ON MINIMAL MAPS OF 2-MANIFOLDS

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Abstract. We prove that a minimal self-mapping of a compact 2-manifold has tree-like fibers (i.e. all points have preimages which are connected, at most 1-dimensional and with trivial shape). We also prove that the only 2-manifolds (compact or not) which admit minimal maps are either finite unions of tori, or finite unions of Klein bottles.

1. Introduction

Minimal maps are one of the main topics of topological dynamics (see e.g. [AU88, Br79, deV93]). A map $f : X \to X$ is *minimal* provided for each $x \in X$ the orbit $\mathcal{O}(x) = \{x, f(x), f^2(x), \dots\}$ is dense in X. For compact spaces this is equivalent to: there exists no proper closed non-empty subset $A \subset X$ such that $f(A) = A$.

An excellent brief introduction into a number of problems related to minimal maps can be found in recent papers [Brkosn03] and [KST00]. One basic problem is whether a given space admits a minimal map or a minimal homeomorphism (see e.g. [Brkosn03, E64, E64, GW79, P74]). Another question, due to Auslander ([AG68], p. 514) is whether on a given space there exist minimal maps which are not homeomorphisms (it is now known that in general such maps exist, see e.g. [AY80, Ree79] as well as [KST00] in which a nice construction using ideas from [Ree79] is suggested). A number of results deal with the case when the space is a Cantor set. For example, in one-dimension it turns out that restrictions of interval maps on the so-called "wild attractors" (whose existence is proven in [BKNS96]) must be minimal [L91], and in the case of negative Schwarzian maps they must be minimal maps of specific type [BL91, BM98]; in all these cases the wild attractor on which the map is minimal is a Cantor set. A lot of other examples and results concerning minimal maps are obtained within the framework of symbolic dynamics (see e.g. [LM95] for general references).

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Our approach to studying minimal maps is the following: minimal maps have several important topological properties which were established in [KST00] (see also [Why42] for alternative ways to establish some of those properties). Since maps with those topological properties were studied in [BOT02], we can apply the tools from [BOT02] together with some dynamical arguments in order to obtain our results.

To state our main results we need to introduce the following two basic notions.

Definition 1.1. A continuous map $f : X \to Y$ is said to be *light* if there exist no connected non-degenerate subsets of X which are collapsed by f into points.

Definition 1.2. A continuous map $f : X \to Y$ is said to be *monotone* if for every $y \in Y$ the set $f^{-1}(y)$ is connected.

We leave the task of suggesting examples of light or monotone maps to the reader. The importance of these maps is explained partly by the existence of the so-called *monotone-light decomposition* of an arbitrary continuous map of compact spaces. Namely, it can be shown that if $f : X \to Y$ is a continuous map of compact metric spaces then there exists a compact metric space Z, a monotone map $g : X \to Z$ and a light map $h : Z \to Y$ such that $f = hg$. The idea of the proof is to define the map g by collapsing all components of preimages of points under f and thus to create a quotient space Z for which g is the factor map; it is easy to see that then h is well-defined and g and h have the required properties.

All spaces considered in this paper are separable and metric. A *continuum* is a compact and connected space. A continuum X is *tree-like* provided for every $\varepsilon > 0$ there exists an open cover of mesh less than ε whose nerve is a finite tree (alternatively, X is homeomorphic to the inverse limit of finite trees). For a subset Y of a topological space X, we denote the *boundary* of Y by $\mathrm{Bd}(Y)$. We also rely upon the standard definition of a *closed manifold* (compact, connected manifold without boundary). Finally, in the case of a compact manifold M *with* boundary its boundary is called the *manifold boundary* of M and denoted by ∂M. Thus, if D is homeomorphic to the closed unit ball in a Euclidean space then $\mathrm{Bd}(D)$ is empty while ∂D is homeomorphic to the corresponding unit sphere. Finally, by a *disk* we mean an open (closed) set homeomorphic to the Euclidean open (closed) unit disk.

2. Preliminaries

We need the following definitions. which are due to Whyburn ([Why42], Chapter VIII, pp. 162-164).

Definition 2.1. Suppose that $f : X \to Y$ is a continuous map of compact metric spaces. Then f is said to be *strongly irreducible* if for every closed proper subset A of X we have $f(A) \neq Y$. If for every proper subcontinuum K of X we have $f(K) \neq Y$ then the map f is called *irreducible.*

An example of an onto map which is irreducible but not strongly irreducible can be an interval map $f : [0,1] \to [0,1]$ such that $f^{-1}\{0,1\} = \{0,1\}$ while there exist two disjoint open intervals U, V such that $f(U) = f(V)$; then clearly $f([0,1] \setminus U) = [0,1]$ while any interval I such that $f(I) = [0,1]$ must contain $0, 1$ and hence must coincide with $[0,1]$.

For other compact *manifolds* such examples are impossible as yielded by the following result from [Why42] (we state a version of Whyburn's theorem convenient for our purposes).

Theorem 2.2. *(*[Why42]*, VII Theorem 10.4) If X is locally connected and the non-cut points of X are dense in X then all irreducible maps $f : X \to Y$ of compacta are strongly irreducible (in particular this holds if X is a compact manifold and not an interval).*

In other words, if for a map $f : X \to Y$ there exists a closed subset $F \subset X, F \neq X$ such that $f(F) = Y$ then there exists a subcontinuum $K \subset X, K \neq X$ such that $f(K) = Y$.

We need several other definitions (here we follow our previous paper [BOT02] and some classical sources such as [Why42] while drawing the reader's attention to the fact that in some papers, e.g. in [KST00], some terms may have slightly different meanings).

Definition 2.3. A map $f : X \to Y$ is *quasi-interior* if for every non-empty open set $U \subset X$, the interior of $f(U)$ is not empty.

For a map $f : X \to Y$ denote by D_f the set of points such that $f^{-1}(f(x)) = x$ and by R_f the set of points $y \in Y$ such that $|f^{-1}(y)| = 1$. Clearly $f(D_f) = R_f$ and $f^{-1}(R_f) = D_f$. Now we can introduce another notion.

Definition 2.4. A map $f : X \to Y$ of compact metric spaces is *almost 1-to-1* if the set D_f is dense in X.

We list a few important properties of minimal maps related to the above introduced notions. These properties are established in [KST00] (we state them using the above terminology).

Theorem 2.5. *Suppose that $f : X \to X$ is a minimal map of a compact metric space. Then f is strongly irreducible, quasi-interior and almost* 1-*to*-1*.*

To clarify terminology let us point out that some of the above notions can be given different but equivalent definitions. As an example, let us look at almost 1-to-1 maps. The following lemma is proven in [BOT02].

Lemma 2.6. *Suppose that $f : M \to N$ is a continuous map of metric σ-compact spaces. Then the set R_f is a G_δ-subset of $f(M)$ and the set D_f is a G_δ-subset of M.*

Maps with the property that D_f is dense are closely related to maps for which R_f is dense, as another lemma from [BOT02] shows.

Lemma 2.7. *Suppose that $f : X \to Y$ is a map of compact metric spaces. Then the following properties are equivalent:*

(1) *D_f is dense in X,*
(2) *R_f is dense in $f(X)$ and f is quasi-interior as a map from X to $R(X)$.*

Hence one can equivalently define almost 1-to-1 maps of compact metric spaces as such maps f that R_f is dense in $f(X)$ and f is quasi-interior (so almost 1-to-1 maps are always quasi-interior). This simple observation has the following surprising corollary (even though it is obvious, we have not been able to find it elsewhere). Observe that the condition of surjectivity of maps in Corollary 2.8 can be weakened, yet we decided not to do this since in this paper it is not important for applications.

Corollary 2.8. *Suppose that $f : X \to Y$ and $g : Y \to Z$ are almost* 1*-to-*1 *onto maps of metric compact spaces. Then their composition $h = g \circ f : X \to Z$ is also almost* 1*-to-*1*.*

Proof. It is enough to show that D_h is dense. Indeed, the sets D_g and R_f are both dense G_δ-subsets of Y, hence so is their intersection. On the other hand, it follows from the definition that $D_h = f^{-1}(R_f \cap D_g)$ and because f is quasi-interior we see that D_h is dense in X as desired. □

Theorem 2.5 allows us to apply techniques developed in [BOT02] to minimal maps. Observe that parts of this theorem can be also deduced from results of Whyburn's. Indeed, the following theorem is proven in his book [Why42].

Theorem 2.9. *(*[Why42]*, VII Theorem 10.2) A continuous onto map $f : X \to Y$ between compacta is strongly irreducible if and only if it is almost 1-to-1.*

Since by the first part of the claim of Theorem 2.5 minimal maps are strongly irreducible we conclude by Theorem 2.9 that they are almost 1-to-1, and hence by Lemma 2.7 they are quasi-interior.

Let us now list the results of [BOT02] dealing with almost 1-to-1 maps. The first one deals with almost 1-to-1 maps of closed manifolds.

Theorem 2.10. *If an almost* 1*-to-*1 *map* $f : M \to N$ *of a closed* n*-manifold* M *into a connected* n*-manifold* N *is light then it is a homeomorphism.*

The next result deals with other types of manifolds.

Theorem 2.11. *If an almost* 1*-to-*1 *map* $f : M \to N$ *of an open (resp. with boundary) manifold* M *to a connected manifold* N *is light then* f *is an open embedding on the entire manifold* M *(resp. on* $M \setminus \partial M$*).*

3. Minimal maps on closed 2-manifolds

As we say in Introduction, it is known that there exist minimal non-invertible maps (see e.g. [AY80]). In the papers [Ree79] and [KST00] (the latter contains a construction which develops a construction from the former) it is shown that there exist minimal maps of closed 2-manifolds which are not homeomorphisms. In the constructions from these two papers all minimal maps act on 2-torus and happen to be monotone and to have tree-like continua as point preimages.

The purpose of this section is to show that *all* minimal maps of closed connected 2-manifolds must be monotone with tree-like point preimages. This allows us to conclude that among closed 2-manifolds only the torus and the Klein bottle admit minimal maps and thus extend well-known results concerning minimal homeomorphisms. In the next sections we will show that in fact 2-manifolds with boundary and non-compact 2-manifolds do not admit minimal maps.

We begin by proving several results concerning almost 1-to-1 maps of closed 2-manifolds. First we need the following definition.

Definition 3.1. A map $f : M \to N$ of 2-manifolds is said to be *non-separating* if every component of any point preimage is a tree-like continuum.

In other words, f is non-separating if and only if components of point preimages are continua which are at most 1-dimensional and of trivial shape.

Now for the sake of completeness we prove a version of Theorem 2.10 for 2-manifolds. We begin by proving the following elementary lemmas.

Lemma 3.2. *If* $A \subset D$ *is a dense subset of the open unit* 2*-disk* D *then for every point* $x \in D$ *there exists an arbitrarily small disk* U *whose boundary* S *is a simple closed curve* S *such that* $A \cap S$ *is dense in* S*.*

Proof. Indeed, because of the density of A we can always choose a small PL simple closed curve S_1 which bounds a disk U containing x such that all vertices of S_1 are points of A. Then we can choose points of A very close to the midpoints of straight line segments on the boundary of S_1 and consider a new PL simple closed curve whose vertices are the vertices of S_1 and the just chosen points, etc. Clearly, in the end of this process we will get a simple closed curve S such that $A \cap S$ is dense in S as desired. □

Lemma 3.3. *Suppose $f : X \to I$ is an almost 1-to-1 map from a continuum X onto an arc I. Then f is monotone. In particular, if f is also light, then f is a homeomorphism.*

Proof. Let $f = g \circ m$ be the monotone-light decomposition which first collapses components of preimages to points. Then g is also almost 1-to-1, and it suffices to show that g is a homeomorphism. Therefore we may assume that f is a light map from the very beginning.

Suppose that there exists $y \in I$ such that $f^{-1}(y)$ is not connected. Then there exist open sets U and V such that $f^{-1}(y) \cap U \neq \emptyset \neq f^{-1}(y) \cap V$, $\overline{U} \cap \overline{V} = \emptyset$ and $f^{-1}(y) \subset U \cup V$. Choose $u \in f^{-1}(y) \cap U$, $v \in f^{-1}(y) \cap V$ and let C_u (respectively C_v) be the component of $\overline{U}$ which contains u (the component of $\overline{V}$ which contains v, respectively). By the Boundary Bumping Theorem C_u meets the boundary of U and C_v meets the boundary of V. Thus both $f(C_u)$ and $f(C_v)$ are non degenerate continua and, hence, meet only in y (otherwise a non-degenerate interval in $[0, 1]$ consists of points having preimages in both C_u and C_v). We may assume that $f(x) \leq y$ for all $x \in C_u$ and $f(x) \geq y$ for all $x \in C_v$.

Clearly, $f(C_u \cup C_v)$ is a neighborhood of y. Choose a smaller neighborhood T of y and consider $U' = U \cap f^{-1}(T)$ and $V' = V \cap f^{-1}(T)$. Then both U' and V' are non-empty disjoint open sets. Let us show that $f(U') \leq y$. Indeed, otherwise there exists a point $u' \in U'$ such that $f(u') > y$. Consider the component R of U' containing u'. Since it is non-degenerate and f is light then $f(R)$ is non-degenerate too. By the choice of T this implies that $f(R) \cap f(C_v)$ contains an interval while on the other hand $R \cap C_v = \emptyset$, a contradiction with f being almost 1-to-1. Similarly it can be shown that $f(V') \geq y$.

Consider the sets $U'' = U' \cup \{u'' : f(u'') < y\}$ and $V'' = V' \cup \{v'' : f(v'') > y\}$. Then it follows that U'' and V'' are open disjoint sets whose union is the entire X, a contradiction. □

Now we are ready to prove our theorem specifying Theorem 2.10 for 2-manifolds.

Theorem 3.4. *Let $f : M \to N$ be an almost* 1*-to-*1 *non-separating map of a closed* 2*-manifold M into a connected* 2*-manifold N. Then f*

is a monotone onto map with tree-like point preimages and M is homeomorphic to N. In particular, if f is light then it is a homeomorphism.

Proof. Let $f = g \circ m$ the monotone-light decomposition of the map f. By [RS38], $m(M) = X$ is homeomorphic to M. Clearly both m and g are almost 1-to-1. It suffices to show that g is one-to-one. Hence suppose there exist $x_1 \neq x_2 \in X$ such that $g(x_1) = g(x_2) = y$. Choose two disjoint closed disks D_i containing x_i in X. Let D be a sufficiently small open disk in N containing y and with simple closed curve boundary S such that $R_g \cap S$ is dense in S and if H_i is the component of $g^{-1}(D)$ containing x_i, then $H_i \subset D_i$. Now H_i is open since D_i is locally connected. Since D_i is unicoherent and $\mathrm{Bd}(H_i)$ separates D_i, there exist components B_i of $\mathrm{Bd}(H_i)$ which separate D_i. Then $g(B_i)$ are non-degenerate sub-continua of S since g is light. Moreover, since g is almost 1-to-1, the intersection of $g(B_1)$ and $g(B_2)$ consists of no more than 2 points (recall that $g(B_1) \subset S$, $g(B_2) \subset S$, and S is homeomorphic to the unit circle). In particular, $g : B_1 \to I \not\subset S$ is an almost 1-to-1 light map of a continuum to an arc. By Lemma 3.3, B_1 is an arc and thus cannot separate M, a contradiction. Hence g is a homeomorphism and X is homeomorphic to N. This completes the proof. $\square$

Observe that in fact an analog of Theorem 3.4 for manifolds with boundary or open manifolds can also be proven (this analog mimics Theorem 2.11 which holds for light maps). However we do not prove it here because we do not need it in this paper.

Now we need the following definition.

Definition 3.5. A locally connected continuum which contains no simple closed curve is called a *dendrite*. A continuum K is said to be a *local dendrite* if every point $x \in K$ has a neighborhood U whose closure is a dendrite (any such neighborhood is then said to be *dendritic*).

Slightly abusing the language we will also use the following definition.

Definition 3.6. A one-dimensional finite connected polyhedron is called a *graph*.

It is known [Kur68, p.303, Theorem 4] that a continuum is a local dendrite if and only if it is locally connected and there exists a finite graph containing all its simple closed curves. To study the topology of local dendrites in more detail we need another definition.

Definition 3.7. Any connected subgraph H of a local dendrite G containing all simple closed curves of G is called a *root* of G. A subgraph which is minimal with respect to containing all simple closed curves is called a *minimal root*; any such subgraph is denoted by R_G (if G contains no simple closed curves, set $R_G = \emptyset$).

The following lemma contains some basic facts about the structure of any local dendrite and also explains our terminology. See [Why42, Chapter 4, 3.1.] for related results.

Lemma 3.8. *Let G be a local dendrite. Then G contains a root, the minimal root R_G of G is unique, and the following holds:*

(1) *the family of components of $G \setminus R_G$ is at most countable;*
(2) *for each component K of the set $G \setminus R_G$ its closure $\overline{K}$ is a dendrite and $\overline{K} \setminus K = \overline{K} \cap R_G$ consists of exactly one point;*
(3) *for two components L, M of $G \setminus R_G$ the intersection $\overline{L} \cap \overline{M}$ consists of at most one point and in this case coincides with $\overline{L} \cap R_G = \overline{M} \cap R_G$;*
(4) *for any subcontinuum T of G the intersection $T \cap R_G$ is connected.*

Proof. Let R be a root of G and let C be a component of $G \setminus R$. Since $G \setminus R$ is open and because G is locally connected, C is open itself. Since it is also connected, it is arcwise connected [Nad92, Theorem 8.26] and contains no simple closed curve. Suppose that $x \neq y \in \overline{C} \setminus C$. Clearly, both x and y belong to R (otherwise they would have belonged to C). Then using small disjoint arcwise connected neighborhoods of x and y, respectively, and two arcs in C and R connecting x and y, it is easy to see that there exists a simple closed curve which is not contained in R. Hence $\overline{C} \cap R = b_C$ is a single point. Then $\overline{C}$ is locally connected because it is such everywhere but at b_C by the definition and it is easy to see that it is locally connected at b_C. Since $\overline{C}$ contains no simple closed curve, it is a dendrite. This proves (2) in the case of any root.

Since G is locally connected, there are at most countably many components of $G \setminus R$ which proves (1) for an arbitrary root. To prove (3) observe that if two distinct components L, M of $G \setminus R$ are such that $\overline{L} \cap \overline{M}$ contains a point not from R than this point must belong to $L \cap M$ and so $L = M$, a contradiction.

Let us show that the intersection of any subcontinuum K of G with R must be connected. Indeed, for any component C of $G \setminus R$ we have that $K \setminus C = K'$ is connected. To see that, observe that if $b_c \notin K$ then either $K \subset C$ or $K \cap C = \emptyset$. Hence we may assume that $K' = L \cup M$ where L, M are disjoint compacts and $b_C \in L$. Then $L \cup (\overline{C} \cap K)$ and M are disjoint compact sets whose union is K, a contradiction. In countably many steps of this process we will get the set $K \cap R$ as the intersection of sets $K \setminus (\cup C_i)$ for components C_i of $G \setminus R$. It is easy to see by way of contradiction that because these intermediate sets are connected, then so is $K \cap R$. Hence, if A and B are both minimal roots, the intersection $A \cap B$ must be a root and $A = B$ by minimality. $\square$

In the situation of the above lemma it is convenient to use the following definition.

Definition 3.9. Given a local dendrite G, its minimal root R_G and a component C of $G \setminus R_G$, call the point $b_C = \overline{C} \cap R_G$ the *basepoint* of C.

So far we have been studying topological questions. Let us now pass on to dynamics. According to Theorem 2.5, a minimal map is strongly irreducible. We expand this a little bit and prove the following lemma.

Lemma 3.10. *Suppose that $f : X \to X$ is a continuous map of compact metric spaces. Then f is minimal if and only if no proper, closed non-empty subset A of X is such that $f(A) \supset A$.*

Proof. Suppose that no proper, closed subset A of X has the property that $f(A) \supset A$. Then there are no proper, closed invariant subsets of X and so f is minimal. Now, suppose that f is minimal. By way of contradiction assume that there exists a proper closed subset A such that $A \subset f(A)$. Put $A = A_0$ and define compact subsets $A_n \subset A_{n-1}$ inductively such that $f(A_n) = A_{n-1}$. Then $A_0 \supset A_1 \supset \dots$ and $f^n(A_n) = A$ for each n. Hence A_n is a non-empty compact subset and $\cap A_n = A_\infty \neq \emptyset$. Clearly A_∞ is invariant and proper contradicting the minimality of f. □

The following definition is related to that of an irreducible map.

Definition 3.11. Say that a map $f : X \to X$ of a topological space is *mobile* if for any proper (perhaps degenerate) subcontinuum Y of X we know that $f(Y)$ does not contain Y.

By Lemma 3.10 any minimal map is mobile; this shows the connection existing between minimal and mobile maps which will be used below. In what follows we study mobile maps of local dendrites and apply our results to minimal maps. In fact we prove the following lemma.

Theorem 3.12. *Suppose that G is a local dendrite which is not a finite graph. Then there are no mobile maps of G into itself.*

Proof. Denote by $n(G)$ the maximal number of distinct simple closed curves in G (or, equivalently, in R_G) and prove the claim by induction on $n(G)$.

If $n(G) = 0$, then G is a dendrite and f must have a fixed point contradicting that f is mobile. Hence suppose that G is a non-trivial local dendrite which admits a mobile map $f : G \to G$ such that $n(G)$ is minimal. Then $n(G) > 0$. Since G is not a graph, R_G is a proper subcontinuum of G and, since f is mobile, $R_G \setminus f(R_G) \neq \emptyset$. Then it is easy to find a connected open set U such that $\overline{U} \cap f(R_G) = \emptyset$, U intersects R_G and the set $G \setminus U$ is connected and is not a graph. Indeed, choose a point $a \in R_G \setminus f(R_G)$ which is on a simple closed curve in R_G and which is not the basepoint of a component of $G \setminus R_G$ and is not a branchpoint of G; it is possible since the union of the set of all basepoints of components of $G \setminus R_G$ and all branchpoints of G is at most countable. Now, choose a small open arc $I \subset R_G \cap U$ in such a way that the following holds:

1. I is an open neighborhood of a in R_G;
2. I contains no vertices of R_G;
3. the endpoints of I are not the basepoints of components of $G \setminus R_G$;
4. if V is defined as the union of I and all components of $G \setminus R_G$ whose basepoints belong to I, then V is disjoint from R_G.

It is easy to see that the choice of the point a enables us to choose I. Moreover, since $n_G > 0$ then the set $R_G \setminus I$ is connected which easily implies that $K = G \setminus V$ is connected too (after all, K is the union of $R \setminus I$ and all components of $G \setminus R_G$ which “grow” from $R_G \setminus I$).

It follows from the minimality of R_G that $n(K) < n(R_G)$. Define a map $g : K \to K$ as follows. First, set $g|_{K \cap R_G} = f|_{K \cap R_G}$. By the choice of K, $g(K \cap R_G) = f(K \cap R_G) \subset K$. For each $x \in K \setminus R_G$, there exists a unique component C' of $G \setminus R_G$ such that $x \in C'$, and $C = \overline{C'}$ is a dendrite which meets R_G in exactly one point c. If $f(c) \in \mathrm{Bd}(V)$, define $g(x) = f(c)$ for all $x \in C$. Otherwise let D_0 be the closure of the component of $C \setminus f^{-1}(\mathrm{Bd}(V))$ containing c and let $\{c, e_\alpha\}_{\alpha \in A(C)}$ be the boundary of D_0 (this defines the set $A(C)$; clearly, all these points are endpoints of D_0, but there may exist endpoints of D_0 which do not belong to $\mathrm{Bd}(D_0)$). Since $c \in R_G$, $f(c) \in K$ and $f(D_0) \subset K$. Moreover for each $\alpha \in A(C)$, $f(e_\alpha) \in \mathrm{Bd}(V)$. Consider the set $P = C \setminus \cup\{e_\alpha\}$; it is easy to see that P has one component containing D_0 and perhaps some other components. For each $\alpha \in A(C)$ denote by D_α the union of all the components of P which contain e_α in their closure. Define $g|_{D_0} = f|_{D_0}$ and $g(D_\alpha) = f(e_\alpha)$ for each $\alpha \in A(C)$. Then $g : K \to K$ is a continuous function. Moreover, for any subcontinuum $X \subset K$, $g(X) \subset f(X)$ and hence $\emptyset \neq X \setminus f(X) \subset X \setminus g(X)$. Then g is a mobile map of a local dendrite K, which is not a graph, and with $n(K) < n(G)$ contradicting the minimality of $n(G)$. □

Since minimal maps are mobile by Lemma 3.10, we conclude that a minimal map of a local dendrite X may only exist if X is a graph. The description of minimal maps now follows from some results from one-dimensional dynamics whose description can be found below.

Suppose that $f : G \to G$ is a map of a graph G which is not an irrational rotation but is monotonically and non-trivially semiconjugate to an irrational rotation. We will call such maps *Denjoy* maps; they serve as examples of mobile but not minimal maps. They also play an important role in Theorem 3.13 which was proven in [Blo84, Blo86a, Blo86b, Blo87] (in the case of the circle this theorem was actually proven in [AK79]).

Theorem 3.13. *Let $f : G \to G$ be a map of a connected graph which has no periodic points. Then either f is an irrational rotation of the circle, or f is a Denjoy map.*

Now we are ready to prove our main result concerning mobile and minimal maps on local dendrites; it follows immediately from Theorem 3.13 and Lemma 3.12.

Theorem 3.14. *Suppose that G is a local dendrite. If $f : G \to G$ is minimal then G is a circle and f is conjugate to an irrational rotation.*

Proof. By Lemma 3.12 G is a graph. By the definition of a minimal map f has no periodic points. Therefore by Theorem 3.13 f is either an irrational rotation or a Denjoy map. Since Denjoy maps are not minimal f is an irrational rotation of a circle as desired. $\square$

To study arbitrary minimal maps of 2-dimensional manifolds we need the following definitions.

Definition 3.15. Let X be a locally connected continuum. We say that a subcontinuum Y of X is a *true cyclic element* if it is maximal with respect to the property of containing no cut point of itself. We say that a continuum X is a *generalized cactoid* if every true cyclic element of X is a closed 2-manifold and only a finite number of these are different from the 2-sphere.

Now we are ready to prove our main result describing the properties of arbitrary minimal maps of closed 2-manifolds.

Theorem 3.16. *Suppose $f : M \to M$ is a minimal map from a closed 2-manifold to itself. Then f is monotone with tree-like point inverses.*

Proof. Suppose $f = g \circ m$ is the monotone-light factorization of f. By Theorem 2.9 f is almost 1-to-1. Clearly, it implies that both g and m are almost 1-to-1. If we can prove that for every $y \in M$ each component of $f^{-1}(y)$ is tree-like then it follows from Theorem 3.4 that f is actually monotone. Therefore it remains to show that point inverses under f are tree-like.

By way of contradiction assume that there are points whose inverses are cyclic. Put $X = m(M)$. By [RS38], X is the image of a generalized cactoid C on which a finite number of identifications was made. We consider the following two cases.

Case 1. $\dim(X) = 2$. Then X is the image of a generalized 2-cactoid C after a finite number of identifications. Hence, there exists a true cyclic element N in C. Since C is a generalized 2-cactoid, N is a closed 2-manifold. Then X contains a copy of the closed manifold N with finitely many points identified. Moreover, the map which identifies finitely many points of N can be precomposed with g thus resulting into a light almost 1-to-1 map $h : N \to M$. By Theorem 3.4 h is a homeomorphism which implies that g is a homeomorphism. It follows that f is a monotone map with tree-like point inverses as desired.

Case 2. $\dim(X) = 1$. Now X is the image of a dendrite under finitely many identifications. Moreover, by the results of [RS38] the number of generators of the fundamental group of X is finite. This implies that in fact X is a local dendrite. Now consider the minimal map $h : X \to X$ defined as $h = m \circ g$. By Theorem 3.14, X is a simple closed curve. Recall that f and hence m is almost 1-to-1. Fix two points $a, x \in X$ such that $m^{-1}(x)$ and $m^{-1}(a)$ are singletons. Then $m^{-1}(\{a, x\})$ is a 2-point set separating M, a contradiction. This completes the proof. $\square$

It turns out that in some cases, in particular in terms of studying their periodic points, the maps with the properties listed in Theorem 3.16 can be dealt with in the same manner as homeomorphisms. To explain this we need the following definition.

Suppose that $f : M \to M$ is a minimal map on a closed 2-manifold with non-zero Euler characteristic. By Theorem 3.16, f is monotone with tree-like point inverses. Then by [Dav86, Theorem 25.1 and Corollary 1A], f can be approximated by homeomorphisms. In particular f is a homotopy equivalence and an isomorphism on homology. Hence by results of Fuller and Halpern [Ful53, Hal68], f cannot be minimal because it has to have periodic points. Thus we have shown:

Corollary 3.17. *The only closed connected 2-manifolds which admit minimal maps are the Klein bottle and the 2-torus.*

4. Minimal maps of compact 2-manifolds with boundary

The aim of this section is to prove that there are no minimal maps of compact 2-manifolds with boundary. In the case of a homeomorphism this is clear. Indeed, if $f : M \to M$ is a minimal homeomorphism of a 2-manifold M into itself then the boundary of M must be invariant, a contradiction with minimality. However in the case of an arbitrary map of a manifold M with boundary the set ∂M does not have to be invariant which does not allow one to use the same arguments as above.

Still, our strategy in the general case develops an idea described in the previous paragraph for homeomorphisms. First of all, it is enough to consider connected manifolds. Now, by Lemma 3.10 if f is a minimal map of a manifold M with boundary then $f(\partial M) \supset \partial M$ is impossible. Using this and the techniques developed in previous sections and in [BOT02] we prove that for some k in the monotone-light decomposition $l_k \circ m_k$ of f^k the monotone quotient space $m_k(M)$ is a local dendrite which leads to a contradiction as before.

We begin with a series of lemmas concerning almost 1-to-1 maps.

Lemma 4.1. *Let $f : A \to B$ be an almost* 1*-to-*1 *map of compact metric spaces. Then a closed $C \subset A$ has interior if and only if so does the set $f(C) \subset B$.*

Proof. If C has interior then so does $f(C)$ because by Lemma 2.7 almost 1-to-1 maps are quasi-interior. Suppose that C has no interior and show that neither does $f(C)$. Indeed, otherwise consider the set $D = f^{-1}(f(C))$. The set D has interior because f is continuous, which together with the fact that C has no interior implies that $D \setminus C = E$ has non-empty interior. Since f is quasi-interior we see that the set $\mathrm{Int}(f(E))$ is an open subset of $f(C)$. Clearly, every point of $f(E)$ has preimages in both E and C, a contradiction. □

Lemma 4.1 implies the following corollary applicable in particular to minimal maps of manifolds with boundary.

Corollary 4.2. *Let $f : M \to M$ be an almost* 1*-to-*1 *onto map of a compact manifold with boundary (e.g., f can be a minimal map). Then the following holds.*

1. *each set of the form $f^{-m}(f^n(\partial M))$ is a closed set with empty interior;*
2. *the grand orbit $\bigcup_{n=0,m=0}^{\infty} f^{-m}(f^n(\partial M))$ is an F_σ-subset of M.*

Proof. It is enough to observe that by Corollary 2.8 the maps f^n and f^m are both almost 1-to-1 and then apply Lemma 4.1 to the set ∂M and these maps. □

Let us denote the grand orbit of a set A by $\Gamma(A)$. As we announced in the beginning of this section, our strategy is to show that in the in the monotone-light decomposition of a possibly existing minimal map of a 2-manifold with boundary the monotone quotient space of the manifold has to be 1-dimensional (in fact it is a local dendrite). Corollary 4.2 allows us to show that even if there exist 2-dimensional pieces of the monotone quotient space in question, they are not coming from the manifold boundary.

Lemma 4.3. *Suppose that $f : M \to N$ is an almost 1-to-1 map of compact 2-manifolds. Let $f = l \circ m$ be its monotone-light decomposition. Then if $A \subset M$ is a closed nowhere dense set then $m(A)$ cannot contain 2-disks. Moreover, the set $m(\Gamma(\partial M))$ is the union of countably many compact sets not containing 2-disks, and therefore cannot contain 2-disks either.*

Proof. Suppose that $m(A)$ contains a 2-disk. Then since l is almost 1-to-1 and light we see that by Theorem 2.11 the set $l(m(A)) = f(A)$ contains a 2-disk too. Thus, the interior of $f(A)$ is non-empty, a contradiction with Lemma 4.1. The rest easily follows from Corollary 4.2. □

With the notation of Lemma 4.3, Corollary 4.2 and Lemma 4.3 now imply that the m-image of the grand orbit of the set ∂M is the union of countably many closed subsets of $m(M)$ none of which contains a 2-disk. Observe that if $f : M \to M$ is an onto almost 1-to-1 map and we replace f by f^k for some k and consider for f^k its monotone-light decomposition $f^k = l_k \circ m_k$, the map m_k will not be the same as $m = m_1$. However, the same conclusion can be made since Corollary 4.2 and Lemma 4.3 are still applicable to f^k by Corollary 2.8.

The next lemma narrows down the possibilities for having 2-disks in $m(M)$ even further. Namely, it turns out that any connected open set in M disjoint from ∂M whose boundary components are collapsed under f has a local dendrite as its m-image.

Lemma 4.4. *Suppose that $f : M \to K$ is an almost 1-to-1 map of a compact connected 2-manifold M with $\partial M \neq \emptyset$ into a connected compact manifold K with $\partial K \neq \emptyset$. Let $f = l \circ m$ be its monotone-light decomposition. Let $V \subset M \setminus \partial M$ be an open connected set such that the f-image of each component of Bd(V) is a point. Then $m(\overline{V})$ is a local dendrite.*

Proof. Let $V \subset M \setminus \partial M$ be a connected open set such that each component of the $\mathrm{Bd}(V)$ is mapped by f to a point. Let O_i be components of $M \setminus \overline{V}$. Since M is a compact manifold there exist finitely many components $O_1, \dots, O_N$ such that $\partial M \subset \cup_{i=1}^N O_i$. Let $U = M \setminus \cup_{i=1}^N \overline{O_i}$. Then U is a connected open set in $M \setminus \partial M$, $V \subset U$ and each component of $\mathrm{Bd}(U)$ is mapped by f into a point. Now it suffices to show that $m(\overline{U})$ is a local dendrite.

Let $f' = f|_{\overline{U}}$ and $f' = l' \circ m'$ be the monotone-light decomposition of f'. We shall prove that there exists an onto map $t : m'(\overline{U}) \to m(\overline{U})$ and a finite set $F \subset m(\overline{U})$ such that for every $y \in m(\overline{U}) \setminus F$ the set

$t^{-1}(y)$ is a point while for each $y \in F$ we have $t^{-1}(y)$ is a finite set. Clearly this would imply that to prove the lemma it is enough to show that $m'(\overline{U})$ is a local dendrite.

For $x \in m'(\overline{U})$ put $t(x) = m \circ m'^{-1}(x)$. Observe that for every $w \in m'(\overline{U})$ there exists a unique $s \in m(\overline{U})$ such that $m'^{-1}(w) \subset m^{-1}(s)$. Hence t is a well-defined continuous function. Let us show that it has the desired properties. Indeed, for points w not contained in the m'-image of $\mathrm{Bd}(U)$ the map t is trivial. On the other hand, for the remaining finitely many points the map t is finite-to-one map.

Define an equivalence relation $\sim$ on $\overline{U}$ by setting $x \sim y$ if and only if $x = y$ or x and y lie in the same component of $\mathrm{Bd}(U)$. Then $\sim$ is upper semicontinuous. Then $N = \overline{U}/\sim$ is a closed manifold. Let $g : \overline{U} \to N$ be the quotient map. Let $h : N \to m'(\overline{U})$ be such that $m'|_{overlineU} = h \circ g$; then h is monotone. Now by [RS38], Theorem 5, either $m'(\overline{U}) = h(N)$ contains a subset P' homeomorphic to a closed 2-manifold with finitely pairs of points identified, or $m'(\overline{U})$ is a local dendrite. Let us show, by way of contradiction, that the former is impossible.

Indeed, suppose that $P' \subset m'(\overline{U})$ is a quotient of a closed 2-manifold P with finitely many identifications. Hence P' is the continuous image of the closed 2-manifold P under an almost 1-to-1 light map $\pi : P \to P'$. Then the composition $l' \circ \pi : P \to K$ is a continuous almost 1-to-1 map. By Theorem 2.11 $l' \circ \pi$ is an embedding. However, it is impossible to embed a closed 2-manifold into a connected 2-manifold with non-empty boundary. □

The above lemmas deal with almost 1-to-1 maps of 2-manifolds with boundary and therefore apply to minimal maps. Now we start studying certain specific properties of minimal maps, so before proceeding further, let us fix notation which will be used from now on. We suppose that $f : M \to M$ is a minimal map of a compact 2-manifold M with $\partial M \neq \emptyset$ (the case of compact 2-manifolds without boundary was covered in Section 3). Our next step is to prove that there exists a power f^k of f such that the following holds: if a point x does not belong to the grand orbit of ∂M then it has a compact neighborhood U such that $m_k(U)$ does not contain 2-disks where $f^k = f_k \circ m_k$ is the monotone-light factorization of f^k. This together with Lemma 4.3 would imply that in fact the entire manifold M can be covered by a countable union of compact sets such that their m_k-images contain no 2-disks. Then by [RS38] the space $m_k(M)$ is a local dendrite, leading to the same contradiction as in the proof of Theorem 3.16.

Lemma 4.5. *Let M be a compact 2-manifold with $\partial M \neq \emptyset$ and let f be a minimal map. Then there is an open connected set $U \subset M \setminus \partial M$ such that every component of the boundary of U is mapped into a point by f.*

Proof. Observe that since f is minimal then by Lemma 3.10 $\partial M \not\subset f(\partial M)$. Choose a point $y \in \partial M \setminus f(\partial M)$. Choose $x \in f^{-1}(y)$ and a sufficiently small open disk neighborhood D of y whose boundary is an arc I, with $R_f \cap I$ dense in I, such that $\overline{D} \cap f(\partial M) = \emptyset$. Let H be the component of $f^{-1}(D)$ containing x, then $\overline{H} \cap \partial M = \emptyset$.

Let $B_1, \dots, B_n$ be the components of ∂M. For each i choose a point $z_i \in B_i$. Let $K_i \subset \mathrm{Bd}(H)$ be an irreducible separator of M between z_i and x. Then K_i is an irreducible separator between x and B_i. Fix i and put $K_i = K$ and $z_i = z$. We will show that each component of K is mapped by f into a point. Since M has finite degree of multicoherence [Sto49], K has finitely many components $C(1), \dots, C(m)$. Since f is almost 1-to-1 and $f(K) \subset I$ we have $|f(C(i)) \cap f(C(j))| \leq 1$ for all $i \neq j$.

Let $g \circ m = f$ be the monotone-light decomposition of f. Since m is monotone, $m(K)$ is an irreducible separator of $m(M)$ between $m(x)$ and $m(z)$. If $f(C(i))$ is non-degenerate, then it follows by Lemma 3.3 that $m(C(i))$ is an arc which is mapped 1-to-1 into I by g. If $m(C(i))$ and $m(C(j))$ are both non-degenerate and if they meet then they must meet in a single point which is an endpoint of both. Indeed, since g is 1-to-1 on each $m(C(t))$ and $|f(C(i)) \cap f(C(j))| \leq 1$, $m(C_i)$ and $m(C_j)$ meet in at most one point. Since $g|_{m(C_i) \cup m(C_j)}$ is almost 1-to-1, this point must be a common endpoint of both. Since g is light and maps $m(K)$ almost 1-to-1 into the arc I, it follows from Lemma 3.3 that each component of $m(K)$ is an arc or a point.

Let N be a closed 2-manifold such that $M \subset N$ and $N \setminus M$ consists of finitely many disjoint open disks. Define an equivalence relation $\sim$ on N by setting $p \sim q$ if and only if $p = q$ or $p, q \in M$ and $m(p) = m(q)$. Then $\sim$ is a monotone upper semicontinuous decomposition. By [RS38] the quotient space $N/\sim$ is a generalized cactoid with finitely many pairs of points identified. Since $K \cap \partial M \subset \overline{H} \cap \partial M = \emptyset$ and $m(K)$ separates $m(M)$ irreducibly between $m(x)$ and $m(z)$, it follows that $m(K)$ separates $N/\sim$ irreducibly. Since $m(K)$ is a finite union of arcs and points which irreducibly separates $N/\sim$, each component of $m(K)$ must be a point. This implies that $m(C(i))$ is a point for each i. Therefore, for each i the set K_i is an irreducible separator between x and B_i, each K_i consists of finitely many components each of which is mapped into a point by f. In particular, $f(\cup_{i=1}^{n} f(K_i)) = F$ is a finite set.

Let U be the component of $M \setminus \cup_{i=1}^{n} K_i$ containing x. Then $U \cap \partial M = \emptyset$, $\mathrm{Bd}(U) \subset \cup_{i=1}^{n} K_i$ and therefore $f(\mathrm{Bd}(U)) \subset F$ is finite. In particular every component of $\mathrm{Bd}(U)$ is mapped to a point. This completes the proof. $\square$

From now on we fix the set U found in Lemma 4.5. Clearly, by Lemma 4.4 the monotone part m of the monotone-light decomposition of $f = l \circ m$ maps $\overline{U}$ onto a local dendrite. In order to use this fact in implementing of our plan we now use the minimality of the map f. Indeed, by the minimality of f and compactness of M there exists a number N such that for any point $y \in M$ one can find a number $r(y) = r \leq N$ such that $f^r(y) \in U$. The major step now is the following lemma.

Lemma 4.6. *Consider f^N. Then every point $y \notin \Gamma(\partial M)$ has a neighborhood E such that its boundary consists of components each of which is collapsed to a point by f^{N+1}.*

Proof. Let $y \notin \Gamma(\partial M)$. Let us begin by describing the basic construction and specify the details as to how certain neighborhoods involved in it should be chosen later on.

Let U be the set given by Lemma 4.5. One can find a number $r(y) = r \leq N$ such that $f^r(y) \in U$. Since $f^r(y) \in U$ then the map m maps the point $f^r(y)$ into a local dendrite $D = m(\overline{U})$ (as follows from Lemma 4.4 the set $m(\overline{U})$ is a local dendrite). By [Nad92, Theorem 10.2], dendrites are regular (i.e., every pair of points can be separated by a finite set). Therefore we can find an arbitrarily small neighborhood V of $m(f^r(y))$ in D such that the boundary of V is finite (the choice of V will be specified later). The m-preimage of V is an open connected neighborhood $W = m^{-1}(V)$ of $f^r(y)$ whose boundary is the union of m-preimages of boundary points of V. Each m-preimage of a boundary point of V is a continuum collapsed by m (and therefore by f), and the union of all such continua separates M. Let us take a component E of $f^{-r}(W)$ containing y. Let us show that V can be chosen so small that $\overline{E}$ is disjoint from ∂M. Indeed, to this end it is enough to choose V so that $\overline{V}$ is disjoint from $m(f^r(\partial M))$. Now, this is possible because $y \notin \Gamma(\partial M)$ and so $f^i(y)$ never belongs to any image of ∂M, in particular $f^{r+1}(y) \notin f^{r+1}(\partial M)$. By the definition of m this implies that $m(f^r(y)) \notin m(f^r(\partial M))$ which guarantees that V with the desired properties can be found. Hence $\overline{E}$ is disjoint from ∂M. Moreover, since components of the boundary of E are mapped into components of the boundary of W by f^r we see that E is a neighborhood of y whose boundary consists of components each of which is collapsed by f^{r+1} and therefore by f^{N+1} as desired. □

We are finally ready to prove the main theorem of this section.

Theorem 4.7. *There are no minimal maps on compact connected* 2-*manifolds with boundary.*

Proof. By way of contradiction assume that there exists a minimal map $f : M \to M$ of a 2-manifold with boundary. It is easy to see that since M is connected then f^i is minimal for every i. Then by Lemma 4.3, Lemma 4.4 and Lemma 4.6 there exists a number N such that the manifold M can be covered by a countable collection of sets each of which has the m_{N+1}-image which contains no 2-disks (where $f^{N+1} = l_{N+1} \circ m_{N+1}$ is the monotone-light decomposition of f^{N+1}). By [RS38] this implies that $m_{N+1}(M)$ is a local dendrite. Thus we find ourselves in the situation of Theorem 3.16. That is, $l_{N+1} \circ m_{N+1} : m_{N+1}(M) \to m_{N+1}(M)$ is a minimal map of a local dendrite. By Theorem 3.14 $m_{N+1}(M)$ is a circle. Choose two distinct points x and y in the circle with unique preimages. Then $m^{-1}_{N+1}(\{x, y\})$ is a 2-point set which separated the connected 2-manifold M, a contradiction. □

5. Proof of Main Theorem

We are now ready to establish the main result of this paper:

Theorem 5.1. *Suppose that $f : M \to M$ is a minimal map of a 2-manifold. Then f is a monotone map with tree-like point inverses and M is either a finite union of tori, or a finite union of Klein bottles which are cyclically permuted by f.*

Proof. It follows immediately from [HK53] that X must be compact. Hence M is a finite union of compact connected 2-manifolds which are permuted by f. Let C be a component of M. Then there exists an $n \geq 1$ such that the map $f^n|_C : C \to C$ is a minimal map. By Theorem 4.7, C is a closed manifold. Hence by Corollary 3.17, C is either a Klein bottle or a 2-torus. Since f is minimal, and M is compact, the space M is the union of finitely many components which are permuted by the map; moreover, the map is surjective, and so each component of M is mapped *onto* the next one. It is enough to show that $f(C)$ is homeomorphic to C. Given a point $x \in f(C)$ observe that $f^{-1}(x) = f^{n-1}(f^{-n}(x))$. On the other hand, $f^{-n}(x)$ is connected by Theorem 3.16. Hence, $f^{-1}(x)$ is connected too, and f is monotone. Now, it is easy to see that $f^{-1}(x)$ is a subset of the set $F = f^{-n}(f^{n-1}(x))$, and since F is a tree-like continuum by Theorem 3.16, we conclude that $f^{-1}(x)$ is a tree-like continuum too. Hence by [RS38], $f(C)$ is homeomorphic to C.

We showed that point inverses are tree-lime continua and that all components of M are either cyclically permuted 2-tori, or cyclically permuted Klein bottles. This completes the proof. $\square$

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