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# Wandering triangles exist

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**Abstract**

W.P. Thurston introduced closed $\sigma_d$-invariant laminations (where $\sigma_d = z^d : S^1 \to S^1$, $d \geqslant 2$) as a tool in complex dynamics. He defined *wandering triangles* as triples $T \subset S^1$ such that $\sigma_d^n(T)$ consists of three distinct points for all $n \geqslant 0$ and the convex hulls of all the sets $\sigma_d^n(T)$ in the plane are pairwise disjoint, and proved that $\sigma_2$ admits no wandering triangles. We show that for every $d \geqslant 3$ there exist uncountably many $\sigma_d$-invariant closed laminations with wandering triangles and pairwise non-conjugate factor maps of $\sigma_d$ on the corresponding quotient spaces. ***To cite this article: A. Blokh, L. Oversteegen, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***


**Résumé**

**L'existence des triangles errants.** Les laminations fermées $\sigma_d$-invariantes (où $\sigma_d = z^d : S^1 \to S^1$, $d \geqslant 2$) ont été introduites par W. P. Thurston comme un outil pour l'étude des systèmes dynamiques dans le plan complexe. Il avait défini les *triangles errants* comme étant des triplets $T \subset S^1$ tels que $\sigma_d^n(T)$ est composé des trois points distincts pour tout $n \geqslant 0$, et les enveloppes convexes de tous les ensembles $\sigma_d^n(T)$ sont deux-à-deux disjointes dans le plan complexe. Il avait démontré que $\sigma_2$ n'admet pas des triangles errants. Nous montrons que pour tout $d \geqslant 3$ il existe une collection nondénombrable de laminations fermées $\sigma_d$-invariantes qui ont des triangles errants et des applications-facteurs de $\sigma_d$ non-conjuguées, deux-à-deux distinctes, sur les espaces quotients associés. ***Pour citer cet article : A. Blokh, L. Oversteegen, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***


## Version française abrégée

Les laminations ont été introduites par Thurston [10] comme un outil pour étudier en même temps les polynômes complexes individuels et leur ensemble tout entier. Dans le cas où le degré est $d$, ce dernier se réduit à l'étude de

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l'espace paramétrique des polynômes centrés moniques de degré $d \geqslant 2$ qui sont de la forme $z \mapsto z^d + a_{d-2}z^{d-2} + \cdots + a_0$ [3]. L'ensemble des paramètres pour lesquels l'ensemble de Julia correspondant est connexe est appelé *le lieu géométrique de connexité*. (Si $d = 2$, le lieu géométrique de connexité s'appelle *l'ensemble de Mandelbrot* et l'on utilise la notation $\mathcal{M}$).

Soit $P : \mathbb{C}^* \to \mathbb{C}^*$ un polynôme de degré $d$ qui est défini sur la sphère complexe $\mathbb{C}^*$ et qui possède un ensemble de Julia $J$ connexe. Soit $\theta = z^d : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ (où $\mathbb{D} \subset \mathbb{C}$ est le disque-unité). Il existe un isomorphisme conforme $\Psi : \mathbb{D} \to \mathbb{C}^* \setminus J$ tel que $\Psi \circ \theta = P \circ \Psi$ [4,5]. Si $J$ est localement connexe, alors $\Psi$ se prolonge à une fonction continue $\bar{\Psi} : \overline{\mathbb{D}} \to \overline{\mathbb{C}^* \setminus J}$. Soient $\sigma_d = \theta|_{\partial\mathbb{D}}$, $\psi = \bar{\Psi}|_{\partial\mathbb{D}}$. Pour chaque $y \in J$, soit $C(y)$ l'enveloppe convexe de l'ensemble $\psi^{-1}(y)$ dans le disque-unité, soit $\mathcal{L}_P$ la collection de toutes les cordes contenues dans la frontière de tous les ensembles $C(y)$, $y \in J$. Alors $\mathcal{L}_P$ est un exemple d'une lamination $d$-invariante. Une telle lamination donne une description combinatoire de $J$. D'après Kiwi [7], une construction similaire est possible pour tous les polynômes qui ont des ensembles de Julia connexes et qui n'ont pas des cycles neutres irrationnels.

Dans le cas où $d = 2$, Thurston [10] a démontré que l'espace de toutes les laminations 2-invariantes (*quadratiques*) peut être interprété à travers une « méta-lamination » appelé LMQ (*lamination mineure quadratique*). La relation exacte entre LMQ et $\mathcal{M}$ n'est pas connue (Thurston a conjecturé que le bord de $\mathcal{M}$ est essentiellement la LMQ.) Un ingrédient majeur de la théorie de Thurston est la nonexistence des triangles errants pour les laminations quadratiques. Du point de vue des dynamiques dans l'ensemble de Julia, ceci est equivalent à la nonexistence des points de branchement dans $J$ qui sont nonprécritiques nonprépériodiques, et peut être vu comme une extension naturelle du résultat similaire pour les applications des graphes finis.

Les extensions des résultats de Thurston au delà du cas $d = 2$ ont été entravées par le manque d'information sur l'existence des triangles errants pour $d > 2$. Dans ce qui suit, nous donnons un schéma sur la construction des laminations invariantes ayant des triangles errants, et nous démontrons le théorème suivant.

**Théorème 0.1.** *Pour chaque $d \geqslant 3$ il existe une collection nondénombrable des laminations $d$-invariantes $\mathcal{L}(\alpha)$ ayant un triangle errant telles que les applications induites $f_{\mathcal{L}(\alpha)}|_{J_{\mathcal{L}(\alpha)}}$ sont deux-à-deux non conjuguées.*

Nous donnons les idées principales de la démonstration. Soit $X \subset S^1$, une application $g : X \to g(X) \subset S^1$ est dite *$\sigma$-prolongeable* si $X \cup g(X)$ peut être plongé dans $S^1$ au moyen d'une application (non-nécessairement continue) $\varphi$ préservant l'ordre de telle sorte que l'application induite $g' : \varphi(X) \to g'(\varphi(X)) \subset S^1$ (définie par $g' = \varphi \circ g \circ \varphi^{-1}$) coincide avec l'application $\sigma_d|_{\varphi(X)}$ pour un certain $d$. Un tel $d$ minimal est appelé le *pseudo-degré* de $g$.

L'idée est de construire un ensemble $A \subset S^1$ et une application $\sigma$-prolongeable $g : A \to A$ de pseudo-degré 3 tels que $A$ est l'orbite d'un triplet $T_0$ par l'application $g$, et $T_0$ est un triangle errant de $g$. La construction est flexible et peut être mise en oeuvre de plusieurs manières différentes nondénombrables. En utilisant la définition, nous pouvons alors plonger $A$ dans $S^1$ à l'aide d'une application $\varphi$ qui préserve l'ordre de telle manière que l'application induite sur $\varphi(A)$ coincide avec $\sigma_3$. L'ensemble $\varphi(T_0)$ est un triangle errant pour $\sigma_3$. La lamination $\mathcal{L}'$, qui est $\sigma_3$-invariante dans le futur et qui est composée des cotés de tous les triangles $\varphi(T_i)$, peut être étendue à une lamination cubique nondégénérée $\mathcal{L}$, et les manières nondénombrables de faire cette construction donnent lieu essentiellement à une collection nondénombrable de laminations distinctes $\mathcal{L}$. L'éxtension aux degrés superieurs est basée sur la technique « d'insertion d'un extra wrap ». Ceci achève la démonstration du Théorème 0.1.

## 1. Introduction

Laminations were introduced by Thurston [10] as a tool for studying both individual complex polynomials and the space of all of them. In the case of degree $d$ the latter reduces to studying the parameter space of degree $d \geqslant 2$ monic centered polynomials of the form $z \mapsto z^d + a_{d-2}z^{d-2} + \cdots + a_0$ [3]. The set of parameters for which the corresponding Julia set is connected is called the *connectedness locus* (if $d = 2$ the connectedness locus is called the *Mandelbrot set* and denoted by $\mathcal{M}$).

Let $P : \mathbb{C}^* \to \mathbb{C}^*$ be a degree $d$ polynomial with a connected Julia set $J_P$ acting on the complex sphere $\mathbb{C}^*$. Denote by $K_P$ the corresponding filled-in Julia set. Let $\theta = z^d : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ ($\mathbb{D} \subset \mathbb{C}$ is the unit disk). There exists a conformal isomorphism $\Psi : \mathbb{D} \to \mathbb{C}^* \setminus K_P$ with $\Psi \circ \theta = P \circ \Psi$ [4,5]. If $J_P$ is locally connected, then $\Psi$ extends to a continuous function $\bar{\Psi} : \bar{\mathbb{D}} \to \overline{\mathbb{C}^* \setminus K_P}$. Let $\sigma_d = \theta|_{\partial\mathbb{D}}$, $\psi = \bar{\Psi}|_{\partial\mathbb{D}}$, for each $y \in J_P$ let $C(y)$ be the convex hull of the set $\psi^{-1}(y)$ in the unit disk, and let $\mathcal{L}_P$ be the collection of all chords of $S^1$ contained in the boundary of all the sets $C(y)$, $y \in J_P$ (if $C(y)$ is a point then this point is included in $\mathcal{L}_P$ too). Then $\mathcal{L}_P$ is an example of a $d$-invariant lamination. Such a lamination gives a combinatorial description of $J_P$. By Kiwi [7] a similar construction is possible for all polynomials with connected Julia sets and no irrational neutral cycles.

In the case $d = 2$ Thurston [10] proved that the space of all 2-invariant (*quadratic*) laminations can be interpreted through a 'meta-lamination' called QML (*quadratic minor lamination*). The exact relationship between QML and $\mathcal{M}$ is unknown (Thurston conjectured that the boundary of $\mathcal{M}$ is essentially QML). A major ingredient of Thurston's theory is the non-existence of wandering triangles for quadratic laminations. From the standpoint of the dynamics in the Julia set this is equivalent to the non-existence of non-preperiodic non-precritical branch points in $J$, and can be viewed as a natural extension of the same result for maps of finite graphs.

Extensions of Thurston's results beyond $d = 2$ have been hampered by the lack of information about the existence of wandering triangles for $d > 2$. Here we sketch the construction of invariant laminations with wandering triangles.

A *lamination* $\mathcal{L}$ is a closed set of chords and points in $\bar{\mathbb{D}} \subset \mathbb{C}$ such that any two distinct chords in $\mathcal{L}$ (called *leaves*) intersect at most at a common endpoint; leaves may be degenerate. A leaf with endpoints $p, q \in S^1$ is denoted by $\ell = \overline{pq}$. Denote the union of all leaves in $\mathcal{L}$ by $\mathcal{L}^*$. A *gap* $G$ of $\mathcal{L}$ is the closure of a complementary domain of $\mathcal{L}^*$ in $\bar{\mathbb{D}}$. For each chord $\ell = \overline{pq}$ let $\sigma_d(\ell)$ be the chord joining the points $\sigma_d(p)$ and $\sigma_d(q)$. The lamination $\mathcal{L}$ is *$d$-invariant* if for each $\ell \in \mathcal{L}$ we have $\sigma_d(\ell) \in \mathcal{L}$, there exist $d$ pairwise disjoint leaves $\ell_i \in \mathcal{L}$ ($i = 1, \ldots, d$) with $\sigma_d(\ell_i) = \ell$, and for each gap $G$ either $|\sigma_d(G \cap S^1)| \leqslant 2$, or there exists a gap $H$ of $\mathcal{L}$ such that $\sigma_d|_{G\cap S^1}$ maps $G \cap S^1$ onto $H \cap S^1$ as a covering map with positive orientation; in this case we write $\sigma_d(G) = H$.

Given a lamination $\mathcal{L}$ there exists the finest closed equivalence relation $\approx_{\mathcal{L}}$ (or simply $\approx$) on $S^1$ with the property that if $\overline{pq} \in \mathcal{L}$ then $p \approx q$ (for some laminations $\mathcal{L}$ all of $S^1$ is a single class and $S^1/\approx$ is a point). If $\mathcal{L}$ is $d$-invariant then $\approx$ is $\sigma_d$-invariant, and $\sigma_d$ induces a branched covering map $f_{\mathcal{L}} : J_{\mathcal{L}} \to J_{\mathcal{L}}$, where $J_{\mathcal{L}}$ is the quotient space $S^1/\approx$. For a polynomial $P$ with locally connected Julia set $J_P$ the above defined lamination $\mathcal{L}_P$ gives rise to the equivalence $\approx_P$ with equivalence classes being the sets $C(y) \cap S^1$, $y \in J_P$ so that $P|_{J_P}$ and $f_{\mathcal{L}_P}|_{J_{\mathcal{L}_P}}$ are topologically conjugate. In particular $J_{\mathcal{L}_P}$ is non-degenerate. To avoid ambiguity we from now on consider only *q-laminations*, i.e. closed $d$-invariant laminations $\mathcal{L}$ such that the convex hull of each non-degenerate equivalence class of $\approx$, is either a leaf or a gap of $\mathcal{L}$.

Let us introduce some notions. Assume that $\mathcal{L}$ is a $d$-invariant q-lamination, $\approx$ is its equivalence, and $X \subset S^1$ is an equivalence class of $\approx$. Call $X$ *critical* iff $\sigma_d|_X$ is not 1-to-1 and *precritical* iff $\sigma_d^j(X)$ is critical for some $j \geqslant 0$. Call $X$ *preperiodic* if $\sigma_d^i(X) = \sigma_d^j(X)$ for some $0 \leqslant i < j$. A gap $G$ is a *wandering $n$-gon* if $|G \cap S^1| = n \geqslant 3$ and $G \cap S^1$ is neither preperiodic nor precritical. A wandering 3-gon is a *wandering triangle*.

Now we list some known facts. Kiwi [6] extended Thurston's theorem by showing that every non-preperiodic non-precritical gap in a $d$-invariant lamination is at most a $d$-gon. In [8] Levin showed that laminations with one critical class do not have wandering $n$-gons. Another result was obtained in [1] (see Theorem 1.1). Let $k_{\mathcal{L}}$ be the maximal number of critical classes $X$ of $\approx_{\mathcal{L}}$ with pairwise disjoint infinite $\sigma_d$-orbits such that $\sigma_d(X)$ is a singleton.

**Theorem 1.1.** *Let $\mathcal{L}$ be a $d$-invariant q-lamination and let $\Gamma$ be a non-empty collection of wandering $d_j$-gons ($j = 1, 2, \ldots$) with distinct grand orbits. Then $\sum_j (d_j - 2) \leqslant k_{\mathcal{L}} - 1 \leqslant d - 2$.*

Until now, it has not been known if wandering triangles exist; our main result shows that they do.

**Theorem 1.2.** *For each $d \geqslant 3$ there exists an uncountable collection of $d$-invariant q-laminations $\mathcal{L}(\alpha)$ with a wandering triangle such that the induced maps $f_{\mathcal{L}(\alpha)}|_{J_{\mathcal{L}(\alpha)}}$ are pairwise non-conjugate.*

## 2. Construction

For several obvious reasons we call laminations with wandering triangles *WT-laminations*. Here we outline the construction of one 3-invariant (*cubic*) WT-lamination. The example was inspired by ideas of [1] and [9]. In a later paper we will use the freedom of the construction to prove the full version of Theorem 1.2.

The circle $S^1$ is identified with the factor space $\mathbb{R}/\mathbb{Z}$; points of $S^1$ are denoted by real numbers $x \in [0,1]$ with the induced circular order. By an arc $(p,q)$ in the circle we mean the positively oriented arc from $p$ to $q$. A few necessary conditions for a cubic lamination $\mathcal{L}$ to be a WT-lamination follow from [1] (or from [6]). Indeed, by Theorem 1.1 if $\mathcal{L}$ is a cubic WT-lamination then $k_{\mathcal{L}} = 2$. This implies that the two critical classes are leaves of $\mathcal{L}$ and $J_{\mathcal{L}}$ is a dendrite. Other more dynamical facts about cubic WT-laminations follow from [2].

Let $X \subset S^1$. A map $g : X \to g(X) \subset S^1$ is said to be *$\sigma$-extendable* if $X \cup g(X)$ can be embedded into $S^1$ by means of an order-preserving (not necessarily continuous) map $\varphi$ so that the induced map $g' : \varphi(X) \to g'(\varphi(X)) = \varphi(g(X)) \subset S^1$ (defined as $g' = \varphi \circ g \circ \varphi^{-1}$) coincides with the map $\sigma_d|_{\varphi(X)}$ for some $d$. The minimal such $d$ is said to be the *pseudo-degree* of $g$.

The idea is to construct sets $A \subset A' \subset S^1$ and a $\sigma$-extendable map $g : A \to A'$ of pseudo-degree 3 so that $A$ contains the $g$-orbit of a triple $T_0$ and $T_0$ is a wandering triangle of $g$. The construction is flexible and can be implemented in uncountably many ways. By the definition we can then embed $A'$ into $S^1$ by means of an order-preserving map $\varphi$ so that the induced map on $\varphi(A)$ coincides with $\sigma_3$. The set $\varphi(T_0)$ is a wandering triangle for $\sigma_3$. The $\sigma_3$-forward invariant lamination $\mathcal{L}'$ consisting of the sides of all triangles $\varphi(T_i)$ can be extended to a non-degenerate 3-invariant (*cubic*) lamination $\mathcal{L}$, and the uncountably many implementations of the construction give rise to uncountably many essentially distinct laminations $\mathcal{L}$. The extension onto higher degrees relies upon the techniques of 'inserting an extra wrap' and completes the proof of Theorem 1.2.

Set $B = \{0 < c' < s_0 < u_0 < \frac{1}{2} < v_0 < d' < t_0 < 1\}$ and denote by $\bar{c}_0$ the chord with the endpoints $u_0$ and $v_0$ and by $\bar{d}_0$ the chord with the endpoints $s_0$ and $t_0$. Let the point $u_{-k}$ be the only point such that $u_{-k} \in (u_0, v_0)$, $\sigma_3(u_{-k}) \in (u_0, v_0), \ldots, \sigma_3^k(u_{-k}) = u_0$. Similarly we define points $v_{-k}, s_{-k}, t_{-k}$. Observe that $\lim_{n\to\infty} u_{-n} = 1/2$ and $\sigma_3(u_{-i}) = u_{-i+1}$; similar facts hold for $v_{-n}$, $s_{-n}$, and $t_{-n}$. All these points together with the set $B$ form the set $B'$. The chord connecting $u_{-k}, v_{-k}$ is denoted by $\bar{c}_{-k}$ and the chord connecting $s_{-k}, t_{-k}$ is denoted by $\bar{d}_{-k}$.

The set $B'$ is an initial part of $A$ used to determine the location of other points of $A$ on the circle. Below we will define the triple $T_0 = \{\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0\}$ and the set $X_0 = B' \cup T_0$. On each step a new triple $T_n = \{\mathbf{x}_n, \mathbf{y}_n, \mathbf{z}_n\}$ is added and the set $X_n = X_{n-1} \cup \{\mathbf{x}_n, \mathbf{y}_n, \mathbf{z}_n\}$ is defined. We denote new points added on each step by boldface letters. This explains the notation in the next phrase: the map $g$ on points $x_{n-1}$, $y_{n-1}$, $z_{n-1}$ is defined as $g(x_{n-1}) = \mathbf{x}_n$, $g(y_{n-1}) = \mathbf{y}_n$, $g(z_{n-1}) = \mathbf{z}_n$. Below by a 'triple' we mean one of the sets $T_i$ and by a 'triangle' the convex hull of a triple.

We suggest the following system of notation and rules which are enforced throughout. Suppose $X_{i-1}$ has been defined. The location of the $i$-th triple $T_i$ is determined by points $p, q, r \in X_{i-1}$ with $p < x_i < q < y_i < r < z_i$ and $[(p, x_i) \cup (q, y_i) \cup (r, z_i)] \cap X_{i-1} = \emptyset$. In this case we write $T_i = T(p, \mathbf{x}_i, q, \mathbf{y}_i, r, \mathbf{z}_i)$. If 2 or 3 points of a triple are located between two adjacent points of $X_{i-1}$ then we need less than 6 points to denote $T_i$ – e.g., $T(p, x_i, y_i, q, z_i)$ where $p, q \in X_{i-1}$ means that $p < x_i < y_i < q < z_i$ and $[(p, y_i) \cup (q, z_i)] \cap X_{i-1} = \emptyset$. Define the map $g$ on all points of $(B' \cap [s_{-1}, t_{-1}]) \cup \{0\}$ as $\sigma_3$. Set $g(u_0) = g(v_0) = c'$, $g(s_0) = g(t_0) = d'$. This defines a map $g : B' \setminus \{c', d'\} \to B'$. In what follows the map $g$ is constructed to be order preserving on subsets of $A$ contained in the closures of components of $S^1 \setminus \{0, s_0, u_0, 1/2, t_0, v_0\}$.

Now we introduce locations of some initial triples: $T_0 = T(u_0, \mathbf{x}_0, \mathbf{y}_0, t_{-1}, \mathbf{z}_0)$, $T_1 = T(c', \mathbf{x}_1, \mathbf{y}_1, t_0, \mathbf{z}_1)$, $T_2 = T(v_0, \mathbf{x}_2, \mathbf{y}_2, d', \mathbf{z}_2)$, $T_3 = T(c', \mathbf{x}_3, \mathbf{y}_3, \mathbf{z}_3)$, $T_4 = T(s_{-1}, \mathbf{x}_4, v_{-1}, \mathbf{y}_4, \mathbf{z}_4)$, $T_5 = T(s_0, \mathbf{x}_5, v_0, \mathbf{y}_5, \mathbf{z}_5)$, $T_6 = T(0, \mathbf{x}_6, c', \mathbf{y}_6, \mathbf{z}_6)$, $T_7 = T(x_0, \mathbf{x}_7, \mathbf{y}_7, \mathbf{z}_7)$, $T_8 = (x_1, \mathbf{x}_8, \mathbf{y}_8, \mathbf{z}_8)$, $T_9 = T(x_2, \mathbf{x}_9, \mathbf{y}_9, \mathbf{z}_9)$, $T_{10} = T(x_3, \mathbf{x}_{10}, \mathbf{y}_{10}, \mathbf{z}_{10})$,

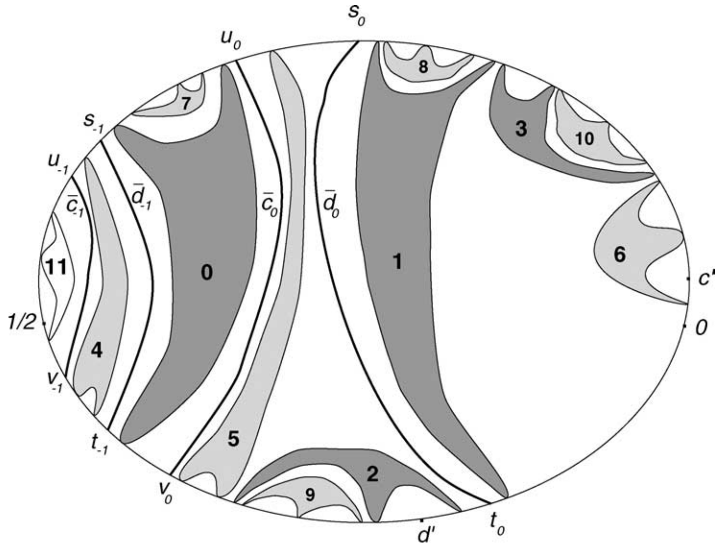


Fig. 1. First twelve triangles.

Fig. 1. Les douze premiers triangles.

$T_{11} = (u_{-1}, \mathbf{x}_{11}, \mathbf{y}_{11}, t_{-2}, \mathbf{z}_{11})$. Our rules force the location of some triples, e.g. the fact that $T_7 \subset (x_0, y_0)$ forces the location of $T_8, T_9, T_{10}$. The segment of triples $\{T_0, \ldots, T_{11}\}$ is the basis of induction (see Fig. 1).

Given two disjoint chords $p, q$ denote by $S(p, q)$ the strip enclosed by $p, q$ and arcs of the circle. Since $T_{11}$ is contained in $S(\bar{d}_{-2}, \bar{c}_{-1})$, its image must be contained in $S(\bar{d}_{-1}, \bar{c}_0)$; set $T_{12} = T(y_0, \mathbf{x}_{12}, \mathbf{y}_{12}, t_{-1}, \mathbf{z}_{12})$. Thus, $T_{12}$ separates the chord $\bar{d}_{-1}$ from $T_0$ (inside the unit disk). Our rules then imply that $T_{13} = T(y_1, \mathbf{x}_{13}, \mathbf{y}_{13}, t_0, \mathbf{z}_{13})$ separates the chord $\bar{d}_0$ from $T_1$. Moreover, this fact together with our rules forces the location of forthcoming triples $T_{14}, T_{15}, \ldots$ with respect to $X_{13}, X_{14}, \ldots$ for some time. The first time when the location of the triple is not forced is when $T_1$ is mapped onto $T_{11}$ and $T_{13}$ is mapped onto $T_{23}$. At this moment our rules guarantee that $T_{23}$ must be located in the arc $(y_{11}, z_{11})$, but otherwise its location is not forced. The freedom of choice of the location of $T_{23}$ at this point, and the similar variety of options which will be available later on at similar moments, is exactly the reason why the construction yields not just one, but uncountably many types of behavior of a wandering triangle.

However here we are only interested in one example, so we choose the location of $T_{23}$ as

$$T_{23} = T(s_{-2}, \mathbf{x}_{23}, v_{-2}, \mathbf{y}_{23}, \mathbf{z}_{23})$$

(in particular $T_{23} \subset S(\bar{c}_{-2}, \bar{d}_{-2})$). This implies that $T_{24}$ must be located inside $S(\bar{c}_{-1}, \bar{d}_{-1})$, and we choose its location so that $T_{24}$ separates $\bar{c}_{-1}$ from $T_4$ in the disk. This forces the location of $T_{25}$ which separates $\bar{c}_0$ from $T_5$. As before with $\bar{d}_0, T_1$ and $T_{13}$, this determines the location of the triples $T_{26}, T_{27}, \ldots$ relative to $X_{25}, X_{26}, \ldots$ for some time until the choice for the location of the triple is not forced. This happens exactly at the moment when $T_5$ maps into $T_{23}$ and $T_{25}$ maps into $T_{43}$. We choose the location for $T_{43}$ as $T_{43} = T(u_{-2}, \mathbf{x}_{43}, \mathbf{y}_{43}, t_{-3}, \mathbf{z}_{43})$ to mimic already existing triples $T_0$ and $T_{11}$. Then we choose $T_{44}$ so that it separates $\bar{d}_{-2}$ and $T_{11}$ and proceed with the construction as before. The step from $T_{11}$ to $T_{43}$ is the first inductive step in the construction.

Let us now describe the induction in general. Step $n$ begins at a moment $i_n$ with a triple

$$T_{i_n} = T(u_{-n}, \mathbf{x}_{i_n}, \mathbf{y}_{i_n}, t_{-n-1}, \mathbf{z}_{i_n}) \subset S(\bar{d}_{-n-1}, \bar{c}_{-n}).$$

It is followed by a segment of triples which comply with our rules and are contained in strips $S(\bar{d}_{-n}, \bar{c}_{-n+1})$, $S(\bar{d}_{-n+1}, \bar{c}_{-n+2}), \ldots$ closer to chords $\bar{d}_{-n}, \bar{d}_{-n+1}, \ldots$ than previously existing triples until the triple $T_{j_n}$ whose triangle is located to the right of $\bar{d}_0$ and separates $\bar{d}_0$ from $T_{j_{n-1}}$ in the disk. From that time on the behavior of $T_{j_n}$ is forced by our rules and behavior of $T_{j_{n-1}}$ until at a later moment the triple $T_{j_{n-1}}$ maps into $T_{i_n}$ and $T_{j_n}$ maps onto a new triple $T_{k_n}$, closer to $1/2$ than previous triples. We choose $T_{k_n}$ as $T_{k_n} = T(s_{-n-1}, \mathbf{x}_{k_n}, v_{-n-1}, \mathbf{y}_{k_n}, \mathbf{z}_{k_n}) \subset S(\bar{c}_{-n-1}, \bar{d}_{-n-1})$. We now follow this triple by a series of triples contained in the strips $S(\bar{c}_{-n}, \bar{d}_{-n}), S(\bar{c}_{-n+1}, \bar{d}_{-n+1}), \ldots$ closer to the chords $\bar{c}_{-n}, \bar{c}_{-n+1}, \ldots$ than previously existing triples. This series of triples ends with a triple $T_{l_n}$ whose triangle separates $\bar{c}_0$ from $T_{l_{n-1}}$ and whose future behavior is forced by that of $T_{l_{n-1}}$ until the moment when $T_{l_{n-1}}$ maps onto $T_{k_n}$. At this moment $T_{l_n}$ maps onto $T_{i_{n+1}}$ and the construction repeats.

This leads to a set $A' = B' \cup (\bigcup_{i=0}^{\infty} T_i) =\subset S^1$ such that all $T_i$'s have pairwise disjoint convex hulls. Moreover, the map $g$ is defined on $A = A' \setminus \{c', d'\}$. We then prove that in fact $g : A \to A'$ is $\sigma$-extendable of pseudo-degree 3. Thus, we can embed $A'$ into $S^1$ by means of an order-preserving map $\varphi$ so that the induced map on $\varphi(A)$ coincides with $\sigma_3$. The set $\varphi(T_0)$ is a wandering triangle for $\sigma_3$. This forward invariant non-closed lamination can be completed to a closed cubic invariant lamination. By the construction, there are countably many available choices as to in what strips the triangles $T_{i_n}$ and $T_{k_n}$ can be placed. That leads to uncountably many cubic WT-laminations whose induced maps on the quotient spaces are non-conjugate. The extension onto higher degrees relies upon the techniques of 'inserting an extra wrap' and completes the proof of Theorem 1.2.

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