NON-DEGENERATE LOCALLY CONNECTED MODELS FOR PLANE CONTINUA AND JULIA SETS

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ABSTRACT. Every plane continuum admits a finest locally connected model. The latter is a locally connected continuum onto which the original continuum projects in a monotone fashion. It may so happen that the finest locally connected model is a singleton. For example, this happens if the original continuum is indecomposable. In this paper, we provide sufficient conditions for the existence of a non-degenerate model depending on the existence of subcontinua with certain properties. Applications to complex polynomial dynamics are discussed.

1. **Introduction.** A natural approach to studying a topological space X is to model X using simpler and easier to deal with spaces. By this we mean finding a factor space of X such that both the quotient map $m : X \to L$ and the model space L are manageable. In this paper we consider only *plane continua*; in that setting we view *monotone* maps and *locally connected continua* as manageable. This leads to the concept of the *finest locally connected model under a monotone map* of a plane continuum X.

The concept was inspired by Jan Kiwi who approached the problem of modeling from the point of view of (complex) dynamical systems. To state Kiwi's results we need a few definitions.

Definition 1.1 (Semiconjugacy of maps)**.** Two maps $f : X \to X$ and $g : Y \to Y$ are said to be *semiconjugate* if there exists a map $\psi : X \to Y$ such that $\psi \circ f = g \circ \psi$. In other words, the following diagram is commutative:

$$\begin{CD} X @>f>> X \\ @V\psi VV @VV\psi V \\ Y @>>g> Y \end{CD}$$

We also need to define a concept of a *monotone map*.

Definition 1.2 (Monotone map)**.** A surjective map $f : X \to Y$ of continua is *monotone* provided for each $y \in Y$, the fiber $f^{-1}(y)$ is connected.

In what follows let $\mathbb{C}$ be the complex plane and let $\widehat{\mathbb{C}}$ be the complex sphere. In his paper [9] Kiwi proves that if a polynomial P with connected Julia set $J(P)$ has no periodic points with multipliers which are complex numbers of modulus 1 and irrational argument then P can be semiconjugate to a so-called *topological polynomial* $f_P : \mathbb{C} \to \mathbb{C}$. The semiconjugacy $\varphi : \mathbb{C} \to \mathbb{C}$ is a monotone map which is one-to-one outside the Julia set $J(P)$; thus, basically φ collapses some subcontinua of $J(P)$ (*fibers* of φ) to points. The topological polynomial f_P is a branched covering map such that $\varphi(J(P))$ is a locally connected continuum with well-understood structure and dynamics described by so-called *laminations*.

As mentioned above, Kiwi's approach to the problem was based upon dynamical systems' considerations. Later on in [2] it was discovered that an approach based upon continuum theory yields results that extend those of [9] while also being applicable in a purely topological setting. We need a few definitions.

Definition 1.3. Let X be a continuum. A continuum Y is a *finest locally connected model for X* if there exists a monotone map $m : X \to Y$ so that for any monotone map $f : X \to Z$, where Z is a locally connected continuum, there exists a monotone map $g : Y \to Z$ so that $g \circ m = f$; then we will call the map m a *finest monotone map*.

We consider this notion on the plane in the context of so-called *unshielded* continua.

Definition 1.4. Given a compact set X in the plane, let U_X^∞ denote the unbounded complementary domain of X. The set $\mathrm{TH}(X) = \mathbb{C} \setminus U_X^\infty$ is called the *topological hull* of X. A compact set X in the plane is *unshielded* provided X coincides with the boundary ∂U_X^∞ of the unbounded complementary domain U_X^∞ of X. Observe that any subcontinuum of an unshielded continuum is unshielded.

The following theorem shows that a finest locally connected model and a finest monotone map are well-defined for unshielded plane continua (in [3] the result was extended to plane compacta).

Theorem 1.5 ([2])**.** *Every unshielded plane continuum X has a finest locally connected model Y and a finest monotone map m. Moreover, any two finest locally connected models of an unshielded continuum X are homeomorphic. Furthermore, m can be extended to a monotone map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which maps ∞ to ∞, in $\mathbb{C} \setminus X$ collapses only those complementary domains to X whose boundaries are collapsed by m, and is a homeomorphism elsewhere in $\widehat{\mathbb{C}} \setminus X$.*

By Theorem 1.5 we can talk about *the* finest locally connected model of an unshielded continuum and *the* finest monotone map. It follows that if an unshielded

plane continuum X has the finest locally connected model which is non-degenerate then its topological hull $\mathrm{TH}(X)$ also has a non-degenerate model.

In particular, the connected Julia set of a polynomial admits a finest locally connected model. However, in some cases the finest locally connected model is a single point; in this case we say that the finest locally connected model is *degenerate* while otherwise we call such model *non-degenerate*. Obviously, if the finest model is degenerate, all information regarding the continuum X is lost while otherwise some of the structure of X is preserved in its model. This shows the importance of the fact that the finest locally connected model of an unshielded continuum is non-degenerate. In the present paper we will study conditions under which the finest locally connected model is non-degenerate. Moreover, in the final section we apply this result to polynomial dynamics.

2. **Statement of main results and applications.** In this section we assume knowledge of basic concepts of continuum theory and complex dynamics (all necessary definitions are given in detail later in the sections of the paper containing the proofs of our main results). Denote the open unit disk by $\mathbb{D}$ and the disk at infinity (i.e., $\mathbb{C} \setminus \overline{\mathbb{D}}$) by $\mathbb{D}^\infty$. We will identify the unit circle $\mathbb{S} = \partial\mathbb{D} = \partial\mathbb{D}^\infty$ with $\mathbb{R}/\mathbb{Z}$ and call the induced order on $\mathbb{S}$ the *circular order*. Note that the circular order is not defined for a pair of points in $\mathbb{S}$, but if $x, y, z \in \mathbb{S}$ are three distinct points, then $x < y < z$ in the circular order if, when traveling from x in the positive direction along $\mathbb{S}$, we encounter y before z. Thus, from now on a single point $x \in \mathbb{S}$ will be denoted by the corresponding angle, i.e. by a number $\alpha \in [0, 1)$ with $x = e^{2\pi\alpha i}$.

If X is a plane continuum, then by the Riemann mapping theorem there exists a conformal map $\psi_X : \mathbb{D}^\infty \to U_X^\infty$ with derivative converging to a real number as $|z| \to \infty$. *External rays* of X foliate U_X^∞ and serve as a major tool in studying the topology of ∂U_X^∞.

Definition 2.1 (External rays). Let X be a plane continuum. By an *external ray of X* we mean the image of the radial line segment with argument $2\pi\alpha$ under the Riemann map ψ_X; in what follows, this image will be denoted by $R_X(\alpha)$. In other words,

$$R_X(\alpha) = \psi_X(\{r\, e^{2\pi\alpha i} \mid r > 1\}).$$

If we do not want to emphasize the argument we denote an external ray of X by R_X. We say that the external ray $R_X(\alpha)$ *lands at* $x_\alpha \in X$ provided $\overline{R_X(\alpha)} \setminus R_X(\alpha) = x_\alpha$.

We will mostly consider external rays for unshielded plane continua X (in that case $X = \partial U_X^\infty$) such as connected Julia sets of complex polynomials, however sometimes we work with external rays of other plane continua (e.g., we consider external rays of connected *filled* Julia sets). Observe that the unbounded complementary domain U_X^∞ of a continuum X coincides with the unbounded complementary domain $U_{\mathrm{TH}(X)}^\infty$ of its topological hull. Therefore we can (and will) interchangeably talk about external rays of X and/or external rays of $\mathrm{TH}(X)$.

Definition 2.2 (Strategically placed subcontinua). Suppose that Y is a subcontinuum of an unshielded continuum X in the complex plane. Then we say that Y is *strategically placed in X* provided that there exists a dense set $\mathcal{A}(Y, X) = \mathcal{A} \subset \mathbb{S}$ so that:

1. for each $\alpha \in \mathcal{A}$, $R_Y(\alpha)$ lands at a point $y_\alpha \in Y$,
2. the set of points $\{y_\alpha\}_{\alpha \in \mathcal{A}}$ is dense in Y,

3. there exists a circle order preserving function $p : \mathcal{A} \to \mathbb{S}$ so that for each $\alpha \in \mathcal{A}$ the ray $R_X(p(\alpha))$ lands at y_α.

In this case we say that $\mathcal{A}$ is an *anchor set* (of Y) and $p : \mathcal{A} \to \mathbb{S}$ is an *external connecting function* (of Y).

Since p preserves order, p is one-to-one but we do not assume that p is continuous. Easy examples show that p is also not necessarily one-sided continuous. For instance, let $Y = [-1, 1]$ be the interval in the x-axis and let X be the union of Y and the two intervals connecting the points $(1, 1)$ and $(-1, 1)$ with the origin O. Then it is easy to see that Y is strategically placed in X. Moreover, denote by α the angle in the anchor set $\mathcal{A}$ such that $R_Y(\alpha)$ lands at O "from above". Then it is easy to see that if the map p is defined so that $R_X(p(\alpha))$ is the positive y-axis, then p is neither continuous from the left nor continuous from the right at α.

Theorem A is our main continuum theory result. It shows that in some cases the fact that a subcontinuum has a non-degenerate finest locally connected model implies that the same can be said about the continuum itself.

Theorem A. *Let X be an unshielded plane continuum. If Y is strategically placed in X, and Y has a non-degenerate finest locally connected model M_Y, then X has a non-degenerate finest locally connected model M_X. Moreover, there exists a canonical embedding of M_Y into M_X.*

The main applications of this result are in complex dynamics. Namely, the following theorem holds.

Theorem B. *Suppose that $P : \mathbb{C} \to \mathbb{C}$ is a polynomial, and $J^* \subset J(P)$ is the Julia set of a polynomial-like map obtained as a restriction of P^n for some $n > 0$. If J^* has a non-degenerate finest locally connected model, then so does $J(P)$.*

In Subsection 4.3, we rely upon [3] and prove a version of Theorem B for disconnected Julia sets. The authors are indebted to the referee for suggesting the "moreover" part of Theorem A.

3. **Proof of Theorem A.** In the first subsection of this section we give various standard definitions. Then we prove Theorem A.

3.1. **Basic definitions.** The notion of the *principal set* is used in studying the limit behavior external rays.

Definition 3.1 (Principal set). Given an external ray $R_X(\alpha)$ of an unshielded continuum X we denote by $\mathrm{Pr}_X(\alpha)$ the set $\overline{R_X(\alpha)} \setminus R_X(\alpha)$ and call it the *principal set of the ray $R_X(\alpha)$*. If $\mathrm{Pr}_X(\alpha)$ is a single point y, then $R_X(\alpha)$ lands at y.

More generally, let $T \subset U_X^\infty$ be an image of $\mathbb{R}_+ = (0, \infty)$ under a continuous map $\psi : \mathbb{R}_+ \to \mathbb{C}$ such that $\lim_{t \to \infty} |\psi(t)| = \infty$ while $\varnothing \neq \overline{T} \setminus T \subset X$. Then we say that $T = \psi(\mathbb{R}_+)$ *accumulates* in X, denote $\overline{T} \setminus T$ by $\mathrm{Pr}_X(T)$ and call it the *principal set of the curve T which accumulates in X*. If $\mathrm{Pr}_X(T)$ is a single point y we say that the curve T *lands at y*.

Another important definition is that of a *crosscut* (see, e.g., [11] for details).

Definition 3.2 (Crosscuts). A *crosscut* C of X is an open arc $C \subset U_X^\infty$ so that its closure is a closed arc with two distinct endpoints both of which belong to X. A *fundamental chain* $\{C_i\}$ (of crosscuts) is a sequence of crosscuts C_i of X such that the following holds:

1. $\overline{C_i} \cap \overline{C_j} = \varnothing$ if $i \neq j$,
2. for each i, the crosscut C_i separates C_{i+1} from infinity in U_X^∞, and
3. $\lim \operatorname{diam}(C_i) = 0$.

For each crosscut C of X its *shadow* S_C is the closure of the bounded complementary domain of $\mathbb{C} \setminus [X \cup C]$ whose boundary contains C.

Note that every fundamental chain $\{C_i\}$ corresponds to a unique point $\alpha \in \mathbb{S}$ defined by $\lim(\varphi_X)^{-1}(C_i) = \alpha$ and in this case we say that $\{C_i\}$ is a *fundamental chain for α*.

Definition 3.3 (Impressions). The $(X$-$)impression$ $\operatorname{Imp}_X(\alpha)$ is defined as

$$\operatorname{Imp}_X(\alpha) = \bigcap S_{C_i}, \text{ where } \{C_i\} \text{ is a fundamental chain for } \alpha.$$

It is easy to see that both $\operatorname{Pr}_X(\alpha)$ and $\operatorname{Imp}_X(\alpha)$ are continua, that $\operatorname{Pr}_X(\alpha) \subset \operatorname{Imp}_X(\alpha)$ and that $\operatorname{Imp}_X(\alpha)$ is independent of the choice of the fundamental chain for α. Moreover, let X be an unshielded continuum. Then, even though $\bigcup_\alpha \operatorname{Pr}_X(\alpha)$ can be a proper subset of the continuum X, $\bigcup_\alpha \operatorname{Imp}_X(\alpha) = X$.

3.2. **Proof of Theorem A.** Let us recall that the notion of a subcontinuum Y strategically placed in an unshielded continuum X was introduced in Definition 2.2. A part of this definition is a function p (so-called *external connecting function*) of a dense set $\mathcal{A} \subset \mathbb{S}$ to $\mathbb{S}$ which preserves circle order and maps angles such that for each $\alpha \in \mathcal{A}$, both the ray $R_Y(\alpha)$ and the ray $R_X(p(\alpha))$ land at a point $y_\alpha \in Y$. We will show below that the choice of the function p is severely restricted. Moreover, the condition in Lemma 3.4 characterizes the situation in which a subcontinuum is strategically placed in an unshielded continuum (and so this characteristic can be used as an alternative definition of the fact that Y is strategically placed in X).

Lemma 3.4. *Suppose that $Y \subset X$ are unshielded planar continua. Then the following are equivalent:*

1. *Y is strategically placed in X with anchor set $\mathcal{A}$,*
2. *There exists a dense set $\mathcal{A} \subset \mathbb{S}$ so that the following holds:*
   (a) *for $\alpha \in \mathcal{A}$ the ray $R_Y(\alpha)$ lands at the point $y_\alpha \in Y$ so that the set of all points $y_\alpha, \alpha \in \mathcal{A}$, is dense in Y;*
   (b) *for each $\alpha \in \mathcal{A}$ there exists $\beta(\alpha) = \beta \in \mathbb{S}$ so that the ray $R_X(\beta)$ also lands at y_α and the rays $R_Y(\alpha)$ and $R_X(\beta)$ are homotopic in $\{y_\alpha\} \cup \mathbb{C} \setminus Y$ under a homotopy which fixes the landing point $y_\alpha \in Y$.*

Proof. Suppose that Y is strategically placed in X with anchor set $\mathcal{A}$ and $p : \mathcal{A} \to \mathbb{S}$ as the external connecting function. Suppose that $\alpha \in \mathcal{A}$ and the ray $R_Y(\alpha)$ lands at $y_\alpha \in Y$. Clearly, $R_X(p(\alpha))$ can be viewed as a curve in $\mathbb{C} \setminus Y$ which accumulates in Y; more precisely, we can say that $R_X(p(\alpha))$ lands at y_α. Thus, $R_X(p(\alpha))$ is homotopic to some external ray $R_Y(\beta)$ in $\mathbb{C} \setminus Y$ under a homotopy which fixes y_α (so that the ray $R_Y(\beta)$ lands at y_α too). If $\alpha \neq \beta$, then both components of $\mathbb{C} \setminus \overline{R_Y(\alpha) \cup R_Y(\beta)}$ intersect Y (because two *distinct* external rays of Y which land at the same point of Y cannot be homotopic outside Y).

Choose $\gamma_1, \gamma_2 \in \mathcal{A}$ so that $\alpha < \gamma_1 < \beta < \gamma_2 < \alpha$ and $R_Y(\gamma_1)$ and $R_Y(\gamma_2)$ land in different components C_1, C_2 of $\mathbb{C} \setminus \overline{R_Y(\alpha) \cup R_Y(\beta)}$, respectively. Let $R_Y(\gamma_1)$ land on a point $y_{\gamma_1} \in C_1 \cap Y$ and let $R_Y(\gamma_2)$ land at a point y_{γ_2} in $C_2 \cap Y$. Then $R_X(p(\gamma_2))$ is an external ray of X which also lands at y_{γ_2}. Since $\alpha < \gamma_1 < \gamma_2$, then $p(\alpha) < p(\gamma_1) < p(\gamma_2)$.

Consider the set E of angles in $p(\mathcal{A})$ which belong to $(p(\alpha), p(\gamma_2))$. Consider also the component $\widetilde{E}$ of the set

$$\mathbb{C} \setminus [R_X(p(\alpha)) \cup \{y_\alpha\} \cup R_X(p(\gamma_2) \cup (C_2 \cap Y)]$$

containing external rays of X with arguments from E. The external rays of X with argument in E can only land at points from the boundary of $\widetilde{E}$ but not on points from other external rays of X; thus, the external rays of X with argument in E can only land at points from $[C_2 \cap Y] \cup \{y_\alpha\}$. In particular this must be true for the ray $R_X(p(\gamma_1))$. However by definition this ray must land at the point $y_{\gamma_1} \in C_1 \cap Y$, a contradiction.

Suppose next that condition (2) holds. It suffices to show that the map $p : \alpha \to \beta(\alpha)$ preserves circular order. Recall that by $\psi_Y : \mathbb{D}^\infty \to U_Y^\infty$ we denote the conformal map with derivative converging to a real number as $|z| \to \infty$. Similarly, let $\psi_X : \mathbb{D}^\infty \to U_X^\infty$ be the corresponding Riemann map from the complement of the closed unit disk to the unbounded component of X. Assume that $\alpha_1 < \alpha_2 < \alpha_3 \in \mathcal{A}$ and let $\beta_j = \beta(\alpha_j)$. Since the rays $R_Y(\alpha)$ and $R_X(\beta)$ are homotopic in $\{y_\alpha\} \cup \mathbb{C} \setminus Y$ under a homotopy which fixes the landing point $y_\alpha \in Y$, the ray $\psi_Y^{-1}(R_X(\beta_j)) = \widehat{R}_j$ lands at the point $e^{2\pi\alpha_j i} = x_j \in \mathbb{S}$ and $x_1 < x_2 < x_3$. Let S_r be the circle $\psi_Y^{-1} \circ \psi_X(\{z \in \mathbb{C} \mid |z| = r\})$ with an induced circular order $<$. As $r \searrow 1$, the circle S_r intersects $\widehat{R}_j$ at a unique point $y_j(r)$, and $\lim_{r \searrow 1} y_j(r) = x_j$. This implies that $y_1(r) < y_2(r) < y_3(r)$ as required. □

Suppose that $Y \subset X$ is strategically placed in X with anchor set $\mathcal{A}$. Then Lemma 3.4 implies that for any $\alpha \in \mathcal{A} \subset \mathbb{S}$ the ray $\psi_Y^{-1}(R_X(p(\alpha)))$ lands at $\alpha \in \mathbb{S}$. This visualization is useful in the proof of the next lemma that describes intersections between closures of components of $X \setminus Y$ and Y. It follows easily from the assumptions that X is unshielded and Y is strategically placed in X.

Lemma 3.5. *Suppose that X is an unshielded continuum and $Y \subset X$ is a continuum strategically placed in X. If C is a component of $X \setminus Y$, then*

$$|\overline{\psi_Y^{-1}(C)} \cap \partial \mathbb{D}| = 1.$$

In particular, if $\alpha = \overline{\psi_Y^{-1}(C)} \cap \partial \mathbb{D}$, then $\overline{C} \cap Y \subset \mathrm{Imp}_Y(\alpha)$.

Proof. Observe that $\psi_Y^{-1}(C)$ is a connected subset of $\mathbb{C} \setminus \overline{\mathbb{D}}$ (because X is unshielded). It follows that if $\overline{\psi_Y^{-1}(C)} \cap \partial \mathbb{D}$ is non-degenerate then there exists a non-degenerate arc $[p, q] \subset \partial \mathbb{D}$ such that *any* (not necessarily radial) ray $T \in \mathbb{C} \setminus \overline{\mathbb{D}}$ which lands at $\beta \in (p, q)$ must intersect $\psi_Y^{-1}(C)$. Choose $\alpha \in (p, q) \cap \mathcal{A}$. By Lemma 3.4, $\psi_Y^{-1}(R_X(p(\alpha))$ lands at α. Then $\psi_Y^{-1}(C) \cap \psi_Y^{-1}(R_X(p(\alpha)) \neq \varnothing$ and, hence, $C \cap R_X(p(\alpha)) \neq \varnothing$, a contradiction. To prove the last claim of the lemma choose a fundamental system of crosscuts B_i such that $\psi_Y^{-1}(B_i)$ converge to $\alpha = \overline{\psi_Y^{-1}(C)} \cap \partial \mathbb{D}$. Then by definition their shadows converge to $\mathrm{Imp}_Y(\alpha)$. Since all these shadows contain $\overline{C} \cap Y$, it follows that $\overline{C} \cap Y \subset \mathrm{Imp}_Y(\alpha)$ as desired. □

Lemma 3.5 motivates the following definition.

Definition 3.6 (Angles associated with components of $X \setminus Y$). Suppose that X is an unshielded continuum and $Y \subset X$ is a continuum strategically placed in X. Given a component C of $X \setminus Y$ we call the angle α such that $\overline{\psi_Y^{-1}(C)} \cap \partial \mathbb{D} = \{\alpha\}$ the *angle associated with C* and denote it by $\alpha(C)$ which defines a map from the family of components of $X \setminus Y$ to the unit circle. We also define the function C

which associates to any point $x \in X \setminus Y$ the component $C(x)$ of $X \setminus Y$ such that $x \in C(x)$. Finally, we consider a function $\alpha : X \setminus Y \to \mathbb{S}$ defined as $\alpha(x) = \alpha(C(x))$ for every $x \in X \setminus Y$.

Using the terminology introduced in Definition 3.6 we can restate Lemma 3.5 as follows: if X is an unshielded continuum and $Y \subset X$ is a continuum strategically placed in X then for every component C of $X \setminus Y$ we have $\overline{C} \cap Y \subset \text{Imp}_Y(\alpha(C))$.

Let us now prove a version of a particular case of a Kuratowski's result dealing with planar continua. More precisely, by [10, Theorem 61.2] the following holds. Suppose that Y and Z are two planar continua. Then it follows from [10, Theorem 61.2] that if $Y \cap Z$ is disconnected then there exists a bounded complementary domain of the union $Y \cup Z$. We extend this result a tiny bit in the lemma below. Recall that *topological hulls* of planar compacta are defined in Definition 1.4. First we prove a simple lemma concerning unshielded continua and their subcontinua.

Lemma 3.7. *Suppose that X is an unshielded continuum and $Y \subset X, Z \subset X$ are two continua. Then $\text{TH}(Y) \cap \text{TH}(Z) = \text{TH}(Y \cap Z)$.*

Proof. Take a point $v \in \text{TH}(Y) \cap \text{TH}(Z)$ which does not belong to $Y \cap Z$. It follows that $v \notin X$ (if $v \in X$ then v can only belong to $\text{TH}(Y)$ if $v \in Y$, and v can only belong to $\text{TH}(Z)$ if $v \in Z$, a contradiction with $v \notin Y \cap Z$). Hence there exists a bounded complementary domain V of X such that $v \in V$, and it is clear that $v \in \text{TH}(Y) \cap \text{TH}(Z)$ implies that $\partial V \subset Y \cap Z$. Thus, points of $\text{TH}(Y) \cap \text{TH}(Z)$ either belong to $Y \cap Z$, or belong to the interiors of boundary complementary domains V to X such that $\partial V \subset Y \cap Z$. Therefore $\text{TH}(Y) \cap \text{TH}(Z) = \text{TH}(Y \cap Z)$. $\qquad\square$

The next corollary follows from [10, Theorem 61.2].

Corollary 3.8. *Suppose that X is an unshielded continuum and $Y \subset X, Z \subset X$ are two continua such that $Y \cap Z$ is disconnected. Then there exists a bounded complementary domain V of the union $Y \cup Z$ such that ∂V contains points of $Y \setminus Z$ and points of $Z \setminus Y$.*

In other words, among the bounded complementary domains of $Y \cup Z$, there must exist a "new" domain which is neither a bounded complementary domain of Y nor a bounded complementary domain of Z.

Proof. By [10, Theorem 61.2] there are boundary complementary domains of $Y \cup Z$. Suppose that for any such domain U we have either $\partial U \subset Y$, or $\partial U \subset Z$ (i.e., no boundary complementary domain of $Y \cup Z$ is "new"). Let us add these domains to $Y \cup Z$. This will result into the continuum $\text{TH}(Y \cup Z)$, the topological hull of $Y \cup Z$, which has no bounded complementary domains at all. On the other hand, our assumption about all complementary domains of $Y \cup Z$ being not "new" implies that $\text{TH}(Y \cup Z) = \text{TH}(Y) \cup \text{TH}(Z)$. Since $\text{TH}(Y \cup Z)$ has no bounded complementary domains it now follows from [10, Theorem 61.2] that $\text{TH}(Y) \cap \text{TH}(Z)$ must be connected. Let us show that this contradicts the original assumption about $Y \cap Z$ being disconnected (observe that so far we have not used the fact that Y and Z are subcontinua of an unshielded continuum X). Indeed, since $Y \cap Z$ is not connected then by Lemma 3.7 $\text{TH}(Y) \cap \text{TH}(Z) = \text{TH}(Y \cap Z)$ is not connected either, a contradiction. $\qquad\square$

We are ready to prove Lemma 3.9, which is used in the "moreover" part of Theorem A.

Lemma 3.9. *Suppose that X is an unshielded continuum and $Y \subset X$ is a continuum strategically placed in X. Let $Z \subset X$ be a continuum. Then $Z \cap Y$ is a continuum too.*

Proof. Suppose that $Z \cap Y$ is disconnected. Then by Corollary 3.8 there exists a bounded complementary domain V of the union $Y \cup Z$ such that ∂V contains points of $Y \setminus Z$ and points of $Z \setminus Y$. Choose an open set T of points on ∂V contained in $Y \setminus Z$. It follows that there exists a point $y_\alpha \in T$ for some angle $\alpha \in \mathcal{A}(Y)$ so that an arc of the ray $R_Y(\alpha)$ is contained in V and lands at y_α. Clearly, this contradicts Lemma 3.4. □

We will also need the following geometric lemma.

Lemma 3.10. *Suppose that X is an unshielded continuum and $Y \subset X$ is a continuum strategically placed in X. Let $\{x_i\}$ be a sequence of points of $X \setminus Y$ such that $x_i \to x$ and $\alpha(x_i) \to \beta$. Then either $x \in \mathrm{Imp}_Y(\beta)$ or $x \in X \setminus Y$ and $\alpha(x) = \beta$. In particular, the map $\alpha : X \setminus Y \to \mathbb{S}$ is continuous.*

Proof. Since impressions are upper semi-continuous and because $\overline{C(x_i)} \cap Y \subset \mathrm{Imp}_Y(\alpha(x_i))$ by Lemma 3.5, we have that

$$\limsup \overline{C(x_i)} \cap Y \subset \limsup \mathrm{Imp}_Y(\alpha(x_i)) \subset \mathrm{Imp}_Y(\beta).$$

If angles $\theta, \theta', \gamma', \gamma \in \mathcal{A}$ are close to β and $\theta < \theta' < \beta < \gamma' < \gamma$ then for sufficiently large i we have that $\alpha(x_i) = \alpha_i \in (\theta, \gamma)$, and by Lemma 3.4 all components $C(x_i)$ are contained in the same appropriately chosen component $Z(\theta, \gamma)$ of $\mathbb{C} \setminus Y \cup R_X(p(\theta)) \cup R_X(p(\gamma))$ containing external rays of X with arguments from $(p(\theta), p(\gamma))$. Since the set

$$Q(\theta, \gamma) = Z(\theta, \gamma) \cup R_X(p(\theta)) \cup R_X(p(\gamma)) \cup Y$$

is closed this implies that $x \in Q(\theta, \gamma)$. Consider now two possibilities.
1. Suppose that $x \notin Y$ but $\alpha(x) \neq \beta$. Then we can choose angles θ and γ so that $\alpha(x) \notin [\theta, \gamma]$ and therefore $C(x)$ is disjoint from $Q(\theta, \gamma)$, a contradiction with the fact that $x \in Q(\theta, \gamma)$. Thus, if $x \notin Y$ then $\alpha(x) = \beta$.
2. Suppose that $x \in Y$. Let us show that then $x \in \mathrm{Imp}_Y(\beta)$. Indeed, choose angles $\theta, \theta', \gamma', \gamma$ as above. Draw crosscuts $T(\theta, \gamma) = T$ and then $T(\theta', \gamma') = T'$ inside the shadow S_T of T. Then for some $\varepsilon > 0$ every point $z \notin S_T$ of a component C of $X \setminus Y$ with $\alpha(C) \in (\theta', \gamma')$ is at least ε-distant from Y. In particular, if $x_i \notin S_T$ then the distance between x_i and Y is at least ε. Since $x_i \to x \in Y$, it follows that $x_i \in S_T$ for sufficiently large i, and hence that $x \in \mathrm{Imp}_Y(\beta)$.
This completes the proof of the lemma. □

The following lemma is proven in [2].

Lemma 3.11 ([2]). *Let $K \subset \mathbb{C}$ be an unshielded continuum and $m : K \to Z$ be a monotone map of K to a locally connected continuum Z. Then all fibers of m are unions of K-impressions (equivalently, m collapses any K-impression to a point). In particular, this holds for the finest monotone map m_K of K.*

We will need the θ-*curve Theorem* due to Kuratowski ([10]). By a θ-*curve* one means the union of three closed arcs having the same endpoints and no other common points (such is the letter θ).

Theorem 3.12 (Theorem 3 [10], p. 329). *Every locally connected continuum X containing no θ-curve has a basis such that the boundary of each element of the basis is finite.*

We will also need another result from [10].

Theorem 3.13 (Theorem 1 [10], p. 283)**.** *Every connected space which has a basis such that all its elements have finite boundary is locally connected.*

Theorems 3.12 and 3.13 imply the following corollary.

Corollary 3.14. *Suppose that Y is a subcontinuum of a locally connected unshielded continuum X. Then Y is locally connected.*

Proof. Since X is unshielded, then Y is unshielded too. Hence Y contains no θ-curve. By Theorem 3.12 Y has a basis all of whose elements have finite boundaries. By Theorem 3.13, Y is locally connected. □

We are ready to prove Theorem A.

Theorem A. *Let X be an unshielded plane continuum. If Y is strategically placed in X, and Y has a non-degenerate finest locally connected model M_Y, then X has a non-degenerate finest locally connected model M_X. Moreover, there exists a canonical embedding of M_Y into M_X.*

Proof. By Theorem 1.5 it suffices to show that there exists a monotone map from X to a non-degenerate locally connected continuum L. Since Y has a non-degenerate finest locally connected model, then there exists the finest monotone map $m_Y : Y \to L$ so that L is a non-degenerate locally connected continuum. We will extend the map m_Y to a monotone map $m : X \to L$ as follows: for every $x \in X \setminus Y$ set $m(x) = m_Y(\mathrm{Imp}_Y(\alpha(x)))$. Observe that since by Lemma 3.11 the map m_Y collapses all Y-impressions to points, then the map $m(x)$ is well-defined. Let us show that this map has the desired properties.

First we show that m is continuous. To see that, we first show that if $x_i \to x$ then one can find a subsequence x_{i_j} such that $m(x_{i_j}) \to m(x)$. This is obvious if infinitely many points x_i belong to Y because m_Y is continuous. Thus we may assume that $x_i \in X \setminus Y$ for every i. Choose a subsequence x_{i_j} so that $\alpha(x_{i_j}) \to \beta$. Then by Lemma 3.10 either $x \in \mathrm{Imp}_Y(\beta)$, or $x \in X \setminus Y$ and $\alpha(x) = \beta$. In either case $m(x) = m(\mathrm{Imp}_Y(\beta))$ while $m(x_{i_j}) = m(\mathrm{Imp}_Y(\alpha(x_{i_j})))$. Since impressions are upper semi-continuous and m is continuous, then $m(\mathrm{Imp}_Y(\alpha(x_{i_j}))) = m(x_{i_j}) \to m(\mathrm{Imp}_Y(\beta)) = m(x)$ as desired.

We claim this implies continuity of m. Indeed, suppose that $x_i \to x$ but $m(x_i) \not\to m(x)$. Refining our sequence we may assume that $m(x_i) \to t \neq m(x)$. However by the previous paragraph we can find a subsequence x_{i_j} of x_i such that $m(x_{i_j}) \to m(x)$, a contradiction.

Since for $y \in Y$, $m^{-1}(y)$ is the union of $(m_Y)^{-1}(y)$ and all components of $X \setminus Y$ whose closure intersects $(m_Y)^{-1}(y)$, $m^{-1}(y)$ is connected. Hence $m : X \to L$ is the desired monotone map.

Let us now prove the "moreover" part of Theorem A. We will use a monotone map $m : X \to L$ of X to the finest model L of Y constructed above. Recall that $m|_Y$ coincides with the finest map m_Y of Y to its finest model L. Let us denote the finest monotone model of X by T and the finest monotone map from X to T by $m_X : X \to T$. The fact that m is a monotone map implies that there exists a map $\psi : T \to L$ such that $m = \psi \circ m_X$.

Let us show that $m_X|_Y$ is monotone. Indeed, take a point $t \in m_X(Y)$. Then $m_X^{-1}(t)$ is a continuum because m_X is monotone. By Lemma 3.9 $m_X^{-1}(t) \cap Y$ is a continuum. Since $m_X^{-1}(t) \cap Y$ is the preimage of the point t under the restriction

$m_X|_Y$ of m_X onto Y, it follows that this restriction is monotone. Hence $m_Y = (\psi|_{m_X(Y)}) \circ (m_X|_Y)$ (observe that by Corollary 3.14 $m_X(Y)$ is locally connected). However m_Y is the finest monotone map for Y that implies that $\psi|_Y$ must be a homeomorphism. Thus, the restriction of the finest map m_X for X onto Y maps Y onto the finest model L of Y as desired.   □

4. **Applications.** In this section we apply our results to complex dynamics.

4.1. **Preliminaries from complex dynamics.** We rely upon basic facts discussed, e.g., in [11]. Let us fix a polynomial P of degree at least two.

Definition 4.1 (Periodic points). A periodic point p of period n is *repelling* if $(P^n)'(x) = re^{2\pi i\alpha}$ with $r > 1$ and *parabolic* if $(P^n)'(p) = e^{2\pi i \frac{p}{q}}$, with $p, q \in \mathbb{N}$. A periodic point p of P of period n and $(P^n)'(p) = e^{2\pi i\alpha}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is a *Siegel point* if there exists an open disk U containing p so that $P^n|_U$ is analytically conjugate to the rigid rotation $R(z) = e^{2\pi i\alpha}z$ of the open unit disk and a *Cremer point* if such a disk does not exist.

Periodic points play a crucial role in complex dynamics; in particular, they are used in one of the standard equivalent definitions of the *Julia set* of P.

Definition 4.2 ((Filled) Julia set). The Julia set $J(P)$ of a polynomial P is the closure of the set of repelling periodic points of P; it is known that $J(P)$ is compact. The set $\mathbb{C} \setminus U^\infty_{J(P)} = \mathrm{TH}(J(P))$ is called the *filled Julia set* and is denoted by $K(P)$.

The Julia set $J(P)$ coincides with the boundary $\partial U^\infty_{J(P)}$ of the open set $U^\infty_{J(P)}$ and, hence, $J(P)$ is unshielded. The dynamics of P outside the filled Julia set $K(P)$ is rather predictable.

Definition 4.3 ((Non-)escaping points). Points attracted to infinity under iterations of P are called *escaping*. Otherwise points are said to be *non-escaping*.

It is known that the unbounded complementary domain $U^\infty_{J(P)}$ of $J(P)$ is in fact the set of all escaping points while its complement $K(P)$ is in fact the set of all non-escaping points. The set $U^\infty_{J(P)} = U^\infty_{K(P)}$ is therefore called the *basin of attraction of infinity*.

The Julia set $J(P)$ is a continuum if and only if all critical points of P are non-escaping (in other words, the orbits of all critical points of P are contained in $K(P)$). We will first assume that $J(P)$ (equivalently, $K(P)$) is connected. Then it is known that all repelling and parabolic periodic points of P (and all their pre-images) are the landing points of finitely many rays $R_{J(P)}(\alpha)$ with $\alpha \in Q$.

In a vast majority of cases the connected Julia set of a polynomial is either locally connected, or at least admits a non-degenerate finest locally connected model. However, this is not always the case. To give an example we need the following alternative definition of a Cremer point.

Definition 4.4. Let P be a polynomial. Suppose that a is a periodic point of P of period n such that $(P^n)'(a) = e^{2\pi i\theta}$ with θ irrational. Moreover, suppose that a belongs to the Julia set $J(P)$ of P. Then a is a *Cremer* periodic point of P.

The main result of [6] shows that in some cases the finest locally connected model of a connected Julia set is degenerate.

Theorem 4.5 ([6]). *For the Julia set of a quadratic polynomial with a fixed* Cremer *point the finest locally connected model is a point.*

In general the existence of a subcontinuum with a non-degenerate finest locally connected model provides no information about such a model for the entire unshielded continuum. However, if the subcontinuum is strategically placed, then Theorem A shows that a non-degenerate model for the entire space does exist. A natural choice of a subcontinuum of $J(P)$ on which one can hope to have a non-degenerate finest locally model is that of a connected Julia set of a *polynomial-like* map which is a power of P. This is another application of polynomial-like maps that are a powerful tool in complex dynamics introduced by Douady and Hubbard [8].

Definition 4.6 (Polynomial-like maps)**.** A *polynomial-like* map of degree d is a triple (U, V, f) where U and V are open subsets of $\mathbb{C}$ isomorphic to discs, with U relatively compact in V, and $f : U \to V$ is a proper analytic map of degree d.

Similar to polynomials, one can define the (filled) Julia set of a polynomial-like map.

Definition 4.7 ((Filled) Julia set of a polynomial-like map)**.** If $f : U \to V$ is a polynomial-like map of degree d, we will denote

$$K_f = \bigcap_{n \geq 0} f^{-n}(U),$$

the compact set of points $z \in U$ such that $f^n(z)$ is defined and belongs to U for all $n \in \mathbb{N}$. The set K_f is called the *filled Julia set* of f. The *Julia set* J_f of f is the boundary of K_f.

Given a polynomial P, we will often say that $P^n|_{K^*} : K^* \to K^*$ (or $P^n|_{J^*} : J^* \to J^*$) is a polynomial-like map meaning that there exist open sets U and V as in Definition 4.6 such that K^* is the filled Julia set (or J^* is the Julia set) of the corresponding polynomial-like map (P^n, U, V).

The term *polynomial-like* maps is justified by the Straightening Theorem stated below. However first we need one more definition.

Definition 4.8 (Hybrid equivalence [8])**.** Two polynomial-like maps $f : U \to V$ and $g : U' \to V'$ are *hybrid equivalent* if there is a quasi-conformal map $\varphi : U \to U'$ conjugating f to g such that φ is conformal almost everywhere on $K(f)$ (in other words, φ is such that $\varphi \circ f = g \circ \varphi$ near K_f). The map φ is called a *straightening map*.

An important result of [8] is given below; this theorem allows us to talk about finest locally connected models of connected *polynomial-like* Julia sets.

Straightening Theorem ([8])**.** *Let $f : U \to V$ be a polynomial-like map. Then f is hybrid equivalent to a polynomial P. Moreover, if $K(f)$ is connected, then P is unique up to (global) conjugation by an affine map.*

4.2. **Main applications in the connected case.** Suppose that the connected Julia set $J(P)$ of a polynomial P contains a subcontinuum K^* so that $P^n|_{K^*}$ is a polynomial-like map. Then by the Straightening Theorem $P^n|_{K^*} : K^* \to K^*$ is hybrid equivalent to a polynomial g with connected filled Julia set $K(g)$. In particular, under the hybrid equivalence appropriate arcs contained in external rays of $K(g)$ correspond to arcs inside U which accumulate to the corresponding polynomial-like Julia set J^* (the open set U is defined as in Definition 4.6). Slightly abusing the language we will call these arcs *polynomial-like rays* and will denote them in

the same way as we would have denoted external rays of K^* (or equivalently, of J^*), i.e. $R_{K^*}(\alpha)$ where α is the argument of the external ray of the polynomial g corresponding to $R_{K^*}(\alpha)$.

Recall that an external ray $R_J(\alpha)$ is said to *accumulate* in J^* if $\mathrm{Pr}(\alpha) \subset J^*$. Also, it is easy to see that the property of a point being repelling or parabolic is preserved under hybrid equivalence. By Definition 4.2 this allows one to conclude that repelling periodic points of P are dense in J^*. Moreover, it follows that if p is a repelling or parabolic periodic point of P, then only finitely many external rays $R_{J^*}(\alpha)$ of J^* and finitely many external rays $R_J(\beta)$ of J land at p.

Suppose that $Y \subset X$ are unshielded plane continua. Above in Lemma 3.4 we considered a map $p : \mathcal{A} \to \mathbb{S}$; this map associated to a ray $R_Y(\alpha)$ the ray $R_X(p(\alpha))$ so that both rays landed on the same point $y_\alpha \in Y$ and were homotopic outside Y by a homotopy fixing y_α. In the case of polynomials f and polynomial-like maps f^* it is easier to first consider the "inverse" map which associates rays $R_{J(f)}(\beta)$ which land at a point $y_\beta \in J(f^*)$ to rays $R_{J(f^*)}(\nu(\beta))$ which land at y_β and are homotopic to $R_X(\beta)$ outside Y by a homotopy which fixes y_β. This is accomplished in Lemma 4.9.

In what follows, given a map, we call a point *preperiodic* if it is not periodic but eventually maps to a periodic point, and (*pre*)*periodic* if it is periodic or preperiodic. Recall that if the Julia set $J(P)$ of a polynomial P is connected and an angle α is (pre)periodic then the external ray $R_{J(P)}$ lands at a (pre)periodic (in the sense of P) point in $J(P)$ [11]. Given a set $T \subset \mathbb{S}$ we say that a map $\Psi : T \to \mathbb{S}$ is *extendably monotone* if Ψ has a monotone (but not necessarily continuous!) extension $m : \mathbb{S} \to \mathbb{S}$.

Lemma 4.9. *Suppose that P is a polynomial of degree d with connected Julia set J and $J^* \subset K$ is a subcontinuum of J such that $P^n|_{J^*}$ is a polynomial-like map with filled Julia set K^* and Julia set J^*. Suppose that $P^n|_{J^*}$ is hybrid equivalent to a polynomial Q of degree k. Let $\mathcal{B} \subset \mathbb{S}$ be the set of **all** angles β so that $R_J(\beta)$ lands at a point $y_\beta \in J^*$. Then there exists a extendably monotone continuous map $\nu : \mathcal{B} \to \mathbb{S}$ such that:*

1. *for each $\beta \in \mathcal{B}$ the ray $R_{J^*}(\nu(\beta))$ lands at the same point y_β and the rays $R_J(\beta)$ and $R_{J^*}(\nu(\beta))$ are homotopic outside K^* under a homotopy which fixes the point y_β,*
2. *if $\mathcal{B}' \subset \mathcal{B}$ is the set of all (pre)periodic angles, then $\nu(\mathcal{B}')$ is dense in $\mathbb{S}$,*
3. *$\nu \circ \sigma_d = \sigma_k \circ \nu$.*

Notice that the continuity of ν on $\mathcal{B}$ only means that ν is continuous at points of $\mathcal{B}$ and does not imply that ν can be extended to a continuous monotone map of the circle to itself.

Proof. Since $P^n|_{J^*}$ is polynomial-like, there exist Jordan disks $U \subset \overline{U} \subset V$ such that $J^* \subset U$ and $P^n : U \to V$ is polynomial-like. Denote $P^n|_U$ by P^*.

Let $R_J(\beta)$ be an external ray of J which lands at a point $y_\beta \in J^*$. Consider the inverse $\xi : U_{J^*}^\infty \to \mathbb{D}^\infty$ of the corresponding Riemann map from $\mathbb{D}^\infty$ to $U_{J^*}^\infty$ with derivative converging to a real number at infinity. Then $\xi(R_J(\beta))$ is a curve which accumulates at a point $z \in \mathbb{S}$. Choose the polynomial-like ray $R_{J^*}(\alpha)$ of J^* whose ξ-image is the radial ray to $\mathbb{D}^\infty$ landing at z (the argument of this radial ray and hence the argument of the corresponding polynomial-like ray is denoted by α). Since in the $\mathbb{D}^\infty$-plane the radial ray to z and $\xi(R_J(\alpha))$ are homotopic, it follows that $R_J(\beta)$ and $R_{J^*}(\alpha)$ are homotopic outside J^* by a homotopy which fixes y (the

homotopy carries over to $\mathbb{C} \setminus J^*$ under the Riemann map). Define $\nu(\beta) = \alpha$. Since this construction goes through for all angles $\beta \in \mathcal{B}$, this defines a map $\nu : \mathcal{B} \to \mathbb{S}$.

To see that ν is extendably monotone suppose that $\nu(\beta_1) = \nu(\beta_2)$. Then $\xi(R_J(\beta_1))$ and $\xi(R_J(\beta_2))$ are two curves which land at the same point $z \in \mathbb{S}$. Denote by T the component of $\mathbb{D}^\infty \setminus [\xi(R_J(\beta_1)) \cup \xi(R_J(\beta_2))]$ whose closure meets $\mathbb{S}$ only in the point $z \in \mathbb{S}$ (in other words, T is the wedge between $\xi(R_J(\beta_1))$ and $\xi(R_K(\beta_2))$ which does not contain the unit disk). Then any external ray $R_J(\gamma)$ with $\xi(R_J(\gamma)) \subset T$ that lands at a point of J^* must land at y so that $\xi(R_J(\gamma))$ lands at z.

This implies that there exists an arc $A_z \subset \mathbb{S}$ so that $\nu^{-1}(z) = A_z \cap \mathcal{B}$. To see that there exists a monotone extension of ν it remains to observe that circular orientation among points of $\mathcal{B}$ is preserved under ν in the following sense: if $\beta_1 < \beta_1 < \beta_3$ then it is impossible that $\nu(\beta_1) < \nu(\beta_3) < \nu(\beta_2)$ as otherwise some external rays of K will have to intersect. Thus, the arcs A_z constructed above for all points $z \in \nu(\mathcal{B})$ have the same circular order as the points $z \in \mathbb{S}$ which implies the desired claim.

Now, choose an angle $\beta \in \mathcal{B}$ such that $y_\beta \in J^*$, the landing point of the external ray $R_J(\beta)$, is preperiodic. Set $\alpha = \nu(\beta)$. Properties of polynomials (and hence of polynomial-like maps) imply that the family of all polynomial-like rays which are preimages of $R_{J^*}(\alpha)$ is such that their arguments are dense in $\mathbb{S}$. Each such polynomial-like ray R^* with argument α' is a unique pullback of $R_{J^*}(\alpha)$ under the appropriate branch of the inverse function to P^* (recall that y_β is not periodic). If we simultaneously pull back $R_J(\beta)$ under the same branch of the inverse function of P^* we will obtain an external ray $R_J(\beta')$ of J with argument β' which lands at the same point as R^* and is homotopic to R^* outside K^*. Denote the argument of R by α', then $\nu(\beta') = \alpha'$. This shows that (2) holds.

To see that ν is continuous consider a sequence $\beta_1 < \beta_2 < \ldots$ in $\mathcal{B}$ so that $\lim \beta_i = \beta_\infty \in \mathcal{B}$. Consider the landing points z_i of the curves $\xi(R_J(\beta_i))$ and the landing point z_∞ of $\xi(R_J(\beta_\infty))$. The fact that ν is extendably monotone implies that $z_1 \leq z_2 \leq \cdots \leq z_\infty$. We claim that $z_\infty = \lim z_i$. Indeed, otherwise we have that $z_1 \leq \lim z_i = t < z_\infty$. By (2) we can choose a (pre)periodic angle $\hat{\beta} \in \mathcal{B}'$ such that $t < \nu(\hat{\beta}) < z_\infty$. Since ν is extendably monotone this contradicts the fact that $\lim \beta_i = \beta_\infty$. Thus, $z_\infty = \lim z_i$ as desired. The last claim of the lemma is left to the reader. $\square$

The following corollary easily follows from definitions, Lemma 3.4 and Lemma 4.9

Corollary 4.10. *Suppose that the connected Julia set $J(P)$ of a polynomial P contains a subcontinuum J^* so that $P^n|_{J^*}$ is a polynomial-like map for some $n \geq 1$. Then J^* is strategically placed in $J(P)$.*

Proof. Let us use the notation from Lemma 4.9. Set $\mathcal{A} = \nu(\mathcal{B})$. Then by Lemma 4.9 the set $\mathcal{A}$ is dense in $\mathbb{S}$. Moreover, by Lemma 4.9 conditions listed in Lemma 3.4(2) are satisfied for $\mathcal{A} \subset \mathbb{S}$ and $J^* \subset J(P)$. Hence J^* is strategically placed in $J(P)$. $\square$

Lemma 4.10 allows one to conclude that connected polynomial-like Julia sets with non-degenerate finest locally connected models force the existence of non-degenerate finest locally connected models of containing them connected polynomial Julia sets.

Theorem B. *Suppose that $P : \mathbb{C} \to \mathbb{C}$ is a polynomial and $J^* \subset J(P)$ is the polynomial-like Julia set of a suitable restriction of P^n for some $n > 0$. If J^* has a non-degenerate finest locally connected model, then so does $J(P)$.*

Proof. Indeed, by Lemma 4.10 Theorem A implies the desired. $\square$

Note that if K^* is a filled polynomial-like Julia set of a polynomial P, then K^* is a component of $P^{-n}(K^*)$. As it turns out this is almost sufficient (the proof of Lemma 4.11 uses some ideas communicated by M. Lyubich to the third named author). For convenience we state these results in the case that $n = 1$.

Theorem 4.11 (Theorem B [4])**.** *Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial, and $Y \subset \mathbb{C}$ be a full P-invariant continuum. The following assertions are equivalent:*

1. *the set Y is the filled Julia set of some polynomial-like map $P : U^* \to V^*$ of degree k,*
2. *Y is a component of the set $P^{-1}(P(Y))$, and, for every attracting or parabolic point y of P in Y, the immediate attracting basin of y or the union of all parabolic domains at y is a subset of Y.*

The following corollary is now almost immediate.

Corollary 4.12. *Suppose that $K^* \subset K$ is an invariant subcontinuum of the filled Julia set of a polynomial P such that K^* is a component of $P^{-1} \circ P(K^*)$ containing all immediate parabolic and attracting basins of all attracting and parabolic points in K^*. Then if $\partial U_{K^*}^\infty$ has a non-degenerate finest locally connected model, then $J(P)$ has a non-degenerate finest locally connected model.*

Proof. By [4], $P|_{K^*} : K^* \to K^*$ is a polynomial-like map. Hence the result follows from Theorem B. $\square$

4.3. **Models for non-connected spaces.** Models for non-connected spaces were studied in [3]. A *compactum* is a compact metric space. Since a compactum with infinitely many distinct components is always not locally connected at some point, we need to replace the condition of local connectedness of the model by a suitable notion.

A compactum X is called *finitely Suslinian* if, for every $\varepsilon > 0$, every collection of disjoint subcontinua of X with diameters at least ε is finite. By Lemma 2.9 [1], unshielded planar locally connected continua are finitely Suslinian and vice versa. Thus, in the unshielded case, the notion of finitely Suslinian generalizes the notion of local connectivity. This motivates us to look for good finitely Suslinian models of planar compacta.

Definition 4.13. Let X be a compactum. A *finest finitely Suslinian model* for X is a finitely Suslinian compactum S and a monotone map $m : X \to S$ so that for each monotone map $f : X \to Y$ to a finitely Suslinian compactum Y there exists a monotone map $g : S \to Y$ with $g \circ m = f$. Then the map $m : X \to S$ is called a *finest finitely Suslinian model map*. We say that a compactum X has a *non-degenerate finitely Suslinian model* S if at least one component of S is non-degenerate.

Observe that by definition of a monotone map it follows that if m is monotone then distinct components of X map to distinct components of $m(X)$. Observe also that the above introduced notion of a degenerate finitely Suslinian model agrees with the notion of a degenerate locally connected model in the case of continua.

By [3] all finest finitely Suslinian models of a compactum X are homeomorphic and we can talk about *the* finest finitely Suslinian models of compacta. It was shown in [3] that every planar unshielded compactum X has a finest finitely Suslinian model S (which is unique up to homeomorphisms). As previously in the case of

continua, the finest finitely Suslinian model S of X may be degenerate (i.e., the finest finitely Suslinian model monotone map $m : X \to S$ may well collapse *all* components of X to points). The following theorem is the main result of [3] concerning finest finitely Suslinian models of polynomial Julia sets (this time including disconnected Julia sets).

Theorem 4.14 (Theorem 6 [3]). *The finest finitely Suslinian model monotone map $m : J(P) \to S$ of the Julia set $J(P)$ of a polynomial P coincides on each component X of $J(P)$ with the finest monotone map m_X of X to a locally connected continuum. In particular, the following holds:*

1. *the finest finitely Suslinian monotone model of $J(P)$ is non-degenerate if and only if there exists a periodic component of $J(P)$ whose finest finitely Suslinian monotone model is non-degenerate;*
2. *the Julia set $J(P)$ is finitely Suslinian if and only if all periodic non-degenerate components of $J(P)$ are locally connected.*

Hence, the following theorem immediately follows.

Theorem 4.15. *Suppose that J is the Julia set of a polynomial P and $J^* \subset J$ is a subcontinuum so that, for some integer r, $P^r|_{J^*} : J^* \to J^*$ is a polynomial-like map and J^* has a non-degenerate finest locally connected model. Then J has the finest finitely Suslinian model.*

Proof. Suppose that K^* is contained in the component C of J. Then C must be periodic of some period n. By a result of [5], $P^n|_C : C \to C$ is a polynomial-like map. Hence $P^n|_C$ is hybrid equivalent to a polynomial g. Since $J^* \subset C$ it follows from Theorem B that C has a non-degenerate finest locally connected model. Hence, by Theorem 4.14, J has the finest finitely Suslinian model. $\qquad\square$

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