatou domain, and all periodic points on its boundary are not cutpoints of the Julia set $J_P$. Suppose that there is a non-(pre)periodic critical point on its boundary. Then there exists a pullback $E'$ of $E$, attached to $E$ at $c$. As follows from the definition, $E'$ must be contained in $F_{\mathbb{C}}(p)$ too. Moreover, appropriate pullbacks of $E'$ will also have to be contained in $F_{\mathbb{C}}(p)$ because they will not be separated from $p$ by a cut generated by a point of $R$. Thus, in this case $F_{\mathbb{C}}(p)$ includes not only $E$ but the entire family of pullbacks of $E$ which are attached to it.

## 4. Non-repelling cycles and wandering continua

If $p$ is a CS-point, the orbit of the set $F^{\mathfrak{R}}_{\mathbb{C}}(p)$ is called a *CS-set* and is denoted by $F^{\mathfrak{R}}_{\mathbb{C}}(\mathrm{orb}\, p)$. In this section we use the tools developed in Section 3 in order to study CS-sets in connection with wandering non-(pre)critical cut-continua as well as recurrent critical points. This is necessary for our study because it is through CS-sets that both phenomena which we are interested in – wandering non-(pre)critical cut-continua and recurrent critical points – are related.

### *4.1. Limit behavior of orbits of wandering non-(pre)critical cut-continua*

In this subsection we show that wandering cut-continua cannot live in CS-sets (Theorem 4.1). This is used in Corollary 4.2 which relates geometric prelaminations $\mathcal{L}^{\mathfrak{R}}, \overline{\mathcal{L}}^{\mathfrak{R}}$ and $\mathcal{L}^{\mathcal{B}_{\mathbb{C}}}, \overline{\mathcal{L}}^{\mathcal{B}_{\mathbb{C}}}$.

**Theorem 4.1.** *Let $p \in \mathrm{AN}$ and let $Q \subset J_P$ be a wandering non-(pre)critical cut-continuum. Then the CS-set $F^{\mathfrak{R}}_{\mathbb{C}}(\mathrm{orb}\, p)$ is disjoint from $Q$.*

**Proof.** Consider $\mathcal{B}_{\mathbb{C}} = \{Q\}$ as a wandering "collection" of cut-continua. Assume that $p \in \mathrm{AN}$ and $P(p) = p$. Set $F = F^{\mathfrak{R}}_{\mathbb{C}}(p)$ and $G = F^{\mathfrak{R}}_{\mathbb{D}}(p)$. Then by Lemma 3.18 $P(F) = F$ and, by Lemma 3.21, $\sigma^*(G) = G$. By Lemma 3.6 $|A(Q)| = \mathrm{val}(Q) > 1$ is finite. By way of contradiction assume that $F$ is not disjoint from $Q$. Consider an element $\widehat{Q}$ of the grand orbit $\Gamma(Q)$ of $Q$ and prove a few claims.

*First we show that if $F \cap \widehat{Q} \neq \emptyset$ then $\widehat{Q} \subset F$.* Indeed, let $\widehat{Q} \not\subset F$ and $z \in \widehat{Q} \setminus F$. Then by definition there is a point $y \in \mathfrak{R}$ such that the planar wedges $W^y_{\mathbb{C}}(p)$ and $W^y_{\mathbb{C}}(z)$ are distinct and therefore disjoint. Since $y$ is (pre)periodic, it cannot belong to $\widehat{Q}$ (which is wandering), hence $\widehat{Q} \subset W^y_{\mathbb{C}}(z)$ which implies that $\widehat{Q} \cap F = \emptyset$, a contradiction.

*Next we prove that $A(\widehat{Q}) \subset G'$ and no point of $A(\widehat{Q})$ is an endpoint of a circle arc complementary to $G'$.* By Lemma 3.19(1) if there exists $\alpha \in A(\widehat{Q}) \setminus G'$ then $\mathrm{Pr}(\alpha) \cap F$ is a (pre)periodic point which contradicts $\mathrm{Pr}(\alpha) \subset \widehat{Q}$ and $\widehat{Q}$ being wandering. Also, if $\alpha \in A(\widehat{Q})$ is an endpoint of a boundary leave of $G$ then by Lemma 3.21(3) and Lemma 2.5(3) $\alpha$ is (pre)periodic, contradicting that $\widehat{Q}$ wanders. This proves the claim.

*Now we prove that $\widehat{Q} \subset F$ cuts $F$ into at least* $\mathrm{val}(\widehat{Q})$ *components.* Indeed, by the above, no point of $A(\widehat{Q})$ is an endpoint of an arc complementary to $G'$. Hence, for adjacent angles $\alpha, \beta \in A(\widehat{Q})$ (so that $(\alpha, \beta)\cap$ contains no points of $A(\widehat{Q})$), there is an angle $\gamma \in G' \cap (\alpha, \beta)$ which is an endpoint of an arc complementary to $G'$, (pre)periodic by Lemma 2.5(3). The landing point $z$ of $R_\gamma$ does not belong to $\widehat{Q}$ and can be associated to the arc $(\alpha, \beta)$. Clearly, two points associated to such distinct arcs are separated in $F$ by the set $\mathrm{Tail}(\widehat{Q})$. Hence $\widehat{Q} \subset F$ cuts $F$ into at least $\mathrm{val}(\widehat{Q})$ components.

Consider the laminations $\mathcal{L} = \mathcal{L}^{\mathcal{B}_{\mathbb{C}}, \mathfrak{R}}, \overline{\mathcal{L}} = \overline{\mathcal{L}}^{\mathcal{B}_{\mathbb{C}}, \mathfrak{R}}$ and the set $\Gamma = \Gamma(\mathcal{B}_{\mathbb{C}}, \mathfrak{R})$. Set $\widetilde{G} = F^{\Gamma}_{\mathbb{D}}(p)$. By Lemma 3.18 $P(F^{\Gamma}_{\mathbb{C}}(p)) = F^{\Gamma}_{\mathbb{C}}(p)$ and, by Lemma 3.21(3), $\sigma^*(\widetilde{G}) = \widetilde{G}$. Also, by Lemma 3.21 $\widetilde{G}$ is an infinite invariant gap, and by Lemma 2.5(3) all leaves on the boundary of $\widetilde{G}$ are (pre)periodic. By the construction $F^{\Gamma}_{\mathbb{C}}(p) \subsetneqq F^{\mathfrak{R}}_{\mathbb{C}}(p)$ and $\widetilde{G} \subsetneqq G$.

**Claim A.** *Except for $\widetilde{G}$ and leaves from* $\mathrm{Bd}(G)$, *there are no fixed or periodic disk fibers of $\overline{\mathcal{L}}$ contained in $G$. All periodic points or leaves in* $\mathrm{Bd}(G)$ *which are not contained in* $\mathrm{Bd}(\widetilde{G})$, *are limits of elements of $\mathcal{L}$ from within $G$ which separate these periodic points or leaves from the rest of $G$. Moreover, all periodic leaves in* $\mathrm{Bd}(G)$ *that are not contained in* $\mathrm{Bd}(\widetilde{G})$, *are pairwise disjoint.*

**Proof of Claim A.** Let us first show that *if $E \subset G, E \neq \widetilde{G}$ is a periodic disk fiber of $\overline{\mathcal{L}}$, then $E$ is a leaf from* $\mathrm{Bd}(G)$. Indeed, by Lemma 3.13 $E$ is either a leaf of $\mathcal{L}$ or a gap-leaf of $\overline{\mathcal{L}}$. In the first case the claim follows since $\mathcal{L} = \mathcal{L}^{\mathcal{B}_{\mathbb{C}}, \mathfrak{R}}$ and $\mathcal{B}_{\mathbb{C}}$ is formed by a wandering cut-continuum $Q$. So we may assume that $E$ is a gap-leaf of $\overline{\mathcal{L}}$ which is not a leaf of $\mathcal{L}$. If $E$ is infinite, then by Lemma 3.21(3) $F^{\Gamma}_{\mathbb{C}}(E)$ contains a point $p' \in \mathrm{AN}$. Since $E \neq \widetilde{G}$, then $p' \neq p$. However, by the construction $F^{\Gamma}_{\mathbb{C}}(E) \subset F$ and by Lemma 3.21(3) the set $F^{\mathfrak{R}}_{\mathbb{C}}(p)$ contains a unique point of AN, namely $p$. This contradiction implies that $E$ is finite. Then by Lemma 3.21(2) there is a periodic point $x \in \mathfrak{R}$ such that $E \subset G(x)$. Since $E \subset G$ and by the construction it is easy to see that $E$ is a boundary leaf of $G$.

Suppose that a periodic leaf $\ell \subset \mathrm{Bd}(G)$ or a periodic point $\alpha \in G'$, which is not contained in $\mathrm{Bd}(\widetilde{G})$, is not a limit of elements of $\mathcal{L}$ from within $G$. Then there must exist a periodic gap of $\overline{\mathcal{L}}$ contained in $G$ and containing $\ell$ (or $\alpha$) in its boundary. This contradicts the previous paragraph and shows that all periodic points or leaves in $\mathrm{Bd}(G)$ which do not come from $\mathrm{Bd}(\widetilde{G})$, are limits of elements of $\mathcal{L}$ from within $G$.

Finally, it is easy to see that all periodic leaves in $\mathrm{Bd}(G)$ which do not come from $\mathrm{Bd}(\widetilde{G})$, are pairwise disjoint; indeed, elements of $\mathcal{L}$, which approach a periodic leaf in $\mathrm{Bd}(G)$, cut it off other leaves in $\mathrm{Bd}(G)$, that implies the desired and proves Claim A. $\square$

Since $\widetilde{G} \subsetneqq G$, there are leaves of $\mathrm{Bd}(\widetilde{G})$ *inside* $G$. By Lemma 3.8 they are (pre)periodic. Let $\ell \subset \mathrm{Bd}(\widetilde{G})$ be a (pre)periodic leaf inside $G$; *we show that $\ell$ can be assumed to have fixed endpoints*. Indeed, $\ell$ is a limit leaf of sets $\mathrm{Ch}(A(\widehat{Q}_i))$ where $\widehat{Q}_i$ are elements of the grand orbit of $Q$. By the properties of such sets, established in the beginning of the proof, all sets $A(\widehat{Q}_i) \subset G'$ consist of non-endpoints of complementary to $G'$ arcs. Therefore and by continuity of $\sigma$, $\ell$ can never be mapped to the boundary leaves of $G$. Replacing $\ell$ by its appropriate image and using a power of $\sigma$, we may assume that $\ell$ has fixed endpoints.

Let $\widetilde{Q} \in \Gamma(Q)$ be such that the convex hull $\mathrm{Ch}(A(\widetilde{Q})) \subset G$ is close to $\ell$. Then $A(\widetilde{Q})$ is repelled away from $\widetilde{G}$ to a component of $G \setminus \mathrm{Ch}(A(\widetilde{Q}))$ disjoint from $\widetilde{G}$. Denote by $Z$ this component united with $\mathrm{Ch}(A(\widetilde{Q}))$. Let us now construct a set $\widetilde{Z}$. Denote by $Y_1, \dots, Y_k$ the fixed leaves in $\mathrm{Bd}(Z)$ and the fixed points in $\mathrm{Bd}(Z)$ which are not endpoints of complementary to $G'$ arcs. By Claim A the sets $\{Y_i\}$ are pairwise disjoint. Choose pairwise disjoint elements $\widetilde{S}_i$ of $\mathcal{L}$ contained in $Z$ very close to each $Y_i$ (this is possible by Claim A). Let $F_i$ be the component of $Z \setminus \widetilde{S}_i$ containing $Y_i$. Set $\widetilde{Z} = Z \setminus \bigcup F_i$. By choosing $\widetilde{S}_i$ very close to $Y_i$, we may assume that all $F_i$ are pairwise disjoint with each other and with $\mathrm{Ch}(A(\widetilde{Q}))$ and that their images are contained in $\widetilde{Z}$.

Let $r : G \to \widetilde{Z}$ be a retraction. Define a new map $g = r \circ \sigma^* : \widetilde{Z} \to \widetilde{Z}$. Let $a \in \widetilde{Z}$ be a $g$-fixed point. Then it is easy to see that by the construction $a \notin \mathrm{Bd}(\widetilde{Z})$. Therefore $a$ is actually $\sigma^*$-fixed. If $a$ belongs to the interior of a gap of $\overline{\mathcal{L}}$, then this gap must be $\sigma^*$-invariant which contradicts Claim A. If $a$ belongs to a leaf of $\overline{\mathcal{L}}$, then, since by the construction this leaf cannot belong to $\mathcal{L}$, it follows that there exists a fixed gap-leaf of $\overline{\mathcal{L}}$ containing $a$. This again contradicts Claim A. $\square$

Now assume that $\mathcal{B}_{\mathbb{C}}$ is a wandering collection of non-(pre)critical cut-continua. Lemma 4.1 implies the next corollary.

**Corollary 4.2.** *Every element of $\mathcal{L}^{\mathcal{B}_{\mathbb{C}}}$ is contained in a finite wandering gap of $\overline{\mathcal{L}}^{\mathfrak{R}}$. The infinite gaps of $\overline{\mathcal{L}}^{\mathfrak{R}}$ and $\overline{\mathcal{L}}^{\mathcal{B}_{\mathbb{C}},\mathfrak{R}}$ are the same. Gap-leaves of $\overline{\mathcal{L}}^{\mathfrak{R}}$, and gap-leaves of $\overline{\mathcal{L}}^{\mathcal{B}_{\mathbb{C}},\mathfrak{R}}$ disjoint from leaves of $\mathcal{L}^{\mathcal{B}_{\mathbb{C}}}$, are the same. Any limit leaf of $\overline{\mathcal{L}}^{\mathcal{B}_{\mathbb{C}}}$ is a limit leaf of $\overline{\mathcal{L}}^{\mathfrak{R}}$ from the same side. All-critical gap-leaves of $\overline{\mathcal{L}}^{\mathcal{B}_{\mathbb{C}}}$ are all-critical gap-leaves of $\overline{\mathcal{L}}^{\mathfrak{R}}$.*

**Proof.** First we show that *every element of $\mathcal{L}^{\mathcal{B}_{\mathbb{C}}}$ is contained in a finite wandering gap of $\overline{\mathcal{L}}^{\mathfrak{R}}$*. Clearly, every element of $\mathcal{L}^{\mathcal{B}_{\mathbb{C}}}$ is contained in a gap of $\overline{\mathcal{L}}^{\mathfrak{R}}$. By Theorem 4.1 this gap of $\overline{\mathcal{L}}^{\mathfrak{R}}$ is finite. Thus, if we add $\mathcal{L}^{\mathcal{B}_{\mathbb{C}}}$ to $\overline{\mathcal{L}}^{\mathfrak{R}}$, we can possibly break some finite gaps of $\overline{\mathcal{L}}^{\mathfrak{R}}$ into smaller gaps but otherwise we will not change $\overline{\mathcal{L}}^{\mathfrak{R}}$. Obviously, the finite gaps of $\overline{\mathcal{L}}^{\mathfrak{R}}$, containing wandering gaps from $\mathcal{L}^{\mathcal{B}_{\mathbb{C}}}$, are wandering themselves. This implies that the *infinite gaps of $\overline{\mathcal{L}}^{\mathfrak{R}}$ and $\overline{\mathcal{L}}^{\mathcal{B}_{\mathbb{C}},\mathfrak{R}}$ are the same.*

*Let us prove that gap-leaves of $\overline{\mathcal{L}}^{\mathfrak{R}}$ disjoint from leaves of $\mathcal{L}^{\mathcal{B}_{\mathbb{C}}}$, and gap-leaves of $\overline{\mathcal{L}}^{\mathcal{B}_{\mathbb{C}},\mathfrak{R}}$ disjoint from leaves of $\mathcal{L}^{\mathcal{B}_{\mathbb{C}}}$, are the same.* Clearly, a gap-leaf of $\overline{\mathcal{L}}^{\mathfrak{R}}$, disjoint from leaves of $\mathcal{L}^{\mathcal{B}_{\mathbb{C}}}$, remains a gap-leaf of $\overline{\mathcal{L}}^{\mathcal{B}_{\mathbb{C}},\mathfrak{R}}$. Now, let $G$ be a gap-leaf of $\overline{\mathcal{L}}^{\mathcal{B}_{\mathbb{C}},\mathfrak{R}}$ disjoint from leaves of $\mathcal{L}^{\mathcal{B}_{\mathbb{C}}}$. Then its boundary leaf $\ell$ is either from $\overline{\mathcal{L}}^{\mathfrak{R}}$, or is a limit leaf of $\mathcal{L}^{\mathcal{B}_{\mathbb{C}}}$. In the latter case the elements of $\mathcal{L}^{\mathcal{B}_{\mathbb{C}}}$ which approach $\ell$ are contained in gaps or leaves of $\overline{\mathcal{L}}^{\mathfrak{R}}$ (by the already proven). Hence in any case $\ell \in \overline{\mathcal{L}}^{\mathfrak{R}}$. So, *all* leaves in $\mathrm{Bd}(G)$ belong to $\overline{\mathcal{L}}^{\mathfrak{R}}$ and $G$ is a gap-leaf of $\overline{\mathcal{L}}^{\mathfrak{R}}$.

*Next we show that any limit leaf $\ell'$ of $\overline{\mathcal{L}}^{\mathcal{B}_{\mathbb{C}}}$ is a limit leaf of $\overline{\mathcal{L}}^{\mathfrak{R}}$ from the same side.* By the above, leaves of $\mathcal{L}^{\mathcal{B}_{\mathbb{C}}}$, converging to $\ell'$, are contained in finite gap-leaves of $\overline{\mathcal{L}}^{\mathfrak{R}}$; we may assume that these gap-leaves of $\overline{\mathcal{L}}^{\mathfrak{R}}$ are all distinct. Hence $\ell'$ can be approximated from this side by distinct gap-leaves of $\overline{\mathcal{L}}^{\mathfrak{R}}$, and therefore it can be approximated from the same side by leaves of $\mathcal{L}^{\mathfrak{R}}$. The last claim of the lemma concerning all-critical gap-leaves now follows from this and Lemma 2.5. $\square$

Thus, with the help of Theorem 4.1 we have established a relation between the geometric prelaminations $\mathcal{L}^{\mathfrak{R}}, \overline{\mathcal{L}}^{\mathfrak{R}}$ and $\mathcal{L}^{\mathcal{B}_{\mathbb{C}}}, \overline{\mathcal{L}}^{\mathcal{B}_{\mathbb{C}}}$.

### *4.2. Recurrent critical points in CS-sets*

In this subsection we show that each CS-set contains a recurrent critical point whose limit set contains the mother hedgehog associated to the CS-set. To this end we need a result of [8] (in [8] it was used to study Milnor attractors of rational functions with dendritic critical limit sets).

Let $g$ be a rational function. For a Jordan disk $V$ with a pullback $W$, let the *recurrent* criticality of $W$ be the number of *recurrent* critical points (with multiplicities) in the pullbacks of $V$ all the way to $W$. Given two concentric round disks $D_1 \subset D_2$ of radii $r_1 < r_2$ say that $D_1$ is *$k$-inside* $D_2$ if $r_1/r_2 < k$. Let $\varepsilon > 0, 0 < k < 1, \gamma > 0, r \in \mathbb{N}$. Then by Theorem 3.5 [8] there exists $\delta > 0$ with the following properties. Let $V'$ be a round disk of diameter less than $\delta$, $\gamma$-distant from parabolic and attracting points. If the recurrent criticality of a $g^N$-pullback $V''$ of $V'$ is $r$, then for any disk $U' \subset V'$ which is $k$-inside $V'$, the diameter of any $g^N$-pullback $U'' \subset V''$ of $U'$ is less than $\varepsilon$ and the criticality of $g^N|_{U''}$ is at most $d + r$. A standard argument, based upon the Shrinking Lemma [30], then implies that *the diameter of pullbacks $U''$ of $U'$ tends to zero uniformly with respect to $N$.*

Theorem 4.3 uses notation from Subsection 2.2 and ideas of [12]. It implies Theorem 1.1(1) for connected Julia sets. Recall that $M_{\mathrm{orb}\, p}$ denotes the mother-hedgehog of the orbit $\mathrm{orb}\, p$ of a CS-point $p$.

**Theorem 4.3.** *Let $p$ be a CS-point and $\mathrm{orb}\, p$ be its cycle. Then there exists a recurrent critical point $c_{\mathrm{orb}\, p}$, weakly non-separated from a point $q \in \mathrm{orb}\, p$, such that $\mathrm{Bd}(M_{\mathrm{orb}\, p}) \subset \omega(c_{\mathrm{orb}\, p})$. Distinct CS-cycles correspond to distinct recurrent critical points so that the number of CS-cycles is less than or equal to the number of recurrent critical points of $P$.*

**Proof.** Assume that $P(p) = p$ and, by abuse of notation, $\operatorname{orb} p = p$. By definition the set of all points which are weakly non-separated from $p$ is $F^{\mathfrak{R}}_{\mathbb{C}}(p) \cap K_P$, so we need to find the desired critical point in $F^{\mathfrak{R}}_{\mathbb{C}}(p)$. Clearly, $F^{\mathfrak{R}}_{\mathbb{C}}(p)$ contains $M_p$: if there are hedgehogs, it follows from the fact that $F^{\mathfrak{R}}_{\mathbb{C}}(p)$ contains all hedgehogs at $p$ (recall, that hedgehogs contain no periodic points distinct from $p$), and if $M_p = \overline{\Delta}$ is the closure of a Siegel disk $\Delta$, then it follows from the fact that $\Delta \subset F^{\mathfrak{R}}_{\mathbb{C}}(p)$.

We need the following construction which begins with the choice of constants. Choose $N$ so that if $X$ is the union of sets $\mathrm{Tail}(x)$ over the set of periodic points of $P$ of period less than $N$ then there exists $n$ such that the following holds:

(1) if $A$ is the component of $\mathbb{C} \setminus P^{-n}(X)$ containing $p$ then all critical points of $P|_A$ belong to $F^{\mathfrak{R}}_{\mathbb{C}}(p)$; and
(2) each component of $\mathbb{C} \setminus P^{-n}(X)$ contains at most one Cremer point or Fatou component.

Clearly, if $N$ is big then (1) follows by the definition of $F^{\mathfrak{R}}_{\mathbb{C}}(p)$ while (2) follows from [21]. By definition, $P$ is a proper map of $A$ onto $P(A)$. Moreover, $\mathfrak{R}$ is invariant, and so if $U \subset A$ is a Jordan disk, then its pullbacks are either contained in $A$, or disjoint from $A$. Thus, if we choose a backward orbit of $x \in A$ which consists of points of $A$, then all corresponding pullbacks of $U$ are contained in $A$.

Let the set of recurrent critical points of $P$ in $A$ be $E$; then by (1) we have $E \subset F^{\mathfrak{R}}_{\mathbb{C}}(p)$. Let the union of their limit sets be $\omega(E)$. By way of contradiction suppose that $\mathrm{Bd}(M_p) \not\subset \omega(E)$. Choose a non-parabolic point $x \in \mathrm{Bd}(M_p) \setminus \omega(E)$. By Theorem 3.5 [8], described in the beginning of Subsection 4.2, this implies that a small neighborhood of $x$ has pullbacks inside $A$ which converge to 0 in diameter uniformly with respect to the order of the pullback (alternatively, one can refer here to a similar result of [25]).

However, this contradicts the fact that $P$ on a hedgehog (or, in the case when $M_p = \overline{\Delta}$ is the closure of a Siegel disk, on the closed invariant Jordan disk contained in $\overline{\Delta}$) is a recurrent diffeomorphism (see Subsection 2.2). The contradiction implies that $\mathrm{Bd}(M_p) \subset \omega(E)$.

*Let us show that then there exists at least one critical point $c_p$ with $\mathrm{Bd}(M_p) \subset \omega(c_p)$.* Consider first the case when there are no true hedgehogs and $M_p = \overline{\Delta}$ where $\Delta$ is a Siegel disk. Then there exists a point $x \in \mathrm{Bd}(\Delta)$ with a dense orbit in $\mathrm{Bd}(\Delta)$ (see, e.g., [20]). It is now enough to choose a point $c_p \in E$ such that $x \in \omega(c_p)$. Now, suppose that there are true hedgehogs. Since the map is transitive on each hedgehog, similarly to the above for each hedgehog $H$ there exists at least one critical point $c_H \in E$ such that $H \subset \omega(c_H)$. By way of contradiction assume that there is no critical point $c \in E$ such that $\mathrm{Bd}(M_p) \subset \omega(c)$. This means that for each critical point $c \in E$ there exists a hedgehog $H_c \ni p$ such that $H_c \not\subset \omega(c)$. Consider the set $H' = \bigcup_{c \in E} H_c$ contained (by construction) in $\mathrm{Bd}(M_p)$. Since all hedgehogs are invariant and by the Maximum Principle, the set $H'$ is forward invariant and onto.

We claim that there exists a hedgehog $H \supset H'$. Indeed, consider the case when $p$ is a Cremer fixed point (the case when $p$ is Siegel is similar). Then we claim that the set $H'$ is a continuum which does not separate the plane. Indeed, suppose otherwise. Since each hedgehog is non-separating, this can only happen if there exists a bounded Fatou domain $U$ complementary to $H'$. By Sullivan [50] we may assume that $U$ is periodic. Then by Kiwi [21] there exists a repelling or parabolic point $z$ and two rays landing at $z$ such that their union separates $U$ from $p$. However this would imply that $p \in H'$ contradicting the fact that hedgehogs do not contain repelling or parabolic periodic points. Thus, $H'$ is a continuum which does not separate the plane. We then can choose a tight topological disk $V$ containing $H'$ and not containing any critical points. Clearly, the hedgehog $H = H(U)$ generated by $U$ contains $H'$ as desired.

It remains to observe that by the above there exists a critical point $c \in E$ such that $E \supset H \supset H'$ while on the other hand the construction implies that this is impossible. This contradiction shows that we can find a critical point $c_p \in E \subset F^{\mathfrak{R}}_{\mathbb{C}}(p)$ with $M_p \subset \omega(c_p)$. Observe that by (2) distinct fixed CS-points $p$ correspond to distinct recurrent critical points. The result for periodic CS-points can be proven similarly. Summing up over all CS-cycles we get the last claim of the theorem. □

## 5. Main theorem for connected Julia sets

Section 5 contains the proof of the main theorem in the connected case (see Theorem 5.5). We find an upper bound on the number of dynamical phenomena such as non-repelling cycles and wandering non-(pre)critical branch continua which inevitably has to depend on the degree of the polynomial. We also suggest a bound which depends on specific types of critical points of a map. This is reflected in Theorem 1.1 and Theorem 1.2, where we speak *only* of weakly recurrent critical points and escaping critical points (the latter does not apply in the case of connected Julia sets). As we will see, the critical points which we need to use can be drawn from an even more narrow class.

Below we first study all-critical recurrent gap-leaves of $\overline{\mathcal{L}}^{\mathfrak{R}}$. Note that a disk fiber with a critical leaf on its boundary cannot be a leaf of $\mathcal{L}^{\mathcal{B}_{\mathbb{C}},\mathfrak{R}}$ because there are no critical leaves in $\mathcal{L}^{\mathcal{B}_{\mathbb{C}},\mathfrak{R}}$. Hence by Lemma 3.13 a disk fiber of $\overline{\mathcal{L}}^{\mathcal{B}_{\mathbb{C}},\mathfrak{R}}$ with a critical leaf on its boundary is a gap-leaf of $\overline{\mathcal{L}}^{\mathcal{B}_{\mathbb{C}},\mathfrak{R}}$; by Lemma 2.5 this disk fiber is an all-critical gap-leaf.

**Lemma 5.1.** *Let $G_1, \dots, G_l$ be the all-critical gap-leaves of $\overline{\mathcal{L}^{\mathfrak{R}}}$. Then the following properties hold.*

(1) *For each $i$ and $m$, the sets $F^{\mathfrak{R}}_{\mathbb{C}}((\sigma^*)^m(G_i))$ are disjoint from impressions of all angles not from $(\sigma^*)^m(G'_i)$ and do not contain preimages of points of* AN*; moreover, $P^k(F^{\mathfrak{R}}_{\mathbb{C}}(G_i)) \cap F^{\mathfrak{R}}_{\mathbb{C}}(G_i) = \emptyset$ for any $k > 0$. In particular:*
   (a) *$F^{\mathfrak{R}}_{\mathbb{C}}(G_i)$ contains no periodic points, and if $G_i$ is (pre)periodic then $F^{\mathfrak{R}}_{\mathbb{C}}(G_i) \cap K_P$ is degenerate;*

(b) *any point of any image of $F^{\mathfrak{R}}_{\mathbb{C}}(G^i)$ is weakly separated from any point outside that image.*

(2) *For each $i$ there is at least one critical point in $F^{\mathfrak{R}}_{\mathbb{C}}(G_i)$.*

(3) *$G_i$ is recurrent if and only if any $x \in F^{\mathfrak{R}}_{\mathbb{C}}(G_i)$ is weakly recurrent (in particular, in this case $x$ is not (pre)periodic). If $G_i, G_j$ have distinct grand orbits then any two points from the grand orbits of $F^{\mathfrak{R}}_{\mathbb{C}}(G_i), F^{\mathfrak{R}}_{\mathbb{C}}(G_j)$ can be separated by a set* Tail$(a)$, *$a \in \mathfrak{R}$.*

(4) *If $x_i \in F^{\mathfrak{R}}_{\mathbb{C}}(G_i), x_j \in F^{\mathfrak{R}}_{\mathbb{C}}(G_j)$ with $G_i, G_j$ recurrent, then we have $\omega(G'_i) = \omega(G'_j)$ if and only if for any $a \in \omega(x_i)$ there is $b \in \omega(x_j)$ such that $a$ and $b$ are weakly non-separated and vice versa (in this case call $\omega(x_i)$ and $\omega(x_j)$* weakly equivalent*).*

**Proof.** (1) *We show that, for any $i$ and $m$, $F^{\mathfrak{R}}_{\mathbb{C}}((\sigma^*)^m(G_i))$ contains no preimages of points of* AN. By Lemma 2.5 all boundary leaves of $G_i$ are limit leaves and all $\sigma^*$-images of $G_i$ (which are points because $G_i$ is all-critical) are separated from the rest of the circle by sequences of leaves of $\mathcal{L}^{\mathfrak{R}}$. By Lemma 3.19(1) $F^{\mathfrak{R}}_{\mathbb{C}}((\sigma^*)^m(G_i))$ is disjoint from impressions of all angles not from $(\sigma^*)^m(G'_i)$. So, if $F^{\mathfrak{R}}_{\mathbb{C}}((\sigma^*)^m(G_i))$ contains a point of AN, then by Lemma 3.21(3) there are infinitely many angles with principal sets in $F^{\mathfrak{R}}_{\mathbb{C}}((\sigma^*)^m(G_i))$, a contradiction. Sets $F^{\mathfrak{R}}_{\mathbb{C}}((\sigma^*)^m(G_i))$ are non-separating with no preimages of Cremer points.

*Let us prove the rest of* (1). By Lemma 2.5(1) $\sigma^s(G_i) \cap G_i = \emptyset, s > 0$. By the above, Lemma 3.19(1), and Lemma 3.18, $P^k(F^{\mathfrak{R}}_{\mathbb{C}}(G_i)) \cap F^{\mathfrak{R}}_{\mathbb{C}}(G_i) = \emptyset$ for any $k > 0$. The claims (1a) and (1b) now follow easily.

(2) Since by (1) $\mathrm{Tail}'(F^{\mathfrak{R}}_{\mathbb{C}}(\sigma^*(G_i)))$ is a tree-like continuum, and $P|_{\mathrm{Tail}'(F^{\mathfrak{R}}_{\mathbb{C}}(G_i))} : \mathrm{Tail}'(F^{\mathfrak{R}}_{\mathbb{C}}(G_i)) \to \mathrm{Tail}'(F^{\mathfrak{R}}_{\mathbb{C}}(\sigma^*(G_i)))$ is not one-to-one, by [19] there are critical points in $F^{\mathfrak{R}}_{\mathbb{C}}(G_i)$.

(3) Both claims follow easily from the definitions and Lemma 2.5.

(4) *Let us prove that if $\omega(G'_i) = \omega(G'_j)$ then the sets $\omega(x_i), \omega(x_j)$ are weakly equivalent.* If $a \in \omega(x_i)$, then $P^{s_n}(x_i) \to a$ for a sequence $n_i \to \infty$. Assume that $\sigma^{s_n}(G'_i) \to \alpha \in \omega(G'_i) = \omega(G'_j)$ and choose a sequence $t_n$ such that $\sigma^{t_n}(G'_j) \to \alpha$. Since $G_i$ is recurrent, $\sigma(G'_i) \in \omega(G'_j)$ and hence we may assume that $\sigma^{t_n}(G'_j)$ approach $\alpha$ from the same side as $\sigma^{s_n}(G'_i)$. By compactness we may assume that $P^{t_n}(x_j) \to b$. Let us show that $a$ and $b$ are weakly non-separated. Indeed, otherwise there exists a cut $\mathrm{Cut}^\ell, \ell = \beta\gamma \in \mathcal{L}^{\mathfrak{R}}$ which separates $a$ from $b$. Choose $N$ so large that $P^{s_n}(x_i) \in W^\ell_{\mathbb{C}}(a)$ and $P^{t_n}(x_j) \in W^\ell_{\mathbb{C}}(b)$ for $n \ge N$. Since $a$ and $b$ are separated by $\mathrm{Cut}^\ell$, the open planar wedges $W^\ell_{\mathbb{C}}(a)$ and $W^\ell_{\mathbb{C}}(b)$ are disjoint, and hence disk wedges $W^\ell_{\mathbb{D}}(a)$ and $W^\ell_{\mathbb{D}}(b)$ are disjoint.

Since all points of $P^{s_n}$-images of rays with arguments from $G'_i$ are weakly non-separated from $P^{s_n}(x_i)$, the entire set $P^{s_n}(F_{\mathbb{C}}(G_i))$ is contained in $\widehat{W}^\ell_{\mathbb{C}}(a)$ and hence $\sigma^{s_n}(G'_i)$ belongs to the disk wedge $W^\ell_{\mathbb{D}}(a)$. Analogously, the angles $\sigma^{t_n}(G'_j)$ belong to the disk wedge $W^\ell_{\mathbb{D}}(b)$. This contradicts the fact that $\sigma^{t_n}(G'_j)$ approach $\alpha$ from the *same* side as $\sigma^{s_n}(G'_i)$ and proves that sets $\omega(x_i), \omega(x_j)$ are weakly equivalent.

*Let us now prove that if the limit sets $\omega(x_i), \omega(x_j)$ are weakly equivalent then $\omega(G'_i) = \omega(G'_j)$.* Suppose that $\omega(G'_i) \not\subset \omega(G'_j)$ while sets $\omega(x_i), \omega(x_j)$ are weakly equivalent. Since both $G_i$ and $G_j$ are recurrent, $\omega(G'_i) \not\subset \omega(G'_j)$ implies that $\sigma(G'_i) \notin \omega(G'_j)$. Choose a sequence $\Sigma$ of leaves of $\mathcal{L}^{\mathfrak{R}}$ which converge to $\sigma(G'_i)$ separating it from the rest of $\mathbb{S}^1$. Then leaves of $\Sigma$ eventually separate $\sigma(G'_i)$ from $\omega(G'_j)$ which implies that $P(x_i)$ cannot be weakly non-separated from a point of $\omega(x_j)$, a contradiction with $\omega(x_i), \omega(x_j)$ being weakly equivalent. $\square$

**Definition 5.2.** We introduce the following sets of critical points.

(1) Let $C_{at}$ be the set of critical points, belonging to parattracting periodic Fatou domains.
(2) Let $C_{cs}$ be the set of *recurrent* critical points $c$ belonging to CS-sets.
(3) If $c \in F^{\mathfrak{R}}_{\mathbb{C}}(G)$ is a critical point, where $G$ is an all-critical recurrent gap-leaf of $\mathcal{L}^{\mathfrak{R}}$, then $c$ is called *all-critical (associated to $G$)*; denote by $C^{ac}_{wr}$ the union of all such critical points.

Clearly, $C_{at} \cap C_{cs} = \emptyset$ and $C_{at} \cap C^{ac}_{wr} = \emptyset$. By Theorem 4.1 (see also Lemma 5.1), $C_{cs} \cap C^{ac}_{wr} = \emptyset$. It is clear that $C_{at} \cup C_{cs} \subset C_{wr}$. Since the all critical gap-leaf $G$ is recurrent, $C^{ac}_{wr} \subset C_{wr}$. Now we define an equivalence relation among the limit sets of critical points from $C_{at} \cup C_{cs} \cup C^{ac}_{wr}$ (for points of $C^{ac}_{wr}$ it is already introduced in Lemma 5.1).

**Definition 5.3.** Limit sets $\omega(c), \omega(d)$ of critical points $c, d$ are called *weakly equivalent* if (1) $c, d \in C_{at}$ belong to the same cycle of parattracting Fatou domains, or (2) $c, d \in C_{cs}$ belong to the same CS-set, or (3) $c, d \in C^{ac}_{wr}$ so that for any $a \in \omega(c)$, there is $b \in \omega(d)$ weakly non-separated from $a$, and vice versa.

By Lemma 5.1 the weak equivalence is indeed an equivalence relation.

**Lemma 5.4.** *A critical point $c$ belongs to $C^{ac}_{wr}$ if and only if all its images are weakly separated from points of* AN*, $c$ is weakly recurrent, and $P(c)$ does not belong to a wandering cut-continuum.*

**Proof.** Suppose that $c \in C^{ac}_{wr}$. By Lemma 5.1(1) and by Lemma 3.21(3), all images of $c$ are weakly separated from AN. By the above $c$ is weakly recurrent. Now, by definition there exists an all-critical recurrent gap-leaf $G$ with $c \in F^{\mathfrak{R}}_{\mathbb{C}}(G)$. If $P(c) \in W$ where $W$ is a wandering cut-continuum, then by Lemma 3.1 there are at least two rays with principal sets in $W$. Since by Lemma 5.1 $\sigma(G')$ is the *only* angle whose impression is non-disjoint from $F^{\mathfrak{R}}_{\mathbb{C}}(\sigma(G'))$, then $W \not\subset F^{\mathfrak{R}}_{\mathbb{C}}(\sigma(G'))$. Hence $W$ connects the point $P(c) \in F^{\mathfrak{R}}_{\mathbb{C}}(\sigma(G'))$ to points outside this fiber which implies that $W$ cannot be wandering, a contradiction.

Suppose now that $c$ is a weakly recurrent critical point with all images weakly separated from points of AN, and $P(c)$ does not belong to a wandering cut-continuum. Then $c$ does not map to an attracting or CS-cycle. By definition of weak recurrence $c$ does not map to a parabolic or repelling cycle and $c$ is not (pre)periodic. Let us show that the disk fiber $F^{\mathfrak{R}}_{\mathbb{D}}(P(c))$ is a point separated from the rest of the circle by leaves of $\mathcal{L}^{\mathfrak{R}}$. Indeed, otherwise there are the several cases. First, by Lemma 5.1 $F^{\mathfrak{R}}_{\mathbb{D}}(P(c))$ cannot be an infinite gap because then some image of $c$ will be weakly non-separated from a point of AN. Second, $F^{\mathfrak{R}}_{\mathbb{D}}(P(c))$ cannot be (pre)periodic since otherwise some image of $F^{\mathfrak{R}}_{\mathbb{D}}(P(c))$ is a finite periodic disk fiber which by Lemma 3.21(2) implies that $c$ is (pre)periodic, a contradiction.

Hence $F^{\mathfrak{R}}_{\mathbb{D}}(P(c))$ is a finite wandering disk fiber. If there are more than one angle in its basis, then the associated planar fiber $F^{\mathfrak{R}}_{\mathbb{D}}(P(c))$ is a wandering cut-continuum. Indeed, choose a rational angle in each circle arc adjacent to the basis of $F^{\mathfrak{R}}_{\mathbb{D}}(P(c))$. The corresponding rays have landing points which belong to distinct components of $J_P \setminus F^{\mathfrak{R}}_{\mathbb{C}}(P(c))$, and so $F^{\mathfrak{R}}_{\mathbb{C}}(P(c))$ is a wandering cut-continuum, a contradiction. □

By Lemma 5.4 the set $C^{ac}_{wr}$ can be defined in pure topological terms (without the system of external rays). It is easy to see that the same applies also to the sets $C_{at}$, $C_{cs}$. Thus, in terms of formulations, our results can be viewed as having a topologically dynamical nature. However, of course, the proofs heavily rely upon the combinatorics of the map $\sigma$ and do require constant usage of the system of external rays which allows one to relate this combinatorics and the dynamics of $P$.

Now we prove Theorem 5.5 which implies Theorem 1.1 and Theorem 1.2 in the connected case. The relation between wandering non-(pre)critical branch continua and weak equivalence classes of weakly recurrent critical points is more complicated than that between non-repelling cycles and associated critical points, hence Theorem 5.5 is more quantitative than Theorem 4.3. We use the following notation. For $H \in \{C^{ac}_{wr}, C_{at}, C_{cs}\}$, let $K(H)$ be the number of classes of weak equivalence of grand orbits of points of $H$ and $L(H)$ be the number of classes of weak equivalence of the limit sets of points of $H$.

**Theorem 5.5.** *Consider a non-empty wandering collection $\mathcal{B}_{\mathbb{C}}$ of non-(pre)critical branch continua $\{Q_i\}$. Then*

$$\sum_{\mathcal{B}_{\mathbb{C}}} (\mathrm{val}_{J_P}(Q_i) - 2) \le K(C^{ac}_{wr}) - L(C^{ac}_{wr}) \le K(C^{ac}_{wr}) - 1 \le |C^{ac}_{wr}| - 1$$

*and*

$$N_{FC} = K(C_{at}) + K(C_{cs})$$

*which implies that*

$$\sum_{\mathcal{B}_{\mathbb{C}}} (\mathrm{val}_{J_P}(Q_i) - 2) + N_{FC} \le K(C_{wr}) - 1 \le |C_{wr}| - 1 \le d - 2.$$

**Proof.** By Subsection 3.2, $\mathcal{B}_{\mathbb{C}} = \{Q_i\}$ gives rise to a wandering collection of gaps $\mathrm{Ch}(A(Q_i)) = G(Q_i)$, all non-(pre)critical by Lemma 3.8. Therefore Theorem 2.7 applies to the collection $\{G(Q_i)\} = \mathcal{B}_{\mathbb{D}}$. By Theorem 2.7(1) there are critical leaves which are limits of forward orbits of the sets $G(Q_i)$. By Lemma 3.8 and Corollary 4.2, these leaves come from the boundaries of all-critical gap-leaves of $\overline{\mathcal{L}^{\mathfrak{R}}}$, recurrent by Theorem 2.7(1). Denote the collection of these gap-leaves by $\mathcal{AC}_l$.

If $m$ is the number of distinct grand orbits of elements of $\mathcal{AC}_l$, then by Lemma 5.1(3), (4) $m$ equals the number of classes of weak equivalence of grand orbits of all-critical weakly recurrent points from sets $F^{\mathfrak{R}}_{\mathbb{C}}(H_j), H_j \in \mathcal{AC}_l$. If $l$ is the number of distinct limit sets of elements of $\mathcal{AC}_l$, then by Lemma 5.1(3), (4) $l$ equals the number of classes of weak equivalence of limit sets of all-critical weakly recurrent points from sets $F^{\mathfrak{R}}_{\mathbb{C}}(H_j), H_j \in \mathcal{AC}_l$. By Theorem 2.7(3) $\sum_{\mathcal{B}_{\mathbb{C}}}(\mathrm{val}_{J_P}(Q_i) - 2) \le m - l$.

Now, denote by $\mathcal{AC}$ the collection of *all* all-critical recurrent gap-leaves. By definition and Lemma 5.1(2), (3) *each* all-critical recurrent gap-leaf corresponds to all-critical weakly recurrent point(s) in $J_P$. Again, by Lemma 5.1(3), (4) the number of distinct grand orbits of these gap-leaves equals $K(C^{ac}_{wr})$ and the number of distinct limit sets of these gap-leaves equals $L(C^{ac}_{wr})$. The collection $\mathcal{AC}$ can be obtained by adding new elements to the collection $\mathcal{AC}_l$. Adding one class of weak equivalence of the grand orbit of an all-critical weakly recurrent point to $\mathcal{AC}_l$ increases $m$ by *exactly* 1 and increases the current count for $l$ by *at most* 1. Hence, $\sum_{\mathcal{B}_{\mathbb{C}}}(\mathrm{val}_{J_P}(Q_i) - 2) \le m - l \le K(C^{ac}_{wr}) - L(C^{ac}_{wr})$ as desired. The rest follows from $K(C^{ac}_{wr}) \le |C^{ac}_{wr}|$ and $L(C^{ac}_{wr}) \ge 1$.

The equality $N_{FC} = K(C_{at}) + K(C_{cs})$ follows by definition. Thus,

$$K(C_{at}) + K(C_{cs}) + K(C^{ac}_{wr}) - L(C^{ac}_{wr}) \le K(C_{wr}) - 1 \le |C_{wr}| - 1 \le d - 2;$$

obtained by adding the preceding two inequalities and observing that $C_{at}, C_{cs}$ and $C^{ac}_{wr}$ are pairwise disjoint subsets of $C_{wr}$. □

Let us show how Theorem 1.1 and Theorem 1.2 for connected Julia sets follow from our results (except the parts dealing with disconnected Julia sets). Clearly, Theorem 4.3 implies Theorem 1.1(1) for connected Julia sets (observe that by Lemma 3.4 we can talk about a wandering collection of non-(pre)critical branch continua $Q_i$ and $\mathrm{val}_{J_P}(Q_i)$ instead of eventual continua $\widehat{Q}_i$ and $\mathrm{eval}_{J_P}(\widehat{Q}_i)$). Since points from $C^{ac}_{wr}$ are weakly recurrent and weakly separated from all non-repelling periodic points, then the first inequality of Theorem 5.5 implies Theorem 1.1.

The statement of Theorem 1.2 includes an inequality for connected Julia sets, an inequality concerning phenomena which can happen only in disconnected Julia sets, and their sum. Thus, now it suffices to consider only the first inequality of Theorem 1.2. If $J_P$ is connected and there are no wandering non-(pre)critical branch continua, the constants from Theorem 1.2 are $N_{co} = 0$ and $m = 0$. In this case Theorem 1.2 claims that $N_{FC} \le |C_{wr}|$ and follows from the fact that $N_{FC} = K(C_{at}) + K(C_{cs}) \le |C_{at}| + |C_{cs}| \le |C_{wr}|$. If there is a non-empty wandering collection $\mathcal{B}_{\mathbb{C}}$ of non-(pre)critical branch continua $\{Q_i\}$, then $N_{co} = 1, m > 0$ and Theorem 1.2 claims that $N_{FC} + 1 + \sum_{i=1}^{m}(\mathrm{val}_{J_P}(Q_i) - 2) \le |C_{wr}|$ which is what Theorem 5.5 proves.

In the rest of the paper we deal with disconnected Julia sets. Before we switch to them, we would like to comment on an important difference between the connected and the disconnected cases. As was mentioned in remark (5) in the Introduction, in the connected case the objects involved in the inequality are all of topological nature and can be defined with no regards to the system of external rays. That system plays a crucial role in the proofs, but can be avoided as one states the results in the connected case.

This is not so in the disconnected case. More precisely, there are two notions which simply cannot be defined without invoking the system of external rays. These are the notion of the valence of a wandering component of $J_P$ and the notion of a periodic repelling point at which infinitely many rays land. E.g., the fact that a component $A$ of $J_P$ is wandering, is independent of the system of rays. However the number of rays accumulating in $A$ cannot be defined in a way which does not depend on the system of rays (as the valence in the connected case)

## 6. External rays to periodic components of the Julia set

This section enables us to use the results for connected Julia sets on $p$-periodic non-degenerate components $E$ of a disconnected Julia set. We relate the (polynomial-like) map $P^p$ on a neighborhood of $E$ to a polynomial $f$, with connected Julia set $J_f$, such that $P^p|_E$ and $f|_{J_f}$ are conjugate, and establish a connection between external rays of $P$, with principal sets in $E$, and external rays of $J_f$.

Fix an *arbitrary* polynomial $P$ of degree $d$, with not necessarily connected Julia set. Set $U^\infty = U^\infty(J_P)$. The *equipotential* containing a point $z \in U^\infty$ is defined as the closure of the union of all preimages $P^{-n}(P^n(z))$, $n = 1, 2, \ldots$ [49]. Then $U^\infty$ is foliated by equipotentials defined by the dynamics of $P$. Critical points $c \in U^\infty$ are called *escaping*. Denote by $C_*$ the set of all preimages $P^{-n}(c)$, $n = 0, 1, 2, \ldots,$ of escaping critical points $c$. A component of an equipotential is a smooth curve if and only if it does not contain a point of $C_*$.

The *flow of external rays* of $P$ is defined as the gradient flow to the equipotentials. More precisely, by an *external ray* $R_t$ of $P$ we mean an unbounded curve $R$, such that either $R$ is smooth, crosses every equipotential orthogonally and terminates in the Julia set of $P$, or $R$ is a one-sided limit of such smooth rays (then the ray is called *non-smooth* or *one-sided*). An external ray is smooth if and only if it is disjoint with $C_*$. Every point of $U^\infty$ belongs to an external ray, and smooth external rays are dense in $U^\infty$. *Every* external ray, whether smooth or not, accumulates in one component of $J$.

The argument $t \in \mathbb{R}/\mathbb{Z}$ of $R_t$ is defined uniquely as the angle at which $R_t$ goes asymptotically to infinity. If the ray is non-smooth, then there is precisely one more (non-smooth) external ray with the same argument. Nevertheless, this will not cause ambiguity, because we will be speaking about external rays rather than their arguments. Observe that if a ray is periodic then its argument must be periodic. Vice versa, if an argument of a ray is periodic, then the ray must be periodic. For the general theory of external rays, see [3], and for the theory of external rays of polynomials with disconnected Julia sets, see, e.g., [18,29,28].

The equipotentials and external rays for the polynomial $P_0(z) = z^d$ are standard circles $|z| = \exp(a)$, $a > 0$, and rays $\{r \exp(2\pi it) : r > 1\}$, $t \in \mathbb{R}/\mathbb{Z}$, respectively. A more traditional way to define equipotentials and external rays for an arbitrary polynomial $P$ is as follows. The map $P$ is conjugate to $P_0$ in a neighborhood of infinity by a univalent change of coordinates $B$ (the *Böttcher coordinates*). Then the equipotentials and rays of $P$ near infinity are the preimages by $B$ of the standard circles and rays respectively near infinity. By applying branches of the inverse function $P^{-n}$, the equipotentials and rays are spread over the entire basin of infinity $U^\infty$.

The *level* of a point $z \in U^\infty$ is a positive number $a = a(z)$ defined as follows. If $|z|$ is large enough, then $B(z)$ is well defined, and $a(z)$ is said to be the number $\log |B(z)|$. For any other $z$, we choose $n > 0$, such that $|P^n(z)|$ is large, and set $a(z) = d^{-n}a(P^n(z)) = d^{-n} \log |B(P^n(z))|$. It is easy to see that $a(z)$ is well defined (in fact, $a(z)$ is the so-called *Green's function* of $U^\infty$). The levels of two points are equal if and only if they belong to the same equipotential. Therefore, one can define the *level of an equipotential* as the level of a point of the equipotential.

The level function also defines the direction from infinity to $J_P$ on every external ray. For any external ray $R$, the function $a$ restricted to $R$ decreases monotonically from $+\infty$ near $\infty$ to 0 near the Julia set. In particular, every external ray is homeomorphic to the standard (open) ray $\mathbb{R}_+ = \{x > 0\}$. Every subarc of an external ray starts either at infinity or at a finite point of $U^\infty$, and either ends at another point of $U^\infty$ or accumulates in the Julia set.

The equipotential of level $a_0$ splits the plane into finitely many open components, so that the level of a point in the unbounded component is strictly bigger than $a_0$, and the level of a point in the bounded components is strictly smaller than $a_0$. If two points $z_1$, $z_2$ lie in different bounded components of the complement of an equipotential of a given level, then the subarcs of the external rays through these points between $z_1$ and $J_P$, and between $z_2$ and $J_P$ respectively, are disjoint (even their closures are disjoint).

Obviously, *all* equipotentials as well as external rays are smooth if and only if the Julia set is connected, or, equivalently, the set $C_*$ is empty. In this case $B$ extends to a Riemann map from $U^\infty$ onto the complement of the unit disk, and one can define equipotentials and rays of $P$ directly by taking preimages by $B$ of the standard circles and rays outside the unit disk.

In the rest of Section 6, we assume that $J_P$ is *not* connected. Then $J_P$ has infinitely many components. Consider $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, always understanding it as a circle at infinity (e.g., arguments of external rays belong to $\mathbb{S}^1$). Denote by $\mathbb{D}^*$ the exterior of the closed unit disk, and let $S^1$ be its boundary, always understood as a subset of the plane. As usual, we consider the map $\sigma : z \mapsto z^d$ for $z \in S^1$. We also denote the map $t \mapsto dt$ of $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ to itself by $\sigma$. The following lemma, though simple, serves as a useful tool in what follows.

**Lemma 6.1.** *If two different rays $R, R'$ have a common point, then $R, R'$ are both non-smooth. The intersection $L = R \cap R'$ is connected and can contain a point of $C_*$ only as an endpoint. Furthermore, one and only one of the following cases holds:*

(i) *$L$ is a smooth curve joining infinity and a point of $C_*$,*
(ii) *$L$ is a single point of $C_*$,*
(iii) *$L$ is a smooth closed arc between two points of $C_*$,*
(iv) *$L$ is a smooth curve from a point of $C_*$ to $J_P$ and, moreover, the rays $R, R'$ are not periodic.*

*Except for the last case, the rays $R, R'$ have their principal sets in different components of $J_P$.*

**Proof.** A smooth ray is disjoint from all other rays. Now, assume that two *different* non-smooth rays $R, R'$ are not disjoint. Since rays fill up $U^\infty$ and smooth rays are dense in $U^\infty$, the intersection $L$ of $R$ and $R'$ is a connected set (otherwise there is a "lake", i.e. a component of $\mathbb{C} \setminus [R \cup R']$, unreachable by smooth rays). Hence $L$ is either (i) a smooth curve from infinity to a point in $U^\infty$, or a (ii) single point, or (iii) a smooth closed arc between two points in $U^\infty$, or (iv) a smooth curve from a point of $U^\infty$ to $J_P$ (a smooth curve from infinity to $J_P$ is impossible as $R \neq R'$).

Let us show next that $L$ can only contain points of $C_*$ as endpoints. Let $q \in C_*$ be a point of $L$. Suppose by way of contradiction that $q$ is not an endpoint of $L$. Consider the component $\gamma$ of the equipotential through the point $q$. Then $q$ is a *singular point* (*branch point*) of $\gamma$, and $\mathbb{C} \setminus \gamma$ contains at least two bounded components with the only joint point on their boundaries to be $q$.

Let $U_1, \dots, U_m$ be the bounded components of $\mathbb{C} \setminus \gamma$ containing $q$ in their closures. Let $U_1$ be the component containing points of $L$. Choose a neighborhood $W$ of $q$ such that $W \setminus \overline{\bigcup U_i}$ consists of $m$ open components $V_1, \dots, V_m$. Since $q$ is not an endpoint of $L$, $L$ intersects *only* one of the sets $V_1, \dots, V_m$, say, $V_1$. However, as $L$ is approached by smooth rays converging to $R, R'$ from two *distinct* sides (of $L$), the smooth rays located on distinct sides of $L \cap V_1$ must enter distinct sets $U_i$, a contradiction with $U_1$ being the component containing points of $L$. So, $q$ is an endpoint of $L$.

Note that if $z \in R$ (resp., $z \in R'$) and $z \notin C_*$, then, in a neighborhood of $z$, $R$ (resp., $R'$) is a *smooth* curve. Hence, (i) if $L$ is a smooth curve from infinity to a point in $U^\infty$, then it joins infinity and a point of $C_*$, (ii) if $L$ is a single point, then it is a point of $C_*$, and (iii) if $L$ is a smooth closed arc, then its endpoints belong to $C_*$. The remaining possibility is that $L$ is a smooth curve joining a point of $C_*$ and $J_P$. Let us show that in this case neither $R$ nor $R'$ can be periodic. Indeed if $R$ is periodic and contains a point $q \in C_*$, then $R$ contains infinitely many preimages of $q$ converging to $J_P$. Hence $L$ would contain infinitely many preimages of $q$, a contradiction. $\square$

**Example 6.2.** The cases (i)–(iii) are already possible for quadratic polynomials $z^2 + c$ with $c > 1/4$. Case (i) is realized for the two one-sided rays $R_{0^+} = \lim_{t\to 0^+} R_t$, $R_{0^-} = \lim_{t\to 1^-} R_t$, so that the intersection of $R_{0^+}$ and $R_{0^-}$ is the positive real axis. Case (ii) happens for the rays $R_{0^+}$ and $R_{1/2^+} = \lim_{t\to 1/2^+} R_t$, with $R_{0^+} \cap R_{1/2^+} = \{0\}$. Case (iii) holds if there are two points from $C_*$ on the same ray, e.g., the intersection of $R_{0^-}$ and $R_{1/2^+}$ is an arc joining 0 and the first preimage of 0 in the lower half plane. Finally, if $P(z) = z^2 + c$ with $c > 1/4$, then any non-smooth ray is (pre)periodic which by Lemma 6.1 makes case (iv) impossible for $P$. But it is realized for any $z^2 + c$ with $c$ outside of the Mandelbrot set, for which the external arguments of 0 are not periodic.

Given $E \subset K_P$, let $A(E)$ be the set of the arguments of all external rays with principal sets in $E$ (clearly, these principal sets are in fact contained in $\mathrm{Bd}(E) \subset J_P$). Similarly, for $z \in U^\infty$ let $A(z)$ be the set of the arguments of all external rays containing $z$ (since for every ray its argument is well-defined, the definition is consistent). For $z \in U^\infty$ any angle from $A(z)$ is said to be an *(external) argument* of $z$. Lemma 6.3 is simple and well-known; we add it here for the sake of completeness.

**Lemma 6.3.** *For a component $E$ of the filled-in Julia set $K_P$ of $P$, the set $A(E)$ is a non-empty compact subset of $\mathbb{S}^1$.*

**Proof.** Take the arguments of all external rays that cross a component $\gamma$ of the equipotential of a given level $a > 0$ and enter the bounded component of $\mathbb{C} \setminus \gamma$ which contains $E$. It is a non-empty compact subset $A_a(E)$ of $\mathbb{S}^1$. As $a \to 0$, these compacta shrink to a non-empty compact set, which is the set $A(E)$. □

**Theorem 6.4.** *(See Theorem 1 of [28].) Let $z$ be a repelling or parabolic periodic point of $P$ of period $m$. Then the following claims hold.*

1. *$A(z)$ is a non-empty compact subset of $\mathbb{S}^1$, invariant under $\sigma^m$.*
2. *If $A(z)$ is infinite, then the point $\{z\}$ is a periodic component of $K_P$. The set $A(z)$ contains external arguments $t_q, t'_q$ of a critical point $q \in U^\infty$ of $P^m$. Moreover, the set $A(z)$ is a Cantor set, and every forward $\sigma^m$-orbit in $A(z)$ is dense in $A(z)$.*
3. *If $\{z\}$ is not a component of $K_P$, then $A(z)$ is finite.*
4. *The set $A(z)$ is finite if and only if it contains a periodic point. In this case every $t \in A(z)$ is periodic under $\sigma^m$, all with the same period.*

From now on assume that $E$ is a periodic *non-degenerate* component of $K_P$ of period $p$. It happens if and only if $P^p$ has a critical point in $E$. Since $P$ is a polynomial, by the Maximum Principle, $E$ does not separate the plane. Fix such $E$, and denote by $\psi : \mathbb{C} \setminus E \to \mathbb{D}^*$ the Riemann map of the exterior of $E$ onto the exterior of the unit disk, with $\psi(z) \sim kz$ as $z \to \infty$, for some $k > 0$.

For a non-closed curve $l$ from infinity or a finite point in $\mathbb{C}$ to a bounded region in $\mathbb{C}$, we can define its *principal set* $\mathrm{Pr}(l)$ analogously to how it is done for conformal external rays (see Subsection 2.3). For a continuum $M$, a curve $l$ with $\Pr(l) \subset M$ is called a *curve to* $M$ (e.g., this terminology applies to some rays). If $R$ is an external ray of $P$ then $\psi(R)$ is a curve in $\mathbb{D}^*$; the *argument* of $\psi(R)$ is set to be the argument of $R$. An external ray $R$ (of $P$) to $E$ has $\psi$-image $\hat{R} := \psi(R)$. Then $\hat{R}$ is called an *E-related ray* (see [28]) if and only if $\Pr(\hat{R}) \subset S^1$. Each $E$-related ray is a curve from $\infty$ to $S^1$.

The $E$-related ray $\hat{R}$ is called *(non-)smooth* if and only if the external ray $R$ is (non-)smooth. Fix a simply-connected neighborhood $V$ of $E$ bounded by an equipotential. Choose a component $U$ of $P^{-p}(V)$, that is also a neighborhood of $E$. One can assume further that $P^p$ has no critical points in $\bar{U} \setminus E$. Denote $\hat{V} = \psi(V \setminus E), \hat{U} = \psi(U \setminus E)$. Note that $\hat{V}, \hat{U}$ are "annuli" with the inner boundary $S^1$. Call the intersections of $E$-related rays with $\hat{V}$ *E-related arcs* (of $E$-related rays).

The Riemann map $\psi$ induces a conjugated map $g : \hat{U} \to \hat{V}$ as follows: $g = \psi \circ P^p \circ \psi^{-1}$. It is well known that $g$ extends through $S^1$ to an analytic map in a neighborhood of $S^1$, and, moreover, $g$ is expanding: there are $n > 0$ and $\lambda > 1$, such that, $|(g^n)'(w)| > \lambda$ provided $g^n(w)$ lies in the closure of $\hat{U}$, see [41,14]. (Proof: by the Reflection Principle [2], $g$ extends to a holomorphic (unbranched) covering map $g : A \to B$, where $A \subset B$ are "annuli" containing $S^1$ in their interiors, and $A$ is compactly contained in $B$. Then $g$ is lifted to a univalent map $\hat{g} : \hat{A} \to \hat{B}$ where $\hat{A} \subset \hat{B}$ and $\hat{B}$ is the universal cover of $B$. It follows that the inverse map $\hat{g}^{-1}$ strictly contracts the hyperbolic metric on $\hat{B}$ which implies the expanding property of $g$.)

Now, $g$ maps intersections of $E$-related rays with $\hat{U}$ onto $E$-related arcs. Abusing the notation, say that $g$ *maps E-related rays to E-related rays* (i.e., $g$ maps an $E$-related ray of argument $t$ to an $E$-related ray of argument $\sigma^p(t)$). A curve $l : \mathbb{R} \to \mathbb{D}^*$ with $\lim_{t\to\infty} l(t) = \{w\} \subset S^1$ approaches $w$ *non-tangentially* if for some $T$ the set $l([T, \infty))$ is contained in a sector of angle less than $\pi$ with the vertex at $w$ symmetric with respect to the standard ray through 0 and $w$.

**Lemma 6.5.** *(See Lemma 2.1 of [28].) The following claims hold.*

(1) *Every E-related arc has a finite length, and hence lands at a unique point of $S^1$.*
(2) *Every point $w \in S^1$ is a landing point of at least one E-related ray, and the arguments of the E-related rays landing at $w$ form a compact subset of $\mathbb{S}^1$.*
(3) *An E-related arc $l$ goes to a point $w_l \in S^1$ non-tangentially.*

**Sketch of the proof.** Part (1) holds as $g$ is uniformly expanding, so the local branches of inverses $g^{-k}$ are uniformly exponentially contracting as $k \to \infty$. For part (2) notice, that by Lemma 6.3, there is at least one $E$-related ray. If we take preimages of an $E$-related ray by all branches of $g^{-k}$, we see (since $g$ is expanding) that $E$-related rays land inside every arc on $S^1$. By the intersection of compacta we get a non-empty compact set of $E$-related rays landing at a given point of $S^1$. $\square$

**Theorem 6.6.** *(See Theorem 2 of [28].) If $a \in E$ is accessible from the complement of $E$, then $a$ is accessible by an external ray of $P$. More precisely, if a curve $l \in \mathbb{C} \setminus E$ converges to $a$, then there exists an external ray $R$ of $P$, which lands at $a$ and is such that $l$ and $R$ are homotopic among the curves in $\mathbb{C} \setminus E$ which land at $a$.*

**Sketch of the proof.** Indeed, if a point $a$ of $E$ is accessible by a curve $l$ from outside of $E$, then the curve $\psi(l)$ lands at a point $w$ of $S^1$ and $a \in \mathrm{Bd}(E)$. Consider an $E$-related ray $L$ landing at $w$. By Proposition 6.5(2), it exists, and by Proposition 6.5(3), it tends to $w$ non-tangentially. Hence, by Lindelöf's theorem (see, e.g., Theorem 2.16 of [40]), $\psi^{-1}(L)$ and $l$ tend to the same point $a$. $\square$

Lemma 6.7 studies periodic points of $g|_{S^1}$ and $P^p|_E$.

**Lemma 6.7.** *Let $w \in S^1$ be a periodic point of $g|_{S^1}$. Then the non-tangential limit of $\psi^{-1}$ at $w$ exists and is a repelling or parabolic periodic point of $P$ in $\mathrm{Bd}(E)$. Moreover, the set of $E$-related rays landing at $w$ is finite, and each of them is periodic of the same period.*

**Proof.** Let $l$ be a curve in $\mathbb{C} \setminus E$ with its principal set in $E$, invariant under some iterate $P^k$ of $P$. Then $l$ lands at a periodic point $a \in \mathrm{Bd}(E)$ of $P$ (the proof goes back to Fatou, see [17], p. 81, and also [39,42]). By the Snail Lemma (see, e.g., [34]), $a$ is repelling or parabolic. If $w$ is of period $m$, it is easy to find a $g^m$-invariant curve $\gamma$ landing at $w$; then the curve $l = \psi^{-1}(\gamma)$ is $P^m$-invariant and, by the above, accumulates on a repelling or parabolic point $a \in \mathrm{Bd}(E)$ of $P$. By Lindelöf's theorem, $a$ is the non-tangential limit of $\psi^{-1}$ at $w$. The remaining claim of the lemma follows from Theorem 6.4(3). $\square$

The map $G := P^p : U \to V$ is a polynomial-like map of degree $m \geq 2$, such that $E$ is the (connected) filled-in Julia set $K_G = \{z : G^n(z) \in U, n = 0, 1, 2, \dots\}$ of $G$. The map $g$ defined above is called in [14] the *external map* of $G$. By [14], $G : U \to V$ is *hybrid equivalent* to a polynomial $f$ of degree $m$, i.e. there is a quasiconformal homeomorphism $h$ defined on $V$, which is conformal a.e. on $E$, such that $f \circ h = h \circ G$ in $U$. The map $h$ is called the *straightening map*. The filled-in Julia set $K_f = h(E)$ of $f$ is connected. Hence, the Böttcher coordinate $B$ of $f$ is well defined in the basin of attraction of infinity $\mathbb{C} \setminus K_f$ of $f$. We have there that $B(f(z)) = (B(z))^m$.

Since $K_f$ is connected, external rays of $f$ are smooth. For an external ray $R^f_\tau$ of $f$ of argument $\tau$, its $h^{-1}$-image $l_\tau := h^{-1}(R^f_\tau)$ in $V$ is called the *polynomial-like ray (to $E$)* of argument $\tau$. Fix the straightening map $h$; then the polynomial-like rays are well-defined. As $h : V \to h(V)$ is a homeomorphism, $\Pr(l_\tau) = h^{-1}(\Pr(R^f_\tau))$. Below we refer to different planes and objects in them by the names of maps acting in them. Thus, $E$-related rays lie in the $g$-plane, external rays of $P$ and polynomial-like rays are in the $P$-plane, etc.

The main results of the present section are Theorems 6.8 and 6.9 below. They complete Proposition 6.5 and Theorem 6.6.

**Theorem 6.8.** *Any point $w \in S^1$ is the landing point of precisely one $E$-related ray, except for when one and only one of the following holds:*

(i) *$w$ is the landing point of exactly two $E$-related rays, which are non-smooth and have a common arc that goes from a point of $\psi(C_*)$ to the point $w$;*
(ii) *$w$ is a landing point of at least two disjoint rays in which case $w$ is a (pre)periodic point of $g$ and some iterate $g^n(w)$ belongs to a finite (and depending only on $E$) set $\hat{Y}(E)$ of $g|_{S^1}$-periodic points each of which is the landing point of finitely many, but at least two, $E$-related rays, which are smooth and periodic of the same period depending on the landing point.*

*Moreover, if $w$ is periodic then* (i) *cannot hold.*

**Proof.** Assume that there are two $E$-related rays landing at a point $w \in S^1$ and that (i) does not hold. We need to prove that then (ii) holds. If (i) does not hold, then there exist disjoint $E$-related rays landing at $w$. Let us study this case in detail.

Associate to any such pair of rays an open arc $(\hat{R}_t, \hat{R}_{t'})$ of $\mathbb{S}^1$ ($\mathbb{S}^1$ is viewed as the circle at infinity in the $g$-plane) as follows. Two points of $\mathbb{S}^1$ with the arguments $t, t'$ split $\mathbb{S}^1$ into two arcs. Let the arc $(\hat{R}_t, \hat{R}_{t'})$ be the one of them that contains no arguments of $E$-related rays except for possibly those that land at $w$. Geometrically, it means the following. The $E$-related rays $\hat{R}_t, \hat{R}_{t'}$ together with $w \in S^1$ split the plane into two domains. The arc $(\hat{R}_t, \hat{R}_{t'})$ corresponds to the one of them, disjoint from $S^1$. Let $L(\hat{R}_t, \hat{R}_{t'}) = \delta$ be the angular length of $(\hat{R}_t, \hat{R}_{t'})$. Clearly, $0 < \delta < 1$. Now we make a few observations.

(1) *If $E$-related disjoint rays of arguments $t_1, t_1'$ land at a common point $w_1$ while $E$-related disjoint rays of arguments $t_2, t_2'$ land at a point $w_2 \neq w_1$, then the arcs $(\hat{R}_{t_1}, \hat{R}_{t_1'})$, $(\hat{R}_{t_2}, \hat{R}_{t_2'})$ are disjoint.*

This follows from the definition of the arc $(\hat{R}_t, \hat{R}_t)$.

(2) *If disjoint $E$-related rays $\hat{R}_t, \hat{R}_{t'}$ of arguments $t, t'$ land at a common point $w$, then $E$-related rays $g(\hat{R}_t), g(\hat{R}_{t'})$ are also disjoint and land at the common point $g(w)$. Moreover,*

$$L(g(\hat{R}_t), g(\hat{R}_{t'})) \geq \min\{d^p\delta (\operatorname{mod} 1), 1 - d^p\delta (\operatorname{mod} 1)\} > 0.$$

Indeed, the images $g(\hat{R}_t), g(\hat{R}_{t'})$ are disjoint near $g(w)$, because $g$ is locally one-to-one. By Lemma 6.1, $g(\hat{R}_t) \cap g(\hat{R}_{t'}) = \emptyset$. Since the argument of $g(\hat{R}_t)$ is $\sigma^p(t) = d^p t (\operatorname{mod} 1)$, we get the inequality of (2).

Let us consider the following set $\hat{Z}(E)$ of points in $S^1$: $w \in \hat{Z}(E)$ if and only if there is a pair of disjoint $E$-related rays $\hat{R}, \hat{R}'$, which both land at $w$, and such that $L(\hat{R}, \hat{R}') \geq 1/(2d^p)$. Denote by $\hat{Y}(E)$ a set of periodic points which are in forward images of the points of $\hat{Z}(E)$.

(3) *If the set $\hat{Z}(E)$ is non-empty, then it is finite, and consists of (pre)periodic points.*

Indeed, $\hat{Z}(E)$ is finite by (1). Assume $w \in \hat{Z}(E)$. Then, by (2) some iterate $g^n(w)$ must hit $\hat{Z}(E)$ again.

To complete the proof, choose disjoint $E$-related rays $\hat{R}_t, \hat{R}_{t'}$ landing at $w \in S^1$ and use this to prove that all claims of (ii) hold.

*We show that the orbit* $w, g(w), \ldots$ *cannot be infinite.* Indeed, otherwise by (1)–(2), we have a sequence of non-degenerate pairwise disjoint arcs $(g^n(\hat{R}_t), g^n(\hat{R}_{t'})) \subset \mathbb{S}^1$, $n = 0, 1, \ldots$. By (2), some iterates of $w$ must hit the finite set $\hat{Z}(E)$ and hence $\hat{Y}(E)$ (which are therefore non-empty), a contradiction.

*Hence for some* $0 \le n < m$, $g^n(w) = g^m(w)$; *let us verify that other claims of* (ii) *holds.* Replacing $w$ by $g^n(w)$, we may assume that $w$ is a (repelling) periodic point of $g$ of period $k = m - n$. By (2), $w \in \hat{Y}(E)$. By Theorem 6.4, the set of $E$-related rays landing at $w$ is finite, and each $E$-related ray landing at $w$ is periodic with the same period. By Lemma 6.1, each such ray is also smooth. Hence, (ii) holds. Finally, the last claim of the lemma follows from by Lemma 6.1. □

Let the set $Y(E)$ be the set of non-tangential limits of $\psi^{-1}$ at the points of $\hat{Y}(E)$; by Lemma 6.7 $Y(E)$ is a well-defined finite set of repelling or parabolic periodic points of $P$ in $\mathrm{Bd}(E)$. By Theorem 6.8 all external rays landing at points in $Y(E)$ are smooth, and at each point finitely many, but at least two, land. All rays landing at the same point in $Y(E)$ have the same period.

**Theorem 6.9.** *For each external ray $R$ to $E$ there is exactly one polynomial-like ray $l = \lambda(R)$ with $\mathrm{Pr}(l) = \mathrm{Pr}(R)$ and the curves $l$ and $R$ homotopic in $\mathbb{C} \setminus E$ among curves with the same limit set.*

*Moreover, $\lambda : R \mapsto l$ maps the set of external rays to $E$* **onto** *the set of polynomial-like rays to $E$, and is "almost injective": $\lambda$ is one-to-one except for when one and only one of the following holds. Suppose that $\lambda^{-1}(\ell) = \{R_1, \ldots, R_k\}$ with $k > 1$. Then either:*

(i) *$k = 2$ and both rays $R_1, R_2$ are non-smooth and share a common arc to $E$, or*
(ii) *there is a (pre)periodic point $z$ such that $\mathrm{Pr}(R_i) = \{z\}, i = 1, \ldots, k$, at least two of the rays $R_1, \ldots, R_k$ are disjoint, and, for some $n \ge 0$, $P^{pn}(z)$ belongs to $Y(E)$.*

**Proof.** Let $h$ be a quasiconformal homeomorphism defined on a neighborhood of $E$ which conjugates $P$ (restricted on a smaller neighborhood) to a polynomial $f$ with connected Julia set $h(E)$ restricted to a neighborhood of $h(E)$. We can extend the map $h$ onto the entire $\mathbb{C}$ as a quasiconformal homeomorphism even though the conjugacy between $P$ and $f$ will only hold on a neighborhood of $E$. Let $B : \mathbb{C} \to \mathbb{D}^*$ be the Böttcher uniformization map of $f$.

Consider the map $\Psi := \psi \circ h^{-1} \circ B^{-1} : \mathbb{D}^* \to \mathbb{D}^*$ from the uniformization plane of the polynomial $f$ to the $g$-plane. It is a quasiconformal homeomorphism which leaves $S^1$ invariant. For $c \in S^1$, let $L_c = \Psi(r_c \cap \mathbb{D}^*)$ where $r_c = \{tc : t > 0\}$ is a standard ray in the uniformization plane of $f$.

**Claim A.** *The curve $L_c$ tends non-tangentially to a unique point $w_0$ of the unit circle $S^1$. Moreover, for every $w \in S^1$ there exists a unique $c$ such that $L_c$ lands on $w$.*

**Proof of Claim A.** This follows from properties of quasiconformal mappings [1]. Extend $\Psi$ to a quasiconformal homeomorphism $\Psi^*$ of $\mathbb{C}$, symmetric with respect to $S^1$, by the symmetry $\zeta \mapsto 1/\overline{\zeta}$ with respect to $S^1$. Consider the curve $L_c^* = \Psi^*(r_c)$. It is an extension of the curve $L_c$, which crosses $S^1$ at the point $w_0 = \Psi^*(c)$. As a quasiconformal image of the straight line, the curve $L_c^*$ has the following property [1]: there exists $C > 0$, such that $|w - w_0|/|w - 1/\overline{w}| < C$, for every $w \in L_c^*$. Therefore, $L_c^*$ tends to $w_0$ non-tangentially. The last claim follows from the fact that $\Psi^*$ is a homeomorphism. $\square$

Let $R$ be an external ray to $E$. By Proposition 6.5(3), the $E$-related ray $\hat{R} = \psi(R)$ tends to a point $w_0 \in S^1$ non-tangentially. By Clam A there exists a unique $L_c$ which lands at $w_0$. Set $\lambda(R) = \psi^{-1}(L_c)$. By Lindelöf's theorem, $R = \psi^{-1}(\hat{R})$ and $l = \psi^{-1}(L_c)$ have the same limit set in $E$. Since $\hat{R}$ and $L_c$ are homotopic among the curves which land at $w_0$ non-tangentially, the claim about homotopy follows. By Claim A, the map $\lambda$ is onto. Observe that the conditions that $\Pr(R) = \Pr(\ell)$ and that $R$ and $\ell$ are homotopic outside $E$ among curves with the same limit set, uniquely determine the polynomial-like ray $\lambda(R)$.

It remains to prove the "almost injectivity" of $\lambda$. This is a direct consequence of Theorem 6.8 and the construction above. $\square$

Now we study wandering continua in the disconnected case. Let us make some remarks. If a wandering continuum $W$ is contained in a (pre)periodic component of $J_P$, the situation is like the connected case, thanks to Theorem 6.9; otherwise, the entire component of $J_P$ containing $W$ wanders. A continuum $W \subset J_P$ is called *a wandering cut-continuum (of $J_P$)* if (1) $W$ is a wandering *component* of $J_P$ with at least two external rays accumulating in $W$, or (2) $W \subset E$, where $E$ is a (pre)periodic component of $J_P$ and $W$ is a wandering cut-continuum of $E$. The set $\mathrm{Tail}(W)$ can be defined in the disconnected case as in the connected case (only now some rays accumulating in $W$ may be non-smooth).

Let us now reprove Lemma 3.4 in the disconnected case. For convenience we restate it here with necessary amendments.

**Lemma 6.10.** *If $W$ is a wandering cut-continuum of $J_P$, then $P^n|_{\mathrm{Tail}(W)}$ is not one-to-one if and only if* $\mathrm{Tail}(W)$ *contains a critical point of $P^n$ (in this case there are two rays in* $\mathrm{Tail}(W)$ *mapped to one ray).*

**Proof.** If $W$ is contained in a (pre)periodic component $E$ of $J_P$, then the claim follows from the proof of the original Lemma 3.4 and Theorem 6.9 (recall, that Lemma 3.3 holds for arbitrary Julia sets).

Let $W$ be a wandering component of $J_P$. By Lemma 6.3 the set $A(W)$ of arguments of all rays accumulating in $W$ is non-empty and compact. If $P^n|_{\mathrm{Tail}(W)}$ is not one-to-one, then, as before, by [19] and Lemma 3.3, there is a critical point $c$ of $P^n$ in $\mathrm{Tail}(W)$. Now, let $c \in \mathrm{Tail}(W)$ be a critical point of $P^n$. If $c \in W$, then using $A(W) \neq \emptyset$ and repeating the arguments from Lemma 3.4 we complete the proof. If, however, $c \notin W$ then $c$ belongs to a ray included in $\mathrm{Tail}(W)$, and Lemma 6.1 completes the proof. □

Lemma 6.10 shows that Definition 3.5 can be given in the disconnected case in literally the same way, as in the connected case. That is, a wandering continuum $W \subset J_P$ is said to be *non-(pre)critical* if $\mathrm{Tail}(W)$ is such that for every $n$ the map $P^n|_{\mathrm{Tail}(W)}$ is one-to-one. Equivalently, $W$ is non-(pre)critical if and only if $\mathrm{Tail}(W)$ contains no (pre)critical points (or if and only if no iterate of $\mathrm{Tail}(W)$ contains a critical point of $P$); then, clearly, each ray $R$ with $\mathrm{Pr}(R) \subset W$ is smooth. If $W$ is contained in a (pre)periodic component of $J_P$, this component (which must be non-degenerate) is denoted by $\widehat{E}(W)$, the corresponding component of $K_P$ is denoted by $E(W)$, and $\mathrm{val}'_{J_P}(W)$ is defined as $\mathrm{val}'_{\widehat{E}(W)}(W)$, i.e. the number of components of $\widehat{E}(W) \setminus W$. Recall also, that we define $\mathrm{val}_{J_P}(W)$ as the number of external rays of $P$ with principal sets in $W$.

**Corollary 6.11.** *Let $W \subset J_P$ be a wandering non-(pre)critical cut-continuum contained in a periodic component $\widehat{E}(W)$ of $J_P$. Then $\mathrm{val}'_{J_P}(W) = \mathrm{val}_{J_P}(W) = |A(W)| = M < \infty$. The polygon $B_W$, whose basis is $A(W)$, is wandering and non-(pre)critical under $\sigma$, and if $W_1, W_2$ are two continua as above with disjoint orbits, then the $\sigma$-orbits of the polygons $B_{W_1}, B_{W_2}$ are pairwise unlinked.*

*Moreover, $M$ equals the number of components of $E(W) \setminus W$. Also, if $W'$ is any non-(pre)critical element of the grand orbit $\Gamma(W)$, then $\mathrm{eval}(W') = \mathrm{val}_{\widehat{E}(W)}(W) = \mathrm{val}_{P^n(\widehat{E}(W))}(P^n(W))$ for all $n \geq 0$.*

**Proof.** Let $M = \mathrm{val}_{J_P}(W) = |A(W)|$. Let us consider the relation between polynomial-like rays to $\widehat{E}(W)$ and external rays to $\widehat{E}(W)$. To each polynomial-like ray $T$ to $\widehat{E}(W)$ we associate by Theorem 6.9 a *unique* external ray $R$ homotopic to $T$ outside $\widehat{E}$; the ray $R$ is unique because $W$ is non-(pre)critical (and hence the case (i) from Theorem 6.9 is impossible) and wandering (and hence the case (ii) from Theorem 6.9 is impossible). Since by Theorem 6.9 this describes *all* external rays whose principal sets are in $W$, we see that there is the same number of external rays to $\widehat{E}(W)$ and polynomial-like rays to $\widehat{E}(W)$. Thus, there are $M$ polynomial-like rays to $\widehat{E}(W)$. By Corollary 3.6, $M$ equals $\mathrm{val}'_{J_P}(W)$, the number of components of $\widehat{E}(W) \setminus W$, as desired. Moreover, since $W$ is non-(pre)critical, then by Lemma 6.10, $P^n_{\mathrm{Tail}(W)}$ is one-to-one and $M = \mathrm{val}_{\widehat{E}(W)}(W) = \mathrm{val}_{P^n(\widehat{E}(W))}(P^n(W))$ for all $n \geq 0$. This implies that $\mathrm{eval}(W') = \mathrm{val}_{\widehat{E}(W)}(W)$ for any non-(pre)critical element $W'$ of the grand orbit $\Gamma(W)$.

*We claim that $W$ is disjoint from the boundary of any Fatou domain.* Indeed, suppose otherwise. Then we may assume that $W \cap \mathrm{Bd}(U) \neq \emptyset$ where $U$ is a fixed Fatou domain. Consider two rays $R_1, R_2$ with principal sets in $W$; define $T(R_1, R_2) = T_0$ as the component of $\mathbb{C} \setminus [R_1 \cup R_2 \cup W]$ disjoint from $U$ (we will call such components *wedges*). We can define similar wedges $T(f^i(R_1), f^i(R_2)) = T_i$. Note that $T_i$'s are pairwise disjoint because $W$ is wandering. It follows that there exists $N$ such that for every $n > N$ the wedge $T_n$ contains no critical points. Then $f(T_i) = T_{i+1}$ for all $i > N$. Clearly, this contradicts the expansion on the circle at infinity.

*We claim that $M$ equals the number of components of $E(W) \setminus W$.* Indeed, let $U$ be a Fatou domain of $E(W)$. Then $\mathrm{Bd}(U)$ is a connected set disjoint from $W$. Hence $\mathrm{Bd}(U)$ is contained in exactly one component of $E(W) \setminus W$. Hence $\overline{U}$ is contained in this component, and therefore the number of components of $\widehat{E}(W) \setminus W$ does not change if we add all Fatou components of $E(W)$ to $\widehat{E}(W)$ as desired. □

By Corollary 6.11, for a wandering branch continuum $W \subset J_P$, $\mathrm{eval}_{J_P}(W)$ is well-defined. We will use the following notation.

**Definition 6.12.** A *valence stable wandering collection $\mathcal{B}_{\mathbb{C}}$ of continua* is a finite collection of wandering continua $\{W_1, \dots, W_n\}$ with pairwise disjoint grand orbits such that for each $j$ and $n \geq 0$, $P^n|_{\mathrm{Tail}(p^n(W_j))}$ is one-to-one and $|A(W_j)| \geq 3$. Denote by $\mathcal{B}_{\mathbb{C}}^{\infty}$ elements of $\mathcal{B}_{\mathbb{C}}$ which are wandering components of $J_P$ and by $\mathcal{B}_{\mathbb{C}}^{p}$ elements of $\mathcal{B}_{\mathbb{C}}$ which are contained in a (pre)periodic component of $J_P$.

Note that if $W \in \mathcal{B}_{\mathbb{C}}^{p}$ then $|A(W)| = \mathrm{val}'_{\widehat{E}(W)}(W) = \mathrm{val}_{\widehat{E}(W)}(W) = \mathrm{eval}(W)$.

## 7. The Fatou–Shishikura inequality for polynomials with disconnected Julia sets

In Section 7 we prove Theorem 1.2. Throughout the section we deal with a valence stable wandering collection $\mathcal{B}_{\mathbb{C}}$ as introduced above.

Let $W \in \mathcal{B}_{\mathbb{C}}$ be a wandering component of $J_P$. Then no iterate of $W$ intersects a periodic component of $K_P$. It has recently been shown [24,44] that every wandering component of $J_P$ is a point. However we will not rely on this in our paper. Let $\omega_P(W) = \limsup P^n(W)$ be the set of all limit points of $P^n(W)$.

**Lemma 7.1.** *If $W$ is a wandering component of $J_P$ then $\omega_P(W)$ cannot be contained in a finite union of cycles of components of $K_P$.*

**Proof.** Let $F$ be such a union. Choose a neighborhood $U$ of $F$ bounded by a finite union of equipotentials of the same (small) level, such that, if $P^n(x) \in U$ for all $n \geq 0$, then $x \in F$. Since $W$ never maps in $F$, iterates of $W$ leave $U$ infinitely many times and $\omega_P(W)$ is not contained in $F$ as desired. □

In Theorem 7.2 we associate to a wandering non-(pre)critical component $W$ of $J_P$ specific sets of external arguments and critical points.

**Theorem 7.2.** *If $W \in \mathcal{B}_{\mathbb{C}}$ is a wandering component with $M = |A(W)|$, then $B := \mathrm{Ch}(A(W))$ is a wandering non-(pre)critical $M$-gon under the map $\sigma$ and there exist $M-1$ critical points $c_1, c_2, \dots, c_{M-1}$ with disjoint orbits such that for every $j = 1, \dots, M-1$*

*there exist a component $T_j$ of $J_P$, external arguments $t_j \neq t'_j$, (possibly, one-sided) external rays $R_{t_j}, R_{t'_j}$ of arguments $t_j, t'_j$ with $\sigma(t_j) = \sigma(t'_j)$, and the following claims hold.*

(a) *The leaf $\ell_j = t_j t'_j$ is a limit leaf of a sequence of $\sigma$-iterates of $B$.*
(b) *Either $c_j \in R_{t_j} \cup R_{t'_j}$, or $c_j \in T_j$. Moreover, principal sets of $R_{t_j}$ and $R_{t'_j}$ are contained in $T_j$ and one of the following holds:*
  (b1) *if $c_j \in T_j$, then $\{t_j, t'_j\} \subset A(T_j)$ and $A(T_j)$ is all-critical;*
  (b2) *if $c_j \in R_{t_j} \cup R_{t'_j}$, then $R_{t_j}$ and $R_{t'_j}$ are one-sided rays having a common arc from $c_j$ to $T_j$, and $A(T_j) = \{t_j, t'_j\}$. Also, $P(T_j)$ is a component of $J_P$ and $P(R_{t_j}) = P(R_{t'_j}) = R_{\sigma(t_j)}$ is a unique (smooth) ray which accumulates in $P(T_j)$.*
(c) *$T_j$ is a wandering component of $J_P$.*

**Proof.** Set $B' := A(W)$. Then $\sigma^n(B') \cap \sigma^m(B') = \emptyset$ if $m \neq n$. Indeed, otherwise let $\alpha \in \sigma^n(B') \cap \sigma^m(B')$. Since $W$ is non-(pre)critical, the ray $R_\alpha$ is smooth and $\Pr(R_\alpha) \subset P^n(W) \cap P^m(W) \neq \emptyset$, a contradiction.

As in Lemma 3.2, $B$ is a wandering non-(pre)critical $M$-gon under the map $\sigma$. Take the grand orbit $\Gamma(W)$ (see Subsection 3.2.1), and associate to each $W' \in \Gamma(W)$ the sets of arguments $A(W')$ of rays to $W'$ and the polygons $\mathrm{Ch}(A(W'))$. Then the set $\mathrm{Tail}(W)$ is wandering. Indeed, since $W$ is wandering and by the previous paragraph $\mathrm{Tail}(W)$ is non-wandering only if two distinct forward images of rays from $\mathrm{Tail}(W)$ intersect. By Lemma 6.1 then there are non-smooth rays to some image of $W$, and $W$ is *not* non-(pre)critical, a contradiction.

The pullbacks of sets from the forward orbit of $\mathrm{Tail}(W)$ form the *grand orbit* $\Gamma(\mathrm{Tail}(W))$ of $\mathrm{Tail}(W)$. We consider the set $\mathrm{Tail}(W)$ instead of $W$ because in the disconnected case there are critical points outside $J_P$, hence to catch *all* criticality which shows along the orbit of $W$ we have to consider $W$ *together with* external rays to $W$.

Associate to all sets from $\Gamma(\mathrm{Tail}(W))$ the sets of the arguments of rays in them. As in Subsection 3.2 this collection $\mathcal{L}^B$ of polygons is a geometric prelamination without critical leaves (see Lemma 3.8).

Consider the family $L^B_{\lim}$ of limit leaves of polygons from $\mathcal{L}^B$ (including degenerate leaves). By Theorem 2.7 there exist at least $M-1$ recurrent critical leaves $\ell_1, \dots, \ell_{M-1}$ in $L^B_{\lim}$ with pairwise disjoint infinite orbits and the same $\omega$-limit set $X$ (since $\sigma$-images of $\ell_i$'s are points on the circle, $X \subset \mathbb{S}^1$), such that $X$ *intersects every leaf in* $L^B_{\lim}$.

By Lemma 2.5(1), applied to the geometric prelamination $\mathcal{L}^B$, for each $j, 1 \le j \le M-1$ the leaf $l_j = t_j t'_j$ is contained in an all-critical gap-leaf $C_j$ of $\overline{\mathcal{L}}^B$. If $\sigma^{n_i}(B)$ approach $\ell_j$, then the gaps $\sigma^{n_i+1}(B)$ separate the point $\sigma(\ell_j)$ from the rest of the circle. Now we prove a few claims.

(i) *The rays $R_{t_j}, R_{t'_j}$ have limit sets in the same component $T_j$ and hence are included in* $\mathrm{Tail}(T_j)$. This follows from the connectedness of the set $\limsup(P^{n_i}(W))$,

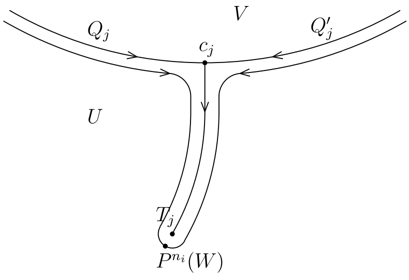


**Fig. 6.** An illustration for the proof of Theorem 7.2.

where lim sup is taken over a sequence $n_i$ of iterations of $\sigma$, along which images of $B$ converge to $l_j$.

(ii) *$P(T_j)$ is a component of $J_P$ and $P(R_{t_j}) = P(R_{t'_j}) = R_{\sigma(t_j)}$ is a (unique) smooth ray which accumulates in $P(T_j)$.* Since $\sigma^{n_i+1}(B)$ separate the point $\sigma(\ell_j)$ from the rest of the circle, $R_{\sigma(\ell_j)}$ is the *only* ray accumulating in $P(T_j)$. Moreover, from the properties of the system of external rays described in Section 6 it follows that $R_{\sigma(\ell_j)}$ is smooth.

(iii) *Since $l_j$ has an infinite orbit, $t_j, t'_j$ are not (pre)periodic.*

(iv) *$T_j$ is wandering.* Indeed, otherwise $P(T_j)$ is a (pre)periodic component of $J_P$ such that $R_{\sigma(\ell_j)}$ is a unique ray accumulating in it. This implies, that $\sigma(\ell_j) = \sigma(t_j)$ is (pre)periodic, a contradiction with $\ell_j$ having infinite orbit by Theorem 2.7. This proves (c).

Choose points $u_j \in R_{t_j}$ ($u'_j \in R_{t'_j}$) so that the closed rays $[u_j, \infty)$ ($[u'_j, \infty)$) from $u_j$ ($u'_j$, respectively) to infinity contain no points from $C^*$. Let $A_j = R_{t_j} \setminus (u_j, \infty)$, $A'_j = R_{t'_j} \setminus (u'_j, \infty)$ and $Z' = T_j \cup A'_j \cup A_j$. By (i) $P$ is not one-to-one on $Z'$. By (iv) and Lemma 3.3 $Z'$ is a non-separating continuum with no interior in the plane. As before, by [19], $Z'$ contains a critical point of $P$. Denote it by $c_j$. Then there are two possibilities.

(b1) $c_j \notin R_{t_j} \cup R_{t'_j}$; then $c_j \in T_j$ as required.

(b2) $c_j \in R_{t_j} \cup R_{t'_j}$; by Lemma 6.1, $R_{t_j}$ and $R_{t'_j}$ are one-sided rays sharing an arc from $c_j$ to $T_j$. We show that $A(T_j) = \{t_j, t'_j\}$. Let the closed arcs of $R_{t_j}, R_{t'_j}$ from infinity to $c_j$ be $Q_j$ and $Q'_j$. Then $Q_j \cup Q'_j$ separates $\mathbb{C}$ into components $U$ and $V$ with $U \supset T_j$. An external ray in $V$ has the principal set in $\overline{V}$, disjoint from $T_j$. Also, the closure of an external ray in $U$ is separated from $T_j$ by a forward image $P^{n_i}(W)$ of $W$ with its associated external rays (see Fig. 6). Hence $A(T_j) = \{t_j, t'_j\}$ proving (b). $\square$

**Definition 7.3** *(The set $C^w_\infty$). A critical point $c$ of $P$ lies in $C^w_\infty$* if and only if there exists a wandering component $T_c$ of $J_P$ and two external arguments $t_c, t'_c$, such that:

(i) $t_c, t'_c \in A(T_c)$, and $\sigma(A(T_c))$ is a point (thus, $\sigma(t_c) = \sigma(t'_c)$);
(ii) either $t_c$ or $t'_c$ is recurrent under the map $\sigma$; and
(iii) (a) $c$ belongs to the connected set $R_{t_c} \cup T_c \cup R_{t'_c}$, (b) $P(c) \in R_{\sigma(t_c)} \cup P(T_c)$, (c) $R_{\sigma(t_c)}$ is a unique ray whose closure is non-disjoint from $P(T_c)$ (moreover, $\Pr(R_{\sigma(t_c)}) \subset P(T_c)$ and $R_{\sigma(t_c)}$ is a smooth ray).

By $C'_{wr}$ we denote the number of weakly recurrent critical points in wandering components of $J_P$. By Definition 7.3, $C^w_\infty \subset C'_{wr}$. Denote by $K(C^w_\infty)$ the number of different grand orbits of $t_c$ ($t'_c$), and by $L(C^w_\infty)$ the number of different limit sets of $t_c$ under the map $\sigma$, for $c \in C^w_\infty$. Theorem 2.7, Theorem 7.2, and the inequality $K(C^w_\infty) \le |C^w_\infty| \le |C'_{wr}|$ imply Theorem 7.4 (if $\mathcal{B}^\infty_\mathbb{C} \neq \emptyset$, then $C^w_\infty \neq \emptyset$).

**Theorem 7.4.** *Consider valence stable wandering collection of $m' \ge 0$ components $Q'_j$ of $J_P$. If $m' > 0$, then*

$$\sum_{j=1}^{m'} (\mathrm{val}_{J_P}(Q'_j) - 2) \le K(C^w_\infty) - L(C^w_\infty) \le |C^w_\infty| - 1 \le |C'_{wr}| - 1 \le d - 2.$$

*So, with $\chi(m')$ defined as 1 for $m' > 0$ and 0 otherwise, we have*

$$\chi(m') + \sum_{j=1}^{m'} (\mathrm{val}_{J_P}(Q'_j) - 2) \le \chi(m')|C'_{wr}| \tag{1}$$

Recall, that by $N_{irr}$ we denote the total number of repelling cycles $O$ such that the set $A(O)$ of arguments of external rays landing at points of $O$ contains no periodic angles; by Theorem 6.4(2) such cycles $O$ are exactly the cycles for which $A(O)$ is infinite. Also, by Theorem 6.4(2), each point of $O$ is a component of $J_P$.

**Definition 7.5** *(The set $C^p_\infty$). A critical point $c$ of $P$ belongs to $C^p_\infty$* if and only if there exists a repelling cycle $O$ with infinite set $A(O)$, $c \in U^\infty$ has two external non-(pre)periodic arguments $t_c, t'_c \in A(O)$ with the same $\sigma$-image (by [28] we may assume that there are no periodic external rays landing at $O$ and $t_c, t'_c$ are recurrent with the same infinite minimal limit set).

Recall, that $C_{esc}$ is the set of all escaping critical points; then $C^p_\infty \subset C_{esc}$. As always, denote by $K(C^p_\infty)$ the number of distinct grand orbits of points of $C^p_\infty$. Theorem 6.4(2) and the obvious inequality $K(C^p_\infty) \le |C^p_\infty| \le C_{esc}$ imply Theorem 7.6.

**Theorem 7.6.** *The following inequality holds.*

$$N_{irr} \le K(C^p_\infty) \le |C^p_\infty| \le |C_{esc}| \le d - 1 \tag{2}$$

**Remark 7.7.** If $P_v(z) = z^2 + v$ is a quadratic polynomial with disconnected Julia set, then it can have at most one cycle $O_v$ with an infinite set $A(O_v)$. If this happens, the set $A(O_v)$ contains two external arguments $t_0, t'_0$ as above of the critical point $0 \in U^\infty$ sharing the same image $t_* = \sigma(t_0) = \sigma(t'_0)$. Let $m$ be the period of $O_v$. Assume that the base-$2^m$ representation of $t_* \in (0, 1)$ contains only two digits. (For example, it obviously holds, if $m = 1$.) Then, by [26], as $v$ approaches the Mandelbrot set $M$ along the external ray of $M$

of argument $t_*$, the multiplier of $O_v$ tends to some point $e^{2\pi\nu}$ of the unit circle, where $\nu$ is irrational. Hence, the ray ends at a point $v_*$ of the boundary of a hyperbolic component of $M$ and $O_v$ tends to either a Cremer or a Siegel cycle of $P_{v_*}$. Converse statement is also true and follows essentially from Yoccoz's result about the local connectivity of the Mandelbrot set at the boundaries of hyperbolic components, see e.g. [47]. Note that by Theorem 4.3 the critical point of $P_{v_*}$ is recurrent and weakly non-separated from a point of the (Cremer or Siegel) cycle.

**Corollary 7.8.** *Consider a valence stable wandering collection of $m' \geq 0$ components $Q'_j$ of $J_P$. Then we have*

$$N_{irr} + \chi(m') + \sum_{j=1}^{m'} (\mathrm{val}_{J_P}(Q'_j) - 2) \leq \chi(m')|C'_{wr}| + |C_{esc}| \tag{3}$$

**Proof.** Let us show that $C^w_\infty \cap C^p_\infty = \emptyset$. Indeed, let (i) $c \in C^w_\infty$ and (ii) $c \in C^p_\infty$. Then, because of (i), by Definition 7.3 there is a smooth ray $R$ accumulating in a component $T$ of $J_P$ such that $P(c) \in R \cup T$ and $R \cup T$ is disjoint from closures of all rays other than $R$. However, because of (ii), by Definition 7.5, there must also exist a ray $R'$ such that $c \in R'$ and $\mathrm{Pr}(R')$ is a periodic point at which infinitely many other rays land. Thus, we get a contradiction which shows that $C^w_\infty \cap C^p_\infty = \emptyset$. Now we can add inequalities (1) and (2) which implies the desired inequality. □

Corollary 7.8 proves the second inequality of Theorem 1.2. We prove the first one in Lemma 7.9. Recall that $N_{co}$ is the number of cycles of components of $J_P$ containing non-(pre)critical branch continua.

**Lemma 7.9.** *Suppose that $\mathcal{B}^p_{\mathbb{C}} = \{Q_i\}$ is a valence stable wandering collection of continua which consists of $m$ elements contained in periodic components of $J_P$. Then*

$$N_{FC} + N_{co} + \sum_{i=1}^{m} (\mathrm{val}_{J_P}(Q_i) - 2) \leq |C_{wr}|. \tag{4}$$

**Proof.** *Let us show, that we may deal with a valence stable wandering collection of continua which maximizes* $\sum_{i=1}^{m}(\mathrm{val}_{J_P}(Q_i) - 2)$. Indeed, for such a collection all cycles of components of $J_P$ which contain some wandering non-(pre)critical branch continua must be used in the sense that wandering continua contained in the cycle should be part of the collection (otherwise they can be added to the collection increasing the sum in question). Hence if the collection maximizes $\sum_{i=1}^{m}(\mathrm{val}_{J_P}(Q_i) - 2)$, then it maximizes $N_{co} + \sum_{i=1}^{m}(\mathrm{val}_{J_P}(Q_i) - 2)$, and it suffices to prove the inequality for such a maximal collection.

Take a non-degenerate periodic component $E$ of $K_P$ of period $p$. Suppose that it contains $n_E \geq 0$ elements of a chosen maximal valence stable collection of wandering continua ($n_E = 0$ would mean that it contains no such elements). By Corollary 6.11 and the remarks after that, we may assume that only $E$ contains the continua $Q_i$. By [14], $P^p|_E$ is a polynomial-like map. In particular, there exists a sufficiently tight neighborhood $U$ of $E$ such that $P^p|_U$ is conjugate to $f|_V$ for a polynomial $f$ with connected Julia set $J_f$, filled-in Julia set $K_f$, and a tight neighborhood $V$ of $K_f$.

Any such conjugacy transports wandering continua, Fatou domains and CS-points of $P^p$ to wandering continua, Fatou domains and CS-points of $f$ because these objects are defined topologically. The same holds for critical points of $P^p|_U$ and the valence of subcontinua of $J_P$. Moreover, weakly recurrent critical points are also transported by any conjugacy because so are periodic cutpoints and their preimages, and by definition only the cuts in the Julia set made by periodic cutpoints and their preimages are necessary to define weakly recurrent points.

Therefore Theorem 5.5 implies the inequality

$$\chi(n_E) + \sum_{Q_i \subset E} (\mathrm{val}_{J_P}(Q_i) - 2) + N_{FC}(P^p|_E) \leq K(C_{wr}(P^p|_E)) \tag{5}$$

in which by $N_{FC}(P^p|_E)$ we denote the number of cycles of Fatou domains and Cremer cycles of $P^p|_E$ and by $C_{wr}(P^p|_E)$ we denote all the weakly recurrent critical points of $P^p|_E$ (recall also, that then $K(C_{wr}(P^p|_E))$ denotes the number of grand orbits of critical points from $C_{wr}(P^p|_E)$ under the map $P^p$). It is obvious that $N_{FC}(P^p|_E)$ coincides with the number $N_{FC}(\mathrm{orb}_P(E))$ of cycles of Fatou domains and Cremer cycles in the entire (periodic) orbit $\mathrm{orb}_P(E)$ of $E$. Also, it is easy to see that all critical points of $P^p|_E$ are in fact preimages of critical points of $P$ belonging to $\mathrm{orb}_P(E)$, and weakly recurrent critical points of $P^p|_E$ are in fact preimages of weakly recurrent critical points of $P$ belonging to $\mathrm{orb}_P(E)$. Therefore, $K(C_{wr}(P^p|_E))$ coincides with the number $K(C_{wr} \cap \mathrm{orb}_P(E))$ of grand orbits of weakly recurrent critical points of $P$ belonging to $\mathrm{orb}_P(E)$.

Let us sum up inequality (5) over all cycles of components of $J_P$. The left hand side of the summed up inequality coincides literally with the left hand side of inequality (4). The right hand side will be equal to the number of grand orbit of weakly recurrent critical points belonging to periodic components of $P$, and the latter number is obviously less than or equal to $|C_{wr}|$. This completes the proof of the lemma. □

It remains to make the following observations. The first inequality of Theorem 1.2 is inequality (3) proven in Corollary 7.8. The second inequality of Theorem 1.2 is inequality (4) proven in Lemma 7.9. The sum of these two inequalities leads to the main inequality of Theorem 1.2 (notice that since sets of critical points $C_{wr}, C'_{wr}$ and $C_{esc}$ are obviously pairwise disjoint we have that $|C_{wr}| + |C'_{wr}| + |C_{esc}| \leq d - 1$).

**Acknowledgments**

In this paper we combine estimates concerning wandering branch points in the locally connected case (see Subsection 2.4) with the Fatou–Shishikura inequality, and extend this to all polynomials. That idea was suggested to us by Mitsuhiro Shishikura; we acknowledge this here with gratitude. Also, we would like to express our gratitude to the referee for carefully reading the manuscript and making a number of thoughtful and useful remarks.

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