LAMINATIONS IN THE LANGUAGE OF LEAVES

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Abstract. Thurston defined invariant laminations, i.e. collections of chords of the unit circle $\mathbb{S}$ (called *leaves*) that are pairwise disjoint inside the open unit disk and satisfy a few dynamical properties. To be directly associated to a polynomial, a lamination has to be generated by an equivalence relation with specific properties on $\mathbb{S}$; then it is called a *q-lamination*. Since not all laminations are q-laminations, then from the point of view of studying polynomials the most interesting are those of them which are limits of q-laminations. In this paper we introduce an alternative definition of an invariant lamination, which involves only conditions on the leaves (and avoids gap invariance). The new class of laminations is slightly smaller than that defined by Thurston and is closed. We use this notion to elucidate the connection between invariant laminations and invariant equivalence relations on $\mathbb{S}$.

1. Introduction

Invariant laminations, introduced by Thurston in the early 1980's, are used to study the dynamics of individual polynomials and the parameter space of all polynomials, the latter in the quadratic case (an expanded version of Thurston's preprint recently appeared [Thu09]). Investigating the space of all quadratic invariant laminations played a crucial role in [Thu09]. An important idea of Thurston's was, as we see it, similar to one of the main ideas of dynamics as a whole - to suggest a tool (*laminations*) allowing one to model the dynamics under investigation on a topologically/combinatorially nice object (in case of [Thu09] one models polynomial dynamics on the Julia set by so-called *topological polynomials*, generated by laminations).

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According to Thurston, a *lamination* $\mathcal{L}$ is a closed family of chords inside the open unit disk $\mathbb{D}$. These chords meet at most in a common endpoint and satisfy some dynamical conditions; these chords are called *leaves (of the lamination)* and union of all leaves from $\mathcal{L}$ united with $\mathbb{S}$ is denoted by $\mathcal{L}^*$. A natural *direct* way to associate a lamination to a polynomial P of degree d with a locally connected Julia set is as follows: (1) define an equivalence relation $\sim_P$ on $\mathbb{S}$ by identifying angles if their external rays land at the same point (observe that $\sim_P$ on $\mathbb{S}$ is σ_d-invariant); (2) consider the edges of convex hulls of equivalence classes and declare them to be the *leaves* of the corresponding *lamination* $\mathcal{L}_P$.

By [Kiw04, BCO08] more advanced methods allow one to associate a lamination to some polynomials with non-locally connected Julia sets (by declaring two angles equivalent if impressions of their external rays are non-disjoint and extending this relation by transitivity). We call laminations, generated by equivalence relations similar to $\sim_P$ above, *q-laminations*. They form an important class of laminations, many of which correspond to complex polynomials with connected Julia sets. In all these cases the lamination is found through the study of the topology of the Julia set of the polynomial.

The drawback of this approach is that it fails if the topology of the Julia set is complicated (e.g., if a quadratic polynomial has a fixed *Cremer point* [BO06]). Thus, even though ultimately laminations are a tool which allows one to study both individual polynomials and their parameter space, in some cases it is not obvious as to what laminations (or what equivalence relations on the circle) can be directly connected in a meaningful way to certain polynomials. Hence one needs a non-direct way of associating a lamination (or, more generally, some combinatorial structure) to a polynomial with a complicated Julia set.

A possibility here is as follows. For a polynomial $P_c(z) = z^2 + c$, consider sequences of parameters $c_i \to c$ with $P_{c_i} = P_i$ having locally connected Julia sets and associated lamination $\mathcal{L}_{P_i}$. These laminations $\mathcal{L}_{P_i}$ (systems of chords of $\mathbb{S}$) may converge to another lamination (system of chords of $\mathbb{S}$) in the sense that the continua $\mathcal{L}^*_{P_i}$ may converge to a subcontinuum of $\mathbb{D}$ in the Hausdorff sense, and the limit continuum $\mathcal{L}^*$ then comes from an appropriate lamination $\mathcal{L}$). In this case the lamination $\mathcal{L}$ is called the *Hausdorff limit* of laminations $\mathcal{L}_{P_i}$; one may associate all such Hausdorff limit laminations to c.

Using this notion of convergence one can define the Hausdorff closures of sets of laminations. Hence the space of laminations useful for studying polynomials could be a closed set of laminations which contains the Hausdorff closure of the set of all q-laminations, but is not much bigger.

To describe a candidate set of laminations we introduce a new notion of a *sibling invariant lamination* which is slightly more restrictive than the one given by Thurston. The new definition is given intrinsically (i.e., by only listing properties on the leaves of the lamination). We show that the family of all sibling invariant laminations is closed and contains all q-laminations. The new definition significantly simplifies the verification of the fact that a system of chords of $\mathbb{S}$ is an invariant lamination. Thurston [Thu09] introduced the class of *clean* laminations. We use our tools to show that clean laminations are (up to a finite modification) q-laminations. In Section 6 we apply these ideas to the degree 2 case and show that in this case all clean Thurston invariant laminations are q-laminations.

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2. Laminations: classical definitions

2.1. **Preliminaries.** Let $\mathbb{C}$ be the complex plane, $\mathbb{S} \subset \mathbb{C}$ the unit circle identified with $\mathbb{R}/\mathbb{Z}$ and let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk. Define a map $\sigma_d : \mathbb{S} \to \mathbb{S}$ by $\sigma_d(z) = dz \mod 1$, $d \geq 2$. By a *chord* in the unit disk we mean a segment of a straight line connecting two points of the unit circle. A *prelamination* $\mathcal{L}$ is a collection of chords in $\mathbb{D}$, called *leaves*, such that any two leaves of $\mathcal{L}$ meet at most in a point of $\mathbb{S}$. If all points of the circle are elements of $\mathcal{L}$ (seen as degenerate leaves) and $\bigcup \mathcal{L} = \mathcal{L}^*$ is closed in $\mathbb{C}$, then we call $\mathcal{L}$ a *lamination*. Hence, one obtains a lamination by closing a prelamination and adding all points of $\mathbb{S}$ viewed as degenerate leaves. If $\ell \in \mathcal{L}$ and $\ell \cap \mathbb{S} = \{a, b\}$ then we write $\ell = \overline{ab}$. We use the term "leaf" to refer to a non-degenerate leaf in the lamination, and specify when a leaf may be degenerate, i.e. a point in $\mathbb{S}$.

Given a leaf $\ell = \overline{ab} \in \mathcal{L}$, let $\sigma_d(\ell)$ be the chord with endpoints $\sigma_d(a)$ and $\sigma_d(b)$. If $\sigma_d(a) = \sigma_d(b)$, call ℓ a *critical leaf* and $\sigma_d(a)$ a *critical value.* Let $\sigma_d^* : \mathcal{L}^* \to \overline{\mathbb{D}}$ be the linear extension of σ_d over all the leaves in $\mathcal{L}$. It is not hard to check that σ_d^* is continuous. Also, σ_d is locally one-to-one on $\mathbb{S}$, and σ_d^* is one-to-one on any given non-critical leaf. Note that if $\mathcal{L}$ is a lamination, then $\mathcal{L}^*$ is a continuum.

Definition 2.1 (Gap)**.** A *gap* G of a lamination $\mathcal{L}$ is the closure of a component of $\mathbb{D} \setminus \mathcal{L}^*$; its boundary leaves are called *edges (of a gap).* We also say that a leaf ℓ is an *edge* of ℓ.

For each set $A \subset \overline{\mathbb{D}}$ we denote $A \cap \mathbb{S}$ by $\partial(A)$. If G is a leaf or a gap of $\mathcal{L}$, it follows that G coincides with the convex hull of $\partial(G)$. If G is a leaf or a gap of $\mathcal{L}$ we let $\sigma_d(G)$ be the convex hull of $\sigma_d(\partial(G))$. Also, by $\mathrm{Bd}(G)$ we denote the topological boundary of G. Notice that the topological boundary of G is a Jordan curve which consists of leaves and points on $\mathbb{S}$, so that $\mathrm{Bd}(G) \cap \mathbb{S} = G \cap \mathbb{S} = \partial(G)$. A gap G is called *infinite* if and only if $\partial(G)$ is infinite. A gap G is called *critical* if $\sigma_d|_{\partial G}$ is not one-to-one. Observe that there are two types of degenerate leaves of $\mathcal{L}^*$ which are not endpoints of non-degenerate leaves: (1) certain vertices of gaps, (2) points of $\mathbb{S}$, separated from other points of $\mathbb{S}$ by a sequence of leaves of $\mathcal{L}$.

2.2. **q-laminations.** Let P be a complex polynomial with locally connected Julia set J. Then J is connected and there exists a conformal map $\varphi : \mathbb{C}^* \setminus \overline{\mathbb{D}} \to \mathbb{C}^* \setminus K$, where K is the *filled-in Julia set* (i.e., the complement of the unbounded component of J in $\mathbb{C}$). One can choose φ so that $\varphi'(0) > 0$ and $P \circ \varphi = \varphi \circ \sigma_d$, where $\sigma_d(z) = z^d$ and d is the degree of P. Since J is locally connected, φ extends over the boundary $\mathbb{S}$ of $\mathbb{D}$. We denote the extended map also by φ. Define an equivalence relation $\approx_P$ on $\mathbb{S}$ by $x \approx_P y$ if and only if $\varphi(x) = \varphi(y)$. Since J is the boundary of $\mathbb{C}^* \setminus K$, then J is homeomorphic to $\mathbb{S}/{\approx_P}$. Clearly, the map σ_d induces a map $f_d : \mathbb{S}/{\approx_P} \to \mathbb{S}/{\approx_P}$ and the maps $P|_J$ and f_d are conjugate. It is known that all equivalence classes of $\approx_P$ are finite. The collection of boundary edges of convex hulls of all equivalences classes of $\approx_P$ is a lamination denoted by $\mathcal{L}_P$.

Equivalence relations analogous to $\approx_P$ can be introduced with no reference to polynomials [BL02]. Let $\sim$ be an equivalence relation on $\mathbb{S}$. Equivalence classes of $\sim$ will be called *($\sim$-)classes* and will be denoted by Gothic letters. Also, given a closed set $A \subset \mathbb{C}$, let $\mathrm{CH}(A)$ denote the convex hull of the set A in $\mathbb{C}$.

Definition 2.2. An equivalence relation $\sim$ is a *(d-)invariant laminational* equivalence relation if:
(E1) $\sim$ is *closed*: the graph of $\sim$ is a closed set in $\mathbb{S} \times \mathbb{S}$;
(E2) $\sim$ is *unlinked*: if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are distinct $\sim$-classes, then their convex hulls $\mathrm{CH}(\mathfrak{g}_1), \mathrm{CH}(\mathfrak{g}_2)$ in the unit disk $\mathbb{D}$ are disjoint,
(D1) $\sim$ is *forward invariant*: for a class $\mathfrak{g}$, the set $\sigma_d(\mathfrak{g})$ is a class too which implies that
(D2) $\sim$ is *backward invariant*: for a class $\mathfrak{g}$, its preimage $\sigma_d^{-1}(\mathfrak{g}) = \{x \in \mathbb{S} : \sigma_d(x) \in \mathfrak{g}\}$ is a union of classes;
(D3) for any $\sim$-class $\mathfrak{g}$ with more than two points, the map $\sigma_d|_{\mathfrak{g}} : \mathfrak{g} \to \sigma_d(\mathfrak{g})$ is a *covering map with positive orientation*, i.e., for every connected component (s, t) of $\mathbb{S} \setminus \mathfrak{g}$ the arc in the circle $(\sigma_d(s), \sigma_d(t))$ is a connected component of $\mathbb{S} \setminus \sigma_d(\mathfrak{g})$;
(D4) all $\sim$-classes are finite.

There is an important connection between laminations and (invariant laminational) equivalence relations.

Definition 2.3. Let $\mathcal{L}$ be a lamination. Define the equivalence relation $\approx_{\mathcal{L}}$ by declaring that $x \approx_{\mathcal{L}} y$ if and only if there exists a finite concatenation of leaves of $\mathcal{L}$ joining x to y.

Now we are ready to define *q-laminations.*

Definition 2.4. A lamination $\mathcal{L}$ is called a *q-lamination* if the equivalence relation $\approx_{\mathcal{L}}$ is an invariant laminational equivalence relation and $\mathcal{L}$ consists exactly of boundary edges of the convex hulls of all $\approx_{\mathcal{L}}$-classes together with all points of $\mathbb{S}$. If an invariant laminational equivalence relation $\sim$ is given and $\mathcal{L}$ is formed by all edges from the convex hulls of all $\sim$-classes together with all points of $\mathbb{S}$ then $\mathcal{L}$ is called the *q-lamination (generated by $\sim$)* and is denoted by $\mathcal{L}_\sim$. Clearly, if $\mathcal{L}$ is a q-lamination, then it is a q-lamination generated by $\approx_{\mathcal{L}}$.

Let $\sim$ be a laminational equivalence relation and $p : \mathbb{S} \to J_\sim = \mathbb{S}/\sim$ be the quotient map of $\mathbb{S}$ onto its quotient space $J_\sim$, let $f_\sim : J_\sim \to J_\sim$ be the map induced by σ_d. We call $J_\sim$ a *topological Julia set* and the induced map $f_\sim$ a *topological polynomial.* It is easy to see from definition 2.2 that leaves of $\mathcal{L}_\sim$ map to leaves of $\mathcal{L}_\sim$ under σ_d; moreover, the map σ_d acting on leaves and gaps of $\mathcal{L}_\sim$ has also other more specific properties analogous to (D1) - (D4) above. This leads to the abstract notion of an invariant lamination [Thu09] that allows for laminations which are not directly associated to a laminational equivalence relation and, hence, do not correspond (directly) to a polynomial.

2.3. Invariant laminations due to Thurston.

Definition 2.5 (Monotone Map)**.** Let X, Y be topological spaces and $f : X \to Y$ be continuous. Then f is said to be *monotone* if $f^{-1}(y)$ is connected for each $y \in Y$. It is known that if f is monotone and X is a continuum then $f^{-1}(Z)$ is connected for every connected $Z \subset f(X)$.

Definition 2.6 is due to Thurston [Thu09]; recall that gaps are defined in Definition 2.1.

Definition 2.6 (Thurston Invariant Lamination [Thu09])**.** A lamination $\mathcal{L}$ is *Thurston d-invariant* if it satisfies the following conditions.

(1) Forward d-invariance: for any leaf $\ell = \overline{pq} \in \mathcal{L}$, either $\sigma_d(p) = \sigma_d(q)$, or $\overline{\sigma_d(p)\sigma_d(q)} = \sigma_d(\ell) \in \mathcal{L}$.
(2) Backward invariance: for any leaf $\overline{pq} \in \mathcal{L}$, there exists a collection of d **disjoint** leaves in $\mathcal{L}$ (this collection of leaves may not be unique), each joining a pre-image of p to a pre-image of q.

(3) Gap invariance: For any gap G, the convex hull H of $\sigma_d(G \cap \mathbb{S})$ is a gap, a leaf, or a single point (of $\mathbb{S}$).

If H is a gap, $\sigma_d^*|_{\mathrm{Bd}(G)} : \mathrm{Bd}(G) \to \mathrm{Bd}(H)$ must map as the composition of a monotone map and a covering map to the boundary of the image gap, with positive orientation (the image of a point moving clockwise around $\mathrm{Bd}(G)$ must move clockwise around the image $\mathrm{Bd}(H)$ of G).

3. Sibling Invariant Laminations

3.1. **An alternative definition.** Note that in Definition 3.1 we do not require the invariance of gaps.

Definition 3.1 (Sibling d-Invariant Lamination [Mim10])**.** A (pre)lamination $\mathcal{L}$ is *sibling d-invariant* if:

(1) for each $\ell \in \mathcal{L}$ either $\sigma_d(\ell) \in \mathcal{L}$ or $\sigma_d(\ell)$ is a point in $\mathbb{S}$,
(2) for each $\ell \in \mathcal{L}$ there exists a leaf $\ell' \in \mathcal{L}$ such that $\sigma_d(\ell') = \ell$,
(3) for each $\ell \in \mathcal{L}$ such that $\sigma_d(\ell)$ is a non-degenerate leaf, there exist **d disjoint** leaves $\ell_1, \dots, \ell_d$ in $\mathcal{L}$ such that $\ell = \ell_1$ and $\sigma_d(\ell_i) = \sigma_d(\ell)$ for all i.

We need to make a few remarks. Given a continuum or a point $K \subset \mathcal{L}^*$ which maps one-to-one onto $\sigma_d^*(K)$, we call a continuum or a point $T \subset \mathcal{L}^*$ a *sibling (of K)* if $K \cap T = \emptyset$ and T maps onto $\sigma_d^*(K)$ in a one-to-one fashion too (thus, siblings are homeomorphic and disjoint). E.g., the leaves $\ell_2, \dots, \ell_d$ from Definition 3.1 are siblings of ℓ. The collection $\{\ell, \ell_2, \dots, \ell_d\}$ of leaves from Definition 3.1 is called a *full sibling collection (of ℓ)*. In general, for a continuum or a point $K \subset \mathcal{L}^*$ which maps one-to-one onto $\sigma_d^*(K)$, a collection of d sets made up of K and its pairwise disjoint siblings is called a *full sibling collection (of K)*. Often we talk about siblings without assuming the existence of a full sibling collection (e.g., in the context of Thurston d-invariant laminations).

Let $\mathcal{L}$ be a sibling d-invariant lamination. Then by Definition 3.1 (1) we see that Definition 2.6 (1) is satisfied. Now, let $\ell \in \mathcal{L}$ be a leaf. Then by Definition 3.1 (2) and (3) there are d pairwise disjoint leaves of $\mathcal{L}$ which map onto ℓ; thus, Definition 2.6 (2) is satisfied. Therefore both sibling d-invariant laminations and Thurston d-invariant laminations satisfy conditions (1) and (2) of Definition 2.6, i.e. are *forward d-invariant* and *backward d-invariant*. Both conditions deal with *leaves* and in that respect are intrinsic to $\mathcal{L}$ which is defined as a collection of leaves. However having these conditions is not enough to define a meaningful dynamic collection of leaves; there are examples of laminations satisfying conditions (1)-(2) of Definition 2.6 which are not gap invariant. Therefore one needs to add an extra condition to forward and backward invariance.

The choice made in Definition 2.6 deals with gaps, i.e. closures of components of the complement $\mathbb{D}\setminus\mathcal{L}^*$. This is a straightforward way to ensure that σ_d^* has a nice extension over the plane. However a drawback of this approach is that while $\mathcal{L}$ otherwise is defined as a family of chords of $\mathbb{D}$ (leaves), in gap invariance we directly talk about other objects (gaps). One can argue that gap invariance of $\mathcal{L}$ under σ_d is not sufficiently intrinsic since $\mathcal{L}$ is defined as a collection of leaves. As a consequence it is often more cumbersome to verify gap invariance. This justifies the search for a similar definition of an invariant lamination which deals only with leaves. Above we propose the notion of a *sibling (d)-invariant lamination.*

3.2. **Sibling invariant laminations are gap invariant.** Now we show that any sibling d-invariant lamination is a Thurston d-invariant lamination. Some complications below are caused by the fact that we do not yet know that the lamination is gap invariant. E.g., extending the map σ^* over $\mathbb{D}$ to a suitably nice map (i.e., the composition of a monotone and open map) is impossible if the lamination is not gap invariant.

Theorem 3.2. *Suppose that G is a gap of a sibling d-invariant lamination $\mathcal{L}$. Then either*

(1) *$\sigma_d(G)$ is a point in $\mathbb{S}$ or a leaf of $\mathcal{L}$, or*
(2) *there is a gap H of $\mathcal{L}$ such that $\sigma_d(G) = H$, and the map $\sigma_d^*|_{\mathrm{Bd}(G)} : \mathrm{Bd}(G) \to \mathrm{Bd}(H)$ is the positively oriented composition of a monotone map $m : \mathrm{Bd}(G) \to S$, where S is a simple closed curve, and a covering map $g : S \to \mathrm{Bd}(H)$.*

Thus, any sibling d-invariant lamination is a Thurston d-invariant lamination.

To prove Theorem 3.2 we prove a few lemmas. Given a point x, call any point $\hat{x} \in \mathbb{S}$ with $\sigma_d(\hat{x}) = x$ an *x-point*. If a lamination is given, by an *$\hat{a}\hat{b}$-leaf*, we mean a leaf that maps to $\overline{ab}$. The word "chord" is used in lieu of "leaf" in reference to a chord of $\mathbb{D}$ which may not be a leaf of $\mathcal{L}$. By $[a,b]_{\mathbb{S}}$, $a,b \in \mathbb{S}$ we mean the closed arc of $\mathbb{S}$ from a to b, and by $(a,b)_{\mathbb{S}}$ we mean the open arc of $\mathbb{S}$ from a to b. The direction of the arc, clockwise (*negative*) or counterclockwise (*positive*), will be clear from the context; also, sometimes we simply write $[a,b], (a,b)$ etc. By $<$ we denote the positive (circular) order on $\mathbb{S}$. If we say that points are *ordered* on $\mathbb{S}$ we mean that they are either positively or negatively ordered. Proposition 3.3 is left to the reader; observe, that in Proposition 3.3 we do not assume the existence of a lamination.

Proposition 3.3. *Suppose that $a_1 < b_1 < a_2 < b_2 < \dots < a_n < b_n$ are $2n$ points in the circle. Then for any point a_i and b_j either component of $\mathbb{S} \setminus \{a_i, b_j\}$ contains the same number of a-points and b-points. In particular, if $\hat{a}, \hat{b} \in \mathbb{S}$ are such that $a = \sigma_d(\hat{a}) \neq \sigma_d(\hat{b}) = b$, then either component of $\mathbb{S} \setminus \{\hat{a}, \hat{b}\}$ contains the same number of a-points and b-points.*

Since 2-invariant laminations are invariant under the rotation by $\frac{1}{2}$, then, given a 2-invariant lamination we see that its siblings are rotations of each other. Even though this is not typically true for laminations of higher degree (see Figure 1), Lemma 3.6 states that sibling leaves must connect in the same order. To state it we need Definition 3.4.

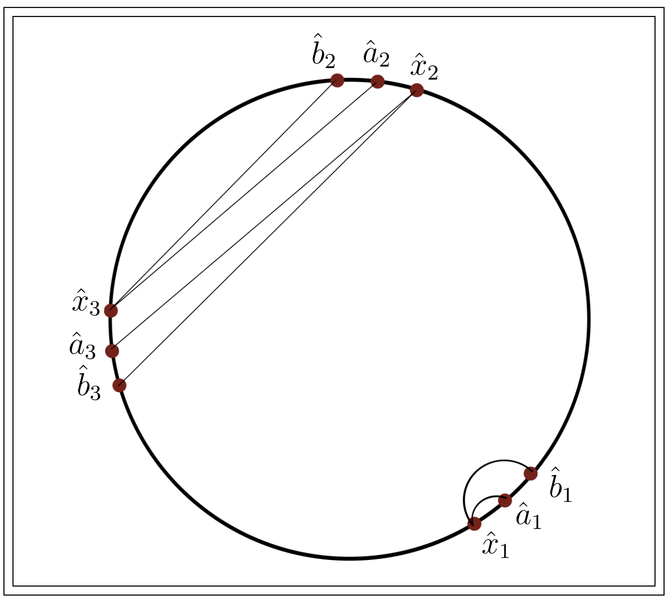

FIGURE 1. Sibling "arcs"
This is an example of sibling "arcs" under σ_3. Notice that while the arcs connect points in different "patches" and are not found by rigid rotation, the manner in which the endpoints connect preserves order.

Definition 3.4. Consider two disjoint sets $A, B \subset \mathbb{S}$ such that $\sigma_d(A) = \sigma_d(B) = C$ is the one-to-one image of A and B under σ_d. Then A, B and C are said to have the *same orientation* if for any three points $x, y, z \in A$ their siblings $x', y', z' \in B$ and their images $\sigma_d(x), \sigma_d(y), \sigma_d(z)$ have the same circular orientation as x, y, z. As we walk along the circle in the positive direction from a point $u \in A$, its sibling $u' \in B$, and its image $\sigma_d(u)$, we will meet points of A, their siblings in B, and their images in C in the same order.

Any two two-point sets have the same orientation; this is not necessarily true for sets with more points. The fact that sets have the same orientation sometimes implies "structural" conclusions. For a set $A \subset \mathcal{L}^*$ we write $A \sqsubset \mathcal{L}^*$ if $A \cap \mathbb{S}$ is zero-dimensional.

Definition 3.5. A *triod* is a homeomorphic image of the *simple triod* τ (the union of three arcs which share a common endpoint). Denote by $B(T)$ the union of the endpoints and the vertex of a triod T. In what follows we *always* consider triods $T \subset \mathbb{D}$ with $B(T) \subset \mathbb{S}$. The edge of T, which separates (inside $\overline{\mathbb{D}}$) the endpoints of T non-belonging to it, is called the *central edge* of T while the other edges of T are said to be *sides* of T. Similarly, if A is the union (the concatenation) of two leaves $\overline{av} \cup \overline{vb}$ we set $B(A) = \{a, v, b\}$.

To avoid ambiguity we call a subarc of $\mathbb{S}$ a *circle arc*. By an *arc in $\mathcal{L}^*$* we mean a *topological arc* (a one-to-one image of an interval). Given a set $A \subset \mathcal{L}^*$ we sometimes need to consider a topological arc in A with endpoints x, y (it will always be clear which arc we actually consider); such an arc will be denoted by $[x, y]_A$. Thus, $[a, b]_{\mathbb{S}}$ is always a circle arc. By default arcs $[a, b], (a, b)$ etc. are circle arcs.

Lemma 3.6. (1) *Let $x_1 < a_1 < b_1 < x_2 < \dots < x_n < a_n < b_n < x_1$ be $3n$ points in $\mathbb{S}$. If for each i there exists $r(i), m(i) \in \{1, \dots, n\}$ such that arcs $A_i = \overline{x_i a_{r(i)}} \cup \overline{x_i b_{m(i)}}$ are pairwise disjoint $(1 \le i \le n)$ then $x_i < a_{r(i)} < b_{m(i)}$ for each i.*

(2) *Let $x_1 < a_1 < b_1 < c_1 < x_2 < \dots < x_n < a_n < b_n < c_n < x_1$ be $4n$ points in $\mathbb{S}$. If for each i there exist $r(i), m(i), l(i) \in \{1, \dots, n\}$ such that triods $T_i = \overline{x_i a_{r(i)}} \cup \overline{x_i b_{m(i)}} \cup \overline{x_i c_{l(i)}}$ are pairwise disjoint $(1 \le i \le d)$ then $x_i < a_{r(i)} < b_{m(i)} < c_{l(i)}$ for each i.*

Proof. (1) Let $x_1 < b_{m(1)} < a_{r(1)}$. Then the arc $(x_1, b_{m(1)})$ contains $m(1) - 1$ points $x_2, \dots, x_{m(1)}$, $m(1) - 1$ points $b_1, \dots, b_{m(1)-1}$, but $m(1)$ points $a_1, \dots, a_{m(1)}$. Clearly, this contradicts the existence of sets A_j.

(2) Follows from (1) applied to parts of the triods T_i. □

We will mostly apply the following corollary of Lemma 3.6.

Corollary 3.7. *Let $\mathcal{L}$ be a sibling d-invariant lamination and $T \sqsubset \mathcal{L}^*$ be an arc consisting of two leaves with a common endpoint v or a triod* *consisting of three leaves with a common endpoint v. Suppose that $S \sqsubset \mathcal{L}^*$ is an arc (triod) such that $\sigma_d^*(S) = T$ and $\sigma_d^*|_S$ is one-to-one. Then the circular orientation of the sets $B(T)$ and $B(S)$ is the same.*

Proof. Let the endpoints of T be a, b (a, b, c if T is a triod). Then the set of all preimages of points of $B(T)$ consists of d triples (if T is an arc) or quadruples (if T is a triod) of points denoted by $B_1, \dots, B_d$ and such that (1) each B_i is contained in a circle arc J_i so that these arcs are disjoint, (2) the circular order of points in B_i is the same as the circular order of σ_d-images of these points.

Take the leaves which comprise T (two leaves if T is an arc and three leaves if T is a triod). Consider the corresponding leaves comprising S. Each leaf of T gives rise to its full sibling collection (here we use the fact that $\mathcal{L}$ is sibling invariant). Then leaves from those collections "grow" out of points $v_1, \dots, v_d$ which are preimages of v. This gives rise to d unlinked sets $S_1, \dots, S_d$ where S_i is a union of two (three) leaves growing out of v_i (indeed, no leaf of S_i can coincide with a leaf of S_j where $i \neq j$ while distinct leaves must be disjoint by the properties of laminations). Moreover, we may assume that $S_1 = S$. It now follows from Lemma 3.6 and the above paragraph that all the sets $B(S_i)$ of endpoints of S_i united with x_i have the same circular orientation coinciding with the circular orientation of the set of their σ_d-images, i.e. the set $B(T)$. □

Corollary 3.7 shows that all pullbacks of certain sets have the same orientation as the sets themselves. However it also allows us to study images of some sets. Indeed, by Corollary 3.7, if $S \sqsubset \mathcal{L}^*$ is a triod mapped by σ_d^* one-to-one into T then the central edge of S maps into the central edge of T.

Lemmas 3.6 and Corollary 3.7 are useful in comparing the orientation of arcs (triods) and their images *in the absence of critical leaves*. In the case when there are critical leaves in the arcs and triods we need additional lemmas. In what follows by a *preimage collection (of a chord $\overline{ab}$)* we mean a collection A of several *pairwise disjoint* chords with the same *non-degenerate* image-chord $\overline{ab}$; here we do not necessarily assume the existence of a lamination. However if we deal with a lamination $\mathcal{L}$ then we always assume that preimage collections consist of leaves of $\mathcal{L}$ and often call them *preimage leaf collections*. If there are d disjoint chords in A we call it *full*. If X is a preimage collection of a chord $\overline{ab}$, we denote the endpoints of chords of X by the same letters as for the endpoints of their images but with a hat and distinct subscripts, and call them correspondingly (a-points, b-points etc). Finally, recall that $\partial(X)$ is the union of all endpoints of chords from X.

Lemma 3.8. *Let X be a full preimage collection of a chord $\overline{ab}$ and $\overline{\hat{a}_1\hat{b}_1}$, $\overline{\hat{a}_2\hat{b}_2}$ be two chords from X. Then the number of chords from X crossing the chord $\overline{\hat{a}_1\hat{a}_2}$ inside $\mathbb{D}$ is even if and only if either $\hat{a}_1 < \hat{b}_1 < \hat{a}_2 < \hat{b}_2$ or $\hat{a}_1 < \hat{b}_2 < \hat{a}_2 < \hat{b}_1$. In particular, if there exists a concatenation Q of chords connecting $\hat{a}_1$ and $\hat{a}_2$, disjoint with chords of X except the points $\hat{a}_1, \hat{a}_2$, then either $\hat{a}_1 < \hat{b}_1 < \hat{a}_2 < \hat{b}_2$ or $\hat{a}_1 < \hat{b}_2 < \hat{a}_2 < \hat{b}_1$.*

The fact that either $\hat{a}_1 < \hat{b}_1 < \hat{a}_2 < \hat{b}_2$ or $\hat{a}_1 < \hat{b}_2 < \hat{a}_2 < \hat{b}_1$ is equivalent to the fact that $\overline{\hat{a}_1\hat{a}_2}$ separates $\overline{\hat{a}_1\hat{b}_1} \setminus \{\hat{a}_1\}$ from $\overline{\hat{a}_2\hat{b}_2} \setminus \{\hat{a}_2\}$ in $\mathbb{D}$.

Proof. See Figure 2. First let us show that if, say, $\hat{a}_1 < \hat{b}_1 < \hat{a}_2 < \hat{b}_2$ then the number of chords from X crossing the chord $\overline{\hat{a}_1\hat{a}_2}$ inside $\mathbb{D}$ is even. Indeed, by Proposition 3.3 there are, say, k a-points and k b-points in $(\hat{b}_1, \hat{a}_2)$. Suppose that among chords of X there are m chords with both endpoints in $(\hat{b}_1, \hat{a}_2)$. Then there are $2k-2m$ a- and b-points in $(\hat{b}_1, \hat{a}_2)$ which are exactly all the endpoints of chords from X which cross $\overline{\hat{a}_1\hat{a}_2}$. inside $\mathbb{D}$. This implies that the number of chords from X crossing the chord $\overline{\hat{a}_1\hat{a}_2}$ inside $\mathbb{D}$ is even.

On the other hand, suppose that the number of chords from X crossing the chord $\overline{\hat{a}_1\hat{a}_2}$ inside $\mathbb{D}$ is even. As before, for definiteness assume that $\hat{a}_1 < \hat{b}_1 < \hat{b}_2 < \hat{a}_1$. For any $Z \subset X$ consider a function $\varphi(I, Z)$ of an arc $I \subset \mathbb{S}$, defined as the difference between the number of a-points in $Z \cap I$ and the number of b-points in $Z \cap I$ taken modulo 2. Clearly, for some k the arc $(\hat{b}_1, \hat{b}_2)$ contains k b-points and $k+1$ a-points; similarly, for some l the arc $(\hat{a}_2, \hat{a}_1)$ contains l a-points and $l+1$ b-points. Hence, $\varphi((\hat{b}_1, \hat{b}_2), X) = \varphi((\hat{a}_2, \hat{a}_1), X) = 1$.

Remove the chords from X connecting the arcs $(\hat{b}_1, \hat{b}_2)$ and $(\hat{a}_2, \hat{a}_1)$ from X to get a new set of chords Y. As we remove one such chord, we increase the value of φ on $(\hat{b}_1, \hat{b}_2)$ by 1, and we increase the value of φ on $(\hat{a}_2, \hat{a}_1)$ by 1 as well. By the assumption, to get the set Y we need to remove an even number of chords. Thus, we see that $\varphi((\hat{b}_1, \hat{b}_2), Y) = \varphi((\hat{a}_2, \hat{a}_1), Y) = 1$. However, the remaining points of $\partial(Y) \cap (\hat{b}_1, \hat{b}_2)$ are endpoints of a certain number of chords from X and hence there must be an equal number of a-points and b-points among them, a contradiction.

If there exists a concatenation Q of chords connecting $\hat{a}_1$ to $\hat{a}_2$ so that $Q \cap X = \{\hat{a}_1, \hat{a}_2\}$ then no chord of X can cross the chord $\overline{\hat{a}_1\hat{a}_2}$ inside $\mathbb{D}$. Hence the desired result follows from the first part. $\square$

To prove Lemma 3.10 we need more definitions.

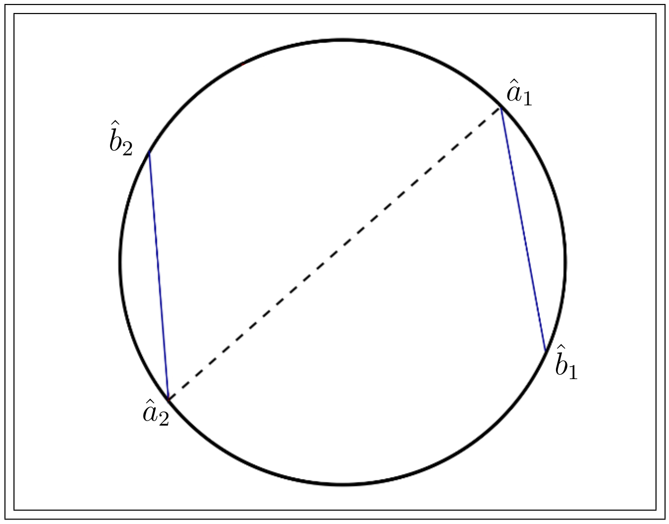

FIGURE 2. Siblings and critical leaves
Siblings must be on opposite sides of the chord $\overline{\hat{a}_1\hat{a}_2}$ which is not crossed by leaves from X.

Definition 3.9. Let $I \sqsubset \mathcal{L}^*$ be an arc (the image of a homeomorphism $\varphi : [0,1] \to I$). We call I a *monotone* arc if its endpoints $\varphi(0), \varphi(1)$ belong to $\mathbb{S}$ and there is a circle arc $T = [\varphi(0), \varphi(1)]_{\mathbb{S}}$ which contains $\partial(I)$ (this implies that the map $\varphi^{-1}|_{\partial(I)}$ is monotone with respect to the circular order on T). Likewise, a triod $T \sqsubset \mathcal{L}^*$ is *monotone* if all its edges are monotone arcs.

As an example of a monotone arc one can consider a single leaf of $\mathcal{L}$ or a subarc of the boundary of a gap of $\mathcal{L}$.

We are ready to prove the following lemma.

Lemma 3.10. *Let $\mathcal{L}$ be a sibling d-invariant lamination. Suppose that σ_d^* monotonically maps a monotone arc I onto the union of the two sides of a monotone triod T with vertex v whose central edge is a leaf $\overline{vm}$. Then there exists $\hat{v} \in \partial(I)$ such that $\sigma_d(\hat{v}) = v$ and there exists a leaf $Q = \overline{\hat{v}\hat{m}}$ such that $I \cup Q$ is a monotone triod with vertex u and central edge Q.*

Proof. Denote the endpoints of T distinct from m by a and b. Then the endpoints of I map to a and b; denote them $\hat{a}$ and $\hat{b}$, respectively. For the sake of definiteness assume that $v \in (a, b)$. Consider $\sigma_d|_{(\hat{b},\hat{a})}$. Clearly, as we move from $\hat{b}$ to $\hat{a}$ we first encounter several semi-open subarcs of $(\hat{b}, \hat{a})$ which wrap around the circle in the one-to-one fashion. Then the last arc which we encounter connects a b-point with an a-point and maps onto $[b, a]$ in the one-to-one fashion. Hence there is one more m-point in $(\hat{b}, \hat{a})$ than v-points in $(\hat{b}, \hat{a})$. This implies that one m-point belonging to $(\hat{b}, \hat{a})$ (denote it by $\hat{m}$) must be connected with a leaf to a v-point belonging to $(\hat{a}, \hat{b})$ (denote it by $\hat{v}$). This completes the proof. □

By a *polygon* we mean a finite convex polygon. In what follows by a *collapsing polygon* we mean a polygon P with edges which are chords of $\mathbb{D}$ such that their images are the same *non-degenerate* chord (thus as we walk along the edges of P, their σ_d-images walk back and forth along the same non-degenerate chord). When we say that Q is a *collapsing polygon of a lamination* $\mathcal{L}$, we mean that **all** edges of Q are leaves of $\mathcal{L}$; we also say that $\mathcal{L}$ *contains a collapsing polygon* Q. However, this does not necessarily imply that Q is a gap of $\mathcal{L}$ as Q might be further subdivided by leaves of $\mathcal{L}$ *inside* Q.

We often deal with *concatenations of leaves*, i.e. finite collections of pairwise distinct leaves which, when united, form a topological arc in $\mathbb{D}$. The concatenation of leaves $\ell_1, \dots, \ell_k$ is denoted $\ell_1 \cdots \ell_k$. If the leaves are given by their endpoints $x_1, \dots, x_k$, we denote the concatenation by $\overline{x_1 \cdots x_k}$. We do not assume that points $x_1, \dots, x_k$ are ordered on the circle; however if they are, we call $\overline{x_1 \cdots x_k}$ an *ordered* concatenation.

Lemma 3.11. *Suppose that $\mathcal{L}$ is a sibling d-invariant lamination which contains two distinct leaves $\ell_0 = \overline{vx}$ and $\ell_1 = \overline{vy}$ such that $\sigma_d(\ell_0) = \sigma_d(\ell_1) = \ell$ is a non-degenerate leaf. Then $\mathcal{L}$ contains a collapsing polygon P with edges ℓ_0 and ℓ_1 such that $\sigma_d(P) = \ell$; also, it contains a critical gap $G \subset P$ with vertex v such that $\sigma_d(G) = \ell$.*

Proof. First assume that $x < v < y$ and that there are no leaves $\ell' = \overline{vz} \in \mathcal{L}$ with $y < z < x$ and $\sigma_d(\ell') = \ell$. Since $\mathcal{L}$ is a *sibling d-invariant lamination*, we can choose a full sibling collection $A_0 \subset \mathcal{L}$ of ℓ_0. By Lemma 3.8 there exists $u_1 \in (y, x)$ such that $\overline{yu_1} \in A_0$ and ℓ_0 are siblings. Similarly, there exists a full sibling collection $A_1 \subset \mathcal{L}$ of ℓ_1 and a point $u_0 \in (y, x)$ such that $\overline{u_0x} \in A_1$ and ℓ_1 are siblings. Since $\overline{yu_1}$ and $\overline{u_0x}$ are disjoint inside $\mathbb{D}$, then $y < u_1 \le u_0 < x$.

Consider all possible choices of points u_0, u_1 so that the above properties hold: $\overline{yu_1}$ and ℓ_0 are siblings, $\overline{u_0x}$ and ℓ_1 are siblings, and $y < u_1 \le u_0 < x$. Observe that now we do not require that u_0 or u_1 be obtained as endpoints of siblings of ℓ_0 or ℓ_1 coming from full sibling collections, but the existence of such collections shows that the set of the choices is non-empty (see the first paragraph). Choose u_0, u_1 so that the arc $[u_1, u_0]$ is the shortest.

If $u_0 = u_1$ then we obtain a collapsing polygon $P = \mathrm{CH}(vyu_0x)$. Suppose that $u_0 \neq u_1$. Then by the construction and by the choice of u_0 and u_1 no leaf of $\mathcal{L}$ which maps onto ℓ can cross the chords $\overline{vu_0}, \overline{vu_1}$. By Lemma 3.8 applied to A_0 and $\ell_0 \in A_0$, and because of the location of the points found so far, there exists a sibling $\overline{w_0u_0} \in A_0$ of ℓ_0 with $w_0 \in (u_1, u_0)$. Similarly, there exists a sibling $\overline{u_1w_1} \in A_1$ of ℓ_1 with $w_1 \in (u_1, u_0)$. Since leaves $\overline{w_0u_0}$ and $\overline{u_1w_1}$ do not intersect inside $\mathbb{D}$, we see that $u_1 < w_1 \leq w_0 < u_0$. Similar to what we did before, we can choose w_1 and w_0 so that $\overline{w_0u_0}$ is a sibling of ℓ_0, $\overline{w_1u_1}$ is a sibling of ℓ_1, and the arc (w_1w_0) is the shortest possible. We can continue in this manner and obtain a collapsing polygon P with edges ℓ_0 and ℓ_1.

Now, suppose that there are leaves ℓ' between ℓ_0 and ℓ_1 with $\sigma_d(\ell') = \ell$. Let K be the collection of *all* such leaves ℓ' together with ℓ_0 and ℓ_1. By the above we can form collapsing polygons for each pair of adjacent leaves from K. If we unite them and erase in that union all leaves of K except for ℓ_0 and ℓ_1, we will get a collapsing polygon P with edges ℓ_0 and ℓ_1 (leaves of K are diagonals of P). This proves the main claim of the lemma. Let G be any gap of $\mathcal{L}$ contained in P and with edge $\overline{vx}$. Then $\sigma^*(G) = \sigma_d(\overline{vx})$ and, hence, G is critical. □

We need the following definition.

Definition 3.12. Given a leaf $\ell = \overline{xy}$, we define the corresponding *open leaf* to be $\ell^\circ = \ell \setminus \{x, y\}$. For a lamination $\mathcal{L}$, denote its critical leaves by $\bar{c}_i(\mathcal{L}) = \bar{c}_i$. Below we often consider the set $\cup_i \bar{c}_i^\circ$ which is the union of *all* open critical leaves of $\mathcal{L}$.

Lemma 3.13. *Let $\mathcal{L}$ be a sibling d-invariant lamination and $\ell = \overline{ab} \in \mathcal{L}$ be a leaf. If C is a component of $\{(\sigma_d^*)^{-1}(\ell) \setminus \cup_i \bar{c}_i^\circ\}$ and G is the convex hull of $\partial(C)$, then G is a leaf or a collapsing polygon of $\mathcal{L}$.*

Proof. If C is not a leaf then there exists $x \in \partial(C)$ which is a vertex of at least two leaves from C. Choose leaves ℓ' and ℓ'' in C which form an angle containing all other leaves from C with the endpoint x. By Lemma 3.11 ℓ' and ℓ'' are edges of a collapsing polygon P. By properties of laminations all other leaves of C with an endpoint x (if any) are diagonals of P. Choose a collapsing polygon P, which is maximal by inclusion, with edges ℓ' and ℓ''.

Let y be a vertex of P and a leaf $\overline{yz} \subset C$ come out of y and is not contained in P. By properties of a lamination then $\overline{yz}$ is in fact disjoint from P (except for y). Choose the edge ℓ''' of P with an endpoint y so that the triangle formed by the convex hull of ℓ''' and $\overline{yz}$ is disjoint from the interior of P. Then by Lemma 3.11 there exists a collapsing

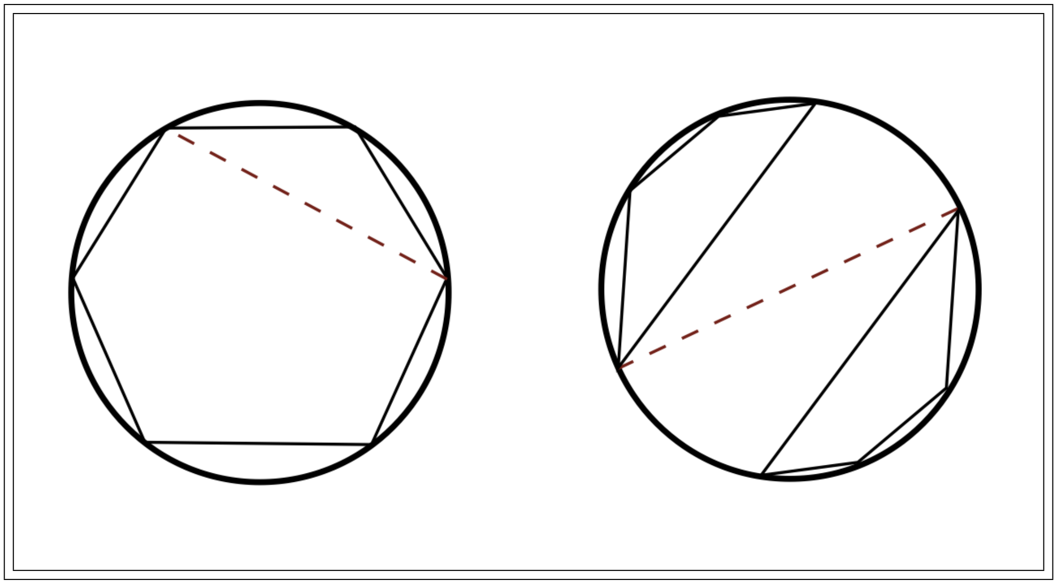

FIGURE 3. Placements of critical leaves
In each picture, the critical leaf is denoted by a dashed line. Notice that in the first example, removing the open critical leaf does not disconnect the polygon, while in the second example, removing the open critical leaf disconnects a previously connected set, resulting in two components.

polygon P' with edges $\overline{yz}$ and ℓ'''. It follows that $P \cup P'$ with ℓ''' removed is a collapsing polygon strictly containing P, a contradiction. Thus, $P = G$ as desired. □

From now on by $P_1(\ell), \dots, P_k(\ell)$, with $\ell = \overline{ab}$ non-degenerate, we mean the $\{(\sigma_d^*)^{-1}(\ell) \setminus \cup_i \overline{c}_i^\circ\}$ (leaves or collapsing polygons) from Lemma 3.13. Note that all edges of the sets $P_i(\ell)$ are leaves of $\mathcal{L}$. We view the set $(\sigma_d^*)^{-1}(\ell)$ as follows. By Definition 3.1, there is a full preimage collection L of ℓ. The endpoints of its leaves are either a-points or b-points (depending on whether they map to a or b). Often these are all the leaves mapped to ℓ. Yet, there might exist other leaves which map into ℓ. Some of these leaves map onto ℓ; a leaf like that connects an a-point from one leaf of L to a b-point from another. Some of these leaves are critical and map to a (b); a leaf like that connects two a-points (b-points) belonging to distinct leaves from L. Consider the sets $P_1(\ell), \dots, P_k(\ell)$. There might exist leaves from $(\sigma_d^*)^{-1}(\ell)$ *inside* $P_i(\ell)$, however we often ignore these leaves (which might be either critical or not). There might also exist other leaves connecting sets $P_i(\ell)$ and $P_j(\ell)$, $i \neq j$. By Lemma 3.13 such leaves must be critical.

Lemma 3.14. *Suppose that $\overline{xv_0 \cdots v_k y} = M$ is an ordered concatenation of leaves of $\mathcal{L}$ such that $\sigma_d(v_i) = w$ for all i and $\sigma_d(x) = \sigma_d(y) \neq w$. Then there exists a collapsing polygon $P_i(\sigma_d^*(\overline{xv_0}))$ which contains M.*

Proof. We see that $\overline{xv_0}$ and $\overline{v_k y}$ have the same non-degenerate image while all leaves $\overline{v_0 v_1}, \overline{v_1 v_2}, \dots$ are critical. Assume that $x = x_0 < v_0 < \cdots < v_k < y = y_0$. Applying Lemma 3.8 to the leaf $\overline{x_0 v_0}$ and its full sibling collection, we get a point $x_1, v_0 < x_1 < v_1$ and a leaf $\overline{x_1 v_1}$ which is a sibling of $\overline{x_0 v_0}$. Similarly we get points $x_2, \dots, x_k$ located between points $v_1, v_2, \dots, v_k$ and leaves $\overline{x_i v_i}$ which are siblings of $\overline{x_0 v_0}$.

Now, apply Lemma 3.11 to leaves $\overline{x_k v_k}$ and $\overline{v_k y_0}$. Then there exists a collapsing polygon P_0 with these leaves as edges. It follows that v_{k-1} is a vertex of P_0 and there is an ordered concatenation of siblings of $\overline{y_0 v_k}$ which begins with $\overline{y_0 v_k}$ and ends with some leaf $\overline{y_1 v_{k-1}}$. We can pair this leaf up with the leaf $\overline{x_{k-1} v_{k-1}}$ and apply the same arguments. In this manner we will discover a "long" ordered concatenation of siblings which begins with $\overline{y_0 v_k}$ and ends with $\overline{v_0 x_0}$. By Lemma 3.13 there exists a collapsing polygon $P_i(\sigma_d^*(\overline{xv_0}))$ from the collection of collapsing polygons described in Lemma 3.13 which contains this concatenation. It follows that $M \subset P_i(\sigma_d^*(\overline{xv_0}))$ as desired. □

A lamination is called *gap-free* if it has no gaps. In the next few lemmas we study such laminations. Lemma 3.15 is left to the reader.

Lemma 3.15. *$\mathcal{L}$ does not contain a collection of leaves with one common endpoint such that their other endpoints fill up an arc $I \subset \mathbb{S}$.*

A continuous interval map $f : I \to I$ is called a *d-sawtooth map* if it has d intervals of monotonicity of length $\frac{1}{d}$ and the slope on each such interval is $\pm d$.

Lemma 3.16. *If $\mathcal{L}$ is gap-free then it consists of a family of pairwise disjoint parallel leaves which fill up the entire disk $\overline{\mathbb{D}}$ except for two diametrically opposed points $a, b \in \mathbb{S}$. The factor map p which collapses each leaf to a point, semiconjugates σ_d to a d-sawtooth map.*

Proof. Let $\ell_0, \ell_1 \in \mathcal{L}$ be leaves with a common endpoint. By Lemma 3.15 we may assume that ℓ_0, ℓ_1 form a wedge with no leaves of $\mathcal{L}$ in it. This implies that $\mathcal{L}$ is not gap-free, a contradiction. Hence all leaves are pairwise disjoint. Consider an equivalence relation on $\mathbb{D}$ identifying every leaf into one class. The absence of gaps implies that then the quotient space is an interval I and that there are only two points $a, b \in \mathbb{S}$ which are not endpoints of leaves from $\mathcal{L}$. Moreover, if $p : \mathbb{S} \to I$ is the corresponding factor map then $p(a), p(b)$ are the endpoints of I. Moreover, all leaves of $\mathcal{L}$ must cross the chord $\overline{ab}$. Indeed, suppose that $\overline{uv}$ is a leaf of $\mathcal{L}$ such that the circular arc $I = [u, v]_{\mathbb{S}}$ contains neither a nor b. Then there must be a gap of $\mathcal{L}$ with vertices in I or a point $t \in (u, v)$ disjoint from all leaves of $\mathcal{L}$, a contradiction,

Since $\mathcal{L}$ is invariant, then either $\sigma_d(a) = a, \sigma_d(b) = b$, or $\sigma_d(a) = b, \sigma_d(b) = a$, or $\sigma_d(a) = \sigma_d(b) = a$, or $\sigma_d(a) = \sigma_d(b) = b$. Consider the case when $\sigma_d(a) = a, \sigma_d(b) = b$ (other cases are similar). Then by continuity it follows that as we travel from a to b along the leaves of $\mathcal{L}$ we meet critical leaves $\bar{c}_1, \bar{c}_2, \dots, \bar{c}_{d-1}$ (in this order). Suppose that a critical leaf $\bar{c}_i$ maps to an endpoint of a leaf $\ell \in \mathcal{L}$. Then some preimage-leaf of ℓ will meet $\bar{c}_i$ at one of its endpoints which is impossible as all leaves are pairwise disjoint. Hence all critical leaves of $\mathcal{L}$ map to a or b. On the other hand, all leaves whose images are non-disjoint from a or b, must be critical.

It follows from the first and second paragraphs of the proof that the critical leaves $\bar{c}_1 = \overline{x_1 y_1}, \dots, \bar{c}_{d-1} = \overline{x_{d-1} y_{d-1}}$ map alternately to b and a and their endpoints form the full preimages of a and b. Assume that the positive circular order is given by $b, x_{d-1}, \dots, x_1, a, y_1, \dots, y_{d-1}$. Then the arc $[x_2, a)$ maps one-to-one onto $\mathbb{S}$ and hence has length $\frac{1}{d}$ while the same applies to the arc $(a, y_2]$. Continuing in the same manner we will see that all critical leaves are in fact perpendicular to the chord $\overline{ab}$ which in fact is a diameter of $\mathbb{D}$. By pulling critical leaves back we complete the proof in the case when $\sigma_d(a) = a, \sigma_d(b) = b$. Other cases can be considered similarly. $\square$

We are ready to prove the main result of this section.

Proof of Theorem 3.2: Let G be a gap of $\mathcal{L}$. If there are two adjacent leaves on the boundary of G which have the same image, then by Lemma 3.11 the gap G maps to a leaf. Moreover, if there are two leaves in $\mathrm{Bd}(G)$ which have the same image and are connected with a finite concatenation of critical leaves in $\mathrm{Bd}(G)$ then by Lemma 3.14 again G maps to a leaf. Hence from now on we may assume that the above two cases do not take place on the boundary of G.

In particular, this implies that the σ_d^*-image of the boundary of G is not an arc. Indeed, otherwise we can choose an endpoint x of $\sigma_d^*(\mathrm{Bd}(G))$ and a point $\hat{x} \in \mathrm{Bd}(G)$ such that $\sigma_d^*(\hat{x}) = x$. Moreover, clearly x must belong to $\mathbb{S}$ and $\hat{x}$ can be chosen to belong to $\mathbb{S}$ too. It is easy to see that under the assumptions made in the first paragraph this is impossible.

Let us show that the σ_d^*-image of the boundary of G is itself the boundary of a gap. To do so, first consider the map m which collapses all critical leaves in $\mathrm{Bd}(G)$ to points and is otherwise one-to-one. Clearly $m(\mathrm{Bd}(G))$ is a simple closed curve and there exists a map g

defined on $m(\mathrm{Bd}(G))$ such that $g \circ m = \sigma_d^*|_{\mathrm{Bd}(G)}$. Let us show that g is locally one-to-one. Clearly g is locally one-to-one on the image of every non-critical leaf, and by the first paragraph g is locally one-to-one at the common endpoint of two concatenated leaves in the boundary of G. If $x \in \mathrm{Bd}(G)$ is not the endpoint of two concatenated leaves, then it follows easily from the fact that σ_d is locally one-to-one that g is locally one-to one at x as well. Set $m(\mathrm{Bd}(G)) = S$.

By Lemma 3.10, there is no monotone arc $I \subset \mathrm{Bd}(G)$ which monotonically maps to *sides* of a monotone triod $T \sqsubset \mathcal{L}^*$ (as opposed to its central edge, see Definition 3.5). Let us show that then $\sigma_d^*(\mathrm{Bd}(G))$ is the boundary of a gap. Choose a bounded component U of the complement of $\sigma_d^*(\mathrm{Bd}(G))$. The boundary $\mathrm{Bd}(U)$ of U is a simple closed curve which contains some leaves. Let $m(\ell) \subset S$ be the m-image of a leaf $\ell \subset \mathrm{Bd}(G)$ which maps by g to a leaf $\sigma_d(\ell) \subset \mathrm{Bd}(U)$. Since there exists no monotone arc $I \subset \mathrm{Bd}(G)$ which monotonically maps to *sides* of a monotone triod $T \sqsubset \mathcal{L}^*$, then, as a point continues moving along S, its g-image must move along $\mathrm{Bd}(U)$. Hence $\sigma_d^*(\mathrm{Bd}(G))$ coincides with the simple closed curve $\mathrm{Bd}(U)$. Moreover, since g is locally one-to-one, σ^* maps subarcs of $\mathrm{Bd}(G)$ monotonically onto subarcs of $\mathrm{Bd}(U)$. Hence, by Lemma 3.10, $\mathrm{Bd}(U)$ is the boundary of a gap of $\mathcal{L}$.

Let a gap H of $\mathcal{L}$ be such that $\sigma_d^*(\mathrm{Bd}(G)) = \mathrm{Bd}(H)$. To show that $\mathcal{L}$ is gap-invariant it suffices to show that $\sigma_d^*|_{\mathrm{Bd}(G)}$ can be represented as the positively oriented composition of a monotone map and a covering map. Consider the map m which collapses all critical leaves in $\mathrm{Bd}(G)$ to points and is otherwise one-to-one. Clearly, there exists a map g defined on $m(\mathrm{Bd}(G))$ such that $g \circ m = \sigma_d^*|_{\mathrm{Bd}(G)}$. We can define a circular order on $m(\mathrm{Bd}(G))$ by choosing three points $x_i \in m(\mathrm{Bd}(G))$ such that $m^{-1}(x_i)$ is a point. Then $x_1 < x_2 < x_3$ if and only if $m^{-1}(x_1) < m^{-1}(x_2) < m^{-1}(x_3)$. Let us show that g preserves orientation.

Consider a point $a \in \partial(G)$ which is not an endpoint of a critical leaf in $\mathrm{Bd}(G)$. By Corollary 3.7 (if a is an endpoint of a leaf in $\mathrm{Bd}(G)$) or because of the fact that σ_d preserves local orientation (if a is not an endpoint of a leaf from $\mathrm{Bd}(G)$ and is hence approached by points of $\partial(G)$ from the appropriate side) it follows that σ_d^* (and hence g) preserves the local orientation at all such points a.

Let us now assume that $C = \overline{x_1 x_2 \cdots x_k} \subset \mathrm{Bd}(G)$ is a maximal by inclusion concatenation of critical leaves in Bd)G). Clearly, $m(C)$ is a point x of $m(\mathrm{Bd}(G))$. Choose a monotone arc $I \subset \mathrm{Bd}(G)$ with endpoints p, q such that $p < x_1 < \cdots < x_k < q$ and $C \subset I$ (we can make these assumptions without loss of generality). If neither x_1 nor x_k is an endpoint of a non-critical leaf from $\mathrm{Bd}(G)$, it follows from the properties of σ_d that g preserves orientation. Suppose that there is a leaf $\bar{a} = \overline{x_k a} \subset I$. Choose a full sibling collection of $\overline{x_k a}$ and let $\overline{x_1 a'}$ be a leaf from this collection. By Lemma 3.8, applied repeatedly, $x_1 < a' < x_2$. It is easy to see that the map σ_d^* preserves orientation on $Q = B \cup \overline{x_1 a'}$ where B is a small subarc of $\mathrm{Bd}(G)$ with an endpoint x_1 otherwise disjoint from the leaf $\overline{x_1 x_2}$ (as before, in the case when x_1 is an endpoint of a leaf $\overline{x_1 b} \subset \mathrm{Bd}(G)$ it follows from Corollary 3.7, and in the case when x_1 is approached by points of $\partial(G)$ it follows from the fact that σ_d locally preserves orientation). Therefore g preserves orientation at $m(x_1)$ as desired. □

The basic property defining d-invariant sibling laminations is that every non-critical leaf can be extended to a collection of d pairwise disjoint leaves with the same image (a full sibling collection). We conclude this subsection by showing that this implies the same result for arcs as long as their images are monotone arcs.

Lemma 3.17. *Suppose that σ_d^* homemorphically maps an arc $A \sqsubset \mathcal{L}^*$ to a monotone arc $B \sqsubset \mathcal{L}^*$. Then there are d pairwise disjoint arcs $A_1 = A, A_2, \dots, A_d$ such that for each i the map σ_d^* homemorphically maps the arc A_i to B.*

Proof. Suppose that the endpoints of B are p, q and that $B \subset [p, q] = I$. Denote by $J_1, \dots, J_d$ circle arcs which map one-to-one to I. Since B is a monotone arc, there are no more than countably many leaves in B. We can order them so that they form a sequence of leaves $\ell_n \subset B$ with $\mathrm{diam}(\ell_n) \to 0$. Given a leaf $\ell_n \subset B$ we can choose its unique preimage $\ell_n^1 \subset A$, and then choose a full sibling collection of ℓ_n^1 consisting of leaves $\ell_n^1, \ell_n^2, \dots, \ell_n^d$. In other words, we choose full preimage collection of leaves for each leaf from B so that this preimage collection includes a leaf from A. Call ℓ_n *long* if there exists a leaf ℓ_n^j with endpoints coming from distinct sets J_r and J_s.

We claim that here are no more than finitely many long leaves in B. Indeed, suppose otherwise. By continuity we can then choose a subsequence for which we may assume that 1) there is a sequence of leaves $t_n \subset B$ which converges to a point $x \in \mathbb{S}$ from one side, and 2) there is a sequence of their pullback-leaves $\hat{t}_n$ (i.e., $\sigma_d^*(\hat{t}_n) = t_n$) which converges to a critical leaf $\hat{x}$ from one side. Clearly, this is impossible since B is a monotone arc.

Suppose that $\ell_{n_1} = \overline{a_{n_1} b_{n_1}}, \dots, \ell_{n_k} = \overline{a_{n_k} b_{n_k}}$ are all long leaves in B. Without loss of generality we may assume that $a_{n_1} < b_{n_1} \le a_{n_2} < \dots < b_{n_k}$. Denote closures of components of $B \setminus \bigcup \ell_{n_i}$ by $S_1, S_2, \dots, S_{k+1}$ numbered in the natural order on the circle so that S_1 precedes a_{n_1}, S_2

is located between b_{n_1} and a_{n_2}, etc (some of these sets may be empty, e.g. if $p = a_{n_1}$ then S_1 is empty).

Then each S_j has d pullbacks, each of which is a monotone arc $\widehat{S}_j^r$ such that $\partial(\widehat{S}_j^r) \subset I_r$. Let us consider the union of all such pullbacks with all leaves from previously chosen full preimage collections of all leaves $\ell_{n_1}, \dots, \ell_{n_k}$ (i.e., of all long leaves in B). Basically, we consider all preimage collections of leaves of B and then take the closure of their union, representing it in a convenient form. We claim that each component of this union is a monotone arc which maps onto B in a one-to-one fashion. Indeed, by the construction each such component X can be extended both clockwise and counter-clockwise until it reaches points mapped to p and q respectively. It follows from the construction that these components are pairwise disjoint and that one of them is the originally given arc $A = A_1$. This completes the proof. □

3.3. **The space of all sibling invariant laminations.** Our approach is to describe laminations in the "language of leaves". The main idea is to use sibling invariant laminations for that purpose. By Theorem 3.2 this does not push us outside the class of Thurston invariant laminations. According to the philosophy, explained in the Introduction, now we need to verify if the set of all sibling invariant laminations contains all q-laminations and is Hausdorff closed. The first step here is made in Lemma 3.18 in which we relate sibling invariant laminations and q-laminations. Observe that it is well-known (and easy to prove) that d-invariant q-laminations are Thurston d-invariant laminations.

Lemma 3.18. *d-invariant q-laminations are sibling d-invariant.*

Proof. Assume that $\sim$ is a d-invariant laminational equivalence relation. Conditions (1) and (2) of Definition 3.1 are immediate. To check condition (3) of Definition 3.1, assume that ℓ is a non-critical leaf of $\mathcal{L}_\sim$ and verify that there are d pairwise disjoint leaves $\ell = \ell_1, \dots, \ell_d$ with the same image. To do so, consider the collection $\mathcal{A}$ of all $\sim$-classes which map to the $\sim$-class of $\sigma_d(\ell)$. If a $\sim$-class $\mathfrak{g} \in \mathcal{A}$ does not contain ℓ and is not critical, then $\mathrm{Bd}(\mathfrak{g})$ contains only a unique sibling of ℓ. If $\mathfrak{g}$ is critical, it maps to the $\sim$-class of $\sigma_d(\ell)$, say, k-to-1, and we can choose k pairwise disjoint siblings leaves of ℓ on the boundary of $\mathrm{CH}(\mathfrak{g})$. If ℓ is an edge of the set $\mathrm{CH}(\mathfrak{h})$ where $\mathfrak{h}$ is a critical class we can still find the appropriate number of its pairwise disjoint siblings in the boundary of $\mathrm{CH}(\mathfrak{h})$. The endpoints of leaves from the thus created list exhaust the list of all points which are preimages of endpoints of $\sigma_d(\ell)$. Thus, we get d siblings one of which is ℓ as desired. □

We prove that all limits of q-laminations are sibling d-invariant. Thus, if there is a Thurston d-invariant lamination $\mathcal{L}$ which is not sibling d-invariant, then it is not a Hausdorff limit of q-laminations which shows that the class of Thurston d-invariant laminations is too wide if we are interested in Hausdorff limits of q-laminations. This justifies our interest in the next example illustrated on Figure 4.

Suppose that points $\hat{x}_1, \hat{y}_1, \hat{z}_1, \hat{x}_2, \hat{y}_2, \hat{z}_2$ are positively ordered on $\mathbb{S}$ and $H = \mathrm{CH}(\hat{x}_1\hat{y}_1\hat{z}_1\hat{x}_2\hat{y}_2\hat{z}_2)$ is a critical hexagon of an invariant q-lamination $\mathcal{L}$ such that $\sigma_d^* : H \to T$ maps H in the 2-to-1 fashion onto the triangle $T = \mathrm{CH}(xyz)$ with $\sigma_d(\hat{x}_i) = x, \sigma_d(\hat{y}_i) = y, \sigma_d(\hat{z}_i) = z$. Now, add to the lamination $\mathcal{L}$ the leaves $\overline{\hat{x}_1\hat{z}_1}$ and $\overline{\hat{x}_1\hat{x}_2}$ and all their pullbacks along the backward orbit of H under σ_d^*. Denote the thus created lamination $\mathcal{L}'$. It is easy to see that $\mathcal{L}'$ is Thurston d-invariant but not sibling d-invariant because $\overline{\hat{x}_1\hat{z}_1} = \ell$ cannot be completed to a full sibling collection (clearly, H does not contain siblings of ℓ).

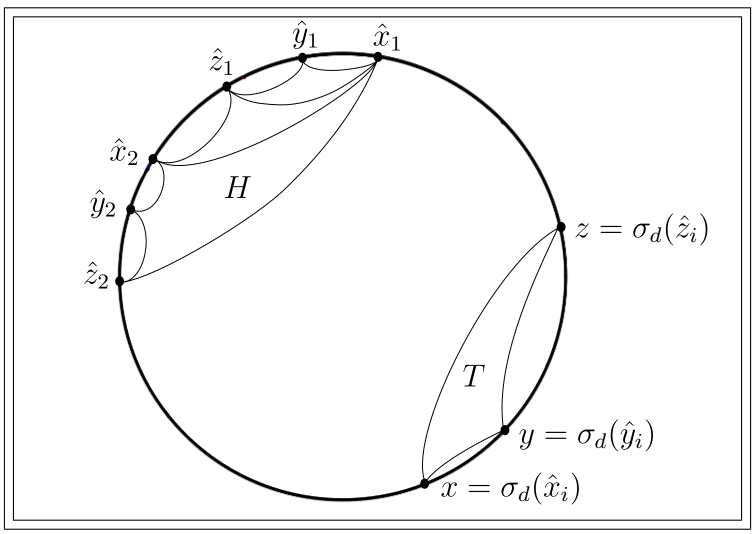

FIGURE 4. An example of a Thurston invariant lamination which is not sibling invariant. The leaf $\overline{\hat{x}_1\hat{z}_1}$ has no siblings in H.

Lemma 3.19. *Take sequences of d sibling leaves $\overline{\hat{a}_i^j\hat{b}_i^j}$, $1 \le j \le d$, $i = 1, 2, \dots$ such that $\overline{\hat{a}_i^1\hat{b}_i^1} \to \overline{\hat{a}^1\hat{b}^1} = \ell^1$, $\overline{\hat{a}_i^2\hat{b}_i^2} \to \overline{\hat{a}^2\hat{b}^2} = \ell^2, \dots, \overline{\hat{a}_i^d\hat{b}_i^d} \to \overline{\hat{a}^d\hat{b}^d} = \ell^d$ and $\sigma_d(\ell^1)$ is not degenerate. Then ℓ^j, $1 \le j \le d$ are siblings with non-degenerate image.*

Proof. By continuity, $\sigma_d(\ell^j) = \sigma_d(\ell^1)$ for all j. To show that leaves $\ell^j, 1 \le j \le d$ are pairwise disjoint consider i_0 and $\varepsilon > 0$ such that for each $i \geq i_0$ and each j, $1 \leq j \leq d$ we have that $|\overline{\hat{a}_i^j \hat{b}_i^j}| \geq \varepsilon$. Then it follows that for a fixed i the pairwise distance between points of sets $\{\hat{a}_i^j, \hat{b}_i^j\}$, for $1 \leq j \neq k \leq d$ is bounded away from 0 ($\hat{a}_i^j, \hat{a}_i^k$ cannot be too close because their images coincide and $\hat{a}_i^j, \hat{b}_i^k$ cannot be too close because their images are too far apart). Hence the leaves ℓ^j, $1 \leq j \leq d$ are disjoint as desired. □

Suppose that $\mathcal{L}$ is a prelamination. Then by its closure $\overline{\mathcal{L}}$ we mean the set of chords in $\overline{\mathbb{D}}$ which are limits of sequences of leaves of $\mathcal{L}$. It is easy to see that $\overline{\mathcal{L}}$ is a closed lamination. Corollary 3.20 shows that it is enough to verify the property of being sibling invariant on dense prelaminations. It immediately follows from Lemma 3.19 (so that it is given here without proof).

Corollary 3.20. *If $\mathcal{L}$ is a sibling d-invariant prelamination, then its closure $\overline{\mathcal{L}}$ is a sibling d-invariant lamination.*

Theorem 3.21 follows from Lemma 3.18 and Lemma 3.19.

Theorem 3.21. *The Hausdorff limit of a sequence of sibling invariant laminations is a sibling invariant lamination. The space of all sibling invariant laminations is closed in the Hausdorff sense and contains all q-laminations.*

4. Proper laminations

Clearly, not all (sibling, Thurston) invariant laminations are q-laminations (e.g., a lamination with two finite gaps with a common edge is not a q-lamination). In this section we address this issue and describe Thurston d-invariant laminations which almost coincide with appropriate q-laminations. According to the adopted approach, we use the "language of leaves" in our description.

Definition 4.1 (Proper lamination)**.** Two leaves with a common endpoint v and the same image which is a leaf (and not a point) are said to form a *critical wedge* (the point v then is said to be its vertex). A lamination $\mathcal{L}$ is *proper* if it contains no critical leaf with periodic endpoint and no critical wedge with periodic vertex.

Proper laminations are instrumental for our description of laminations which almost coincide with q-laminations.

Lemma 4.2. *Any q-lamination is proper.*

Proof. Suppose that A is either a critical wedge or a critical leaf which contains a periodic point of period n. Then A is contained in a finite class $\mathfrak{g}$ such that $|\sigma_d(\mathfrak{g})| < |\mathfrak{g}|$ while on the other hand $\sigma_d^n(\mathfrak{g})$ must coincide with $\mathfrak{g}$, a contradiction. □

The exact inverse of Lemma 4.2 fails. However it turns out that proper laminations are very close to q-laminations. To show this we need a few technical definitions.

Definition 4.3. Suppose that A is a polygon with vertices in $\mathbb{S}$. It is said to be *d-wandering* if for any $m \neq n$ we have $\mathrm{CH}(\sigma_d^m(A \cap \mathbb{S})) \cap \mathrm{CH}(\sigma_d^n(A \cap \mathbb{S})) = \emptyset$.

In Definition 4.3 we do not require that A be a part of some lamination or even that the circular orientation of vertices of A be preserved under σ_d. Still, Childers was able to generalize Kiwi's results [Kiw02] and prove in [Chi07] that A cannot have too many vertices.

Theorem 4.4 ([Kiw02, Chi07])**.** *Suppose that A is a polygon with more than d^d vertices. Then it is not d-wandering.*

In Definition 4.3 we assume that images of A have pairwise disjoint convex hulls. As Lemma 4.5 shows, one can slightly weaken this condition and still obtain useful conclusions.

Lemma 4.5. *Suppose that one the following holds.*

(1) *A is a polygon, $\partial(A) \subset \mathbb{S}$, and for any $m \neq n$ the interiors of the convex hulls $\mathrm{CH}(\sigma_d^m(A\cap\mathbb{S}))$ and $\mathrm{CH}(\sigma_d^n(A\cap\mathbb{S}))$ are disjoint;*
(2) *A is a chord of $\mathbb{S}$ and for any $m \neq n$ two image chords $(\sigma_d^*)^m(A)$ and $(\sigma_d^*)^n(A)$ are disjoint inside $\mathbb{D}$.*

Then, if $\sigma_d^n|_{\partial(A)}$ is one-to-one for all n, then either A is wandering, or A is a chord with a preperiodic endpoint.

Proof. Suppose that $A = \overline{pq}$ is a non-wandering chord and p, q have infinite orbits. We may assume that $\sigma_d^*(A) = \overline{qr}$ and $\sigma_d(p) = q$. Set $A_k = (\sigma_d^*)^k(A)$. Take closures of the two components of $\mathbb{D} \setminus A_k$. It follows from the assumptions that $\bigcup_{i>k} A_i$ is contained in one of them. Hence A_i converge to either a σ_d-fixed point on $\mathbb{S}$ or a σ_d^*-invariant chord. However this contradicts the fact that σ_d is repelling.

Suppose now that A is an non-wandering polygon. We may assume that $\sigma_d^*(A)$ and A intersect either (1) at a common vertex x, or (2) along a common edge $\ell = \overline{xy}$. Choose the vertex u of A with $\sigma_d(u) = x$ and consider the chord $\overline{ux}$. The chord $\overline{ux}$ satisfies the assumptions of the theorem and is non-wandering. Then by the above u is preperiodic. We may assume that u is fixed. Consider the two edges $\overline{uv_0}$ and $\overline{uw_0}$ of A and the arc $[v_0, w_0]$ in $\mathbb{S}$ not containing u. Similarly, for each n set $v_n = \sigma_d^n(v_0)$ and $w_n = \sigma_d^n(w_0)$. Since the interiors of the convex hulls of $\sigma_d^n(A)$ are pairwise disjoint, the open arcs (v_n, w_n) are also pairwise disjoint and hence their diameter must converge to zero, a contradiction with the fact that σ_d is expanding. $\square$

Lemma 4.6. *Suppose that $\mathcal{L}$ is a Thurston invariant lamination. Then there are at most finitely many points x such that there is a critical leaf with an endpoint x or a critical wedge with a vertex x.*

Proof. Clearly there are at most finitely many critical leaves. Hence we may suppose that there are infinitely many points x_i such that there are leaves $\overline{a_i x_i}$, $\overline{b_i x_i}$ with $\sigma_d(a_i) = \sigma_d(b_i) \neq \sigma_d(x_i)$; we may assume that the sets $\overline{\sigma_d(a_i)\sigma_d(x_i)}$ are all distinct and the points $\sigma_d(x_i)$ are all distinct. Clearly, the chords $\overline{a_i b_i}$ and $\overline{a_j b_j}$ are disjoint inside $\mathbb{D}$. Since each such chord is critical, we have a contradiction. $\square$

Let $E(v)$ be the set of *all* endpoints of leaves with a common endpoint v (if $E(v)$ accumulates upon v we include v in $E(v)$). Then $E(v)$ is a closed set. Let $C(v)$ be the family of *all* leaves connecting v and a point of $E(v)$ (it might include $\{v\}$ as a degenerate leaf).

Lemma 4.7. *Suppose that v is a point with infinite orbit and $\mathcal{L}$ is a Thurston d-invariant lamination. Then there are at most finitely many leaves with an endpoint v.*

Proof. Let $E(v)$ be infinite. By Lemma 4.6, we may assume that v and all its images are not endpoints of critical leaves or vertices of critical wedges. If v ever maps to $E(v)$ then by Lemma 4.5 v is preperiodic, a contradiction. Choose $A \subset E(v)$ consisting of d^d points. Consider $\mathrm{CH}(A \cup \{v\})$. By the above for any $n \neq m$, the interiors of $\mathrm{CH}(\sigma_d^n(A \cup \{v\}))$ and $\mathrm{CH}(\sigma_d^m(A \cup \{v\}))$ are disjoint. Then by Lemma 4.5 $\mathrm{CH}(A \cup \{v\})$ is wandering, contradicting Lemma 4.4. $\square$

Lemma 4.8. *Suppose that $\mathcal{L}$ is a Thurston d-invariant lamination and A is a concatenation of leaves $A = \bigcup \overline{x_i x_{i+1}}$, $i = 0, 1, \dots,$ with $\overline{x_i x_{i+1}} \in \mathcal{L}$ (the set $\partial(A)$ is infinite). Then A has preperiodic vertices.*

Proof. Consider the convex hulls of sets $\overline{\partial(A)} = B_0$, $\overline{\partial(\sigma_d(A))} = B_1, \dots$. Suppose that all such convex hulls have disjoint interiors. There are numbers n such that $\sigma_d|_{B_n}$ is not one-to-one. This means that there is a critical chord ℓ_n inside the convex hull of B_n. Since we assume that it is the interiors of sets $\mathrm{CH}(B_n)$ which are disjoint, one critical chord can correspond to at most two sets B_n; otherwise two critical chords ℓ_n and ℓ_m cannot intersect. It follows that there are at most finitely many critical chords ℓ_i constructed as above and that for large enough n the map $\sigma_d|_{B_i}$ is one-to-one. By Lemma 4.5 this is impossible. Hence convex hulls of sets B_n have non-disjoint interiors which implies that we can make the following assumption: there exists m, n such that $\sigma_d^m(x_0) = x_n$. We may also assume that $x_n \neq x_0$.

Denote the concatenation of leaves $\overline{x_0x_1}, \dots, \overline{x_{n-1}x_n}$ by C. Then $\sigma_d(C)$ is a concatenation of leaves attached to C etc. If for some k we have that $\sigma_d^k(C)$ is a point we can choose the minimal such k which implies that $\sigma_d^{k-1}(C)$ is a concatenation of leaves whose image is one of its own vertices y. Hence y is periodic as desired. Otherwise we may assume that the number of vertices of C does not drop under application of the map σ_d. Observe that C may have self intersections. In this case we may refine C to get a concatenation with no self-intersections still connecting x_0 and x_n.

We can optimize the situation even more. Indeed, it is not necessarily so that C only intersects itself when $\sigma_d(C)$ gets concatenated to C at $\sigma_d(x_0) = x_n$. Thus we may assume that C is the shortest subchain of leaves in C which ever intersects itself. This implies that if there are no preperiodic vertices of C then the only way images of C may intersect is by being concatenated to each other at their ends. It now follows that $\lim \sigma_d^n(C)$ is either a leaf in $\mathcal{L}$ or a point of $\mathbb{S}$. In either case this contradicts that σ_d is expanding. □

Recall that $\approx_{\mathcal{L}}$ was the equivalence relation defined by $x \approx_{\mathcal{L}} y$ if and only if there exists a *finite* concatenation of leaves of $\mathcal{L}$ connecting x and y. Theorem 4.9 specifies properties of $\approx_{\mathcal{L}}$.

Theorem 4.9. *Let $\mathcal{L}$ be a proper Thurston invariant lamination. Then $\approx_{\mathcal{L}}$ is an invariant laminational equivalence relation.*

Proof. Let us show that any point $v \in \mathbb{S}$ is the endpoint of at most finitely many leaves of $\mathcal{L}$. Otherwise by Lemma 4.7 we may assume that v is fixed. Take the infinite invariant set $E' = E(v) \cup \{v\}$. Since σ_d is expanding, E' contains points x, x' with $\sigma_d(x) = \sigma_d(x')$ contradicting the fact that $\mathcal{L}$ is proper.

Suppose next that A is an infinite concatenation of leaves $A = \bigcup \overline{x_ix_{i+1}}$, $i = 0, 1, \dots$, with $\overline{x_ix_{i+1}} \in \mathcal{L}$. By Lemma 4.8 we may assume that $x_0 = x$ is fixed. Let us show that if $\ell_1 = \overline{xe_1}, \dots, \ell_k = \overline{xe_k}$ are *all* the leaves with the endpoint x then $\sigma_d(e_i) = e_i$ for all i. Since $\mathcal{L}$ is proper, the leaves $\ell_1, \dots, \ell_k$ have distinct non-degenerate σ_d^*-images. Hence all points $e_i, 1 \leq i \leq k$ are periodic. If $k = 1$ then e_1 is fixed and we are done. If $k = 2$ then ℓ_1, ℓ_2 are edges of some gap G and the fact that the orientation is preserved under σ_d implies that both ℓ_1, ℓ_2 are fixed. Suppose that $k \geq 3$. We may assume that $e_1 < e_2 < \dots < e_k$. Since gaps map to gaps and the orientation is preserved on them, the fact that x is fixed implies that then $\sigma_d(e_1) < \sigma_d(e_2) < \cdots < \sigma_d(e_k)$ and hence in fact $\sigma_d(e_i) = e_i, 1 \le i \le k$. Thus, the leaf $\overline{x_0x_1}$ is fixed, the leaf $\overline{x_1x_2}$ is fixed and, by induction, all the leaves $\overline{x_ix_{i+1}}$ are fixed, a contradiction.

By the above $\mathcal{L}$ contains no infinite cones and no infinite concatenations of leaves. Let us show that all $\approx_{\mathcal{L}}$-classes are finite (and hence closed). Otherwise let E be an infinite $\approx_{\mathcal{L}}$-class and let $x_0 \in E$. For each $y \in E$ fix a concatenation of leaves L_y from x_0 to y containing the least number of leaves. Then there are infinitely many sets $L_y, y \in E$. Since there are only finitely many leaves of $\mathcal{L}$ with an endpoint x_0, we can choose $x_1 \in E$ so that there are infinitely many sets $L_y, y \in E$ whose first leaf is $\overline{x_0x_1}$. Since there only finitely many leaves of $\mathcal{L}$ with an endpoint x_1 we can choose $x_2 \in E$ so that there are infinitely may sets $L_y, y \in E$ whose second leaf is $\overline{x_1x_2}$. Continuing in this manner we will find an infinite concatenation of leaves of $\mathcal{L}$, a contradiction (by the choice of sets $L_y, y \in E$ the points $x_0, x_1, \dots$ are all distinct).

Take convex hulls of $\approx_{\mathcal{L}}$-classes. Clearly, these convex hulls are pairwise disjoint. It follows that if a non-constant sequence of $\approx_{\mathcal{L}}$-classes converges, then it converges to a leaf of $\mathcal{L}$ or a point. Hence $\approx_{\mathcal{L}}$ is a closed equivalence relation. To show that $\approx_{\mathcal{L}}$ is invariant and laminational we have to prove that $\approx_{\mathcal{L}}$-classes map *onto* $\approx_{\mathcal{L}}$-classes in a covering way (i.e., we need to check conditions (D1) and (D3) of Definition 2.2). Let us show that for any $x \in \mathbb{S}$ the $\approx_{\mathcal{L}}$-class maps onto the $\approx_{\mathcal{L}}$-class of $\sigma_d(x)$. Indeed, let y belong to the $\approx_{\mathcal{L}}$-class of $\sigma_d(x)$. Choose a finite concatenation $\ell_1\ell_2\dots\ell_k$ of leaves connecting $\sigma_d(x)$ and y (here x is an endpoint of ℓ_1 and y is an endpoint of ℓ_k). Take a pullback-leaf $\overline{xx_1}$ of ℓ_1 with an endpoint x, then a pullback-leaf $\overline{x_1x_2}$ of ℓ_2, etc until we get a finite concatenation of leaves connecting x and some point y' with $\sigma_d(y') = y$. This implies that the $\approx_{\mathcal{L}}$-class maps onto the $\approx_{\mathcal{L}}$-class of $\sigma_d(x)$ as desired.

It remains to prove that $\approx_{\mathcal{L}}$ satisfies condition (D3) from Definition 2.2 (i.e., that σ is covering on $\approx_{\mathcal{L}}$-classes). Observe that if $\mathcal{L}$ is sibling invariant, this immediately follows from Corollary 3.7. In the case when $\mathcal{L}$ is Thurston invariant we need an extra argument. So, suppose that $\mathfrak{g}$ is a $\approx_{\mathcal{L}}$-class. Some edges of $\mathrm{CH}(\mathfrak{g})$ may well be leaves of $\mathcal{L}$. If ℓ is an edge of $\mathrm{CH}(\mathfrak{g})$ which is *not* in $\mathcal{L}$ then on the side of ℓ, opposite to that where $\mathfrak{g}$ is located, there must lie an infinite gap of $\mathcal{L}$. It is easy to see that if we now add ℓ with its grand orbit, we will get a Thurston invariant lamination. Hence we may assume from the very beginning, that all edges of $\mathrm{CH}(\mathfrak{g})$ are leaves of $\mathcal{L}$.

If $|\mathfrak{g}| = 2$ then (D3) is automatically satisfied. Otherwise let ℓ be an edge of $\mathrm{CH}(\mathfrak{g})$. Then either (1) ℓ is approached from the outside of

$\mathrm{CH}(\mathfrak{g})$ by leaves ℓ_i of $\mathcal{L}$, or (2) there is an infinite gap G on the other side of ℓ, opposite to the side where $\mathrm{CH}(\mathfrak{g})$ is located. Below we refer to these as cases (1) and (2).

Let us now show that if one of the edges of $\mathrm{CH}(\mathfrak{g})$ is critical, then all are critical. Indeed, let $\ell = \overline{ab}$ be a critical edge of $\mathrm{CH}(\mathfrak{g})$. In case (1) images of ℓ_i separate $\sigma_d(\ell)$ from the rest of the circle, hence all points of $\mathfrak{g}$ map to the same point. In case (2) both endpoints of ℓ are limit points of vertices of G because otherwise we could extend the $\approx_\mathcal{L}$ class $\mathfrak{g}$. Since $\mathcal{L}$ is Thurston invariant we conclude that $\sigma_d(\ell)$ is a vertex of an infinite gap $\sigma_d(G)$, approached from either side on $\mathbb{S}$ by vertices of $\sigma_d(G)$. Hence no leaves can come out of $\sigma_d(\ell)$ and again all points of $\mathfrak{g}$ map to the same point. Clearly, in this case (D3) from Definition 2.2 is satisfied.

Suppose now that all edges of $\mathrm{CH}(\mathfrak{g})$ are non-critical. We claim that if ℓ is an edge of $\mathrm{CH}(\mathfrak{g})$ then $\sigma_d(\ell)$ is an edge of $\mathrm{CH}(\sigma_d(\mathfrak{g}))$. Indeed, in the case (1) $\sigma_d(\ell)$ is approached by leaves of $\mathcal{L}$ from one side and in the case (2) it borders an infinite gap of $\mathcal{L}$ from one side. In either case it cannot be a diagonal of the gap $\mathrm{CH}(\sigma_d(\mathfrak{g}))$, and the claim is proved.

It remains to show that as we walk along the boundary of $\mathrm{CH}(\mathfrak{g})$, the σ_d-image of the point walks in the positive direction along the boundary of $\mathrm{CH}(\sigma_d(\mathfrak{g}))$. Indeed, suppose first that $\sigma_d(\mathfrak{g})$ consists of two points. Then by the above there are no critical edges of $\mathrm{CH}(\mathfrak{g})$, and the condition we want to check is automatically satisfied. Otherwise let $\mathrm{CH}(\sigma_d(\mathfrak{g}))$ be a gap. Let $\overline{ab}$ be an edge of $\mathrm{CH}(\mathfrak{g}))$ such that moving from a to b along $\overline{ab}$ takes place in the *positive* direction on the boundary of $\mathrm{CH}(\mathfrak{g})$. Suppose that moving from $\sigma_d(a)$ to $\sigma_d(b)$ along $\overline{\sigma_d(a)\sigma_d(b)}$ takes place in the *negative* direction on the boundary of $\mathrm{CH}(\sigma_d(\mathfrak{g}))$. Then the properties of Thurston laminations imply that in the case (1) images of leaves ℓ_i will have to cross $\mathrm{CH}(\sigma_d(\mathfrak{g}))$, a contradiction. On the other hand, in the case (2) they would imply that the image of the infinite gap G contains $\mathrm{CH}(\sigma_d(\mathfrak{g}))$, a contradiction again. Hence the map is positively oriented on $\mathrm{Bd}(\mathrm{CH}(\mathfrak{g}))$ as desired. □

Theorem 4.9 shows that, up to a "finite" restructuring, a lamination is a q-lamination if and only it is proper; the appropriate claim is made in Corollary 4.10 whose proof is left to the reader.

Corollary 4.10. *A proper Thurston invariant lamination $\mathcal{L}$ is a q-lamination if and only if for each $\approx_\mathcal{L}$-class $\mathfrak{g}$ the edges of its convex hull $\mathrm{CH}(\mathfrak{g})$ belong to $\mathcal{L}$ while no leaf of $\mathcal{L}$ is contained in the interior of $\mathrm{CH}(\mathfrak{g})$.*

5. Clean laminations

Thurston defined clean laminations. In this section we show that every clean Thurston invariant lamination is a proper sibling invariant lamination; thus, up to a minor modification every clean Thurston invariant lamination is a q-lamination. We show in the next section that every clean Thurston 2-invariant lamination is a q-lamination.

Definition 5.1. Let $\mathcal{L}$ be a lamination. Then $\mathcal{L}$ is *clean* if no point of $\mathbb{S}$ is the common endpoint of three distinct leaves of $\mathcal{L}$.

Theorem 5.2. *Let $\mathcal{L}$ be a Thurston d-invariant clean lamination. Then $\mathcal{L}$ is a proper sibling d-invariant lamination.*

Proof. Let $\mathcal{L}$ be a clean Thurston d-invariant lamination. Suppose first that $\mathcal{L}$ contains a critical leaf $\overline{xy}$ with a periodic endpoint. Assume that x is fixed. Then there must exist d disjoint leaves which map to $\overline{xy}$. One of these must have x as an endpoint. Label this leaf $\overline{xz}$ (since $\sigma_d^*(\overline{xy}) = x$, $y \neq z$). Similarly there must exist d leaves which map to $\overline{xz}$ one of which must be $\overline{xw}$ (and, as above, all three leaves are distinct). Hence $\mathcal{L}$ is not clean, a contradiction. The case when $\mathcal{L}$ contains a critical wedge is similar. Thus, $\mathcal{L}$ is proper.

Suppose next that $\ell = \overline{xy} \in \mathcal{L}$ and $\sigma_d(\ell)$ is non-degenerate. To show that $\mathcal{L}$ is sibling d-invariant we need to show that there are $d-1$ siblings of ℓ. Since $\mathcal{L}$ is a Thurston d-invariant lamination, there exists a collection B of d pairwise disjoint leaves $\ell_1, \dots, \ell_d$ so that $\sigma_d(\ell_i) = \sigma_d(\ell)$ for all i. If $\ell = \ell_i$ for some i we are done. Otherwise there exist $i \neq j$ so that $\ell_i \cap \ell \neq \emptyset \neq \ell_j \cap \ell$. Let $\ell_i = \overline{xz}, \ell_j = \overline{yt}$ and consider two cases.

(1) Points z and t are located in distinct components of $\mathbb{S} \setminus \{x, y\}$. Then ℓ_i and ℓ are edges of a certain gap G because $\mathcal{L}$ is clean. Since $\sigma_d^*|_{\mathrm{Bd}(G)}$ is positively oriented in case $\mathrm{CH}(\sigma_d(\partial G))$ is a gap, G must be a finite gap of $\mathcal{L}$, collapsing to a leaf. Hence there exists an edge of G with an endpoint y, contradicting the assumption that $\mathcal{L}$ is clean.

(2) Points z and t belong to the same component of $\mathbb{S} \setminus \{x, y\}$. Similar to (1), there exists a gap G with edges ℓ_i, ℓ, ℓ_j (and possibly other edges), collapsed onto $\sigma_d(\ell)$ under σ_d. Since $\mathcal{L}$ is clean, every leaf of $\mathcal{L}$, which intersects G, is contained in $\mathrm{Bd}(G)$. Hence $\mathrm{Bd}(G)$ consists of $2n$ leaves all of which map to $\sigma_d(\ell)$, and, possibly, some critical leaves.

Let us show that there are no critical edges of G. Suppose that $\overline{uv}$ is a critical edge of G such that all vertices of G are contained in the circle arc $I = [v, u]$. Each leaf of $\mathcal{L}$ close to $\overline{uv}$ and with endpoint from (u, v) will have the image which crosses $\sigma_d(\ell)$. Hence there are no such leaves and $\overline{uv}$ is an edge of a gap H whose vertices belong to $[u, v]$. Since $\mathcal{L}$ is clean, there are no edges of H through u or v except for $\overline{uv}$. Hence there exist sequences $u_i \in \partial(H)$ converging to u and $v_i \in \partial(H)$ converging to v. Then points $\sigma_d(u_i)$ and $\sigma_d(v_i)$ are on opposite sides of $\sigma_d(u)$. It follows that the leaf $\sigma_d(\ell)$ cuts the image of H, a contradiction with the assumption that $\mathcal{L}$ is a Thurston d-invariant lamination. Thus, $\mathrm{Bd}(G)$ consists of $2n$ leaves all of which map to $\sigma_d(\ell)$.

This implies that in the collection $\{\ell_1, \dots, \ell_d\} = B$ there are exactly n edges of G; denote their collection by A. Since $\mathcal{L}$ is clean, for each k either $\ell_k \cap G = \emptyset$ or $\ell_k \subset \mathrm{Bd}(G)$; hence there are $d - n$ leaves in the collection $\ell_1, \dots, \ell_d$ which are disjoint from G. Now, starting with ℓ, select n disjoint siblings of ℓ from $\mathrm{Bd}(G)$ and unite them with leaves from $B \setminus A$ to get a full set of siblings of ℓ. As this can be done for any ℓ, we see that $\mathcal{L}$ is sibling d-invariant. □

Suppose that $\mathcal{L}$ is a clean Thurston d-invariant lamination and let $\approx_{\mathcal{L}}$ be the equivalence relation defined in Definition 2.3; by Theorem 5.2 $\approx_{\mathcal{L}}$ is a d-invariant laminational equivalence relation. By Corollary 4.10 and since $\mathcal{L}$ is clean, $\mathcal{L}$ is a q-lamination if and only if every chord in the boundary of the convex hull of an equivalence class of $\approx_{\mathcal{L}}$ is a leaf of $\mathcal{L}$. We further study the possible difference between the two laminations. For an equivalence class $\mathfrak{g}$, denote by $A_{\mathfrak{g}}$ the union of all leaves of $\mathcal{L}$ which join points of $\mathfrak{g}$. Since $\mathcal{L}$ is clean, each $A_{\mathfrak{g}}$ is either a point, a simple closed curve, a single leaf, or a an arc which contains at least two leaves. In all but the last case all leaves of $\mathcal{L}$ which are contained in $A_{\mathfrak{g}}$ are also leaves of $\mathcal{L}_{\approx_{\mathcal{L}}}$. It follows that $[\mathcal{L} \setminus \mathcal{L}_{\approx_{\mathcal{L}}}] \cup [\mathcal{L}_{\approx_{\mathcal{L}}} \setminus \mathcal{L}]$ is contained in the countable union of the convex hulls of equivalence classes $\mathfrak{g}_i$ so that $A_{\mathfrak{g}_i}$ is an arc containing at least two leaves. We further specify this set in Corollary 5.3.

Corollary 5.3. *Let $\mathcal{L}$ be a clean Thurston d-invariant lamination and $\mathfrak{g}$ an equivalence class of $\approx_{\mathcal{L}}$ such that $A_{\mathfrak{g}}$ is an arc which contains at least two leaves of $\mathcal{L}$. Suppose that $\overline{ab} \subset \mathrm{CH}(\mathfrak{g})$. Then if $\ell = \overline{ab} \in \mathcal{L}_{\approx_{\mathcal{L}}} \setminus \mathcal{L}$, then there exists an infinite gap U of $\mathcal{L}$ so that $\ell \setminus \{a, b\}$ is contained in the interior of U and the subarc of A which connects a and b is a maximal concatenation of leaves in $\mathrm{Bd}(U)$. Vice versa, if $\ell = \overline{ab} \in \mathcal{L} \setminus \mathcal{L}_{\approx_{\mathcal{L}}}$, then $\ell \setminus \{a, b\}$ is contained in the interior of $\mathrm{CH}(\mathfrak{g})$ and ℓ is the intersection of two infinite gaps of $\mathcal{L}$.*

Proof. Suppose that $\overline{ab} \subset \mathrm{CH}(\mathfrak{g})$. and $\ell = \overline{ab} \in \mathcal{L}_{\approx_{\mathcal{L}}} \setminus \mathcal{L}$. Since $\mathfrak{g}$ is finite, no leaf of $\mathcal{L}$ can intersect the chord ℓ inside $\mathbb{D}$ and there exists a gap U of $\mathcal{L}$ such that $\ell \setminus \{a, b\}$ is contained in the interior of U. If U is finite, then $\mathrm{Bd}(U) \subset A_{\mathfrak{g}}$, a contradiction. Since $\mathcal{L}$ is clean, the subarc $[a, b]_{A_{\mathfrak{g}}}$ of $A_{\mathfrak{g}}$ is contained in the boundary of U. Moreover, since ℓ is an edge of $\mathrm{CH}(\mathfrak{g})$, $[a, b]_{A_{\mathfrak{g}}}$ is a maximal concatenation of leaves in $\mathrm{Bd}(U)$.

Conversely, suppose that $\ell = \overline{ab} \in \mathcal{L} \setminus \mathcal{L}_{\approx_{\mathcal{L}}}$. Then $\ell \setminus \{a, b\}$ is contained in the interior of $\mathrm{CH}(\mathfrak{g})$. Hence ℓ is isolated and there exist two gaps U, V of $\mathcal{L}$ so that $\ell = U \cap V$. If one of these gaps is finite, then its boundary is a subset of $A_{\mathfrak{g}}$, a contradiction. □

6. Quadratic invariant laminations

In this section we study quadratic laminations. First we show that Corollary 4.10 can be made more precise in the quadratic case.

If a 2-invariant q-lamination $\mathcal{L}$ has a finite critical gap L then one can insert a critical diameter connecting two vertices of L and then pull it back along the backward orbit of L. Also, if L has six vertices or more, one can insert a critical (collapsing) quadrilateral inside L and then pull it back along the backward orbit of L; one can also insert in L a quadrilateral which itself splits into two triangles by a diameter and then pull it back along the backward orbit of L. In this way one can create proper sibling invariant laminations which are not q-laminations. In fact, a lamination may already exhibit the above described phenomena. Thus, if a lamination contains a finite critical polygon L which contains a critical leaf (collapsing quadrilateral) in the interior of its convex hull, then we say that it has a *critical splitting by a leaf* (resp. *quadrilateral*). Corollary 6.1 shows that these two mechanisms are *the only* ways a proper quadratic lamination can be a non-q-lamination.

Corollary 6.1. *A quadratic sibling invariant lamination is a q-lamination if and only if it is proper and does not have a critical leaf (quadrilateral) splitting.*

Proof. Clearly every q-lamination is proper and has no critical splitting (leaf or quadrilateral). Assume next that $\mathcal{L}$ is a proper sibling invariant lamination which does not have a critical splitting (leaf or quadrilateral). Define $\approx_{\mathcal{L}}$ as in Definition 2.3. Let us show that for each $\approx_{\mathcal{L}}$-class $\mathfrak{g}$ the edges of its convex hull $\mathrm{CH}(\mathfrak{g})$ belong to $\mathcal{L}$. Suppose that for a $\approx_{\mathcal{L}}$-class $\mathfrak{g}$ there is an edge of $\mathrm{CH}(\mathfrak{g})$ not included in $\mathcal{L}$. By definition, there are finite concatenations of edges of $\mathcal{L}$, connecting all points of $\mathfrak{g}$. Hence $\mathrm{CH}(\mathfrak{g})$ cannot be a leaf and $\mathfrak{g}$ consists of more than two points. Then by Thurston's No Wandering Triangles Theorem [Thu09] $\mathfrak{g}$ is either (pre)periodic or (pre)critical (observe that $\mathfrak{g}$ can first map into a critical class of $\approx_{\mathcal{L}}$ and then into a periodic class of $\approx_{\mathcal{L}}$, but not vice versa because $\mathcal{L}$ is proper).

Consider cases. Suppose that $\mathfrak{g}$ is (pre)periodic but not (pre)critical. Then for some n the $\approx_{\mathcal{L}}$-class $\sigma_2^n(\mathfrak{g})$ is periodic. By an important result of [Thu09] the edges of $\mathrm{CH}(\sigma_2^n(\mathfrak{g}))$ form one periodic orbit of edges. Since at least one of them is in $\mathcal{L}$, they all are in $\mathcal{L}$. Since $\mathfrak{g}$ maps onto $\sigma_2^n(\mathfrak{g})$ one-to-one by our assumptions, and because $\mathcal{L}$ is a sibling (and hence, by Theorem 3.2, a Thurston) invariant lamination, then all edges of $\mathrm{CH}(\mathfrak{g})$ are in $\mathcal{L}$ as desired.

Now, suppose that $\mathfrak{g}$ is precritical and $\sigma_2^n(\mathfrak{g})$ is critical. Again, we may assume that $\mathrm{CH}(\mathfrak{g})$ is not a leaf. Since $\sigma_2^n(\mathfrak{g})$ is a critical $\approx_{\mathcal{L}}$-class, it must have $2k$-edges and must map onto its image two-to-one. It follows that the edges of $\sigma_2^n(\mathfrak{g})$ are limits of sequences of $\approx_{\mathcal{L}}$-classes. Indeed, otherwise there are gaps of $\approx_{\mathcal{L}}$ sharing common edges with $\sigma_2^n(\mathfrak{g})$. By construction this would mean that these gaps are infinite and hence a forward image of one of these gaps is a critical $\approx_{\mathcal{L}}$-class. Since we deal with quadratic laminations and $\mathfrak{g}$ is also critical, it is easy to see that this is impossible. Thus, the edges of $\sigma_2^n(\mathfrak{g})$ are limits of sequences of $\approx_{\mathcal{L}}$-classes which implies that edges of $\sigma_2^n(\mathfrak{g})$ are leaves of $\mathcal{L}$. As before, since $\mathcal{L}$ is a Thurston invariant lamination, then all edges of $\mathfrak{g}$ are leaves of $\mathcal{L}$. Thus, in any case if $\mathfrak{g}$ is a $\approx_{\mathcal{L}}$-class then its edges are leaves of $\mathcal{L}$.

It remains to show that $\mathrm{CH}(\mathfrak{g})$ cannot contain any leaves of $\mathcal{L}$ in its interior. Indeed, suppose otherwise. We may assume that $\mathfrak{g}$ has at least 4 vertices. Suppose that $\mathfrak{g}$ is (pre)critical and $\sigma_2^n(\mathfrak{g})$ is critical. Let us show that any leaf inside $\sigma_2^n(\mathfrak{g})$ must have the image which is an edge or a vertex of $\sigma_2^{n+1}(\mathfrak{g})$. Indeed, it suffices to consider the case when $\sigma_2^n(\mathfrak{g})$ has at least six vertices and $\sigma_2^{n+1}(\mathfrak{g})$ is a gap. By No Wandering Triangles Theorem [Thu09] it is (pre)periodic and $\sigma_2^{n+m}(\mathfrak{g})$ is periodic. By the above quoted result of [Thu09] the edges of $\mathrm{CH}(\sigma_2^{n+m}(\mathfrak{g}))$ form one periodic orbit of edges. Hence if there is a leaf of $\mathcal{L}$ inside $\mathrm{CH}(\sigma_2^{n+m}(\mathfrak{g}))$, it will cross itself under the appropriate power of σ_2, a contradiction. Thus, any leaf inside $\sigma_2^n(\mathfrak{g})$ must have the image which is an edge or a vertex of $\sigma_2^{n+1}(\mathfrak{g})$.

We show next that such a leaf cannot exist. In other words, since $\mathcal{L}$ does not admit a critical leaf (quadrilateral) splitting, we need to show that no other splitting of $\mathrm{CH}(\sigma_2^n(\mathfrak{g}))$ by leaves of $\mathcal{L}$ is possible either. Indeed, suppose that there are leaves of $\mathcal{L}$ inside $\mathrm{CH}(\sigma_2^n(\mathfrak{g}))$. It cannot be just one critical leaf as then $\mathcal{L}$ would admit a critical leaf splitting. Neither can it be a quadrilateral or a quadrilateral with a critical leaf inside (because $\mathcal{L}$ does not admit a critical quadrilateral splitting). Now, suppose that there is a unique leaf ℓ of $\mathcal{L}$ inside $\sigma_2^n(\mathfrak{g})$ such that $\sigma_2(\ell)$ is an edge of $\sigma_2^{n+1}(\mathfrak{g})$. Then it has to have a sibling leaf which will also be a leaf inside $\sigma_2^n(\mathfrak{g})$. Hence $\sigma_2^n(\mathfrak{g})$ contains a collapsing quadrilateral, a contradiction. As these possibilities exhaust all possibilities for leaves inside $\mathrm{CH}(\sigma_2^n(\mathfrak{g}))$, it follows that there are no leaves inside $\mathrm{CH}(\sigma_2^n(\mathfrak{g}))$ and hence no leaves inside $\mathrm{CH}(\mathfrak{g})$ as desired. □

Corollary 6.2. *Suppose that $\mathcal{L}$ is a clean Thurston 2-invariant lamination. Then $\mathcal{L}$ is a q-lamination.*

Proof. Suppose that $\mathcal{L}$ is a clean, Thurston 2-invariant lamination. By Theorem 5.2, $\mathcal{L}$ is proper and sibling invariant. Moreover, since $\mathcal{L}$ is clean, it does not have a critical leaf (quadrilateral) splitting. Hence the result follows from Corollary 6.1. □

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