MODELS FOR SPACES OF DENDRITIC POLYNOMIALS

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ABSTRACT. Complex 1-variable polynomials with connected Julia sets and only repelling periodic points are called *dendritic*. By results of Kiwi, any dendritic polynomial is semi-conjugate to a topological polynomial whose topological Julia set is a dendrite. We construct a continuous map of the space of all cubic dendritic polynomials onto a laminational model that is a quotient space of a subset of the closed bidisk. This construction generalizes the "pinched disk" model of the Mandelbrot set due to Douady and Thurston. It can be viewed as a step towards constructing a model of the cubic connectedness locus.

1. INTRODUCTION

The Introduction assumes basic knowledge of complex dynamics and especially its combinatorial part; some concepts are introduced informally and are formalized later in the main body of the paper.

The *parameter space* of complex degree d polynomials is by definition the space of affine conjugacy classes of these polynomials. Equivalently, one can talk about the space of all *monic centered polynomials* of degree d, i.e., polynomials of the form $z^d + a_{d-2}z^{d-2} + \cdots + a_0$. Any polynomial is affinely conjugate to a monic centered polynomial. An important set is the *connectedness locus* $\mathcal{M}_d$ consisting of classes of all degree d polynomials P, whose Julia sets $J(P)$ (equivalently, whose *filled Julia sets* $K(P)$) are connected. General properties of the connectedness locus $\mathcal{M}_d$ have been studied for quite some time. For instance, it is known that $\mathcal{M}_d$ is a compact cellular set in the parameter space of complex degree d polynomials. This was proven in [BrHu88] in the cubic case and in [Lav89] for higher degrees, see also [Bra86]. By definition, following M. Brown [Bro60, Bro61], a

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subset of a Euclidean space $\mathbb{R}^n$ is *cellular* if its complement in the sphere $\mathbb{R}^n \cup \{\infty\}$ is an open topological cell.

For $d = 2$, a monic centered polynomial takes the form $P_c(z) = z^2 + c$, and the parameter space of quadratic polynomials can be identified with the plane of complex parameters c. Clearly, $P_c(z)$ has a unique critical point 0 and a unique critical value c in $\mathbb{C}$. Thus, we can say that polynomials $P_c(z)$ are parameterized by their critical values. The quadratic connectedness locus is the famous *Mandelbrot set* $\mathcal{M}_2$, identified with the set of complex numbers c not escaping to infinity under iterations of the polynomial $P_c(z)$. The Mandelbrot set $\mathcal{M}_2$ has a complicated self-similar structure.

1.1. **A combinatorial model for $\mathcal{M}_2$.** The "pinched disk" model for $\mathcal{M}_2$ is due to Douady and Thurston [Dou93, Thu85]. To describe their approach to the problem of modeling $\mathcal{M}_2$, we first describe *laminational* models of polynomial Julia sets (we follow [BL02]).

Let $\mathbb{S}$ be the unit circle in $\mathbb{C}$, consisting of all complex numbers of modulus one. We write $\sigma_d : \mathbb{S} \to \mathbb{S}$ for the restriction of the map $z \mapsto z^d$. We identify $\mathbb{S}$ with $\mathbb{R}/\mathbb{Z}$ by the mapping taking an *angle* $\theta \in \mathbb{R}/\mathbb{Z}$ to the point $e^{2\pi i\theta} \in \mathbb{S}$. Under this identification, we have $\sigma_d(\theta) = d\theta$. We will write $\mathbb{D}$ for the open unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$.

Given a complex polynomial P, we let $U_\infty(P)$ denote the set $\mathbb{C} \setminus K(P)$. This set is called the *basin of attraction of infinity* of P. Clearly, $\overline{U_\infty(P)} = U_\infty(P) \cup J(P)$. If the Julia set $J(P)$ is locally connected, then it is connected, and the Riemann map $\Psi : \mathbb{C} \setminus \overline{\mathbb{D}} \to U_\infty(P)$ can be continuously extended to a map $\overline{\Psi} : \mathbb{C} \setminus \mathbb{D} \to \overline{U_\infty(P)}$. This gives rise to a map $\psi = \overline{\Psi}|_{\mathbb{S}}$, which semiconjugates $\sigma_d : \mathbb{S} \to \mathbb{S}$ with $P|_{J(P)}$. Define an equivalence relation $\sim_P$ on $\mathbb{S}$ so that $x \sim_P y$ if and only if $\psi(x) = \psi(y)$. Then $\mathbb{S}/\sim_P$ and $J(P)$ are homeomorphic, and the homeomorphism in question conjugates the map $f_{\sim_P}$ induced on $\mathbb{S}/\sim_P$ by σ_d, and $P|_{J(P)}$. It is not hard to see that the convex hulls of $\sim_P$-classes are disjoint in $\overline{\mathbb{D}}$.

A productive idea is to consider equivalences relations $\sim$ whose properties are similar to those of $\sim_P$. These properties will be stated precisely later. Such equivalence relations are called *laminational equivalence relations of degree d*. The maps $f_\sim : \mathbb{S}/\sim \to \mathbb{S}/\sim$ induced by σ_d are called *topological polynomials* of degree d. Degree two objects (laminational equivalence relations, topological polynomials, etc.) are referred to as *quadratic*. Similarly, degree three objects are referred to as *cubic*. The quotient space $\mathbb{S}/\sim$ is denoted $J_\sim$ and is called the *topological Julia set (of $f_\sim$)*. For brevity, in what follows, we will talk about "$\sim$-classes" instead of "classes of equivalence of $\sim$".

An important geometric representation of a laminational equivalence relation $\sim$ is as follows. For any $\sim$-class $\mathfrak{g}$, take its convex hull $\mathrm{CH}(\mathfrak{g})$. Consider the edges of all such convex hulls; add all points of $\mathbb{S}$ to this collection of chords. The obtained collection of (possibly, degenerate) chords in the unit disk is denoted by $\mathcal{L}_\sim$ and is called a *geodesic lamination generated by* $\sim$. In general, a *geodesic lamination* in $\mathbb{D}$ is a closed collection of chords in $\mathbb{D}$ that are disjoint in $\mathbb{D}$; the collection is assumed to include all degenerate chords. For brevity, in what follows, we sometimes write "lamination" instead of "geodesic lamination". Observe that often hyperbolic geodesics are used instead of chords; we use chords for the sake of brevity and simplicity.

Clearly, $\mathcal{L}_\sim$ is a closed family of chords. Let $\overline{ab}$ denote the chord connecting points a, $b \in \mathbb{S}$. We will never use this notation for pairs of points not in $\mathbb{S}$. Recall that points in $\mathbb{S}$ are identified with their "angles". Thus, $\overline{0\frac{1}{2}}$ always means the chord of $\mathbb{S}$ connecting the points with angles 0 and $\frac{1}{2}$. For any chord $\ell = \overline{ab}$ in the closed unit disk $\overline{\mathbb{D}}$ set $\sigma_d(\ell) = \overline{\sigma_d(a)\sigma_d(b)}$. For any $\sim$-class $\mathfrak{g}$ and, more generally, for any closed set $\mathfrak{g} \subset \mathbb{S}$, we set $\sigma_d(\mathrm{CH}(\mathfrak{g})) = \mathrm{CH}(\sigma_d(\mathfrak{g}))$.

Recall the construction of Douady and Thurston. Suppose that a quadratic polynomial P_c has locally connected Julia set. We will write G_c for the convex hull of the $\sim_{P_c}$-class corresponding to the critical value c. A fundamental theorem of Thurston [Thu85] is that $G_c \neq G_{c'}$ implies that G_c and $G_{c'}$ are disjoint in $\overline{\mathbb{D}}$ (we will later state a more general and precise version of Thurston's result). Consider the collection of all G_c and take its closure. The thus obtained collection of chords and inscribed polygons defines a geodesic lamination QML introduced by Thurston in [Thu85] and called the *quadratic minor lamination*. The lamination QML corresponds to an equivalence relation $\sim_{\mathrm{QML}}$ on $\mathbb{S}$ [Thu85]. The corresponding quotient space $\mathcal{M}_2^{comb} = \mathbb{S}/\sim_{\mathrm{QML}}$ is a combinatorial model for the boundary of $\mathcal{M}_2$. It is called the *combinatorial Mandelbrot set*. Conjecturally, the combinatorial Mandelbrot set is homeomorphic to the boundary of $\mathcal{M}_2$. This conjecture is equivalent to the famous MLC conjecture: the Mandelbrot set is locally connected.

1.2. **Dendritic polynomials.** When defining the combinatorial Mandelbrot set, we used a partial association between parameter values c and laminational equivalence relations $\sim_{P_c}$. In order to talk about $\sim_{P_c}$, we had to assume that $J(P_c)$ was locally connected. Recall that a *dendrite* is a locally connected continuum that does not contain Jordan curves. Recall also that a continuous map from a continuum to a continuum is called *monotone* if, under this map, point-preimages (*fibers*) are connected.

Definition 1.1. A complex polynomial P is said to be *dendritic* if it has connected Julia set and all cycles repelling. A topological polynomial is said to be *dendritic* if its Julia set is a dendrite. In that case, the corresponding laminational equivalence relation and the associated geodesic lamination are also said to be *dendritic*.

There are dendritic polynomials with non-locally connected Julia sets. Nevertheless, by [Kiw04], for *every* dendritic polynomial P of degree d, there is a monotone semiconjugacy m_P between $P : J(P) \to J(P)$ and a certain topological polynomial $f_{\sim_P}$ such that the map m_P is one-to-one on all periodic and pre-periodic points of P. Moreover, by [BCO11], the map m_P is unique and can be defined in a purely topological way. Call a monotone map φ_P of a connected polynomial Julia set $J(P) = J$ onto a locally connected continuum L the *finest monotone map of* $J(P)$ *onto a locally connected continuum* if, for any monotone $\psi : J \to J'$ with J' locally connected, there is a monotone map $h : L \to J'$ with $\psi = h \circ \varphi_P$. It is proven in [BCO11] that, for *any polynomial* P with $J(P)$ connected, the finest monotone map of $J(P)$ onto a locally-connected continuum semiconjugates $P|_{J(P)}$ to a topological polynomial $f_{\sim_P}$ on its topological Julia set $J_{\sim_P}$ generated by a laminational equivalence relation possibly with infinite classes $\sim_P$, and that in the dendritic case this semiconjugacy identifies with the map m_P constructed by Kiwi in [Kiw04]. Clearly, this shows that m_P is unique up to post-composition with a homeomorphism.

Thus, P gives rise to a corresponding laminational equivalence relation $\sim_P$ even if $J(P)$ is not locally connected. If $P_c(z) = z^2 + c$ is a quadratic dendritic polynomial, then G_c is defined, and is either a finite-sided polygon inscribed into $\mathbb{S}$, or a chord, or a point. A parameter value c is said to be *quadratic dendritic* if P_c is dendritic. The fundamental results of Thurston [Thu85] imply, in particular, that G_c and $G_{c'}$ are either the same or disjoint, for all pairs c, c' of dendritic parameter values. Moreover, the mapping $c \mapsto G_c$ is *upper semi-continuous* (if a sequence of dendritic parameters c_n converges to a dendritic parameter c, then the limit set of the corresponding convex sets G_{c_n} is a subset of G_c). We call G_c the *tag associated to* c.

Now, consider the union of the tags of all quadratic dendritic polynomials. This union is naturally partitioned into individual tags (distinct tags are pairwise disjoint!). Thus the space of tags can be equipped with the quotient space topology induced from the union of tags. On the other hand, take the set of quadratic dendritic parameters. Each such parameter c maps to the polygon G_c, i.e. to the tag associated to c. Thus each quadratic dendritic parameter maps to the corresponding point of the space of tags. This provides for a combinatorial (or laminational) model for the set of quadratic dendritic polynomials (or their parameters).

In this paper, we extend these results to cubic dendritic polynomials.

1.3. **Mixed tags of cubic polynomials.** Recall that monic centered quadratic polynomials are parameterized by their critical values. A combinatorial analog of this parameterization is the association between topological polynomials and their tags. Tags of quadratic topological polynomials are post-critical objects of the corresponding laminational equivalences. Monic centered cubic polynomials can be parameterized by a critical value and a co-critical point. Recall that the *co-critical* point ω^* of a cubic polynomial P corresponding to a simple critical point ω of P is defined as a point different from ω but having the same image under P as ω. If ω is a multiple critical point of P, then we set $\omega^* = \omega$. In any case we have $P(\omega^*) = P(\omega)$. Let c and d be the two critical points of P (if P has a multiple critical point, then $c = d$). Set $a = c^*$ and $b = P(d)$. Assuming that P is monic and central, we can parameterize P by a and b:

$$P(z) = b + \frac{a^2(a - 3z)}{4} + z^3.$$

For P in this form, we have $c = -\frac{a}{2}$, $d = \frac{a}{2}$. Similarly to parameterizing cubic polynomials by pairs (a, b), we will use the so-called *mixed tags* to parameterize topological cubic dendritic polynomials.

Consider a cubic dendritic polynomial P. By the above, there exists a laminational equivalence relation $\sim_P$ and a monotone semiconjugacy $m_p : J(P) \to \mathbb{S}/\sim_P$ of P_{J_P} with the topological polynomial $f_{\sim_P}$. Given a point $z \in J(P)$, we associate with it the convex hull $G_{P,z}$ of the $\sim_P$-equivalence class represented by the point $m_P(z) \in \mathbb{S}/\sim_P$. The set $G_{P,z}$ is a convex polygon with finitely many vertices, a chord, or a point; it should be viewed as a combinatorial object corresponding to z. For any points $z \neq w \in J(P)$, the sets $G_{P,z}$ and $G_{P,w}$ either coincide or are disjoint.

By definition, a (*critically*) *marked* (cf [Mil12]) cubic polynomial is a triple (P, c, d), where P is a cubic polynomial with critical points c and d. If P has only one (double) critical point, then $c = d$, otherwise we require that $c \neq d$. In particular, if $c \neq d$, then the triple (P, c, d) and the triple (P, d, c) are viewed as two distinct critically marked cubic polynomials. We will sometimes write P instead of (P, c, d). Critically marked polynomials do not have to be dendritic (in fact, the notion is used by Milnor and Poirier [Mil12] for hyperbolic polynomials, i.e., in the situation diametrically opposite to that of dendritic polynomials). Convergence in the space of marked polynomials is understood as convergence of the coefficients and of the marked critical points.

Let $\mathcal{MD}_3$ be the space of all critically marked cubic dendritic polynomials. With every marked dendritic polynomial (P, c, d), we associate the corresponding *mixed tag*

$$\mathrm{Tag}(P, c, d) = G_{c^*} \times G_{P(d)} \subset \overline{\mathbb{D}} \times \overline{\mathbb{D}}.$$

Here c^* is the co-critical point corresponding to the critical point c.

A similar construction can be implemented for any cubic dendritic laminational equivalence relation $\sim$. Let C and D denote the convex hulls of its *critical classes*, i.e., classes, on which the map σ_3 is not one-to-one. Then either $C = D$ is the unique critical $\sim$-class, or $C \neq D$ are disjoint. The sets C and D are called the *critical objects* of $\sim$. By a (*critically*) *marked cubic laminational equivalence relation* we mean a triple $(\sim, C, D)$. If $C \neq D$, then we define $C^* = \mathrm{co}(C)$ as the convex hull of the unique $\sim$-class that is distinct from the class $C \cap \mathbb{S}$ but has the same σ_3-image. If $C = D$, then we set $C^* = C$. The set C^* is called the *co-critical set of* C. For a marked laminational equivalence relation $(\sim, C, D)$, define its *mixed tag* as

$$\mathrm{Tag}_l(\sim, C, D) = C^* \times \sigma_3(D) \subset \overline{\mathbb{D}} \times \overline{\mathbb{D}}$$

Let $\mathfrak{C}(\overline{\mathbb{D}})$ denote the set of all compact subsets of $\overline{\mathbb{D}}$. Clearly the range of the map Tag_l is a subset of $\mathfrak{C}(\overline{\mathbb{D}}) \times \mathfrak{C}(\overline{\mathbb{D}})$.

The subscript l in Tag_l stands for "laminational". We distinguish the map Tag_l from the map Tag, which acts on polynomials. These two maps are closely related though: for any marked dendritic cubic polynomial (P, c, d) and the corresponding marked laminational equivalence relation $(\sim_P, G_c, G_d)$, we have $\mathrm{Tag}(P, c, d) = \mathrm{Tag}_l(\sim_P, G_c, G_d)$.

1.4. **Statement of the main result.** Consider the collection of the sets $\mathrm{Tag}(P)$ over all $P \in \mathcal{MD}_3$. By [Kiw04, Kiw05], for any dendritic laminational equivalence relation $\sim$, there exists a dendritic complex polynomial P with $\sim = \sim_P$. Thus, equivalently, we can talk about the collection of mixed tags of all dendritic laminations $\mathcal{L}_\sim$. In Theorem 4.15, we show that the mixed tags $\mathrm{Tag}(P)$ are pairwise disjoint or equal. Let us denote this collection of sets by $\mathrm{CML}(\mathcal{D})$ (for *cubic mixed lamination of dendritic polynomials*). Note, that $\mathrm{CML}(\mathcal{D})$ can be viewed as (non-closed) "lamination" in $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$ whose elements are products of points, leaves or gaps. One can consider $\mathrm{CML}(\mathcal{D})$ as the higher-dimensional analog of Thurston's QML restricted to dendritic polynomials.

Theorem 4.15, in addition, establishes the fact that the collection of sets $\mathrm{CML}(\mathcal{D})$ is upper semi-continuous. Let the *union* of all sets in $\mathrm{CML}(\mathcal{D})$ be denoted by $\mathrm{CML}(\mathcal{D})^+ \subset \overline{\mathbb{D}} \times \overline{\mathbb{D}}$. It follows that the quotient space of $\mathrm{CML}(\mathcal{D})^+$, obtained by collapsing all elements of $\mathrm{CML}(\mathcal{D})$ to points, is a separable metric space, which is denoted by $\mathcal{MD}_3^{comb}$. Denote by $\pi : \mathrm{CML}(\mathcal{D})^+ \to \mathcal{MD}_3^{comb}$ the corresponding quotient map.

Main Theorem. *Mixed tags of critically marked polynomials from $\mathcal{MD}_3$ are either disjoint or coincide. The map $\pi \circ \mathrm{Tag} : \mathcal{MD}_3 \to \mathcal{MD}_3^{comb}$ is continuous.*

Hence $\mathcal{MD}_3^{comb}$ is a combinatorial model for $\mathcal{MD}_3$. This theorem can be viewed as a partial generalization of Thurston's results [Thu85] to cubic polynomials. Indeed, Thurston establishes the existence of tags of laminational equivalence relations that are pairwise disjoint and form an upper-semicontinuous family of subsets of the closed unit disk. We extend this to the cubic dendritic case by suggesting a new method of tagging such polynomials that guarantees that if two tags are distinct then they are actually disjoint. Choosing such tags and showing that they have the just mentioned properties is, in our view, an important step towards constructing a combinatorial model the cubic connectedness locus.

1.5. **Previous work and organization of the paper.** Lavaurs [Lav89] proved that $\mathcal{M}_3$ is not locally connected. Epstein and Yampolsky [EY99] proved that the bifurcation locus in the space of real cubic polynomials is not locally connected either. This makes the problem of defining a combinatorial model of $\mathcal{M}_3$ very delicate. There is no hope that a combinatorial model would lead to a precise topological model. Schleicher [Sch04] constructed a geodesic lamination modeling the space of *unicritical* polynomials, that is, polynomials with a unique multiple critical point. We have heard of an unpublished old work of D. Ahmadi and M. Rees, in which cubic geodesic laminations were studied, however, we have not seen it. The present paper is based on the results obtained in [BOPT16]. These results are applicable to invariant laminations of any degree.

The paper is organized as follows. In Section 2, we discuss basic properties of geodesic laminations and laminational equivalence relations. In Section 3, we recall the results of [BOPT16] adapting them to the cubic case. Finally, Section 4 is dedicated to the proof of the main result.

2. Laminations and their properties

By a *chord* we mean a closed segment connecting two points of the unit circle. If these two points coincide, then the chord is said to be *degenerate*.

Definition 2.1 (Geodesic laminations)**.** A *geodesic lamination* is a collection $\mathcal{L}$ of chords called *leaves* that satisfy the following properties:

(1) different leaves do not intersect in $\mathbb{D}$;
(2) all degenerate chords (points of $\mathbb{S}$) are leaves;
(3) the set $\mathcal{L}^+ = \bigcup_{\ell \in \mathcal{L}} \ell$ is compact.

Gaps of $\mathcal{L}$ are the closures of the components of $\mathbb{D} \setminus \mathcal{L}^+$.

Given a compact metric space X, the space of all its compact subsets with the Hausdorff metric is denoted by $\mathfrak{C}(X)$. Any leaf of a geodesic lamination is an element of $\mathfrak{C}(\overline{\mathbb{D}})$. Thus a lamination itself can be regarded as a compact subsect of $\mathfrak{C}(\overline{\mathbb{D}})$, i.e., as an element of $\mathfrak{C}(\mathfrak{C}(\overline{\mathbb{D}}))$. In what follows, convergence of laminations is always understood in the sense of the Hausdorff distance on $\mathfrak{C}(\mathfrak{C}(\overline{\mathbb{D}}))$.

In the introduction, we discussed laminational equivalence relations. We now give a precise definition.

Definition 2.2 (Laminational equivalence relations)**.** An equivalence relation $\sim$ on the unit circle $\mathbb{S}$ is said to be *laminational* if the following holds:

(E1) the graph of $\sim$ is a closed subset of $\mathbb{S} \times \mathbb{S}$;
(E2) the convex hulls of distinct equivalence classes are disjoint;
(E3) each equivalence class of $\sim$ is finite.

Let $d \geqslant 2$ be a integer. A laminational equivalence relation $\sim$ is called (σ_d-)*invariant* if:

(D1) it is *forward invariant*: for a $\sim$-class $\mathfrak{g}$, the set $\sigma_d(\mathfrak{g})$ is a $\sim$-class;
(D2) for any $\sim$-equivalence class $\mathfrak{g}$, the map $\sigma_d : \mathfrak{g} \to \sigma_d(\mathfrak{g})$ extends to $\mathbb{S}$ as an orientation preserving covering map such that $\mathfrak{g}$ is the full preimage of $\sigma_d(\mathfrak{g})$ under this covering map.

As in the introduction, we write $\mathcal{L}_\sim$ for the lamination generated by $\sim$. Recall that it consists of edges of the convex hulls of all $\sim$-classes. Equivalently, $\overline{ab} \in \mathcal{L}_\sim$ if $a \sim b$, and the points a, b are not separated in $\mathbb{S}$ by elements of the same equivalence class. A geodesic lamination is called a σ_d-invariant q-*lamination* (q from e**q**uivalence) if it has the form $\mathcal{L}_\sim$, where $\sim$ is a σ_d-invariant laminational equivalence.

Definition 2.3. A σ_d-invariant *limit lamination* is defined as a limit of σ_d-invariant q-laminations.

Below, we list the most important properties of σ_d-invariant q-laminations $\mathcal{L}$ with references.

Forward leaf invariance: for every non-degenerate leaf $\ell \in \mathcal{L}$, we have $\sigma_d(\ell) \in \mathcal{L}$. This property is straightforward from the definition. It is a part of the original definition of an invariant lamination by Thurston [Thu85].

Backward leaf invariance: for every non-degenerate leaf $\ell \in \mathcal{L}$, there is a leaf $\ell^* \in \mathcal{L}$ such that $\sigma_d(\ell^*) = \ell$. This property is straightforward from the definition. It is a part of the original definition of an invariant lamination by Thurston [Thu85].

Forward gap invariance: if G is a gap of $\mathcal{L}$, then $H = \sigma_d(G)$ is a leaf of $\mathcal{L}$ (possibly degenerate), or a gap of $\mathcal{L}$. In the latter case, the map $\sigma_d : G \cap \mathbb{S} \to H \cap \mathbb{S}$ extends to a map of the boundary of G onto the boundary of H so that the extended map is an orientation preserving composition of a monotone map and a covering map. This property is proved in [BMOV13]. It is a part of the original definition of an invariant lamination by Thurston [Thu85].

Sibling property: for every $\ell \in \mathcal{L}$ such that $\sigma_d(\ell)$ is a non-degenerate leaf, there exist d **pairwise disjoint** leaves $\ell_1, \dots, \ell_d$ in $\mathcal{L}$ such that $\ell_1 = \ell$, and $\sigma_d(\ell_i) = \sigma_d(\ell)$ for all $i = 2, \dots, d$. This property is proved in [BMOV13]. It is a part of the notion of a *sibling invariant lamination*.

Call a leaf ℓ^* such that $\sigma_d(\ell^*) = \ell$ a *pullback* of ℓ. A *sibling of* ℓ is defined as a leaf $\ell' \neq \ell$ with $\sigma_d(\ell') = \sigma_d(\ell)$. The backward leaf invariance property stipulates the existence of pullbacks of non-degenerate leaves. The sibling property is equivalent to saying that every leaf ℓ with non-degenerate image has $d-1$ siblings that are disjoint from each other and from ℓ. For $d = 2$, the sibling property means that, for any $\ell \in \mathcal{L}$, the chord obtained from ℓ by a half-turn around the center of the disk $\mathbb{D}$ also belongs to $\mathcal{L}$. Observe that, since leaves are closed segments, pairwise disjoint siblings cannot intersect even on the unit circle.

For brevity we often talk about laminations meaning σ_d-invariant limit geodesic laminations. Clearly, the limit of a sequence of σ_d-invariant limit laminations is again a σ_d-invariant limit lamination.

Definition 2.4 (Linked chords)**.** Two **distinct** chords of $\mathbb{D}$ are *linked* if they intersect in $\mathbb{D}$. We will also sometimes say that these chords *cross each other*. Otherwise two chords are said to be *unlinked*.

A gap G is said to be *infinite* (*finite, uncountable*) if $G \cap \mathbb{S}$ is infinite (finite, uncountable). Uncountable gaps are also called *Fatou* gaps. For a closed convex set $H \subset \mathbb{C}$, straight segments in the boundary $\mathrm{Bd}(H)$ of H are called *edges* of H.

Definition 2.5 (Critical sets)**.** A *critical chord* (*leaf*) $\overline{ab}$ of $\mathcal{L}$ is a chord (leaf) of $\mathcal{L}$ such that $\sigma_d(a) = \sigma_d(b)$. A gap is *all-critical* if all its edges are critical. An all-critical gap or a critical leaf (of $\mathcal{L}$) is called an *all-critical set* (of $\mathcal{L}$). A gap G of $\mathcal{L}$ is said to be *critical* if it is an all-critical gap or there is a critical chord contained in the interior of G except for its endpoints. A *critical set* of $\mathcal{L}$ is by definition a critical leaf or a critical gap. We also define a *critical object* of $\mathcal{L}$ as a maximal by inclusion critical set. See Figure 1 for illustrations of various critical sets.

2.1. **Dendritic laminations.** We now consider dendritic laminations and corresponding topological polynomials.

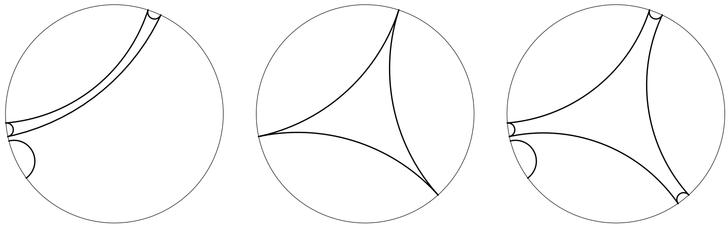

FIGURE 1. From left to right: a critical quadrilateral and its image leaf, an all-critical triangle, a critical hexagon of degree 3 and its image leaf (all critical sets are for σ_3)

Definition 2.6. A q-lamination $\mathcal{L}_\sim$ is called *dendritic* if all its gaps are finite. Then the corresponding topological Julia set $\mathbb{S}/\sim$ is a dendrite. The laminational equivalence relation $\sim$ and the topological polynomial $f_\sim$ are said to be *dendritic* too.

Recall that, by [Kiw04], with every dendritic polynomial P one can associate a dendritic topological polynomial $f_{\sim_P}$ so that $P|_{J(P)}$ is monotonically semi-conjugate to $f_{\sim_P}|_{J(f_{\sim_P})}$. By [Kiw05], for every dendritic topological polynomial f, there exists a polynomial P with $f = f_{\sim_P}$. Below, we list some well-known properties of dendritic geodesic laminations.

Definition 2.7 (Perfect parts of geodesic laminations [BOPT16])**.** Let $\mathcal{L}$ be a geodesic lamination considered as a subset of $\mathfrak{C}(\overline{\mathbb{D}})$. Then the maximal perfect subset $\mathcal{L}^p$ of $\mathcal{L}$ is called the *perfect part* of $\mathcal{L}$. A geodesic lamination $\mathcal{L}$ is called *perfect* if $\mathcal{L} = \mathcal{L}^p$. Equivalently, this means that all leaves of $\mathcal{L}$ are non-isolated in the Hausdorff metric.

Observe that $\mathcal{L}^p$ must contain $\mathbb{S}$.

Lemma 2.8. *Dendritic geodesic laminations $\mathcal{L}$ are perfect.*

Proof. Indeed, otherwise two gaps G, H of $\mathcal{L} = \mathcal{L}_\sim$ meet along a common edge that is an isolated leaf of $\mathcal{L}$. However by definition they are convex hulls of classes of $\sim$ which means that the corresponding two classes are non-disjoint, a contradiction. □

We will need Corollary 6.6 of [BOPT16], which reads:

Corollary 2.9. *Let $\mathcal{L}$ be a perfect limit lamination. Then the critical objects of $\mathcal{L}$ are pairwise disjoint and are either all-critical sets, or critical sets whose boundaries map exactly k-to-1, $k > 1$, onto their images.*

By Lemma 2.8, Corollary 2.9 applies to dendritic geodesic laminations. Moreover, by properties of dendritic geodesic laminations, all their critical objects are finite.

3. LINKED QUADRATICALLY CRITICAL GEODESIC LAMINATIONS

Now we will review results of [BOPT16] that are essential for this paper. Let us emphasize that results of [BOPT16] hold for any degree. However, we will adapt them here to degree three, omitting the general formulations. By *quadratic* (respectively, *cubic*) laminations, we mean σ_2-invariant (respectively, σ_3-invariant) limit laminations.

Consider a quadratic lamination $\mathcal{L}$ with a critical quadrilateral Q. Thurston [Thu85] associates to $\mathcal{L}$ its minor $\mathfrak{m} = \sigma_2(Q)$. Then $Q \cap \mathbb{S}$ is the full σ_2-preimage of $\mathfrak{m} \cap \mathbb{S}$. Thurston proves that different minors obtained in this way never cross in $\mathbb{D}$. Observe that two minors cross if and only if the vertices of the corresponding critical quadrilaterals alternate in $\mathbb{S}$. Thurston's result can be translated as follows in terms of critical quadrilaterals. If two quadratic laminations generated by laminational equivalences have critical quadrilaterals whose vertices strictly alternate, then the two laminations are the same. This motivates Definition 3.1.

Definition 3.1. Let A and B be two quadrilaterals with vertices in $\mathbb{S}$. Say that A and B are *strongly linked* if the vertices of A and B can be numbered so that

$$a_0 \leqslant b_0 \leqslant a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant a_3 \leqslant b_3 \leqslant a_0,$$

where a_i, $0 \leqslant i \leqslant 3$, are vertices of A and b_i, $0 \leqslant i \leqslant 3$ are vertices of B. The inequalities refer to the circular order on $\mathbb{S}$.

By definition, a *critical chord* is a chord $\overline{ab}$ with $a \neq b$ such that $\sigma_3(a) = \sigma_3(b)$.

Definition 3.2. A *(generalized) critical quadrilateral* Q is a circularly ordered quadruple $[a_0, a_1, a_2, a_3]$ of points $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant a_0$ in $\mathbb{S}$, where $\overline{a_0a_2}$ and $\overline{a_1a_3}$ are critical chords called *spikes*; critical quadrilaterals $[a_0, a_1, a_2, a_3]$, $[a_1, a_2, a_3, a_0]$, $[a_2, a_3, a_0, a_1]$ and $[a_3, a_0, a_1, a_2]$ are viewed as equal.

We will often say "critical quadrilateral" when talking about the convex hull of a critical quadrilateral. Clearly, if all vertices of a critical quadrilateral are distinct, or if its convex hull is a critical leaf, then the quadrilateral is uniquely defined by its convex hull. However, if the convex hull is a triangle, this is no longer true. For example, let $\mathrm{CH}(a, b, c)$ be an all-critical triangle. Then $[a, a, b, c]$ is a critical quadrilateral, but so are $[a, b, b, c]$ and $[a, b, c, c]$. If all vertices of a critical quadrilateral Q are pairwise distinct, then we call Q *non-degenerate*. Otherwise Q is called *degenerate*. Vertices a_0 and a_2 (a_1 and a_3) are called *opposite*.

Lemma 3.3 (Lemma 5.2 [BOPT16])**.** *The family of all critical quadrilaterals is closed in $\mathfrak{C}(\mathbb{D})$. The family of all critical quadrilaterals that are critical sets of cubic laminations is closed too.*

Being strongly linked is a closed condition on two quadrilaterals: if two sequences of critical quadrilaterals A_i, B_i are such that A_i and B_i are strongly linked and $A_i \to A$, $B_i \to B$, then A and B are strongly linked critical quadrilaterals.

In [BOPT16], quadratically critical portraits are defined for any degree d. Below, we adapt this definition for cubic laminations. Consider distinct critical quadrilaterals Q^1, Q^2 whose relative interiors are disjoint. (Recall that the *relative interior* of a subset $X \subset \mathbb{C}$ is the interior of X in the affine hull of X). The pair (Q^1, Q^2) is called a *quadratically critical portrait*. If $\mathcal{L}$ is a cubic lamination such that Q^1, Q^2 are leaves or gaps of $\mathcal{L}$, then we say that (Q^1, Q^2) is a *quadratically critical portrait of $\mathcal{L}$*. Observe that not all cubic laminations admit quadratically critical portraits. For example, if $\mathcal{L}$ has a unique critical object that is not all-critical (say, if this critical object is a hexagon that maps forward in the three-to-one fashion), then $\mathcal{L}$ has no quadratically critical portrait. If $\mathcal{L}$ has two disjoint critical objects, then it admits a quadratically critical portrait if and only if both critical objects are (possibly degenerate) critical quadrilaterals.

Assume that $\mathcal{L}$ has an all-critical triangle Δ. Then possible quadratically critical portraits of $\mathcal{L}$ are:

(1) pairs of distinct edges of Δ; and
(2) pairs consisting of Δ and an edge of it.

Now we define linked quadratically critical portraits.

Definition 3.4. Let (Q^1_1, Q^2_1) and (Q^1_2, Q^2_2) be quadratically critical portraits. These two portraits are said to be *linked or essentially equal* if one of the following holds.

(1) For every $j = 1, 2$, the quadrilaterals Q^j_1 and Q^j_2 are strongly linked. If Q^j_1 and Q^j_2 share a spike for every $j = 1, 2$, then the two portraits are said to be essentially equal.
(2) We have that $\mathrm{CH}(Q^1_1 \cup Q^2_1) = \mathrm{CH}(Q^1_2 \cup Q^2_2)$ is an all-critical triangle. In this case, the two portraits are also said to be essentially equal.

If (1) but (Q^1_1, Q^2_1) and (Q^1_2, Q^2_2) are not essentially equal, then the two portraits are said to be *linked*.

Critically marked polynomials, topological polynomials, and laminational equivalence relations were defined in the introduction. Let us now define critically marked cubic laminations. Suppose that $\mathcal{L}$ is a cubic lamination and an ordered pair of critical sets (gaps or leaves) C, D of $\mathcal{L}$ is chosen so that on the boundary of each component E of $\overline{\mathbb{D}} \setminus (C \cup D)$ the map σ_3 is one-to-one (except for the endpoints of possibly existing critical edges of such components). Then we call $(\mathcal{L}, C, D)$ a *critically marked* lamination. For brevity, we often talk about *marked* (topological) polynomials and laminations meaning *critically marked* ones. Let $(\mathcal{L}, C^1, C^2)$ be a marked cubic lamination. Then (C^1, C^2) is called a *critical pattern* of $\mathcal{L}$; when talking about *critical patterns* we mean critical patterns of some marked lamination $\mathcal{L}$ and allow for $\mathcal{L}$ to be unspecified.

Let $\mathcal{L}$ be a dendritic lamination. If $C \neq D$ are its critical sets, then the only two possible critical patterns that can be associated with $\mathcal{L}$ are (C, D) or (D, C). If $\mathcal{L}$ has a unique critical set X that is not an all-critical triangle, then the only possible critical pattern of $\mathcal{L}$ is (X, X). However, if $\mathcal{L}$ has a unique critical set Δ that is an all-critical triangle, then there are more possibilities for a critical pattern of $\mathcal{L}$. Namely, by definition, a critical pattern of $\mathcal{L}$ can be either (Δ, Δ), or Δ and an edge of Δ, or an edge of Δ and Δ, or an ordered pair of two edges of Δ.

A *collapsing quadrilateral* is a critical quadrilateral that maps to a non-degenerate leaf.

Definition 3.5. Marked laminations $(\mathcal{L}_1, C_1^1, C_1^2)$ and $(\mathcal{L}_2, C_2^1, C_2^2)$, and their critical patterns, are said to be *linked* (*essentially equal*) if there are linked (respectively, essentially equal) quadratically critical portraits (Q_1^1, Q_1^2) and (Q_2^1, Q_2^2) such that $Q_i^j \subset C_i^j$ for all $i, j = 1, 2$, and, if Q_i^j is a collapsing quadrilateral, then it shares a pair of opposite edges with C_i^j.

The following is a special case of one the central results of [BOPT16].

Theorem 3.6 (Theorem 9.6 [BOPT16])**.** *Let $(\mathcal{L}_1, C_1^1, C_1^2)$ and $(\mathcal{L}_2, C_2^1, C_2^2)$ be marked laminations. Suppose that $\mathcal{L}_1$ is perfect. If $\mathcal{L}_1$, $\mathcal{L}_2$ are linked or essentially equal, then $\mathcal{L}_1 \subset \mathcal{L}_2$ and $C_1^j \supset C_2^j$ for $j = 1, 2$. In particular, if both laminations are perfect, then $(\mathcal{L}_1, C_1^1, C_1^2) = (\mathcal{L}_2, C_2^1, C_2^2)$.*

In particular, Theorem 3.6 applies when $\mathcal{L}_1$ is dendritic, as follows from Lemma 2.8.

4. Proof of the main result

In the rest of the paper, we define a visual parameterization of the family of all marked cubic dendritic laminations.

4.1. **Convergence of marked laminations.** Let $(\mathcal{L}_i, \mathcal{Z}_i)$ be a sequence of marked cubic laminations with critical patterns $\mathcal{Z}_i = (C_i^1, C_i^2)$. Assume that the sequence $\mathcal{L}_i$ converges to a limit lamination $\mathcal{L}_\infty$. Then the critical sets C_i^1, C_i^2 converge to gaps or leaves C_∞^1, C_∞^2 of $\mathcal{L}_\infty$. We say that the sequence $(\mathcal{L}_i, \mathcal{Z}_i)$ *converges* to $(\mathcal{L}_\infty, C_\infty^1, C_\infty^2)$.

Lemma 4.1. *Suppose that a sequence $(\mathcal{L}_i, \mathcal{Z}_i)$ of marked cubic laminations with finite critical sets converges to $(\mathcal{L}_\infty, C^1_\infty, C^2_\infty)$. Then sets C^1_∞, C^2_∞ are critical and non-periodic, and $(\mathcal{L}_\infty, C^1_\infty, C^2_\infty)$ is a marked lamination.*

Proof. Every vertex of C^1_∞ has a sibling vertex in C^1_∞. It follows that C^1_∞ is critical. If C^1_∞ is periodic of period, say, n, then, since it is critical, it is an infinite gap. Then the fact that $\sigma_d^n(C^1_\infty) = C^1_\infty$ implies that any gap C^1_i sufficiently close to C^1_∞ will have its σ_3^n-image also close to C^1_∞, and therefore coinciding with C^1_i. Thus, C^1_i is σ_3-periodic, which is impossible because C^1_i is finite and critical. Similarly, C^2_∞ is critical and non-periodic.

Let us show that $(\mathcal{L}_\infty, C^1_\infty, C^2_\infty)$ is a marked lamination. To this end we need to show that on the boundary of each component E of $\overline{\mathbb{D}} \setminus (C^1_\infty \cup C^2_\infty)$ the map σ_3 is one-to-one (except for the endpoints of possibly existing critical edges of such components). This follows from definitions and the fact that the same claim holds for all $(\mathcal{L}_i, \mathcal{Z}_i)$. □

Any marked lamination similar to $(\mathcal{L}_\infty, C^1_\infty, C^2_\infty)$ from Lemma 4.1 will be called a *limit marked lamination*. In particular, a marked dendritic lamination is a limit marked lamination (consider a constant sequence).

As was explained in the Introduction, a marked cubic dendritic polynomial always defines a marked cubic lamination. Take a marked dendritic polynomial (P, c^1, c^2) and let $(\mathcal{L}, C^1, C^2)$ be the corresponding marked lamination. Define the map $\Gamma : \mathcal{MD}_3 \to \mathfrak{C}(\overline{\mathbb{D}}) \times \mathfrak{C}(\overline{\mathbb{D}})$ by setting $\Gamma(P, c^1, c^2) = (C^1, C^2)$. Consider a sequence of marked dendritic cubic laminations $(\mathcal{L}_i, C^1_i, C^2_i)$. If $\mathcal{L}_i$ converge, then the limit $\mathcal{L}_\infty$ is itself a cubic lamination, and, by the above, the critical patterns (C^1_i, C^2_i) converge to the critical pattern (C^1_∞, C^2_∞) of $\mathcal{L}_\infty$. We are interested in the case when $\mathcal{L}_\infty$ is in a sense compatible with a dendritic lamination.

Lemma 4.2 (Lemma 6.18 [BOPT16])**.** *Let $(\mathcal{L}_i, C^1_i, C^2_i)$ and $(\mathcal{L}_\infty, C^1_\infty, C^2_\infty)$ be as above. If there exists a dendritic cubic geodesic lamination $\mathcal{L}$ with a critical pattern (C^1, C^2) such that $C^j_\infty \subset C^j$ for $j = 1, 2$, then $\mathcal{L}_\infty \supset \mathcal{L}$.*

Lemma 4.2 says that if critical patterns of dendritic cubic geodesic laminations converge **into** a critical pattern of a dendritic cubic geodesic lamination $\mathcal{L}$, then the limit lamination contains $\mathcal{L}$. Recall that convergence in the space of marked polynomials is understood as convergence of the coefficients and of the marked critical points.

Corollary 4.3 (Corollary 6.20 [BOPT16])**.** *Suppose that a sequence (P_i, c^1_i, c^2_i) of marked cubic dendritic polynomials converges to a marked cubic dendritic polynomial (P, c^1, c^2). Consider corresponding marked laminational equivalence relations $(\sim_{P_i}, C^1_i, C^2_i)$ and $(\sim_P, C^1, C^2)$. If $(\mathcal{L}_{\sim_{P_i}}, C^1_i, C^2_i)$ converges to $(\mathcal{L}_\infty, C^1_\infty, C^2_\infty)$, then we have $\mathcal{L}_\infty \supset \mathcal{L}_{\sim_P}, C^1_\infty \subset C^1, C^2_\infty \subset C^2$. In particular, the map Γ is upper semi-continuous.*

By Corollary 4.3, critical objects of dendritic cubic laminations $\mathcal{L}_{\sim_P}$ associated with polynomials $P \in \mathcal{MD}_3$ cannot explode under perturbation of P (they may implode though).

4.2. **Mixed tags of geodesic laminations.**

Definition 4.4 (Minor set)**.** Let $(\mathcal{L}, C, D)$ be a marked lamination. Then $\sigma_3(D)$ is called the *minor set of* $(\mathcal{L}, C, D)$.

Note that, in Definition 4.5, the set C is not assumed to be critical.

Definition 4.5 (Co-critical set)**.** Let C be a leaf or a gap of a cubic lamination $\mathcal{L}$. Assume that either C is the only critical object of $\mathcal{L}$, or there is exactly one hole of C of length $> \frac{1}{3}$. If C is the only critical object of $\mathcal{L}$, then we set $\mathrm{co}(C) = C$. Otherwise, let H be the unique hole of C of length $> \frac{1}{3}$, let A be the set of all points in H with the images in $\sigma_3(C)$, and set $\mathrm{co}(C) = \mathrm{CH}(A)$. The set $\mathrm{co}(C)$ is called the *co-critical set* of C.

We now define tags of marked laminations.

Definition 4.6 (Mixed tag)**.** Suppose that $(\mathcal{L}, C^1, C^2)$ is a marked lamination. Then we call the set $\mathrm{Tag}_l(C^1, C^2) = \mathrm{co}(C^1) \times \sigma_3(C^2) \subset \overline{\mathbb{D}} \times \overline{\mathbb{D}}$ the *mixed tag* of $(\mathcal{L}, C^1, C^2)$ or of (C^1, C^2).

Sets $\mathrm{co}(C^1)$ (and hence mixed tags) are well-defined. The mixed tag T of a marked lamination is the product of two sets, each of which is a point, a leaf, or a gap. One can think of $T \subset \overline{\mathbb{D}} \times \overline{\mathbb{D}}$ as a higher dimensional analog of a gap/leaf of a geodesic lamination. We show that the union of tags of marked dendritic laminations is a (non-closed) "geodesic lamination" in $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$. The main idea is to relate the non-disjointness of mixed tags of marked dendritic laminations and their limits with the fact that they have "tunings" that are linked or essentially equal.

In Definition 4.7, we mimic Milnor's terminology for polynomials.

Definition 4.7 (Unicritical and bicritical laminations)**.** A marked lamination (and its critical pattern) is called *unicritical* if its critical pattern is of form (C, C) for some critical set C and *bicritical* otherwise.

Clearly, a unicritical marked lamination has a unique critical object. However a lamination $\mathcal{L}$ with unique critical object may have a bicritical critical pattern. By definition this is only possible if $\mathcal{L}$ has an all-critical gap Δ and the critical pattern is either two edges of Δ or Δ and an edge of Δ.

The following lemma is a key combinatorial fact about tags.

Lemma 4.8. *Suppose that two marked laminations have non-disjoint mixed tags. Suppose also that at least one of the two laminations is dendritic. Then either the two marked laminations are linked or essentially equal, or the geodesic laminations are equal and share an all-critical triangle.*

The proof of Lemma 4.8 is mostly non-dynamic and involves checking various cases. We split the proof into propositions. Observe that mixed tags are determined by critical patterns; we do not need laminations to define mixed tags. In Propositions 4.9 — 4.10, we assume that the critical patterns (C_1^1, C_1^2) and (C_2^1, C_2^2) are bicritical and have non-disjoint mixed tags.

Proposition 4.9. *Suppose that some distinct edges of* $\mathrm{co}(C_1^1)$ *and* $\mathrm{co}(C_2^1)$ *cross. Then the two critical patterns are linked or essentially equal.*

Proof. By the assumption, some distinct edges of the sets $\mathrm{co}(C_1^1)$ and $\mathrm{co}(C_2^1)$ cross. Denote these linked edges by $\overline{a_1b_1}$ and $\overline{a_2b_2}$, see Fig. 2. We may choose the orientation so that (a_1, b_1), (a_2, b_2) are in the holes of $\mathrm{co}(C_1^1)$, $\mathrm{co}(C_2^1)$ disjoint from C_1^1, C_2^1 respectively, and so that $a_1 < a_2 < b_1 < b_2$. We claim that (a_1, b_1) is of length at most $\frac{1}{3}$. Indeed, if (a_1, b_1) had length greater than $\frac{1}{3}$, then there would exist a sibling ℓ of $\overline{a_1b_1}$ with endpoints in (a_1, b_1). Evidently, ℓ would be an edge of C_1^1, contradicting the choice of (a_1, b_1). Thus, (a_1, b_1) is of length at most $\frac{1}{3}$ and the restriction $\sigma_3|_{(a_1,b_1)}$ is one-to-one. Similarly, (a_2, b_2) is of length at most $\frac{1}{3}$ and the restriction $\sigma_3|_{(a_2,b_2)}$ is one-to-one.

Let us show now that $\sigma_3(C_1^2) \cap \mathbb{S} \subset [\sigma_3(b_1), \sigma_3(a_1)]$. If $C_1^1 = C_1^2$ is of degree three, this follows immediately. Otherwise, let $a_1' = a_1 + \frac{1}{3}$ and $b_1' = b_1 + \frac{2}{3}$. Then $\overline{a_1'b_1'} \subset C_1^1$. Moreover, since C_1^1 is critical, vertices of C_1^1 partition the arc (a_1', b_1') into open arcs on each of which the map is one-to-one. Hence C_1^2 must have vertices in $[b_1', a_1] \cup [b_1, a_1']$. Since each of these intervals maps onto $[\sigma_3(b_1), \sigma_3(a_1)]$ one-to-one, our claim follows.

We claim that $b_2 \leqslant a_1 + \frac{1}{3}$. Indeed, otherwise $[b_1, a_1 + \frac{1}{3}] \subset [a_2, b_2)$, which implies that $[\sigma_3(b_1), \sigma_3(a_1)] \subset [\sigma_3(a_2), \sigma_3(b_2))$. On the other hand, by the above we have $\sigma_3(C_1^1) \subset [\sigma_3(b_1), \sigma_3(a_1)]$ and $\sigma_3(C_2^1) \subset [\sigma_3(b_2), \sigma_3(a_2)]$. Since $\sigma_3(C_1^2) \cap \sigma_3(C_2^2) \neq \varnothing$, we have in fact $b_1 = a_2$, a contradiction. Thus, the points a_i and b_i for $i = 1, 2$ belong to an arc of length at most $\frac{1}{3}$.

We claim that then $\mathrm{co}(\overline{a_1b_1}) = Q_1^1$ and $\mathrm{co}(\overline{a_2b_2}) = Q_2^1$ are strongly linked collapsing quadrilaterals. Indeed, we have that $a_1 < a_2 < b_1 < b_2 \leqslant a_1 + \frac{1}{3}$. It follows then that

$$a_1 + \frac{1}{3} < a_2 + \frac{1}{3} < b_1 + \frac{1}{3} < b_2 + \frac{1}{3} \leqslant a_1 + \frac{2}{3} < a_2 + \frac{2}{3} < b_1 + \frac{2}{3} < b_2 + \frac{2}{3} \leqslant a_1,$$

and therefore that, indeed, Q_1^1 and $\mathrm{co}(\overline{a_2b_2}) = Q_2^1$ are strongly linked collapsing quadrilaterals. Moreover, since $\overline{a_1b_1}$ and $\overline{a_2b_2}$ are edges of $\mathrm{co}(C_1^1)$ and $\mathrm{co}(C_2^1)$ it follows that the quadrilateral Q_1^1 shares two edges with the set C_1^1, and the quadrilateral Q_2^1 shares two edges with the set C_2^1.

Note that all vertices of C_1^2 and C_2^2 are in $[b_2, a_1'] \cup [b_2', a_1]$, where $a_1' = a_1 + \frac{1}{3}$ and $b_2' = b_2 + \frac{2}{3}$. The restriction of σ_3 to each of the arcs $[b_2, a_1']$, $[b_2', a_1]$ is injective. Therefore, a pair of linked edges (or a pair of coinciding

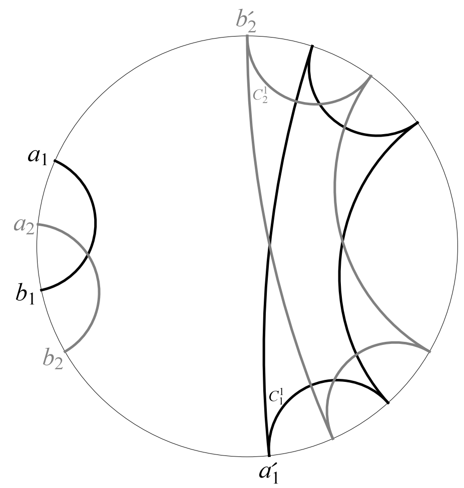

FIGURE 2. This figure illustrates Proposition 4.9.

vertices) of $\sigma_3(C_1^2)$ and $\sigma_3(C_2^2)$ gives rise to a pair of linked quadrilaterals Q_1^2 and Q_2^2 in C_1^2 and C_2^2, respectively, so that if these quadrilaterals share edges with containing them critical sets. $\square$

Now we simply assume that $\mathrm{co}(C_1^1)$ and $\mathrm{co}(C_2^1)$ intersect.

Proposition 4.10. *If $\mathcal{L}_1$ is dendritic, and $(\mathcal{L}_2, C_2^1, C_2^2)$ is a limit marked lamination, then at least one of the following holds:*

(1) *the two critical patterns are linked or essentially equal;*
(2) *$\mathcal{L}_1 = \mathcal{L}_2$ share an all-critical triangle Δ.*

Proof. We will use the same notation as in the proof of Proposition 4.9. If $\mathrm{co}(C_1^1)$ and $\mathrm{co}(C_2^1)$ have distinct edges that cross in $\mathbb{D}$, then Proposition 4.9 applies. Assume now that $\mathrm{co}(C_1^1)$ and $\mathrm{co}(C_2^1)$ share a vertex a. Clearly, there is a unique critical chord ℓ such that $\mathrm{co}(a) = \ell$. Then $C_1^1 \cap C_2^1 \supset \ell$, and we may set $Q_1^1 = Q_2^1 = \ell$.

Both sets C_i^2 have vertices in the closed arc A of length $\frac{2}{3}$ bounded by the endpoints of ℓ. By our assumption, $\sigma_3(C_1^2) \cap \sigma_3(C_2^2) \neq \varnothing$. If the sets $\sigma_3(C_i^2), i = 1, 2$ have a pair of linked edges or share a vertex $z \neq \sigma_3(\ell)$, then these edges or z can by pulled back to $\mathrm{CH}(A)$ as a pair of linked critical quadrilaterals. Assume now that $\sigma_3(C_1^2) \cap \sigma_3(C_2^2) = \{\sigma_3(\ell)\}$.

Clearly, $a \in A$. Set $\Delta = \mathrm{CH}(a, \ell)$. We claim that Δ is a gap of $\mathcal{L}_1$. Indeed, the set C_1^2 contains at least two vertices of Δ and is non-disjoint from C_1^1. Since $\mathcal{L}_1$ is dendritic, $\mathcal{L}_1$ has a unique critical object E. If $E \neq \Delta$, then by definition the critical pattern of $\mathcal{L}_1$ is (E, E), a contradiction with the assumption that $\mathcal{L}_1$ is bicritical. Thus, Δ is a gap of $\mathcal{L}_1$.

We claim that Δ is a gap $\mathcal{L}_2$. We prove first that there is an edge $\ell^* \neq \ell$ of Δ such that one of the sets C_2^1, C_2^2 contains ℓ while the other one contains ℓ^*. This is obvious if C_2^2 contains an edge $\ell^* \neq \ell$ of Δ. Otherwise $C_2^2 \supset \ell$. Then ℓ must be an edge of C_2^2 because otherwise the sets C_2^1 and C_2^2 will have either non-disjoint interiors, or one of them is contained in the interior of the other one, a contradiction. Similarly, ℓ is an edge of C_2^1. It follows that one of the sets C_2^1, C_2^2 is ℓ while the other one is a critical gap $G \neq \Delta$ with ℓ as an edge.

By the above, ℓ and ℓ^* are either leaves of $\mathcal{L}_2$ or are contained in gaps of $\mathcal{L}_2$. Moreover endpoints of ℓ and ℓ^* are not periodic since Δ is a gap of a dendritic lamination $\mathcal{L}_1$. Hence ℓ and ℓ^* can be pulled back in a unique way and its pullbacks either will be contained in gaps of $\mathcal{L}_2$ or will be leaves of $\mathcal{L}_2$. This yields a new lamination $\widehat{\mathcal{L}}_2 \supset \mathcal{L}_2$ and a marked lamination $(\widehat{\mathcal{L}}_2, \ell, \ell^*)$. Consider also the marked lamination $(\mathcal{L}_1, \ell, \ell^*)$. Since these two marked laminations are essentially equal, Theorem 3.6 implies that $\mathcal{L}_1 \subset \widehat{\mathcal{L}}_2$. Hence Δ is a gap of $\widehat{\mathcal{L}}_2$ and, moreover, leaves shared by $\mathcal{L}_1$ and $\widehat{\mathcal{L}}_2$ approximate all edges of Δ from outside of Δ.

It follows that Δ is a subset of a gap G of $\mathcal{L}_2$. Let us show that $G = \Delta$. By Lemma 4.1, G is not periodic. Hence pullbacks of ℓ and ℓ^* do not reenter G, and so an edge of Δ contained in the interior of G (except for the endpoints) remains isolated in both $\mathcal{L}_2$ and $\widehat{\mathcal{L}}_2$. However in the previous paragraph we concluded that it is not isolated in $\widehat{\mathcal{L}}_2$, a contradiction. We conclude that Δ is a gap of $\mathcal{L}_2$.

Let us show that $\mathcal{L}_1 = \mathcal{L}_2$. We can adjust the critical pattern of $\mathcal{L}_2$ so that it coincides with the critical pattern of $\mathcal{L}_1$. By Theorem 3.6, we then have $\mathcal{L}_2 \supset \mathcal{L}_1$. Moreover, no leaves of $\mathcal{L}_2$ are contained in the unique critical set Δ of $\mathcal{L}_1$. By [Kiw02], any periodic gap of $\mathcal{L}_1$ has a single cycle of edges. We conclude that no leaves of $\mathcal{L}_2$ are contained in periodic or preperiodic gaps of $\mathcal{L}_1$. Finally, by [BL02] there are no wandering gaps of $\mathcal{L}_1$. This implies that $\mathcal{L}_2 = \mathcal{L}_1$, as claimed. □

This proves Lemma 4.8 for two bicritical marked laminations. Consider unicritical marked laminations.

Lemma 4.11. *Suppose that $(\mathcal{L}_1, C_1, C_1)$ and $(\mathcal{L}_2, C_2, C_2)$ are marked unicritical laminations with non-disjoint mixed tags. Then $(\mathcal{L}_1, C_1^1, C_1^2)$ and $(\mathcal{L}_2, C_2^1, C_2^2)$ are linked or essentially equal where C_i^j either equals C_i or is a critical chord contained in C_i.*

Proof. Suppose that $\mathcal{L}_1$ has an all-critical triangle Δ (and so $C_1 = \Delta$). Since the mixed tags intersect, then $\sigma_3(C_1) \in \sigma_3(C_2)$ and hence $C_1 \subset C_2$. Choosing two edges of Δ as a quadratically critical portrait in C_1 and in C_2, we see that by definition $(\mathcal{L}_1, C_1^1, C_1^2)$ and $(\mathcal{L}_2, C_2^1, C_2^2)$ are essentially equal. Suppose that neither sibling invariant geodesic lamination has an all-critical triangle. If $\sigma_3(C_1) \cap \sigma_3(C_2)$ contains a point $x \in \mathbb{S}$, then the entire all-critical triangle $\mathrm{CH}(\sigma_3^{-1}(x)) = \Delta$ is contained in $C_1 \cap C_2$; we can choose the same two edges of Δ as a quadratically critical portrait for both laminations. Otherwise, we may assume that an edge ℓ_1 of $\sigma_3(C_1)$ crosses an edge ℓ_2 of $\sigma_3(C_2)$. This implies that the hexagons $\sigma_3^{-1}(\ell_1) \subset C_1$ and $\sigma_3^{-1}(\ell_2) \subset C_2$ have alternating vertices and proves the lemma in this case too. □

Proof of Lemma 4.8. Denote laminations in question by $\mathcal{L}_1$ and $\mathcal{L}_2$. If both laminations are bicritical, then the result follows from Proposition 4.10. If both laminations are unicritical, then the result follows from Lemma 4.11. It remains to consider the case where the first critical pattern (C_1, C_1) is unicritical, and the second one (C_2^1, C_2^2) is bicritical.

Either an edge of $\sigma_3(C_1)$ crosses an edge of $\sigma_3(C_2^2)$ or a vertex of $\sigma_3(C_1)$ lies in $\sigma_3(C_2^2)$. In either case, there are two sibling edges or sibling vertices of C_2^2 that are linked or coincide with edges or vertices of C_1. Taking convex hulls of these pairs of (possibly degenerate) leaves, we obtain $Q_1^2 \subset C_1$ and $Q_2^2 \subset C_2^2$. By construction, these are strongly linked quadrilaterals. Similarly, there is a (possibly degenerate) leaf in $\mathrm{co}(C_2^1)$ that is linked or equal to a leaf in C_1. It follows that the two siblings of this leaf in C_2^1 are linked or equal to some leaves in C_1. As above, this leads to strongly linked quadrilaterals $Q_1^1 \subset C_1$ and $Q_2^1 \subset C_2^1$. It is easy to see that (Q_1^1, Q_1^2) and (Q_2^1, Q_2^2) are linked or essentially equal quadratically critical portraits. □

We are ready to prove Theorem 4.12.

Theorem 4.12. *If $(\mathcal{L}_1, C_1^1, C_1^2)$ and $(\mathcal{L}_2, C_2^1, C_2^2)$ are marked laminations and $\mathcal{L}_1$ is dendritic, then they have non-disjoint mixed tags if and only if* (1) *or* (2) *holds:*

(1) *$\mathcal{L}_1 = \mathcal{L}_2$ has an all-critical triangle Δ, it is not true that C_1^1 and C_2^1 are distinct edges of Δ, and either $C_1^1 \supset C_2^1$, or $C_2^1 \supset C_1^1$;*
(2) *there is no all-critical triangle in $\mathcal{L}_1 \subset \mathcal{L}_2$, and $C_1^j \supset C_2^j$ for $j = 1, 2$ (in particular, if $\mathcal{L}_2$ is dendritic then $\mathcal{L}_1 = \mathcal{L}_2$).*

Proof. If the mixed tags of $(\mathcal{L}_1, C_1^1, C_1^2)$ and $(\mathcal{L}_2, C_2^1, C_2^1)$ are non-disjoint, then, by Lemma 4.8, either $\mathcal{L}_1 = \mathcal{L}_2$ share an all-critical triangle Δ, or these marked laminations are linked or essentially equal. In the first case consider several possibilities for the critical patterns. One can immediately see that the only way the mixed tags are disjoint is when C_1^1 and C_2^1 are distinct edges of Δ; since the mixed tags are known to be non-disjoint we see that this corresponds to case (1) from the theorem. In the second case the fact that our marked laminations are linked or essentially equal implies, by Theorem 3.6, that case (2) of the theorem holds. The opposite direction of the theorem follows from definitions. □

4.3. **Upper semi-continuous tags.**

Definition 4.13. A collection $\mathcal{E} = \{E_\alpha\}$ of compact and disjoint subsets of a metric space X is *upper semi-continuous* (*USC*) if, for every E_α and every open set $U \supset E_\alpha$, there exists an open set V containing E_α so that, for each $E_\beta \in \mathcal{E}$, if $E_\beta \cap V \neq \varnothing$, then $E_\beta \subset U$. A decomposition of a metric space is said to be *upper semi-continuous* (*USC*) if the corresponding collection of sets is upper semi-continuous.

Upper semi-continuous decompositions are studied in [Dav86].

Theorem 4.14 ([Dav86])**.** *If $\mathcal{E}$ is an upper semicontinuous decomposition of a separable metric space X, then the quotient space $X/\mathcal{E}$ is also a separable metric space.*

Consider a marked cubic lamination $(\mathcal{L}_\sim, C_1, C_2)$. Suppose that $\mathcal{L}_\sim$ is generated by a laminational equivalence relation $\sim$. Observe that $(\sim, C_1, C_2)$ does not have to be a marked laminational equivalence relation. Indeed, if the critical object of $\mathcal{L}_\sim$ is an all-critical triangle Δ, then the only marked laminational equivalence corresponding to $\sim$ is $(\sim, \Delta, \Delta)$. However, C_1, C_2 can be two distinct edges of Δ. Despite this discrepancy, mixed tags of laminational equivalence relations coincide with the mixed tags of the corresponding geodesic laminations. Thus our results apply to mixed tags of laminational equivalence relations.

Recall that the map Tag_l was defined in Definition 4.6. To a marked laminational equivalence relation $(\sim, C, D)$, or to its critical pattern (C, D), the map Tag_l associates the corresponding *mixed tag* $\mathrm{Tag}_l(\sim, C, D) = \mathrm{co}(C) \times \sigma_3(D) \subset \overline{\mathbb{D}} \times \overline{\mathbb{D}}$.

Theorem 4.15. *The family $\{\mathrm{Tag}_l(C^1, C^2)\} = \mathrm{CML}(\mathcal{D})$ of mixed tags of cubic marked dendritic laminational equivalence relations forms an upper semi-continuous decomposition of the union $\mathrm{CML}(\mathcal{D})^+$ of all these tags.*

Proof. If $(\sim_1, C_1^1, C_1^2)$ and $(\sim_2, C_2^1, C_2^2)$ are cubic marked dendritic laminational equivalence relations, and $\mathrm{Tag}_l(C_1^1, C_1^2)$ and $\mathrm{Tag}_l(C_2^1, C_2^2)$ are non-disjoint, then, by Theorem 4.12 applied to the marked geodesic laminations $(\mathcal{L}_{\sim_1}, C_1^1, C_1^2)$ and $(\mathcal{L}_{\sim_1}, C_2^1, C_2^2)$, we have that the corresponding marked laminational equivalence relations are equal, i.e. $(\mathcal{L}_{\sim_1}, C_1^1, C_1^2) = (\mathcal{L}_{\sim_2}, C_2^1, C_2^2)$. Hence the family $\{\mathrm{Tag}_l(C^1, C^2)\}$ forms a decomposition of $\mathrm{CML}(\mathcal{D})^+$.

Suppose next that $(\sim_i, \mathcal{Z}_i)$ is a sequence of marked dendritic laminational equivalence relations with $\mathcal{Z}_i = (C_i^1, C_i^2)$. Assume that there is a limit point of the sequence of their tags $\mathrm{co}(C_i^1) \times \sigma_3(C_i^2)$ that belongs to the tag of a marked dendritic laminational equivalence $(\sim_D, \mathcal{Z}_D)$ where $\mathcal{Z}_D = (C_D^1, C_D^2)$. By [BMOV13] and Lemma 4.1, we may assume that the sequence $(\mathcal{L}_{\sim_i}, \mathcal{Z}_i)$ converges to a marked lamination $(\mathcal{L}_\infty, C_\infty^1, C_\infty^2)$ with critical pattern $\mathcal{P}_\infty = (C_\infty^1, C_\infty^2)$. By the assumption, $\mathrm{Tag}_l(\mathcal{Z}_D) \cap \mathrm{Tag}(\mathcal{P}_\infty) \neq \varnothing$. By Theorem 4.12, we have $\mathcal{L}_D \subset \mathcal{L}_\infty$ and $C_\infty^j \subset C_D^j$ for $j = 1, 2$. Hence, $\mathrm{Tag}_l(\mathcal{L}_\infty, \mathcal{P}_\infty) \subset \mathrm{Tag}_l(\mathcal{L}_D, \mathcal{Z}_D)$. □

Denote the quotient space of $\mathrm{CML}(\mathcal{D})^+$, obtained by collapsing every element of $\mathrm{CML}(\mathcal{D})$ to a point, by $\mathcal{MD}_3^{comb}$ (elements of $\mathrm{CML}(\mathcal{D})$ are mixed tags of critical patterns of marked dendritic laminational equivalence relations). Let $\pi : \mathrm{CML}(\mathcal{D})^+ \to \mathcal{MD}_3^{comb}$ be the quotient map. By Theorem 4.14, the topological space $\mathcal{MD}_3^{comb}$ is separable and metric. We show that $\mathcal{MD}_3^{comb}$ can be viewed as a combinatorial model for $\mathcal{MD}_3$. Recall that the map $\Gamma : \mathcal{MD}_3 \to \mathfrak{C}(\overline{\mathbb{D}}) \times \mathfrak{C}(\overline{\mathbb{D}})$ was defined right before Lemma 4.2.

Theorem 4.16. *The composition $\pi \circ \mathrm{Tag}_l \circ \Gamma : \mathcal{MD}_3 \to \mathcal{MD}_3^{comb}$ is a continuous surjective map.*

Proof. By definition and Corollary 4.3, the map Γ is upper semi-continuous and surjective. Also, Tag_l is continuous with respect to the Hausdorff distance and preserves inclusions. Finally, π is continuous by definition. Thus, $\pi \circ \mathrm{Tag}_l \circ \Gamma : \mathcal{MD}_3 \to \mathcal{MD}_3^{comb}$ is a continuous surjective map, as desired. □

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