ASYMPTOTIC BEHAVIOR OF THE ENTROPY OF INTERVAL MAPS

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Abstract. We obtain upper estimates on the entropy of interval maps of given modality and Sharkovskii type. Following our results we formulate a conjecture on asymptotic behavior of the entropy of interval maps.

1. Introduction

The topological entropy is a well-known and widely used characteristic of a dynamical system. It is often considered as a measure of chaos exhibited by the system. The fact that it is a topological invariant makes it even more important. Therefore obtaining estimates of the topological entropy of a map is a useful task. In one-dimensional dynamics an important (but by no means unique) setup here is as follows. One considers a map with certain restrictions on dynamics of its individual points (for example, one may consider interval maps with periodic points of given periods). Then the question is whether anything can be said about the topological entropy of such a map. It turns out that the majority of estimates obtained in such situations for the topological entropy are those from below (the list of references here is prohibitively long, so we refer the reader to [1] and references therein). This is quite understandable as to obtain such an estimate for an interval map from a certain class it is enough to show that each map from this class exhibits a certain type of behavior (*pattern*) which can be described combinatorially. Since such combinatorial description usually yields a guaranteed lower estimate of the entropy, the problem then is solved.

The story with upper estimates on the entropy is quite different. The problem is that the approach described above fails - the fact that the

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map exhibits a certain type of behavior does not imply upper estimates of the entropy.

In fact, we know of only one paper where upper estimates of the entropy of a class of interval maps are done. In [4] (see also [3]) the second author investigated the topological entropy of green maps with prescribed (combinatorial) *complexity* - a characteristic not restricting modality but covering properties of laps of a piecewise-monotone map. He proved that there is a close coherence between the best upper bounds of the entropy of such maps and *bifurcation values* of the parameters determining *strictly monotone solutions* to special difference equations.

In the majority of cases upper estimates of the entropy are not even possible because if the maps are not assumed polymodal, or if they are polymodal but the modality is not bounded, then usually one can give examples of maps from a given class whose entropy is arbitrarily large.

Therefore, a good approach to the problem of upper estimates of the entropy is to assume that maps which exhibit certain type of dynamics are polymodal with bounded modality and use this modality as one of the parameters involved in the estimates. This paper is devoted to obtaining upper estimates on the entropy of maps with two types of restrictions on the dynamics of their points; to describe the restrictions we need to introduce a few known notions in one-dimensional dynamics.

A useful characteristic of dynamics of interval maps is the so-called *Sharkovskii type* $Sh(f)$ of the map f, that is the strongest - in the sense of the Sharkovskii ordering - period [10] of a periodic point of the map. The Sharkovskii ordering is the following ordering among natural numbers:

$$3 \prec 5 \prec 7 \prec \cdots \prec 2 \cdot 3 \prec 2 \cdot 5 \prec 2 \cdot 7 \prec \cdots \prec 8 \prec 4 \prec 2 \prec 1$$

In the case when $n \prec m$ we will say that n is *stronger than* m *(in the sense of the Sharkovskii ordering)*. Denote by $P(f)$ the set of periods of cycles of an interval map f. Also, denote by $S(k)$ the set of all such integers m that $k \prec m$ or $k = m$ and by $S(2^\infty)$ the set $\{1, 2, 4, 8, \dots\}$. The importance of the Sharkovskii ordering is explained by the famous Sharkovskii Theorem on co-existence of periods of cycles of interval maps, a major result in one-dimensional dynamics stated below.

Theorem 1.1. *Let* $g : [0, 1] \to [0, 1]$ *be a continuous map. Then there exists such* $k \in \mathbb{N} \cup 2^\infty$ *that* $P(g) = S(k)$*. Moreover for any such* k *there exists a map* $g : [0, 1] \to [0, 1]$ *with* $P(g) = S(k)$*.*

In the original proof of the Sharkovskii Theorem [10] another fact about interval maps is used as a technical tool. Following Sharkovskii, we say that an interval map f has an *L-scheme* if there exists a fixed point a and a point b such that either $f^2(b) \le a < b < f(b)$, or $f^2(b) \ge a > b > f(b)$ (because of the symmetry of the notion we will assume below that if f has an L-scheme then in fact $f^2(b) \le a < b < f(b)$). In this case we will also say that the points a, b *form* an *L*-scheme. One of the results of [10] is that a map which has an L-scheme has periodic points of all periods.

Observe, that if f has an L-scheme then the intervals $I = [a, b]$, $J = [b, f(b)]$ have disjoint interiors and are such that their images cover their union. In other words, they form the so-called *2-horseshoe* (the importance of horseshoes for interval maps became clear after papers [6, 7, 8, 9], see also [1] and references therein), and in fact it is easy to see that if a map has a 2-horseshoe then it must have an L-scheme.

Our aim is to estimate the entropy of an interval map from above assuming that it does not have any L-scheme and that its *modality* (see below) is known. We need some definitions. A continuous map is said to be *monotone* if pointwise preimages are connected (this is actually a general definition which applies to maps of any topological spaces). Thus, in the interval case we understand monotonicity in the non-strict sense. An interval map $f : I \to I$ is said to be piecewise-monotone if there are finitely many points dividing I into subintervals on each of which f is monotone. The least number q of such (inner) points is called the *modality* of f and denoted by $\mathrm{mod}(f)$, the map f then is said to be *q-modal*, and $q + 1$ closed intervals into which I gets partitioned are then called *laps* of f.

Theorem 1.2. *Let f be a q-modal map without L-scheme (e.g., this is true if $Sh(f) \ne 3$). Then* $h(f) \le \ln(\sqrt{.75q^2 + 1.5q + 1.25} + .5)$

Denote the supremum of the entropy of a map f with $Sh(f) = s$ and modality q by $h(s, q)$. Concerning the numbers $h(s, q)$ one can ask various questions on their (asymptotic) behavior with respect to Sharkovskii type and modality. The simplest observation can be done with the help of Figure 1. Using a suitable extension of a unimodal map of type s one can construct a map of type s and modality q. It implies that the number $h(s, q)$ exists for any pair s, q. Similar argument shows that the sequence $\{h(s, q)\}_q$ is non-decreasing for any type s. On the other hand, proofs of some properties require more subtle technics and detailed analysis (for instance, the fact that $h(s, q)$ need not be a maximum (as it is for $s = 3$), answering the question if $h(s, q) < h(s, q+1)$ for each modality q, resp. $h(s, q) > h(s', q)$ whenever $s \prec s'$ and s is not a power of two). In this paper we prefer rather (new) different approach - to get some information on the asymptotic behavior of

$h(s,q)$ with fixed s as $q \to \infty$. In this direction Theorem 1.2 enables us to estimate the asymptotical difference $\liminf_{q\to\infty} h(3,q) - h(5,q)$. Indeed, by the result of Misiurewicz and Szlenk [9], $h(3,q) = \ln(q+1)$ and from Theorem 1.2 we get

$$\ln(\sqrt{.75q^2 + 1.5q + 1.25} + .5) \geq h(5,q)$$

for each q hence we see that $\liminf_{q\to\infty} h(3,q) - h(5,q) > 0$ - see the last theorem of this section. This can be interpreted as an indication of the fact that the dynamical restriction upon a map which is introduced as we fix its Sharkovskii type is significant - after all, it can be detected through the entropy as we let the modality to tend to infinity.

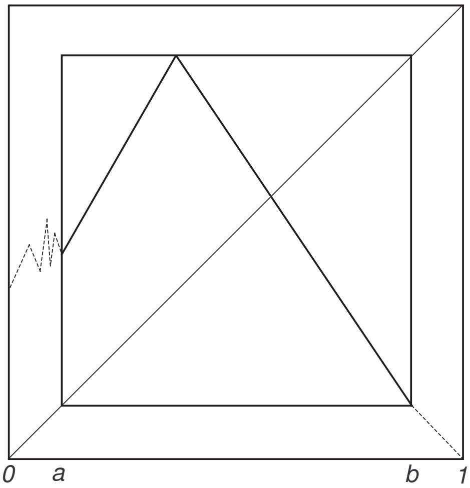


Figure 1. For $0 < a < b < 1$ and $q' > q$, any q-modal map $f\colon [a,b] \to [a,b]$ of type s can be extended to a q'-modal map $\tilde{f}\colon [0,1] \to [0,1]$ of the same Sharkovskii type and the entropy.

We conjecture that in fact the same should hold for all other fixed Sharkovskii types of interval maps except for the case when both types are powers of 2 (including 2^∞, and hence the entropy is zero, see [6, 7, 8, 9]). More precisely, we would like to formulate here the following conjecture.

Conjecture 1.3. *Given two Sharkovskii types $s' \prec s''$ which are not both equal to powers of* 2 *we have that* $\liminf_{q\to\infty} h(s',q) - h(s'',q) > 0$.

In fact, in a simple case when $s' = 3\cdot 2^k \prec s'' = 3\cdot 2^\ell$ have different powers of 2 involved in their prime decompositions, Conjecture 1.3 follows easily from the lemma below which can be obtained using the same tools as the ones used in the proof of Theorem 1.2.

Lemma 1.4. *Let f be a q-modal map with $Sh(f) = (2s+1)2^k$, $s \geq 0$, $k \geq 0$. Then $h(f) \leq \ln(1 + 2^{-k}q)$.*

We finish this section by theorem that summarizes the cases when we are able to estimate the asymptotical differences of entropies from our conjecture.

Theorem 1.5. *Let the numbers $h(s, q)$ be as above. Then*

(1) $\liminf_{q\to\infty} h(3, q) - h(5, q) \geq \ln(2/\sqrt{3})$.
(2) *If* $0 \leq k \leq \ell$ *then* $\liminf_{q\to\infty} h(3 \cdot 2^k, q) - h(3 \cdot 2^\ell, q) \geq \ln 2^{\ell-k}$.

In conclusion we want to say, that even though in general the verification of Conjecture 1.3 may require a set of tools finer than the ones employed in the present paper, we hope that developing our tools will be helpful in fulfilling this task.

2. Preliminaries

First of all, we would like to make some simplifications concerning the notion of L-scheme. Suppose that points x, y form an L-scheme and $f^2(y) \leq x = f(x) < y < f(y)$. Choose the closest to y from the left fixed point x'. Since $f^2(y) < y < f(y)$ then there is a fixed point between y and $f(y)$; let u be the closest from the right to y fixed point. Then $f^2(y) \leq x \leq x' < y < u < f(y)$ and $f(z) > z$ for all $z \in (x', u)$. Choose a preimage q of x' among points of the interval $[u, f(y)]$, then choose a preimage p of q among points of $[x', u]$. We conclude that $f^2(p) = x' = f(x') < p < f(p)$ and $f(z) > z$ for any $z \in (x', p)$. From now on when saying that points p, q form an L-scheme we will mean only this behavior - or its exact symmetric copy. Also, it is clear that to estimate the topological entropy from above it is sufficient to consider only interval maps with positive entropy.

We need a result of [2]; to state it we need to introduce a few notions. An interval I is said to be *periodic (of period n)* if $I, f(I), \ldots, f^{n-1}(I)$ are pairwise disjoint while $f^n(I) = I$. Set $\mathrm{orb}(I) = \cup_{j=0}^{n-1} f^j(I)$. A map restricted on the orbit of a periodic interval (of period n) is said to be *non-strictly periodic (of period n)*; such a map is said to be *transitive* if there exists a point x whose orbit is dense. Finally, according to how we define monotonicity, a map from a union of disjoint closed intervals onto a union of disjoint closed intervals is monotone if and only if for each point in the range its preimage is a closed interval.

Theorem 2.1. [2] *Suppose that f is a piecewise-monotone interval map of positive entropy. Then there exists a periodic interval I of period n such that the following holds:*

(1) *$f|_{orb(I)}$ is semiconjugate by a monotone map φ to a transitive non-strictly periodic map g of period n;*
(2) *$h(f) = h(f|_{orb(I)}) = h(g)$.*

Observe that since the semiconjugacy above is monotone, preimages of points are closed intervals and so if g has a point of some period then so does f. In what follows given a piecewise-monotone interval map f of positive entropy we fix *some* choice of the objects whose existence is proven in Theorem 2.1 and denote them as follows: the periodic interval I is denoted by I_f, the semiconjugacy φ is denoted by φ_f, the period of I is denoted by n_f, the transitive non-strictly periodic map g is denoted g_f and the periodic interval on the orbit of which the map g_f is defined is denoted M_f (we assume that $\varphi_f(I_f) = M_f$).

We will also need another result which deals with transitive interval maps. To do so we need another definition: a non-strictly periodic map is said to be *(topologically) mixing* if for any open U, V there exists $n \geq 0$ such that for any $i \geq n$ the sets $f^i(U)$ and V are non-disjoint.

Lemma 2.2. [2] *Suppose that $f : [0,1] \to [0,1]$ is a transitive interval map. Then exactly one of the two cases holds:*

(1) *the map f is mixing and $Sh(f) = 2n+1, n \geq 1$;*
(2) *there exists a point $a \in (0,1)$ such that $f[0,a] = [a,1], f[a,1] = [0,a]$, restrictions $f^2|_{[0,a]}, f^2|_{[a,1]}$ are mixing, and $Sh(f) = 6$.*

Upper estimates of the entropy of interval maps are often times based upon the following proposition whose proof we only sketch. We need yet a few definitions. Given a piecewise-monotone q-modal map f with laps $K_0, \dots, K_q$ define the *dynamic matrix* $\mathrm{Dyn}_f = (a_{ij})$ of f by the following rule: $a_{ij} = 1$ if $f(K_i) \cap K_J \neq \emptyset$ and $a_{ij} = 0$ if $f(K_i) \cap K_j = \emptyset$. Also, consider the *dynamic oriented graph* G_f of f with vertices $K_0, \dots, K_q$ which corresponds to Dyn_f and is defined as follows: an oriented edge (an *arrow*) in G_f from K_i to K_j exists if and only if $f(K_i) \cap K_j \neq \emptyset$. Denote by $r(A)$ the spectral radius of a matrix A.

Proposition 2.3. *Suppose that $f : [0,1] \to [0,1]$ is piecewise-monotone map. Then $h(f) \leq r(Dyn_f)$.*

Proof. By the results of Misiurewicz and Szlenk [9] (see also [6, 7, 8]), the entropy of f can be computed as follows. Consider the modality $\mathrm{mod}(f^n)$ of f^n; then the limit of the sequence $n^{-1}\ln(\mathrm{mod}(f^n)+1)$ exists and equals the topological entropy $h(f)$ of f. In other words, the entropy is defined by the asymptotic behavior of the number of laps of f^n as n tends to infinity. Consider the graph G_f. Clearly the number of laps of f^n is no more than the number of oriented paths of length $n-1$ in G_f (i.e., paths consisting of $n-1$ consecutive arrows). On the other hand, according to our definition of the dynamic matrix Dyn_f of f, the number of all such paths equals the sum of all entries in Dyn_f^{n-1}. By the well-known properties of the spectral radius of matrices it follows that the entropy of f is no more than the spectral radius of Dyn_f as desired. □

We also need estimates on eigenvalues of matrices. Given an $n \times n$ matrix A, denote by a_{ij} its entries and by $r(A)$ its spectral radius. Also, given i denote by m_i the maximum of the absolute value of a non-diagonal element of the i-th row in A (by the *diagonal* of a matrix we always mean its *main diagonal*). With this notation, set

$$b(A) = \sum_{i=1}^{n} (m_i^{-1} \cdot \sum_{j=1, j \neq i}^{n} |a_{ij}|).$$

The following lemma is proven in [5].

Lemma 2.4. *Let A be an $n \times n$ matrix. Choose a number $\alpha > 0$ such that $\alpha(\alpha+1) \geq b(A)$. Then all eigenvalues of A belong to the union of disks $|a_{ii} - z| \leq \alpha m_i$.*

Corollary 2.5. *Let A be a matrix whose entries are equal to either 0 or 1, in each row there is at least one non-zero non-diagonal entry, and there are R non-zero entries outside the diagonal. Then $r(A) \leq \sqrt{R + .25} + .5$.*

Proof. In the situation of the corollary, $m_i = 1, 1 \leq i \leq n$. Hence $b(A) = R$. The minimal α satisfying the inequality $\alpha(1+\alpha) \geq R$ is given by $\hat{\alpha} = \sqrt{R + .25} - .5$. Since $|a_{ii}| \leq 1, 1 \leq i \leq n$, we see that by Lemma 2.4 $r(A) \leq \sqrt{R + .25} + .5$ as desired. □

3. Main Theorem

In order to prove our main theorem as well as Lemma 1.4 we need three more lemmas.

Lemma 3.1. *Let $f : [0,1] \to [0,1]$ be a transitive map without L-scheme. Then f has a unique fixed point $a \in (0,1)$ such that $f(x) > x$ for any $x < a$, and $f(x) < x$ for any $x > a$.*

Proof. Let us show that f cannot have two fixed points. Indeed, suppose otherwise. Then we may assume that a and b are fixed points and $f(x) > x$ for any $x \in (a,b)$. Then by transitivity $b < 1$. Let us show that $a \in f([b,1])$. Indeed, otherwise $[b,1] \cup f([b,1])$ is an invariant subinterval of $[0,1]$ not containing a which contradicts transitivity.

Now, take the smallest $c \in [b, 1]$ with $f(c) = a$. Then there exists $d \in (a, c)$ such that $f(d) = c$ because otherwise $[a, c]$ is invariant which by transitivity implies $a = 0, c = 1$ and by surjectivity for some d we must have $f(d) = c$ with $d \neq c$ because $f(c) \neq c$. We conclude that $a = f(a) = f^2(d) < d < f(d) = c$ which means that f has an L-scheme, a contradiction. □

Before we state the next lemma observe that the notion of modality can be introduced for non-strictly periodic map similarly as it is done for an interval map (the least number of points dividing its domain into subintervals on each of which a map is monotone). Say that a closed interval J is *weakly periodic (for f of period l)* if $\text{int}(J)$, $\text{int}(f(J)), \dots, \text{int}(f^{l-1}(J))$ are pairwise disjoint while $f^l(J) = J$. Observe that in the case of a weakly periodic interval J the union of all iterates of J (the orbit $\text{orb}(J)$) is a finite forward invariant union of closed intervals.

Lemma 3.2. *Let f be a non-strictly periodic map of modality q and J be a closed interval weakly periodic of period l. Then $h(f|_{orb(J)}) \leq \ln(1 + q/l)$.*

Proof. By the quoted above result of Misiurewicz and Szlenk we can estimate the entropy of $f|_{\text{orb}(J)}$ for J from above as follows. Let n_i be the modality of $f|_{f^i(J)}$. Then by [6, 7, 8, 9] we have that

$$h(f|_{\text{orb}(J)}) \leq l^{-1} \ln[\Pi_{i=0}^{l-1}(n_i + 1)].$$

Since f has modality q then $\sum_{i=0}^{l-1}(n_i + 1) \leq l + q$. Together with the above inequality this implies that

$$h(f|_{\text{orb}(J)}) \leq l^{-1} \ln[(\frac{l+q}{l})^l] = \ln(1 + q/l).$$

□

The following lemma combines and specifies results of Theorem 2.1 and Lemma 2.2.

Lemma 3.3. *Let f be a piecewise-monotone interval map of Sharkovskii type $Sh(f) = (2s + 1)2^k, s \geq 1$. Then there exists a number $l \geq 2^k$, a closed interval $J \subset I_f$ and a subinterval $K \subset M_f$ weakly periodic for g_f of period l such that $h(f^l|_J) = h(f^l)$, $f^l|_J$ is monotonically semi-conjugate to a mixing interval map $H = g_f^l|_K$ of the same entropy. Moreover, one of the following holds:*

1. *$l = n_f, J = I_f$, and f has a point of period $(2r + 1)l, r \geq 1$;*
2. *$l = 2n_f$, H has an L-scheme, and f has a point of period $3l$.*

Proof. Consider the two cases corresponding to the cases from Lemma 2.2. Set $G = g_f^{n_f}|_{M_f}$; by Theorem 2.1 G is transitive.

(1) Suppose that G is mixing. Set $H = G$, $J = I_f$, $K = M_f$ and $l = n_f$. Since the semiconjugacy is monotone, preimages of points are closed intervals and so if H has a point of some period then so does $f^{n_f} = f^l$. By Lemma 2.2 H has a point of an odd period $2r+1$, $r \geq 1$. Since $J = I_f$ is an f-periodic interval and all its iterates until $f^l(J) = J$ are pairwise disjoint then f has a point of period $(2r+1)l, r \geq 1$. Since $l < 2^k$ implies $(2r+1)l \prec (2s+1)2^k = Sh(f)$, we obtain $l \geq 2^k$ as desired.

(2) Suppose that G is not mixing. We may assume that $M_f = [0, 1]$. Then there exists a point $a \in (0, 1)$ such that $G[0, a] = [a, 1], G[a, 1] = [0, a]$ and both restrictions $G^2|_{[0,a]}, G^2|_{[a,1]}$ are mixing. Set $H = G^2|_{[0,a]}$. Since a is a fixed endpoint of H, Lemma 3.1 implies that H has an L-scheme, hence also a point of period 3. Set $J = \varphi_f^{-1}([0, a])$, $K = [0, a]$ and $l = 2n_f$. It follows that $f^l|_J$ is monotonically semiconjugate by φ_f to a mixing map H. The interval J maps back into itself by f^l, still it is not f-periodic of period l because $f^{l/2}(J)$ intersects J over a subset of the interval $R = \varphi_f^{-1}(a)$. However *all other* iterates of J until $f^l(J) = J$ are disjoint from J. Clearly, f has a point of period $3l$. Since $l < 2^k$ implies $3l \prec (2s+1)2^k = Sh(f)$, we obtain $l \geq 2^k$ as desired. □

In what follows, J, l, K, g_f and H are as in Lemma 3.3. It is easy to see that Lemma 3.3, Lemma 3.2 and Lemma 2.2 imply Lemma 1.4.

Proof of Lemma 1.4. The conclusion holds when $s = 0$, since then $h(f) = 0$ [1]. For $s \geq 1$, we want to apply Lemma 3.2 to g_f. The modality of g_f is at most q. On the other hand, $\varphi_f(\mathrm{orb}(I_f)) = \mathrm{orb}(K)$, and K is a weakly periodic interval for g_f of period $l \geq 2^k$. Hence by Lemma 3.2 $h(f) = h(g_f|_{\mathrm{orb}(K)}) \leq \ln(1 + q/l) \leq \ln(1 + 2^{-k}q)$ as desired. □

We are ready now to prove Theorem 1.2.

Proof of Theorem 1.2. Set $K = [0, 1]$. By Lemma 1.4 and Lemma 3.3 $h(f) \leq \ln(1+q/l)$. If $l > 1$ then $h(f) \leq \ln(1+q/l) \leq \ln(1+q/2) < \ln(\sqrt{.75q^2 + 1.5q + 1.25} + .5)$ (the last inequality is easy to check) as desired. Thus it suffices to consider the case when $l = 1$. In this case by Lemma 3.3 the map f on an invariant interval J is semiconjugate by a monotone map φ_f to a mixing map $H : [0, 1] \to [0, 1]$. Since φ_f is monotone, H does not have an L-scheme. Indeed, suppose that it does. Then we may assume that there are points x and y such that

$H^2(y) = x = H(x) < y < H(y)$ and $H(z) > z$ for any $z \in (x, y)$. The φ_f-preimage of x is an interval $[a, b]$, and the φ_f-preimage of y is an interval $[c, d]$. Since $H(z) > z$ for any $z \in (x, y)$ then because of monotonicity of φ_f we see that $f(z') > z'$ for $z' \in (b, c)$. By continuity then $f(b) \geq b$. Since $f([a, b]) \subset [a, b]$ we conclude that $f(b) = b$. It follows that b, c form an L-scheme of f, a contradiction. Also, it is clear that the modality of H is at most q. Let us now use these facts to estimate $h(H)$.

Let $J_1 < J_2 < \cdots < J_r, r \leq q + 1$ be laps of H. Consider the matrix $A^H = \mathrm{Dyn}_H$. By Proposition 2.3 $h(H) \leq \ln r(A^H)$. Let us now estimate $r(A^H)$. By Lemma 3.1 we may assume that d is a unique fixed point of H; observe also that H has no flat spots since it is mixing. Then d cannot be a turning point of H. Indeed, otherwise points in a small semi-neighborhood of d stay on the same side of d while being mapped towards d, a contradiction with mixing of H. Hence we may assume that in a small neighborhood of d the map H is strictly decreasing. Let J_s be the interval containing d. If $i < s$ then $H(J_i)$ is disjoint from J_t for each $t < i$, and if $i > s$ then $H(J_i)$ is disjoint from $J_t, t > i$.

Let us construct a matrix B as follows.

(1) If $i < s$, set $b_{it} = 0$ for $t < i$ and $b_{it} = 1$ for $t \geq i$.
(2) If $i = s$, set $b_{it} = 1$ for every t.
(3) If $i > s$, set $b_{it} = 1$ for $t \leq i$ and $b_{it} = 0$ for $t > i$.

Observe, that for every pair i, t we have $a^H_{it} \leq b_{it}$. Hence $r(A^H) \leq r(B)$. Let us estimate $r(B)$ from above using Corollary 2.5 which obviously applies to B. To this end let us estimate the number R of non-diagonal non-zero entries in the matrix B. It follows from the construction of B that

$$R = (\sum_{i=1}^{s-1} r - i) + r + (\sum_{i=s+1}^{r} i - 1) = .5[-2s^2 + 2(r + 1)s + r^2 - r].$$

It is easy to check, that the maximal value of R is $.75r^2 + .25$ and that it is reached if $s = (r + 1)/2$. Since $r \leq q + 1$ we conclude that $R \leq .75q^2 + 1.5q + 1$. By Corollary 2.5 and since $r(A^H) \leq r(B)$ this implies that

$$r(A^H) \leq r(B) \leq \sqrt{R + .25} + .5 \leq \sqrt{.75q^2 + 1.5q + 1.25} + .5$$

Hence

$$h(f) = h(H) \leq \ln(\sqrt{.75q^2 + 1.5q + 1.25} + .5)$$

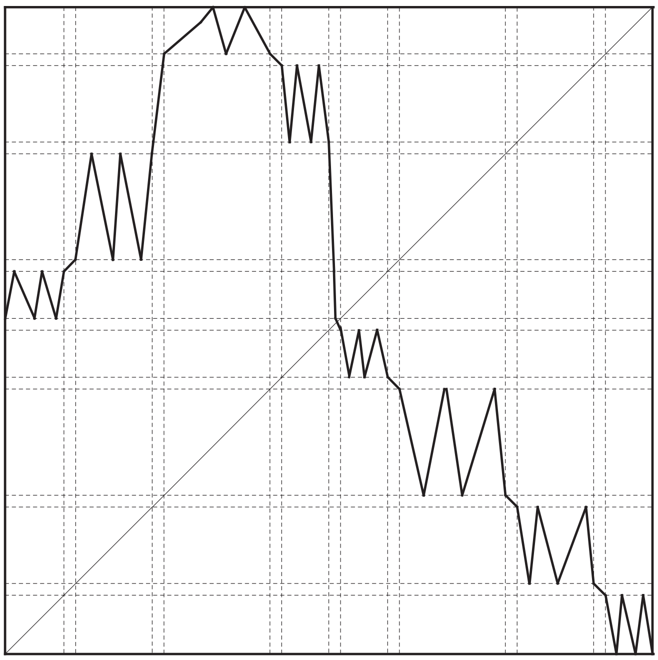
as desired. □

Proof of Theorem 1.5. (1) Since $h(3,q)=\ln(q+1)$ and by Theorem 1.2, $h(5,q)\le \ln(\sqrt{.75q^2+1.5q+1.25}+.5)$, we get easily

$$\liminf_{q\to\infty} h(3,q)-h(5,q)\ge \lim_{q\to\infty}\ln\frac{q+1}{\sqrt{.75q^2+1.5q+1.25}+.5}=\ln\frac{2}{\sqrt{3}}.$$

(2) For $\nu\in\mathbb{N}\cup\{0\}$, let $\pi=(1\ \pi(1)\ \dots\ \pi^{2^\nu-1}(1))$ be a cyclic permutation such that 'connect-the-dots' map f_π has zero topological entropy and for $\nu\ge 2$ is unimodal [1]. Consider intervals $[a_1,b_1]<\dots<[a_{2^\nu},b_{2^\nu}]$ and an interval map $g=g_{\nu,r}:[a_1,b_{2^\nu}]\to[a_1,b_{2^\nu}]$, $r\in\mathbb{N}$, such that $g([a_i,b_i])=[a_{\pi(i)},b_{\pi(i)}]$ and for which ($j=\pi^{-1}(2^\nu)$)

(1) g is increasing on the set $[a_1,a_j]\setminus\cup_{i<j}[a_i,b_i]$,
(2) g is decreasing on the set $[b_j,b_{2^\nu}]\setminus\cup_{j<i}[a_i,b_i]$,
(3) for $i\ne j$, g maps (a_i,b_i) in a $2r{+}1$-to-1 fashion onto $(a_{\pi(i)},b_{\pi(i)})$,
(4) g maps (a_j,b_j) in a $2r$-to-1 fashion onto (a_{2^ν},b_{2^ν}).



The map $g_{3,2}$; $\pi=(1\ 5\ 4\ 7\ 2\ 6\ 3\ 8)$ and $j=\pi^{-1}(8)=3$.

Let us show that $Sh(g_{\nu,r})=3\cdot 2^\nu$. Indeed, the intervals $[a_i,b_i]$ form a cycle of intervals M of period 2^ν, the map g^{2^ν} restricted on each such interval has points of all periods, hence $Sh(g_{\nu,r})$ is $3\cdot 2^\nu$ or $Sh(g_{\nu,r})\prec 3\cdot 2^\nu$. If the latter holds, the corresponding periodic points must be disjoint from M. However by the construction the only periodic points outside M are of periods which are powers of 2, a contradiction.

Moreover, using (1)-(4) we get

$$\mathrm{mod}(g_{\nu,r}) = (2^\nu - 1)2r + (2r-1) = 2^{\nu+1}r - 1,$$

hence by Lemma 1.4,

$$h(g_{\nu,r}) \le \ln(1 + 2^{-\nu}(2^{\nu+1}r - 1)) = \ln(2r + 1 - 2^{-\nu}) = V_1(\nu, r).$$

On the other hand by (1)-(4), each lap of $g^{2^\nu}|_{[a_j,b_j]}$ maps by g^{2^ν} onto $[a_j, b_j]$. Since $h(g) = h(g|_{\mathrm{orb}([a_j,b_j])}) = \frac{1}{2^\nu} h(g^{2^\nu}|_{[a_j,b_j]})$, we get from [9]

$$h(g_{\nu,r}) = \frac{1}{2^\nu}\ln((2r+1)^{2^\nu - 1}2r) = \ln(2r+1) + \frac{1}{2^\nu}\ln\frac{2r}{2r+1} = V_2(\nu, r).$$

By our definition, $h(s,1) \le h(s,2) \le \cdots$; since

$$\mathrm{mod}(g_{\nu,[\frac{q+1}{2^{\nu+1}}]}) \le q \le \mathrm{mod}(g_{\nu,1+[\frac{q+1}{2^{\nu+1}}]})$$

and for each $0 \le k \le \ell$

$$V_2(k, [\frac{q+1}{2^{k+1}}]) \le h(3\cdot 2^k, q), \quad h(3\cdot 2^\ell, q) \le V_1(\ell, 1 + [\frac{q+1}{2^{\ell+1}}])$$

we can write for each q,

$$V_2(k, [\frac{q+1}{2^{k+1}}]) - V_1(\ell, 1 + [\frac{q+1}{2^{\ell+1}}]) \le h(3\cdot 2^k, q) - h(3\cdot 2^\ell, q).$$

Since $\lim_{q\to\infty} V_2(k, [\frac{q+1}{2^{k+1}}]) - V_1(\ell, 1 + [\frac{q+1}{2^{\ell+1}}]) = \ln 2^{\ell-k}$, we have also

$$\liminf_{q\to\infty} h(3\cdot 2^k, q) - h(3\cdot 2^\ell, q) \ge \ln 2^{\ell-k}$$

as desired. $\square$

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