# DYNAMICAL CORES OF TOPOLOGICAL POLYNOMIALS

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ABSTRACT. We define the (dynamical) core of a topological polynomial (and the associated lamination). This notion extends that of the core of a unimodal interval map. Two explicit descriptions of the core are given: one related to periodic objects and one related to critical objects.

## 1. Introduction and the main result

1.1. **Motivation.** Complex dynamics studies, among other topics, limit behavior of points under iterations of complex polynomials. This problem is meaningful if we consider the restriction of a polynomial to its Julia set as elsewhere the limit behavior of points is easy to describe. Since in many cases polynomial Julia sets are one-dimensional continua, one can consider the problem as a far reaching generalization of the similar problem for simple one-dimensional spaces such as an interval.

A popular one-dimensional family is that of *unimodal interval maps*, i.e. interval maps with unique turning point. Often such maps $f$ are considered on $[0,1]$ and normalized by assuming that the turning point in question is a local maximum and that $f(0)=f(1)=0$. It is easy to see that the only case when such map $f$ can exhibit non-trivial dynamics is when $f^2(c)<c<f(c)$. Moreover, all points of $[0,1]$ either eventually map to $[f^2(c),f(c)]$, or converge to a fixed point of $f$. The interval $[f^2(c),f(c)]$ is often called the *core* of $f$; we prefer to call it the *dynamical core* of $f$. In the quadratic polynomial case, when 0 is a

*Date*: June 24, 2011; in the revised form December 8, 2011.

2010 *Mathematics Subject Classification.* Primary 37F20; Secondary 37C25, 37F10, 37F50.

*Key words and phrases.* Complex dynamics; Julia set; Mandelbrot set.

The first named author was partially supported by NSF grant DMS–0901038.

The second named author was partially supported by NSF grant DMS-0906316.

The fourth named author was partially supported by the Deligne fellowship, the Simons-IUM fellowship, RFBR grants 10-01-00739-a, 11-01-00654-a, MESRF grant MK-2790.2011.1, RFBR/CNRS project 10-01-93115-NCNIL-a, and AG Laboratory NRU-HSE, MESRF grant ag. 11 11.G34.31.0023.

repelling fixed point, *all* points of $[0, 1]$ except 0 and 1 are eventually mapped to the core of $f$. A similar notion can be introduced for self-mappings of *graphs* (i.e. one-dimensional branched manifolds).

In these cases the dynamical core is a small invariant subcontinuum which captures the limit sets of all but finitely many points of the space; clearly, subsets smaller than the entire space which nevertheless contain limit sets of all but finitely many points are of interest in dynamics. However, one can also think of the dynamical core as a small invariant subcontinuum which contains the limit sets of all *cutpoints* of the space. In this form the notion of the dynamical core can be extended to polynomials with locally connected Julia sets. Still, one should justify one's interest in the dynamics of cutpoints of connected Julia sets $J$ as then, barring some exceptional cases, the set of cutpoints is not a "big" subspace of $J$ (e.g., in the locally connected case the set of cutpoints is of zero harmonic measure and of first category in $J$).

In our view, one reason for studying the set of cutpoints of $J$, despite its small size, is that the set of cutpoints carries the bulk of the structural information about $J$. Indeed, suppose that $J$ is locally connected and neither an arc nor a Jordan curve. Then it follows from a result of Hausdorff [Hau37], that the set of endpoints of $J$ is always homeomorphic to the set of all irrational numbers (this can also be seen directly by a straightforward argument). Loosely speaking, Julia sets differ inasmuch as their sets of cutpoints differ. This shows the importance of the dynamics of cutpoints and provides a justification for our interest in the dynamical core of a complex polynomial.

### 1.2. **Preliminary version of main results.**

*Topological polynomials* are topological dynamical systems that generalize complex polynomials with locally connected Julia sets restricted to their Julia sets and considered up to topological conjugacy. Note that every complex polynomial $f$ of degree $d$ with locally connected Julia set $J$ gives rise to an equivalence relation $\approx$ on the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ such that two points in $\mathbb{S}^1$ are equivalent if and only if the corresponding external rays land at the same point of $J$. Such an equivalence relation $\approx$ is forward invariant under the map $\sigma_d : \mathbb{S}^1 \to \mathbb{S}^1$, $\sigma_d(z) = z^d$, in the sense that $\sigma_d(z) \approx \sigma_d(w)$ whenever $z \approx w$.

The topological dynamics of $f$ on the Julia set can be recovered from the equivalence relation $\approx$ as follows: we consider the quotient space $J_\approx = \mathbb{S}^1/\approx$ and the map $f_\approx : J_\approx \to J_\approx$ induced by $\sigma_d$. Then the map $f_\approx : J_\approx \to J_\approx$ is topologically conjugate to the map $f|_J : J \to J$. We define a *$\sigma_d$-invariant lamination* $\sim$ as an equivalence relation on

$\mathbb{S}^1$ subject to certain assumptions similar to those satisfied by $\approx$ above (see Section 2 for a more complete description). The set $J_\sim = \mathbb{S}^1/\sim$ is called the *topological Julia set*. Then the map $f_\sim : J_\sim \to J_\sim$ is defined as the map induced on $J_\sim$ by $\sigma_d$ and is called a *topological polynomial*. There is a natural embedding of $J_\sim$ into the plane and a natural extension of $f_\sim$ as a branched self-covering of the plane. We will write $f_\sim$ for both the topological polynomial and its extension to the plane. The components of the complement of $J_\sim$ in the plane are called *Fatou components* (of $f_\sim$).

Define an *atom* of a topological polynomial $f_\sim$ as either a singleton in $J_\sim$ or the boundary of some bounded Fatou component. A *cut-atom* is by definition an atom, whose removal disconnects the topological Julia set. In particular, a point $a \in J_\sim$ is a *cutpoint* if $\{a\}$ is a cut-atom.

An atom $A$ of $J_\sim$ is said to be a *persistent cut-atom* if all its iterated $f_\sim$-images are cut-atoms. A periodic atom $A$ of minimal period $q$ is said to be *rotational* if either $A$ is a cutpoint, and $f^q_\sim$ gives rise to a non-trivial permutation of the germs of complementary components of $A$ in $J_\sim$, or $A$ is the boundary of some Fatou component such that $f^q_\sim : A \to A$ is of degree one and different from the identity.

A continuum $C \subset J_\sim$ is said to be *complete* if, for every bounded Fatou component $U$ of $f_\sim$, the intersection $\mathrm{Bd}(U) \cap C$ is either empty, or a singleton, or the entire boundary $\mathrm{Bd}(U)$. Let $\mathrm{IC}_{f_\sim}(A)$ (or $\mathrm{IC}(A)$ if $\sim$ is fixed) be the smallest complete ***i**nvariant **c**ontinuum in* $J_\sim$ containing a set $A \subset J_\sim$; we call $\mathrm{IC}(A)$ the *dynamical span* of $A$. Recall that the *$\omega$-limit set* $\omega(Z)$ of a set (e.g., a singleton) $Z \subset J_\sim$ is defined as

$$\omega(Z) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} f^i_\sim(Z)}.$$

**Definition 1.1** (Dynamical core)**.** The (*dynamical*) *core* $\mathrm{COR}(f_\sim)$ of $f_\sim$ is the dynamical span of the union of the $\omega$-limit sets of all persistent cut-atoms. The union of all periodic cut-atoms of $f_\sim$ is denoted by $\mathrm{PC}(f_\sim) = \mathrm{PC}$ and is called the *periodic core* of $f_\sim$. Finally, the union of all periodic rotational atoms of $f_\sim$ is denoted by $\mathrm{PC}_{rot}(f_\sim) = \mathrm{PC}_{rot}$ and is called the *periodic rotational core* of $f_\sim$.

One of the aims of our paper is to illustrate the analogy between the dynamics of topological polynomials on their *cutpoints* and *cut-atoms* and interval dynamics. E.g., it is known, that for interval maps periodic points and critical points play a significant, if not decisive, role. The main purpose of this paper is to establish similar facts for topological polynomials. To give a flavor of the main results, below we give a non-technical version of one of them.

**Theorem 1.2.** *The dynamical core of $f_\sim$ coincides with* $\mathrm{IC}(\mathrm{PC}(f_\sim))$. *If $J_\sim$ is a dendrite, then* $\mathrm{COR} = \mathrm{IC}(\mathrm{PC}_{rot}(f_\sim))$.

In Section 2, we state a full version of Theorem 1.2 in which we deal with several types of the dynamical core. Observe, that the so-called *growing trees* [BL02a] are related to the notion of the dynamical core.

Theorem 1.2 is related to the corresponding results for maps of the interval: if $g$ is a piecewise-monotone interval map then the closure of the union of the limit sets of all its points coincides with the closure of the set of its periodic points (see, e.g., [Blo95], where this is deduced from similar results which hold for all continuous interval maps, and references therein). Theorem 1.2 shows the importance of the periodic cores of $f_\sim$. We also introduce the notion of a *critical atom* and prove in Theorem 3.13 that the dynamical cores of a topological polynomial equal the dynamical spans of critical atoms of the restriction of $f_\sim$ onto these cores.

*Acknowledgements.* During the work on this project, the fourth author has visited Max Planck Institute for Mathematics (MPIM), Bonn. He is very grateful to MPIM for inspiring working conditions. All the authors would like to thank the referee for useful and thoughtful suggestions.

## 2. Preliminaries

Let $\mathbb{D}$ be the open unit disk and $\widehat{\mathbb{C}}$ be the complex sphere. For a compactum $X \subset \mathbb{C}$, let $U^\infty(X)$ be the unbounded component of $\mathbb{C} \setminus X$. The topological hull of $X$ equals $\mathrm{Th}(X) = \mathbb{C} \setminus U^\infty(X)$. Often we use $U^\infty(X)$ for $\widehat{\mathbb{C}} \setminus \mathrm{Th}(X)$, including the point at infinity. If $X$ is a continuum, then $\mathrm{Th}(X)$ is a *non-separating* continuum, and there exists a Riemann map $\Psi_X : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to U^\infty(X)$; we always normalize it so that $\Psi_X(\infty) = \infty$ and $\Psi'_X(z)$ tends to a positive real limit as $z \to \infty$.

Consider a polynomial $P$ of degree $d \geq 2$ with Julia set $J_P$ and filled-in Julia set $K_P = \mathrm{Th}(J_P)$. Extend $z^d : \mathbb{C} \to \mathbb{C}$ to a map $\theta_d$ on $\widehat{\mathbb{C}}$. If $J_P$ is connected then $\Psi_{K_P} = \Psi : \mathbb{C} \setminus \overline{\mathbb{D}} \to U^\infty(K_P)$ is such that $\Psi \circ \theta_d = P \circ \Psi$ on the complement of the closed unit disk [DH85a, Mil00]. If $J_P$ is locally connected, then $\Psi$ extends to a continuous function $\overline{\Psi} : \widehat{\mathbb{C}} \setminus \mathbb{D} \to \overline{\widehat{\mathbb{C}} \setminus K_P}$, and $\overline{\Psi} \circ \theta_d = P \circ \overline{\Psi}$ on the complement of the open unit disk; thus, we obtain a continuous surjection $\overline{\Psi}\colon \mathrm{Bd}(\mathbb{D}) \to J_P$ (the *Carathéodory loop*). Identify $\mathbb{S}^1 = \mathrm{Bd}(\mathbb{D})$ with $\mathbb{R}/\mathbb{Z}$.

2.1. **Laminations.** Let $J_P$ be locally connected, and set $\psi = \overline{\Psi}|_{\mathbb{S}^1}$. Define an equivalence relation $\sim_P$ on $\mathbb{S}^1$ by $x \sim_P y$ if and only if $\psi(x) = \psi(y)$, and call it the ($\sigma_d$-invariant) *lamination of* $P$. Equivalence classes of $\sim_P$ are pairwise *unlinked*: their Euclidian convex hulls are disjoint. The topological Julia set $\mathbb{S}^1/\sim_P= J_{\sim_P}$ is homeomorphic to $J_P$, and the topological polynomial $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ is topologically conjugate to $P|_{J_P}$. One can extend the conjugacy between $P|_{J_P}$ and $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ to a conjugacy on the entire plane.

An equivalence relation $\sim$ on the unit circle, with similar properties as $\sim_P$ above, can be introduced abstractly without any reference to the Julia set of a complex polynomial.

**Definition 2.1** (Laminations)**.** An equivalence relation $\sim$ on the unit circle $\mathbb{S}^1$ is called a *lamination* if it has the following properties:
(E1) the graph of $\sim$ is a closed subset in $\mathbb{S}^1 \times \mathbb{S}^1$;
(E2) if $t_1 \sim t_2 \in \mathbb{S}^1$ and $t_3 \sim t_4 \in \mathbb{S}^1$, but $t_2 \not\sim t_3$, then the open straight line segments in $\mathbb{C}$ with endpoints $t_1, t_2$ and $t_3, t_4$ are disjoint;
(E3) each equivalence class of $\sim$ is totally disconnected.

**Definition 2.2** (Laminations and dynamics)**.** A lamination $\sim$ is called ($\sigma_d$-)*invariant* if:
(D1) $\sim$ is *forward invariant*: for a class $g$, the set $\sigma_d(g)$ is a class too;
(D2) for any $\sim$-class $g$, the map $\sigma_d : g \to \sigma_d(g)$ extends to $\mathbb{S}^1$ as an orientation preserving covering map such that $g$ is the full preimage of $\sigma_d(g)$ under this covering map.
(D3) all $\sim$-classes are finite.

Part (D2) of Definition 2.1 has an equivalent version. A *(positively oriented) hole* $(a, b)$ *of a compactum* $Q \subset \mathbb{S}^1$ is a component of $\mathbb{S}^1 \setminus Q$ such that movement from $a$ to $b$ inside $(a, b)$ is in the positive direction. Then (D2) is equivalent to the fact that for a $\sim$-class $g$ either $\sigma_d(g)$ is a point or for each positively oriented hole $(a, b)$ of $g$ the positively oriented arc $(\sigma_d(a), \sigma_d(b))$ is a positively oriented hole of $\sigma_d(g)$.

For a $\sigma_d$-invariant lamination $\sim$ we consider the *topological Julia set* $\mathbb{S}^1/\sim= J_\sim$ and the *topological polynomial* $f_\sim : J_\sim \to J_\sim$ induced by $\sigma_d$. The quotient map $p_\sim : \mathbb{S}^1 \to J_\sim$ extends to the plane with the only non-trivial fibers being the convex hulls of $\sim$-classes. Using Moore's Theorem one can extend $f_\sim$ to a branched-covering map $f_\sim : \mathbb{C} \to \mathbb{C}$ of the same degree. The complement of the unbounded component of $\mathbb{C} \setminus J_\sim$ is called the *filled-in topological Julia set* and is denoted by $K_\sim$. If the lamination $\sim$ is fixed, we may omit $\sim$ from the notation.

A particular case is when $J_\sim$ is a *dendrite* (a locally connected continuum containing no simple closed curve) and so $\widehat{\mathbb{C}} \setminus J_\sim$ is a simply connected neighborhood of infinity. It is easy to see that if a lamination $\sim$ has no domains (i.e., if convex hulls of all $\sim$-classes partition the entire unit disk), then the quotient space $\mathbb{S}^1/\sim$ is a dendrite.

For points $a, b \in \mathbb{S}^1$, let $\overline{ab}$ be the (perhaps degenerate) *chord* with endpoints $a$ and $b$. For $A \subset \mathbb{S}^1$ let $\mathrm{Ch}(A)$ be the *convex hull* of $A$ in $\mathbb{C}$.

**Definition 2.3** (Leaves and gaps of a lamination)**.** If $A$ is a $\sim$-class, we call an edge $\overline{ab}$ of $\mathrm{Bd}(\mathrm{Ch}(A))$ a *leaf* (if $a = b$, we call the leaf $\overline{aa} = \{a\}$ *degenerate*, cf. [Thu85]). All points of $\mathbb{S}^1$ are also called *leaves*. Normally, leaves are denoted as above, or by a letter with a bar above it ($\bar{b}, \bar{q}$ etc), or by $\ell$. The family of all leaves of $\sim$, denoted by $\mathcal{L}_\sim$, is called the *geometric lamination (geo-lamination) generated by* $\sim$. Denote the union of all leaves of $\mathcal{L}_\sim$ by $\mathcal{L}^+_\sim$. The closure of a non-empty component of $\mathbb{D} \setminus \mathcal{L}^+_\sim$ is called a *gap* of $\sim$. If $G$ is a gap, we talk about *edges of* $G$; if $G$ is a gap or leaf, we call the set $G' = \mathbb{S}^1 \cap G$ the *basis of* $G$.

Extend $\sigma_d$ (keeping the notation) linearly over all *individual chords* in $\overline{\mathbb{D}}$, in particularly over leaves of $\mathcal{L}_\sim$. Note, that even though the extended $\sigma_d$ is not well defined on the entire disk, it is well defined on $\mathcal{L}^+_\sim$ (as well as on every individual chord in the disk).

A gap or leaf $U$ is said to be *(pre)periodic* if $\sigma_d^{m+k}(U') = \sigma_d^m(U')$ for some $m \geq 0, k > 0$. If $m$ above can be chosen to be 0, then $U$ is called *periodic*. If $U$ is (pre)periodic but not periodic then it is called *preperiodic*.

Infinite gaps of a $\sigma_d$-invariant lamination $\sim$ are called *Fatou gaps*. Let $G$ be a Fatou gap; by [Kiw02] $G$ is (pre)periodic under $\sigma_d$. If a Fatou gap $G$ is periodic, then by [BL02a] its basis $G'$ is a Cantor set and the quotient map $\psi_G : \mathrm{Bd}(G) \to \mathbb{S}^1$, collapsing all edges of $G$ to points, is such that $\psi_G$-preimages of points are points or single leaves.

**Definition 2.4** (Siegel gaps and gaps of degree greater than 1)**.** Suppose that $G$ is a periodic Fatou gap of minimal period $n$. By [BL02a] $\psi_G$ semiconjugates $\sigma_d^n|_{\mathrm{Bd}(G)}$ to a map $\hat{\sigma}_G = \hat{\sigma} : \mathbb{S}^1 \to \mathbb{S}^1$ so that either (1) $\hat{\sigma} = \sigma_k : \mathbb{S}^1 \to \mathbb{S}^1, k \geq 2$ or (2) $\hat{\sigma}$ is an irrational rotation. In case (1) call $G$ a *gap of degree* $k$. In case (2) $G$ is called a *Siegel gap*. A (pre)periodic gap eventually mapped to a periodic gap of degree $k$ (Siegel) is also said to be of degree $k$ (Siegel). *Domains* (bounded components of the complement) of $J_\sim$ are said to be *of degree $k$ (Siegel)* if the corresponding gaps of $\mathcal{L}_\sim$ are such.

Various types of gaps and domains described in Definition 2.4 correspond to various types of atoms of $J_\sim$; as with gaps and domains, we keep the same terminology while replacing the word "gap" or "domain" by the word "atom". Thus, the boundary of a (periodic) Siegel domain is called a *(periodic) Siegel atom*, the boundary of a (periodic) Fatou domain of degree $k > 1$ is called a *(periodic) Fatou atom of degree* $k$ etc.

2.2. **Complete statement of main results.** Let $f_\sim : J_\sim \to J_\sim$ be a topological polynomial. Recall that an atom is either a point of $J_\sim$ or the boundary of a bounded component of $\mathbb{C} \setminus J_\sim$.

A *persistent cut-atom of degree* 1 is either a non-(pre)periodic persistent cut-atom, or a (pre)periodic cut-atom of degree 1; (pre)periodic atoms of degree 1 are either (pre)periodic points or boundaries of Siegel gaps (recall, that all Siegel gaps are (pre)periodic). A *persistent cut-atom of degree* $k > 1$ is a Fatou atom of degree $k$. A recurring theme in our paper is the fact that in some cases the dynamical span of a certain set $A$ and the dynamical span of the subset $B \subset A$ consisting of all periodic elements of $A$ with some extra-properties (e.g., being a periodic cut-atom, a periodic cut-atom of degree 1 etc) coincide. We call a periodic atom $A$ of period $n$ and degree 1 *rotational* if $\sigma_d^n|_{p_\sim^{-1}(A)}$ has **non-zero** .

**Definition 2.5** (Dynamical cores)**.** The *(dynamical) core* $\mathrm{COR}(f_\sim)$ of $f_\sim$ is the dynamical span of the union of the $\omega$-limit sets of all persistent cut-atoms. The union of all periodic cut-atoms of $f_\sim$ is denoted by $\mathrm{PC}(f_\sim) = \mathrm{PC}$ and is called the *periodic core* of $f_\sim$.

The *(dynamical) core of degree* 1 $\mathrm{COR}_1(f_\sim)$ of $f_\sim$ is the dynamical span of the $\omega$-limit sets of all persistent cut-atoms of degree 1. The union of all periodic cut-atoms of $f_\sim$ of degree 1 is denoted by $\mathrm{PC}_1(f_\sim) = \mathrm{PC}_1$ and is called the *periodic core of degree* 1 of $f_\sim$.

The *(dynamical) rotational core* $\mathrm{COR}_{rot}(f_\sim)$ of $f_\sim$ is the dynamical span of the $\omega$-limit sets of all wandering persistent cutpoints and all periodic rotational atoms. The union of all periodic rotational atoms of $f_\sim$ is denoted by $\mathrm{PC}_{rot}(f_\sim) = \mathrm{PC}_{rot}$ and is called the *periodic rotational core* of $f_\sim$.

Clearly, $\mathrm{COR}_{rot} \subset \mathrm{COR}_1 \subset \mathrm{COR}$ and $\mathrm{PC}_{rot} \subset \mathrm{PC}_1 \subset \mathrm{PC}$. Observe, that in the case of dendrites the notions become simpler and some results can be strengthened. Indeed, first of all in this case we can talk about cutpoints only. Secondly, $\mathrm{COR} = \mathrm{COR}_1$ and $\mathrm{PC} = \mathrm{PC}_1$. A priori, then $\mathrm{COR}_{rot}$ could be strictly smaller than COR, however Theorem 1.2 shows that $\mathrm{COR}_{rot} = \mathrm{COR}$.

**Theorem 2.6.** *The dynamical core of $f_\sim$ coincides with* $\mathrm{IC}(\mathrm{PC}(f_\sim))$*. The dynamical core of degree* 1 *of $f_\sim$ coincides with* $\mathrm{IC}(\mathrm{PC}_1(f_\sim))$*. The rotational dynamical core coincides with* $\mathrm{IC}(\mathrm{PC}_{rot}(f_\sim))$*. If $J_\sim$ is a dendrite, then* $\mathrm{COR} = \mathrm{IC}(\mathrm{PC}_{rot}(f_\sim))$*.*

There are two other main results in Section 3. As the dynamics on Fatou gaps is simple, it is natural to consider the dynamics of gaps/leaves which never map to Fatou gaps, or the dynamics of gaps/leaves which never map to 'maximal concatenations" of Fatou gaps which we call *super-gaps* (these notions are made precise in Section 3). We prove that the dynamical span of limit sets of all persistent cut-atoms which never map to the $p_\sim$-images of super-gaps coincides with the dynamical span of all periodic rotational cut-atoms located outside the $p_\sim$-images of super-gaps (recall that $p_\sim$ is the quotient map generated by $\sim$). In fact, the "dendritic" part of Theorem 2.6 follows from that result.

A result similar to Theorem 2.6, using critical points and atoms instead of periodic ones, is proven in Theorem 3.13. Namely, an atom $A$ is *critical* if either $A$ is a critical point of $f_\sim$, or $f_\sim|_A$ is not one-to-one. In Theorem 3.13 we prove, in particular, that various cores of a topological polynomial $f_\sim$ coincide with the dynamical spans of critical atoms of the restriction of $f_\sim$ onto these cores.

If $J_\sim$ is a dendrite, then critical atoms are critical points and Theorem 2.6 is closely related to the interval, even unimodal, case. Thus, Theorem 2.6 can be viewed as a generalization of the corresponding results for maps of the interval.

2.3. **Geometric laminations.** The connection between laminations, understood as equivalence relations, and the original approach of Thurston's [Thu85], can be explained once we introduce a few key notions. Assume that a $\sigma_d$-invariant lamination $\sim$ and its associated geometric lamination $\mathcal{L}_\sim$ are given.

Thurston's idea was to study similar collections of chords in $\mathbb{D}$ abstractly, i.e., without assuming that they are generated by an equivalence relation on the circle with specific properties.

**Definition 2.7** (Geometric laminations, cf. [Thu85])**.** A *geometric prelamination* $\mathcal{L}$ is a set of (possibly degenerate) chords in $\overline{\mathbb{D}}$ such that any two distinct chords from $\mathcal{L}$ meet at most in a common endpoint; $\mathcal{L}$ is called a *geometric lamination* (*geo-lamination*) if all points of $\mathbb{S}^1$ are elements of $\mathcal{L}$, and $\bigcup \mathcal{L}$ is closed. Elements of $\mathcal{L}$ are called *leaves* of $\mathcal{L}$ (leaves may be degenerate). The union of all leaves of $\mathcal{L}$ is denoted by $\mathcal{L}^+$.

**Definition 2.8** (Gaps of geo-laminations)**.** Suppose that $\mathcal{L}$ is a geo-lamination. The closure of a non-empty component of $\mathbb{D} \setminus \mathcal{L}^+$ is called a *gap* of $\mathcal{L}$. Thus, given a geo-lamination $\mathcal{L}$ we obtain a cover of $\overline{\mathbb{D}}$ by gaps of $\mathcal{L}$ and (perhaps, degenerate) leaves of $\mathcal{L}$ which do not lie on the boundary of a gap of $\mathcal{L}$ (equivalently, are not isolated in $\mathbb{D}$ from either side). Elements of this cover are called *$\mathcal{L}$-sets*. Observe that the intersection of two different $\mathcal{L}$-sets is at most a leaf.

In the case when $\mathcal{L} = \mathcal{L}_\sim$ is generated by an invariant lamination $\sim$, gaps might be of two kinds: finite gaps which are convex hulls of $\sim$-classes and infinite gaps which are closures of *domains* (of $\sim$), i.e. components of $\mathbb{D} \setminus \mathcal{L}^+_\sim$ which are not interiors of convex hulls of $\sim$-classes.

**Definition 2.9** (Invariant geo-laminations, cf. [Thu85])**.** A geometric lamination $\mathcal{L}$ is said to be an *invariant* geo-lamination of degree $d$ if the following conditions are satisfied:

(1) (Leaf invariance) For each leaf $\ell \in \mathcal{L}$, the set $\sigma_d(\ell)$ is a (perhaps degenerate) leaf in $\mathcal{L}$. For every non-degenerate leaf $\ell \in \mathcal{L}$, there are $d$ pairwise disjoint leaves $\ell_1, \dots, \ell_d$ in $\mathcal{L}$ such that for each $i$, $\sigma_d(\ell_i) = \ell$.
(2) (Gap invariance) For a gap $G$ of $\mathcal{L}$, the set $H = \mathrm{Ch}(\sigma_d(G'))$ is a (possibly degenerate) leaf, or a gap of $\mathcal{L}$, in which case $\sigma_d|_{\mathrm{Bd}(G)} : \mathrm{Bd}(G) \to \mathrm{Bd}(H)$ is a positively oriented composition of a monotone map and a covering map (a *monotone map* is a map such that the full preimage of any connected set is connected).

Note that some invariant geo-laminations are not generated by equivalence relations. We will use a special extension $\sigma^*_{d,\mathcal{L}} = \sigma^*_d$ of $\sigma_d$ to the closed unit disk associated with $\mathcal{L}$. On $\mathbb{S}^1$ and all leaves of $\mathcal{L}$, we set $\sigma^*_d = \sigma_d$ (in Definition 2.3, $\sigma_d$ was extended over all chords in $\overline{\mathbb{D}}$, including leaves of $\mathcal{L}$). Otherwise, define $\sigma^*_d$ on the interiors of gaps using a standard barycentric construction (see [Thu85]). Sometimes we lighten the notation and use $\sigma_d$ instead of $\sigma^*_d$. We will mostly use the map $\sigma^*_d$ in the case $\mathcal{L} = \mathcal{L}_\sim$ for some invariant lamination $\sim$.

**Definition 2.10** (Critical leaves and gaps)**.** A leaf of a lamination $\sim$ is called *critical* if its endpoints have the same image. A $\mathcal{L}_\sim$-set $G$ is said to be *critical* if $\sigma_d|_{G'}$ is $k$-to-1 for some $k > 1$. E.g., a periodic Siegel gap is a non-critical $\sim$-set, on whose basis the first return map is not one-to-one because there must be critical leaves in the boundaries of gaps from its orbit. We define *precritical* and *(pre)critical* objects similarly to how (pre)periodic and preperiodic objects are defined above.

We need more notation. Let $a, b \in \mathbb{S}^1$. By $[a, b], (a, b)$ etc we mean the appropriate *positively oriented* circle arcs from $a$ to $b$, and by $|I|$ the length of an arc $I$ in $\mathbb{S}^1$ normalized so that the length of $\mathbb{S}^1$ is 1.

## 2.4. **Stand alone gaps and their basic properties.**

**Definition 2.11** (Return time and related notions)**.** Let $f : X \to X$ be a self-mapping of a set $X$. For a set $G \subset X$, define the *return time* (to $G$) of $x \in G$ as the least positive integer $n_x$ such that $f^{n_x}(x) \in G$, or infinity if there is no such integer. Let $n = \min_{y \in G} n_y$, define $D_G = \{x : n_x = n\}$, and call the map $f^n : D_G \to G$ the *return map* (of $G$).

E.g., if $G$ is the boundary of a periodic Fatou domain of period $n$ of a topological polynomial $f_\sim$ whose images are all pairwise disjoint until $f^n_\sim(G) = G$, then $D_G = G$ and the corresponding return map on $D_G = G$ is the same as $f^n_\sim$.

We have already introduced the notion of a gap of a lamination or of a geo-lamination. Below we will describe a closed convex set in $\overline{\mathbb{D}}$ which has all the properties of a gap of a geo-lamination, but for which no corresponding lamination is specified.

**Definition 2.12** (Stand alone gaps)**.** If $A \subset \mathbb{S}^1$ is a closed set such that all the sets $\mathrm{Ch}(\sigma^i(A))$ are pairwise disjoint, then $A$ is called *wandering.* If there exists $n \geq 1$ such that all the sets $\mathrm{Ch}(\sigma^i_d(A)), i = 0, \dots, n-1$ have pairwise disjoint interiors while $\sigma^n_d(A) = A$, then $A$ is called *periodic* of period $n$. If there exists $m > 0$ such that all $\mathrm{Ch}(\sigma^i(A)), 0 \leq i \leq m+n-1$ have pairwise disjoint interiors and $\sigma^m_d(A)$ is periodic of period $n$, then we call $A$ *preperiodic.* Moreover, suppose that $|A| \geq 3$, $A$ is wandering, periodic or preperiodic, and for every $i \geq 0$ and every hole $(a, b)$ of $\sigma^i_d(A)$ either $\sigma_d(a) = \sigma_d(b)$ or the positively oriented arc $(\sigma_d(a), \sigma_d(b))$ is a hole of $\sigma^{i+1}_d(A)$. Then we call $A$ (and $\mathrm{Ch}(A)$) a *$\sigma_d$-stand alone gap.*

Recall that in Definition 2.4 we defined Fatou gaps $G$ of various degrees as well as Siegel gaps. Given a periodic Fatou gap $G$ we also introduced the monotone map $\psi_G$ which semiconjugates $\sigma_d|_{\mathrm{Bd}(G)}$ and the appropriate model map $\hat{\sigma}_G : \mathbb{S}^1 \to \mathbb{S}^1$. This construction can be also done for stand alone Fatou gaps $G$.

Indeed, consider the basis $G \cap \mathbb{S}^1 = G'$ of $G$ (see Definition 2.3) as a subset of $\mathrm{Bd}(G)$. It is well-known that $G'$ coincides with the union $A \cup B$ of two well-defined sets, where $A$ is a Cantor subset of $G'$ or an empty set and $B$ is a countable set. In the case when $A = \varnothing$, the map $\psi_G$ simply collapses $\mathrm{Bd}(G)$ to a point. However, if $A \neq \varnothing$, one can define a semiconjugacy $\psi_G : \mathrm{Bd}(G) \to \mathbb{S}^1$ which collapses all holes of $G'$ to points. As in Definition 2.4, the map $\psi_G$ semiconjugates $\sigma_d|_{\mathrm{Bd}(G)}$ to a circle map which is either an irrational rotation or the map $\sigma_k, k \geq 2$. Depending on the type of this map we can introduce for periodic infinite stand alone gaps terminology similar to Definition 2.4. In particular, if $\sigma_d|_{\mathrm{Bd}(G)}$ is semiconjugate to $\sigma_k, k \geq 2$ we say that $G$ is a *stand alone Fatou gap of degree* $k$.

**Definition 2.13** (Rotational sets)**.** If $G$ is a periodic stand alone gap such that $G'$ is finite and contains no refixed points, then $G$ is said to be a *finite (periodic) rotational set.* Finite rotational sets and Siegel gaps $G$ are called *(periodic) rotational sets.* If such $G$ is invariant, we call it an *invariant rotational set.*

The maps $\sigma_k$ serve as models of return maps of periodic gaps of degree $k \geq 2$. For rotational sets, models of return maps are non-trivial rotations.

**Definition 2.14** (Rotation number)**.** A number $\tau$ is said to be the *rotation number* of a periodic set $G$ if for every $x \in G'$ the circular order of points in the orbit of $x$ under the return map of $G'$ is the same as the order of points $0, \mathrm{Rot}_\tau(0), \dots$ where $\mathrm{Rot}_\tau : \mathbb{S}^1 \to \mathbb{S}^1$ is the rigid rotation by the angle $\tau$.

It is easy to see that to each rotational set $G$, one can associate its well-defined rotation number $\tau_G = \tau$ (in the case of a finite rotational set, the property that endpoints of holes are mapped to endpoints of holes implies that the circular order on $G'$ remains unchanged under $\sigma$). Since $G'$ contains no points which are fixed under its return map by our assumption, $\tau \neq 0$. Given a topological polynomial $f_\sim$ and an $f_\sim$-periodic point $x$ of minimal period $n$, we can associate to $x$ the rotation number $\rho(x)$ of $\sigma_d^n$ restricted to the $\sim$-class $p_\sim^{-1}(x)$, corresponding to $x$ (recall, that $p_\sim : \mathbb{S}^1 \to J_\sim = \mathbb{S}^1/\sim$ is the quotient map associated to $\sim$); then if $\rho(x) \neq 0$ the set $p_\sim^{-1}(x)$ is rotational, but if $\rho(x) = 0$ then the set $p_\sim^{-1}(x)$ is not rotational.

The following result allows one to find fixed stand alone gaps or points of specific types in some parts of the disk; for the proof see [BFMOT10]. It is similar in spirit to a fixed point result by Goldberg and Milnor [GM93].

**Theorem 2.15.** *Let $\sim$ be a $\sigma_d$-invariant lamination. Consider the topological polynomial $f_\sim$ extended over $\mathbb{C}$. Suppose that $e_1, \dots, e_m \in J_\sim$ are $m$ points, and $X \subset K_\sim$ is a component of $K_\sim \setminus \{e_1, \dots, e_m\}$ such that for each $i$ we have $e_i \in \overline{X}$ and either $f_\sim(e_i) = e_i$ and the rotation number at $e_i$ is zero, or $f_\sim(e_i)$ belongs to the component of $K_\sim \setminus \{e_i\}$ which contains $X$. Then at least one of the following claims holds:*

(1) *$X$ contains an invariant domain of degree $k > 1$;*
(2) *$X$ contains an invariant Siegel domain;*

(3) *$X \cap J_\sim$ contains a fixed point with non-zero rotation number.*

For dendritic topological Julia sets $J_\sim$ the claim is easier as cases (1) and (2) above are impossible. Thus, in the dendritic case Theorem 2.15 implies that there exists a rotational fixed point in $X \cap J_\sim$.

Let $U$ be the convex hull of a closed subset of $\mathbb{S}^1$. For every edge $\ell$ of $U$, let $H_U(\ell)$ denote the hole of $U$ that shares both endpoints with $\ell$ (if $U$ is fixed, we may drop the subscript in the above notation). Notice that in the case when $U$ is a chord there are two ways to specify the hole. The hole $H_U(\ell)$ is called the *hole of $U$ behind (at) $\ell$*. In this situation we define $|\ell|_U$ as $|H_U(\ell)|$.

**Lemma 2.16.** *Suppose that $\ell = \overline{xy}$ is a non-invariant leaf such that there exists a component $Q$ of the complement of its orbit whose closure contains $\sigma_d^n(\ell)$ for all $n \geq 0$. Then the leaf $\ell$ is either (pre)critical or (pre)periodic.*

*Proof.* Suppose that $\ell$ is neither (pre)periodic nor (pre)critical. Then it follows that there are infinitely many leaves in the orbit of $\ell$ such that both complementary arcs of the set of their endpoints are of length greater than $\frac{1}{2d}$. Since the corresponding holes of $Q$ are pairwise disjoint, we get a contradiction. $\square$

## 3. Dynamical core

In Section 3 we fix $\sim$ which is a $\sigma_d$-invariant lamination, study the dynamical properties of the topological polynomial $f_\sim : J_\sim \to J_\sim$, and discuss its dynamical core $\mathrm{COR}(f_\sim)$. For brevity, we write $f$, $J$, $p$, COR etc for $f_\sim$, $J_\sim$, $p_\sim : \mathbb{S}^1 \to J$, $\mathrm{COR}(f_\sim)$, respectively, throughout Section 3. Note that, by definition, every topological Julia set $J$ in this section is locally connected.

3.1. **Super-gaps.** Let $\ell$ be a leaf of $\mathcal{L}_\sim$. We equip $\mathcal{L}_\sim$ with the topology induced by the Hausdorff metric. Then $\mathcal{L}_\sim$ is a compact and metric space. Suppose that a leaf $\ell$ has a neighborhood (in $\mathcal{L}_\sim$!) which contains at most countably many leaves of $\mathcal{L}_\sim$. Call such leaves ***countably isolated*** and denote the family of all such leaves $\mathrm{CI}_\sim = \mathrm{CI}$. Clearly, CI is open in $\mathcal{L}_\sim$. Moreover, CI is countable. To see this note that CI, being a subset of $\mathcal{L}_\sim$, is second countable and, hence, Lindelöf. Hence there exists a countable cover of CI all of whose elements are countable and CI is countable as desired. In terms of dynamics, CI is backward invariant and almost forward invariant (it is forward invariant except for critical leaves in CI because their images are points which are never countably isolated).

It can be shown that if we remove all leaves of CI from $\mathcal{L}_\sim$ (this is in the spirit of *cleaning* of geometric laminations [Thu85]), the remaining leaves (if any) form an invariant geometric lamination $\mathcal{L}^c_\sim$ ("c" coming from "countable cleaning"). One way to see this is to use an alternative definition given in [BMOV11]. A geo-lamination (initially not necessarily invariant in the sense of Definition 2.9) is called a *sibling $d$-invariant lamination* or just *sibling lamination* if (a) for each $\ell \in \mathcal{L}$ either $\sigma_d(\ell) \in \mathcal{L}$ or $\sigma_d(\ell)$ is a point in $\mathbb{S}^1$, (b) for each $\ell \in \mathcal{L}$ there exists a leaf $\ell' \in \mathcal{L}$ with $\sigma_d(\ell') = \ell$, and (c) for each $\ell \in \mathcal{L}$ with non-degenerate image $\sigma_d(\ell)$ there exist $d$ disjoint leaves $\ell_1, \dots, \ell_d$ in $\mathcal{L}$ with $\ell = \ell_1$ and $\sigma_d(\ell_i) = \sigma_d(\ell)$ for all $i$. By [BMOV11, Theorem 3.2], sibling invariant laminations are invariant in the sense of Definition 2.9. Now, observe that $\mathcal{L}_\sim$ is sibling invariant. Since CI is open and contains the full grand orbit of any leaf in it which never collapses to a point and the full backward orbit of any critical leaf in it, $\mathcal{L}^c_\sim$ is also sibling invariant. Thus, by [BMOV11, Theorem 3.2] $\mathcal{L}^c_\sim$ is invariant in the sense of Definition 2.9. Infinite gaps of $\mathcal{L}^c_\sim$ are called *super-gaps of* $\sim$. Note that all finite gaps of $\mathcal{L}^c_\sim$ are also finite gaps of $\mathcal{L}_\sim$.

Let $\mathcal{L}^0_\sim = \mathcal{L}_\sim$ and define $\mathcal{L}^k_\sim$ inductively by removing all isolated leaves from $\mathcal{L}^{k-1}_\sim$.

**Lemma 3.1.** *There exists $n$ such that $\mathcal{L}^n_\sim = \mathcal{L}^c_\sim$; moreover, $\mathcal{L}^c_\sim$ contains no isolated leaves.*

*Proof.* We first show that there exists $n$ such that $\mathcal{L}^{n+1}_\sim = \mathcal{L}^n_\sim$ (i.e., $\mathcal{L}^n_\sim$ contains no isolated leaves). We note that increasing $i$ may only decrease the number of infinite periodic gaps and decrease the number of finite critical objects (gaps or leaves) in $\mathcal{L}^i_\sim$. Therefore we may choose $m$ so that $\mathcal{L}^m_\sim$ has a minimal number of infinite periodic gaps and finite critical objects. If $\mathcal{L}^m_\sim$ has no isolated leaves, then we may choose $n = m$. Otherwise, we will show that we may choose $n = m+1$.

Suppose that $\mathcal{L}^m_\sim$ has an isolated leaf $\ell$. Then $\ell$ is a common edge of two gaps $U$ and $V$. Since finite gaps of $\mathcal{L}_\sim$ (and, hence, of all $\mathcal{L}^i_\sim$) are disjoint we may assume that $U$ is infinite. Moreover, it must be that $V$ is finite. Indeed, suppose that $V$ is infinite and consider two cases. First assume that there is a minimal $j$ such that $\sigma^j_d(U) = \sigma^j_d(V)$. Then $\sigma^{j-1}_d(\ell)$ is critical and isolated in $\mathcal{L}^m_\sim$. Hence $\mathcal{L}^{m+1}_\sim$ has fewer finite critical objects than $\mathcal{L}^m_\sim$, a contradiction with the choice of $m$. On the other hand, if $U$ and $V$ never have the same image, then in $\mathcal{L}^{m+1}_\sim$ their periodic images will be joined. Then $\mathcal{L}^{m+1}_\sim$ would have a periodic gap containing both such images which contradicts the choice of $m$ with the minimal number of infinite periodic gaps. Thus, $V$ is finite.

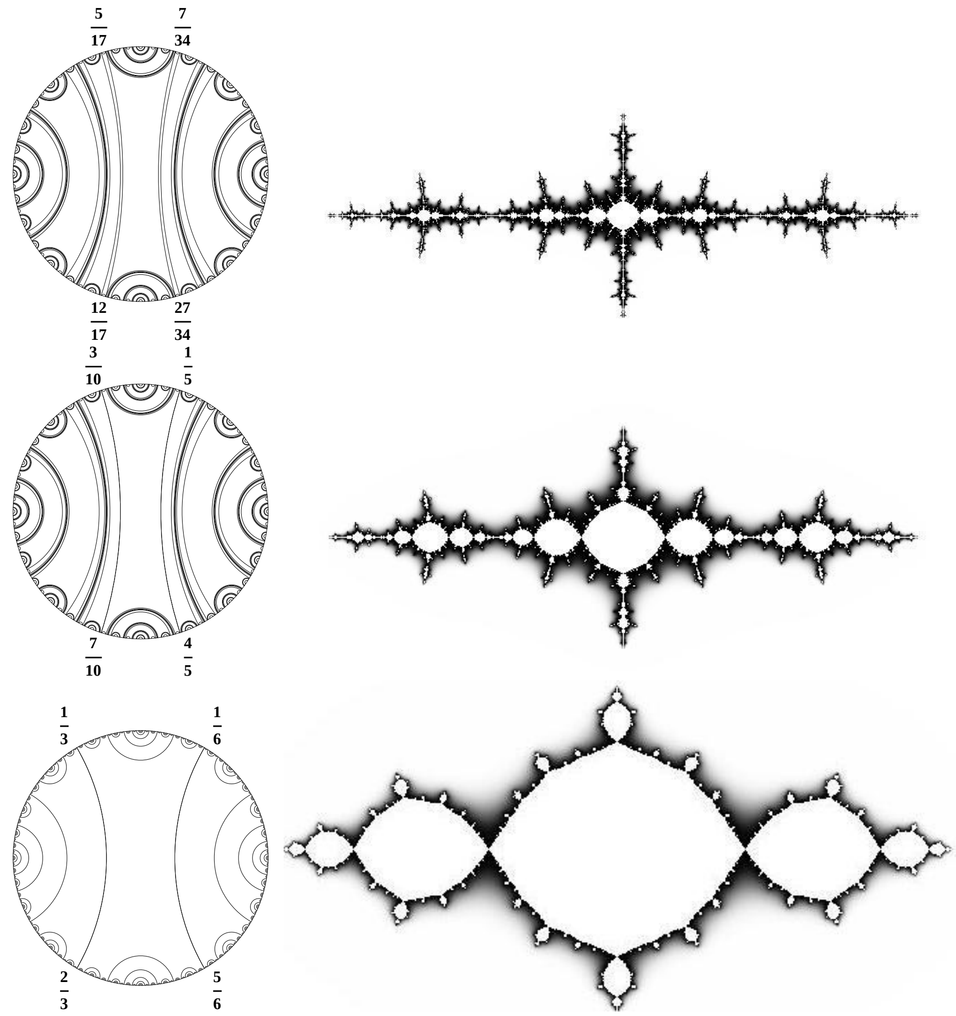


FIGURE 1. An example of 2-step cleaning for quadratic laminations. Above: the quadratic lamination $\mathcal{L}^0$ with a quadratic gap between the leaves $\overline{\frac{5}{17}\frac{12}{17}}$ and $\overline{\frac{7}{34}\frac{27}{34}}$, and the corresponding Julia set (basilica tuned with basilica tuned with basilica). In the middle: the first cleaning $\mathcal{L}^1$ of $\mathcal{L}^0$ — the lamination with a quadratic gap between the leaves $\overline{\frac{1}{5}\frac{4}{5}}$ and $\overline{\frac{3}{10}\frac{7}{10}}$, and the corresponding Julia set (basilica tuned with basilica). Below: the second cleaning $\mathcal{L}^2$ of $\mathcal{L}^0$ — the lamination with a quadratic gap between the leaves $\overline{\frac{1}{3}\frac{2}{3}}$ and $\overline{\frac{1}{6}\frac{5}{6}}$, and the corresponding Julia set (basilica). Note that the next cleaning $\mathcal{L}^3$ is empty and coincides with $\mathcal{L}^c$. Thus, the unique super-gap of $\mathcal{L}^0$ coincides with the entire unit disk.

Furthermore, the other edges of $V$ are not isolated in any lamination $\mathcal{L}^t_\sim, t \geq m$. For if some other edge $\bar{q}$ of $V$ were isolated in $\mathcal{L}^t_\sim$, then it would be an edge of an infinite gap $H$, contained in the same gap of $\mathcal{L}^{t+1}_\sim$ as $U$. If $\sigma^r_d(H) = \sigma^r_d(U)$ for some $r$, then $V$ is (pre)critical and the critical image of $V$ is absent from $\mathcal{L}^{t+1}_\sim$, a contradiction with the choice of $m$. Otherwise, periodic images of $H$ and $U$ will be contained in a bigger gap of $\mathcal{L}^t_\sim$, decreasing the number of periodic infinite gaps and again contradicting the choice of $m$. By definition this implies that all edges of $V$ except for $\ell$ stay in all laminations $\mathcal{L}^t_\sim, t \geq m$ and that $\mathcal{L}^{m+1}_\sim$ has no isolated leaves.

Choose $n$ so that $\mathcal{L}^n_\sim$ contains no isolated leaves. It is well-known that a compact metric space without isolated points is locally uncountable. Hence $\mathcal{L}^n_\sim = \mathcal{L}^c_\sim$ as desired. □

**Lemma 3.2.** *If $\sim$ is a lamination, then the following holds.*

1. *Every leaf of $\mathcal{L}_\sim$ inside a super-gap $G$ of $\sim$ is (pre)periodic or (pre)critical; every edge of a super-gap is (pre)periodic.*
2. *Every edge of any gap $H$ of $\mathcal{L}^c_\sim$ is not isolated in $\mathcal{L}^c_\sim$ from outside of $H$; all gaps of $\mathcal{L}^c_\sim$ are pairwise disjoint. Moreover, gaps of $\mathcal{L}^c_\sim$ are disjoint from leaves which are not their edges.*
3. *There are no infinite concatenations of leaves in $\mathcal{L}^c_\sim$. Moreover, the geo-lamination $\mathcal{L}^c_\sim$ gives rise to a lamination $\sim^c$ such that the only difference between $\mathcal{L}^c_\sim$ and $\mathcal{L}_{\sim^c}$ is as follows: it is possible that one edge of certain finite gaps of $\sim^c$ is a leaf passing inside an infinite gap of $\mathcal{L}^c_\sim$.*
4. *Any periodic Siegel gap is a proper subset of its super-gap.*

*Proof.* (1) An isolated leaf $\ell$ of any lamination $\mathcal{L}^k_\sim$ is either (pre)periodic or (pre)critical. Indeed, since two finite gaps of $\mathcal{L}^k_\sim$ are not adjacent, $\ell$ is an edge of an infinite gap $V$. Then by Lemma 2.16, $\ell$ is (pre)critical or (pre)periodic. Since in the process of constructing $\mathcal{L}^c_\sim$ we remove isolated leaves of laminations $\mathcal{L}^k_\sim$, we conclude that all leaves inside a super-gap of $\sim$ (i.e., a gap of $\mathcal{L}^c_\sim$) are either (pre)periodic or (pre)critical.

Let $G$ be a gap of $\sim$. Since by Lemma 3.1 $\mathcal{L}^c_\sim$ contains no isolated leaves, every edge of $G$ is a limit of leaves from outside of $G$. Hence the only gaps of $\mathcal{L}^c_\sim$ which can contain a critical leaf in their boundaries are those which collapse to a single point (if a gap $H$ of $\mathcal{L}^c_\sim$ has a critical edge $\ell = \lim \ell_i$, then leaves $\sigma_d(\ell_i)$ separate the point $\sigma_d(\ell)$ from the rest of the circle which implies that $\sigma_d(H) = \sigma_d(\ell)$ is a point). Thus, super-gaps have no critical edges, and by Lemma 2.16 all their edges are (pre)periodic.

(2) Every edge of any gap $H$ of $\mathcal{L}^c_\sim$ is not isolated in $\mathcal{L}^c_\sim$ from outside of $H$ by Lemma 3.1. Moreover, no point $a \in \mathbb{S}^1$ can be an endpoint of more than two leaves of $\mathcal{L}_\sim$ because $\mathcal{L}_\sim$ is a geometric lamination generated by an equivalence relation. Hence, no point $a \in \mathbb{S}^1$ can be an endpoint of more than two leaves of $\mathcal{L}^c_\sim$ either. This implies that if $G$ is a gap, then it is disjoint from all leaves which are not its edges. Indeed, suppose otherwise. Then there exists a vertex $a$ of $G$ and a leaf $\ell_1 = \overline{ax}$ which is not an edge of $G$. Then there is an edge $\ell_2$ of $G$ which emanates from $a$ and by the above there is a gap $H$ with edges $\ell_1$ and $\ell_2$ ($H$ is squeezed in-between $G$ and $\ell_1$). This implies that $\ell_2$ is isolated, a contradiction. Thus, two gaps cannot have edges which "touch" at an endpoint. The beginning of the paragraph implies that two gaps cannot have a common edge either. We conclude that all gaps of $\mathcal{L}^c_\sim$ are pairwise disjoint and that all gaps of $\mathcal{L}^c_\sim$ are disjoint from all leaves which are not their edges.

(3) There are no infinite concatenations of leaves in $\mathcal{L}^c_\sim$ because there are no such concatenations in $\mathcal{L}_\sim$. Now it is easy to see that the geometric lamination $\mathcal{L}^c_\sim$ gives rise to a lamination (equivalence relation) which we denote $\sim^c$. Two points $a, b \in \mathbb{S}^1$ are said to be $\sim^c$-*equivalent* if there exists a finite concatenation of leaves of $\mathcal{L}^c_\sim$ connecting $a$ and $b$. Since all leaves of $\mathcal{L}^c_\sim$ are non-isolated, it follows that the geo-lamination $\mathcal{L}^c_\sim$ and the geo-lamination $\mathcal{L}_{\sim^c}$ associated to $\sim^c$ can only differ as claimed.

(4) Clearly a Siegel gap $U$ is contained in a super-gap. To see that it does not coincide with a super-gap, observe that there exists a non-negative integer $k$ such that $\sigma_d^k(U)$ has a critical edge. Then by (1), $\sigma_d^k(U)$ (and hence $U$ itself) is *properly* contained in a super-gap. □

**Proposition 3.3.** *If $X$ is a persistent cut-atom of $J$ of degree one such that $p^{-1}(X)$ is a subset of some super-gap of $\sim$, then either $X$ is the boundary of a Siegel domain, or $X$ is a (pre)periodic point which eventually maps to a periodic cutpoint. In any case, $X$ eventually maps to* $\mathrm{PC}_1$.

*Proof.* We may assume that $X = x$ is a persistent cutpoint. Then the $\sim$-class $p^{-1}(x)$ is non-trivial. If the boundary of this $\sim$-class consists of (pre)critical leaves only, then the entire class gets eventually collapsed, which is a contradiction with $f^n(x)$ being cut-atoms for all $n \geq 0$. Therefore, there is a leaf $\ell$ on the boundary of $p^{-1}(x)$ that is not (pre)critical. Then this leaf is (pre)periodic by Lemma 3.2, hence $p^{-1}(x)$ is also (pre)periodic, and eventually maps to a periodic gap or leaf. □

3.2. **Proof of Theorem 2.6.** Lemma 3.4 studies intersections between atoms and complete invariant continua.

**Lemma 3.4.** *Let $X \subset J$ be an invariant complete continuum and $A$ be an atom intersecting $X$ but not contained in $X$. Then $A \cap X = \{x\}$ is a singleton, $A$ is the boundary of a Fatou domain, and one of the following holds:* (1) *for some $k$ we have $f^i(A) \cap X = \{f^i(x)\}, i < k$ and $f^k(A) \subset X$, or* (2) *$f^i(A) \cap X = \{f^i(x)\}$ for all $i$ and there exists the smallest $n$ such that $f^n(A)$ is the boundary of a periodic Fatou domain of degree $r > 1$ with $f^n(x)$ being a point of $f^n(A)$ fixed under the return map.*

*Proof.* Since $X$ is complete, we may assume that $A \cap X = \{x\}$ is a singleton and $A$ is the boundary of a Fatou domain such that $f^i(A) \cap X = \{f^i(x)\}$ for all $i$. Choose the smallest $n$ such that $f^n(A)$ is periodic. If $f^n(x)$ is not fixed by the return map of $f^n(A)$, then another point from the orbit of $x$ belongs to $f^n(A) \cap X$, a contradiction. □

Lemma 3.5 rules out certain dynamical behavior of points.

**Lemma 3.5.** *Suppose that $x \in J$ is a non-(pre)critical cutpoint. Then there exists $n \geq 0$ such that at least two components of $J \setminus \{f^n(x)\}$ contain forward images of $f^n(x)$.*

*Proof.* An equivalent statement which we will actually work with in the proof can be given as follows: there exists no non-(pre)critical non-degenerate $\sim$-class $X$ such that for every $n$, *all* the sets $f^{n+k}(X), k > 0$ are contained in the same hole of $f^n(X)$.

By way of contradiction suppose that such $\sim$-class $X$ exists. Let us show that then the iterated images of $X$ cannot converge (along a subsequence of iterations) to a critical leaf $\ell$. Indeed, suppose that $\sigma_d^{n_k}(X) \to \ell$ so that $\sigma_d^{n_{k+1}}(X)$ separates $\sigma_d^{n_k}(X)$ from $\ell$. Then $\sigma_d^{n_k+1}(X) \to \sigma_d(\ell)$, where $\sigma_d(\ell)$ is a point of $\mathbb{S}^1$. Since $\sigma_d^{n_{k+1}+1}(X)$ separates $\sigma_d(\ell)$ from $\sigma_d^{n_k+1}(X)$ for a sufficiently large $k$, it follows by the assumption that the entire orbit of $\sigma^{n_k}(X)$ must be contained in a small component of $\overline{\mathbb{D}} \setminus \sigma^{n_k}(X)$, containing $\sigma_d(\ell)$. As this can be repeated for all sufficiently large $k$, we see that the limit set of $X$ has to coincide with the point $\sigma_d(\ell)$, a contradiction. Hence $X$ contains no critical leaves in its limit set.

Note that the assumptions of the lemma imply that $X$ is wandering. By [Chi04] if $X$ is not a leaf then it contains a critical leaf in its limit set. This implies that $X$ must be a leaf. For each image $\sigma_d^n(X)$ let $Q_n$ be the component of $\overline{\mathbb{D}} \setminus \sigma_d^n(X)$ containing the rest of the orbit of $X$. Let $W_n = \bigcap_{i=0}^n \overline{Q_i}$. Then $W_n$ is a set whose boundary consists of finitely many leaves-images of $X$ alternating with finitely many circle arcs. On the next step, the image $\sigma_d^{n+1}(X)$ of $X$ is contained in $W_n$, and becomes a leaf on the boundary of $W_{n+1} \subset W_n$.

Consider the set $W = \bigcap W_n$. For an edge $\ell$ of $W$, let $H_W(\ell)$ be the hole of $W$ behind $\ell$. When saying that a certain leaf is contained in $H_W(\ell)$, we mean that its endpoints are contained in $H_W(\ell)$, or, equivalently, that the leaf is contained in the convex hull of $H_W(\ell)$. If $W$ is a point or a leaf, then the assumptions on the dynamics of $X$ made in the lemma imply that $X$ converges to $W$ but never maps to $W$. Clearly, this is impossible. Thus, we may assume that $W$ is a non-degenerate convex subset of $\overline{\mathbb{D}}$ whose boundary consists of leaves and possibly circle arcs. The leaves in $\mathrm{Bd}(W)$ can be of two types: limits of sequences of images of $X$ (if $\ell$ is a leaf like that, then images of $X$ which converge to $\ell$ must be contained in $H_W(\ell)$), and images of $X$. It follows that the limit leaves from the above collection form the entire limit set of $X$; moreover, by the above there are no critical leaves among them.

Let us show that this leads to a contradiction. First assume that among boundary leaves of $W$ there is a limit leaf $\bar{q} = \overline{xy}$ of the orbit of $X$ (here $(x, y) = H_W(\bar{q})$ is the hole of $W$ behind $\bar{q}$). Let us show that $\bar{q}$ is (pre)periodic. Indeed, since $\bar{q}$ is approached from the outside of $W$ by images of $X$, and since all images of $X$ are disjoint from $W$, it follows that $(\sigma_d(x), \sigma_d(y))$ is the hole of $W$ behind $\sigma_d(\bar{q})$. Then by Lemma 2.16 and because there are no critical leaves on the boundary of $W$ (by the first paragraph of the proof) we see that $\bar{q}$ is (pre)periodic. Let $\ell$ is an image of $\bar{q}$ which is periodic. Since $\ell$ is a repelling leaf, we see that images of $X$ approaching $\ell$ from within $H_W(\ell)$ are repelled farther away from $\ell$ inside $H_W(\ell)$. Clearly, this contradicts the properties of $X$.

Now assume that there are no boundary leaves of $W$ which are limits of images of $X = \overline{uv}$. Then all boundary leaves of $W$ are images of $X$. Let us show that then there exists $N$ such that for any $i \geq N$ we have that if the hole $H_W(\ell)$ of $W$ behind $\ell = \sigma_d^i(X)$ is $(s, t)$ then $H_W(\sigma_d(\ell)) = (\sigma_d(s), \sigma_d(t))$. Indeed, first we show that if $H_W(\ell) = (s, t)$ while $H_W(\sigma_d(\ell)) = (\sigma_d(t), \sigma_d(s))$, then $(s, t)$ contains a $\sigma_d$-fixed point. To see that, observe that in that case $\sigma_d$-image of $[s, t]$ contains $[s, t]$ and the images of $s, t$ do not belong to $(s, t)$. This implies that there exists a $\sigma_d$-fixed point in $[s, t]$. Since there are finitely many $\sigma_d$-fixed points, it is easy to see that the desired number $N$ exists. Now we can apply Lemma 2.16 which implies that $X$ is either (pre)periodic or (pre)critical, a contradiction. $\square$

Now we study dynamics of super-gaps and the map as a whole. Our standing assumption from here through Theorem 3.8 is that $\sim$ is a lamination and $\mathcal{L}_\sim$ is such that $\mathcal{L}_\sim^c$ is not empty (equivalently, $\mathbb{S}^1$ is not a super-gap). Denote the union of all periodic super-gaps by $SG$. Then there are finitely many super-gaps in $SG$, none of which coincides with $\mathbb{S}^1$, and, by Lemma 3.2, they are disjoint. Choose $N_\sim = N$ as the minimal number such that all periodic super-gaps and their periodic edges are $\sigma_d^N$-fixed. By Lemma 3.2 each super-gap has at least one $\sigma_d^N$-fixed edge and all its edges eventually map to $\sigma_d^N$-fixed edges.

Consider a component $A$ of $J \setminus p(SG)$. There are several $\sigma_d^N$-fixed super-gaps bordering $p^{-1}(A)$, and each such super-gap has a unique well-defined edge contained in $\overline{p^{-1}(A)}$. If all these edges are $\sigma_d^N$-fixed, we call $A$ *settled*. By Theorem 2.15, a settled component $A$ contains an element of $\mathrm{PC}_{rot} \setminus p(SG)$ denoted by $y_A$. In this way, we associate elements of $\mathrm{PC}_{rot} \setminus p(SG)$ to all settled components of $J \setminus p(SG)$.

**Lemma 3.6.** *If $\ell$ is not a $\sigma_d^N$-fixed edge of a $\sigma_d^N$-fixed super-gap $H$, then the component of $J \setminus p(\ell)$ which contains $p(H)$, contains a settled component of $J \setminus p(SG)$. In particular, settled components exist.*

*Proof.* Set $\ell_0 = \ell$. Choose a $\sigma_d^N$-fixed edge $\ell_0'$ of $H$ and a component $B_0$ of $J \setminus p(SG)$ such that $\ell_0'$ is contained in the closure of $p^{-1}(B_0)$. If $B_0$ is settled, we are done. Otherwise find a super-gap $H_1$ with an edge $\ell_1$ such that $\ell_1$ is not $\sigma_d^N$-fixed and borders $p^{-1}(B_0)$, then proceed with $\ell_1$ as before with $\ell_0$.

In the end we will find a settled component of $J \setminus p(SG)$ in the component of $J \setminus p(\ell)$ containing $p(H)$. Indeed, on each step we find a new $\sigma_d^N$-fixed super-gap different from the preceding one. Since there are finitely many $\sigma_d^N$-fixed super-gaps, we either stop at some point, or form a cycle. The latter is clearly impossible. Thus, there exists a non-empty collection of settled components $A$. □

Given any subcontinuum $X \subset J$ and $x \in X$, a component of $X \setminus \{x\}$ is called an *$X$-leg of $x$*. An $X$-leg of $x$ is called *essential* if $x$ eventually maps into this leg. An $X$-leg is said to be *critical* if it contains at least one critical atom; otherwise a leg is called *non-critical*.

Recall that by Definition 2.13 we call a periodic atom *rotational* if it is of degree 1 and its rotation number is not zero. Then $\mathrm{PC}_{rot} \setminus p(SG)$ is the set of all periodic rotational atoms $x$ which are not contained in $p(SG)$ (any such $x$ is a point by Lemma 3.2). Finally, define $\mathrm{COR}_s$ as the dynamical span of the limit sets of all persistent cut-atoms $x$ (equivalently, cutpoints) which never map into $p(SG)$. Observe that if $J$ is a dendrite, then $\mathrm{COR}_s = \mathrm{COR}$.

**Lemma 3.7.** *Every $x$-essential $J$-leg of every point $x \in J$ contains a point of* $\mathrm{PC}_{rot} \setminus p(SG)$.

*Proof.* Let $L$ be an $x$-essential $J$-leg of $x$. Denote by $A$ the component of $L \setminus p(SG)$ which contains $x$ in its closure. If $L$ contains no $p$-images of $\sigma_d^N$-fixed super-gaps (which implies that $L = A$), then the claim follows from Theorem 2.15 applied to $\overline{A}$. Otherwise there are finitely many super-gaps $U_1, \dots, U_t$ such that $p(U_i)$ borders $A$. For each $i$ let us take a point $x_i' \in \mathrm{Bd}(p(U_i))$ that separates $x$ from the rest of $p(U_i)$. If all the points $x_i'$ are $g$-fixed, then we are done by Theorem 2.15 applied to $\overline{A}$. If there exists $i$ such that the point $x_i'$ is not $g$-fixed, then, by Lemma 3.6, $x_i'$ separates the point $x$ from some settled component $B$, which in turn contains an element $y_B \in \mathrm{PC}_{rot} \setminus p(SG)$. Thus, in any case *every $x$-essential leg of $x \in J$ contains a point of* $\mathrm{PC}_{rot} \setminus p(SG)$ and the lemma is proven. □

Observe, that by Lemma 3.7 for a non-(pre)periodic persistent cutpoint $x$ there exists $n$ such that $f^n(x)$ separates two points of $\mathrm{PC}_{rot} \setminus p(SG)$ because, by Lemma 3.5, some iterated $g$-image of $x$ has at least two $x$-essential $J$-legs.

**Theorem 3.8.** *If $x$ is a persistent cutpoint that is never mapped to $p(SG)$ then there is $n \ge 0$ such that $f^n(x)$ separates two points of $\mathrm{PC}_{rot} \setminus p(SG)$ and is a cutpoint of $\mathrm{IC}(\mathrm{PC}_{rot} \setminus p(SG))$ so that $\mathrm{COR}_s = \mathrm{IC}(\mathrm{PC}_{rot} \setminus p(SG))$.*

*Moreover, there exist infinitely many persistent periodic rotational cutpoints outside $p(SG)$, $\mathrm{COR}_s \subset \mathrm{COR}_{rot}$, and any periodic cutpoint outside $p(SG)$ separates two points of $\mathrm{PC}_{rot} \setminus p(SG)$ and is a cutpoint of $\mathrm{COR}_s$, of $\mathrm{COR}_1$ and of $\mathrm{COR}$.*

*Proof.* By Lemma 3.7 and the remark after that lemma we only need to consider the case of a $g$-periodic cutpoint $y$ of $J$ outside $p(SG)$ (a priori it may happen that $y$ above is an endpoint of $\mathrm{COR}_s$). We want to show that $y$ separates two points of $\mathrm{PC}_{rot} \setminus p(SG)$. Indeed, a certain power $(\sigma_d^N)^k$ of $\sigma_d^N$ fixes $p^{-1}(y)$ and has rotation number zero on $p^{-1}(y)$. Choose an edge $\ell$ of $p^{-1}(y)$ and consider the component $B$ of $J \setminus \{y\}$ such that $\overline{p^{-1}(B)}$ contains $\ell$.

Then $(\sigma_d^N)^k$ fixes $\ell$ while leaves and gaps in $\overline{p^{-1}(B)}$ close to $\ell$ are repelled away from $\ell$ inside $\overline{p^{-1}(B)}$ by $(\sigma_d^N)^k$. Hence their $p$-images are repelled away from $y$ inside $B$ by the map $g^k$. By Lemma 3.7, there is an element (a point) $t_B \in \mathrm{PC}_{rot} \setminus p(SG)$ in $B$. As this applies to all edges of $p^{-1}(y)$, we see that $y$ separates two points of $\mathrm{PC}_{rot} \setminus p(SG)$. As $\mathrm{PC}_{rot} \setminus p(SG) \subset \mathrm{COR}_s \subset \mathrm{COR}$, this proves that any periodic cutpoint outside $p(SG)$ separates two points of $\mathrm{PC}_{rot} \setminus p(SG)$ and is a cutpoint of $\mathrm{COR}_s$ (and, hence, of $\mathrm{COR}_1$ and of COR). This also proves that for a (pre)periodic persistent cutpoint $x$ there exists $n$ such that $f^n(x)$ separates two points of $\mathrm{PC}_{rot} \setminus p(SG)$; by the above, it suffices to take $r$ such that $g^r(x) = f^{Nr}(x)$ is periodic and set $n = Nr$.

Let us prove that there are infinitely many points of $\mathrm{PC}_{rot}$ in any settled component $A$. Indeed, choose a $g$-periodic point $y \in A$ as above such that a certain power $(\sigma_d^N)^k$ of $\sigma_d^N$ which fixes $p^{-1}(y)$ has rotation number zero on $p^{-1}(y)$. Choose an edge $\ell$ of $p^{-1}(y)$ and consider the component $B$ of $A \setminus \{y\}$ such that $\overline{p^{-1}(B)}$ contains $\ell$. Then leaves and gaps in $\overline{p^{-1}(B)}$, which are close to $\ell$, are repelled away from $\ell$ inside $\overline{p^{-1}(B)}$ by $(\sigma_d^N)^k$. Hence their $p$-images are repelled away from $y$ inside $B$ by $g^k$. By Theorem 2.15, this implies that there exists a $g^k$-fixed point $z \in B$ with non-zero rotation number. Replacing $A$ by a component of $A \setminus z$, we can repeat the same argument. If we do it infinitely many times, we will prove that there are infinitely many points of $\mathrm{PC}_{rot}$ in any settled component. □

Corollary 3.9 follows immediately from Theorem 3.8. Observe that if $J$ is a dendrite, then $SG = p(SG) = \varnothing$, and hence $\mathrm{COR} = \mathrm{COR}_1 = \mathrm{COR}_s$.

**Corollary 3.9.** *If $J$ is a dendrite, then* $\mathrm{COR} = \mathrm{COR}_1 = \mathrm{COR}_s = \mathrm{COR}_{rot} = \mathrm{IC}(\mathrm{PC}_{rot})$. *Furthermore, for any persistent cutpoint $x$ there is $n \ge 0$ such that $f^n(x)$ separates two points from* $\mathrm{PC}_{rot}$ *(thus, at some point $x$ maps to a cutpoint of* COR*). Moreover, any periodic cutpoint separates two points of* $\mathrm{PC}_{rot}$ *and therefore is itself a cutpoint of* COR.

*Proof.* Left to the reader. □

Corollary 3.9 implies the last, dendritic part of Theorem 2.6. The rest of Theorem 2.6 is proven below.

*Proof of Theorem 2.6.* By definition, $\mathrm{IC}(\mathrm{PC}) \subset \mathrm{COR}, \mathrm{IC}(\mathrm{PC}_1) \subset \mathrm{COR}_1$ and $\mathrm{IC}(\mathrm{PC}_{rot}) \subset \mathrm{COR}_{rot}$. To prove the opposite inclusions, observe that by definition in each of these three cases it suffices to consider a persistent cut-atom $X$ which is not (pre)periodic. By Proposition 3.3 and because all Fatou gaps are eventually periodic, this implies that $X$ never maps to $p(SG)$. Hence, in this case, by Theorem 3.8, there exists $n$ such that $f^n(X)$ separates two points of $\mathrm{PC}_{rot} \setminus p(SG)$ and therefore is contained in

$$\mathrm{IC}(\mathrm{PC}_{rot} \setminus p(SG)) \subset \mathrm{IC}(\mathrm{PC}_{rot}) \subset \mathrm{IC}(\mathrm{PC}_1) \subset \mathrm{IC}(\mathrm{PC}),$$

which proves all three inclusions of the theorem. □

3.3. **Critical Atoms.** Theorems 1.2 and 3.8 give explicit formulas for various versions of the dynamical core of a topological polynomial $f$ in terms of various sets of periodic cut-atoms. These sets of periodic cut-atoms are most likely infinite. It may also be useful to relate the sets COR or $\mathrm{COR}_s$ to a finite set of critical atoms.

In the next lemma, we study one-to-one maps on complete continua. To do so we need a few definitions. The set of all critical points of $f$ is denoted by $\mathrm{Cr}_f = \mathrm{Cr}$. The $p$-preimage of Cr is denoted by $\mathrm{Cr}_\sim$. We also denote the $\omega$-limit set of Cr by $\omega(\mathrm{Cr})$ and its $p$-preimage by $\omega(\mathrm{Cr}_\sim)$. A *critical atom* is the $p$-image of a critical gap or a critical leaf of $\mathcal{L}_\sim$. Thus, the family of critical atoms includes all critical points of $f$ and boundaries of all bounded components of $\mathbb{C} \setminus J$ on which $f$ is of degree greater than 1, while boundaries of Siegel domains are not critical atoms. An atom of $J$ is said to be *precritical* if it eventually maps to a critical atom.

**Lemma 3.10.** *Suppose that $X \subset J$ is a complete continuum containing no critical atoms. Then $f|_X$ is one-to-one.*

*Proof.* Otherwise, choose points $x, y \in X$ with $f(x) = f(y)$, and connect $x$ and $y$ with an arc $I \subset X$. If there are no Fatou domains with boundaries in $X$, then $I$ is unique. Otherwise for each Fatou domain $U$ with $\mathrm{Bd}(U) \subset X$, separating $x$ from $y$ in $X$, there are two points $i_U, t_U \in \mathrm{Bd}(U)$, each of which separates $x$ from $y$ in $X$, such that $I$ must contain one of the two subarcs of $\mathrm{Bd}(U)$ with endpoints $i_U, t_U$. Note that $f(I)$ is not a dendrite since otherwise there must exist a critical point of $f|_I$. Hence we can choose a minimal subarc $I' \subset I$ so that $f(I')$ is a closed Jordan curve (then $f|_{I'}$ is one-to-one except for the endpoints $x'$, $y'$ of $I'$ mapped into the same point).

It follows that $f(I')$ is the boundary of a Fatou domain $U$ (otherwise there are points of $J$ "shielded" from infinity by points of $f(I')$ which is impossible). The set $f^{-1}(U)$ is a finite union of Fatou domains, and $I'$ is contained in the boundary of $f^{-1}(U)$. Therefore, there are two possible cases. Suppose that $I'$ is a finite concatenation of at least two arcs, each arc lying on the boundary of some component of $f^{-1}(U)$. Then, as $I'$ passes from one boundary to another, it must pass through a critical point, a contradiction. Suppose now that $I'$ is contained in the boundary of a single component $V$ of $f^{-1}(U)$ which implies that $\mathrm{Bd}(V) \subset X$ is a critical atom, a contradiction. □

Clearly, (pre)critical atoms are dense. In fact, they are also dense in a stronger sense. To explain this, we need the following definition. For a topological space $X$, a set $A \subset X$ is called *continuum-dense* (or *condense*) in $X$ if $A \cap Z \neq \varnothing$ for each non-degenerate continuum $Z \subset X$ (being non-degenerate means containing more than one point). The notion was introduced in [BOT06] in a different context.

Let us introduce a relative version of the notion of a critical atom. Namely, let $X \subset J$ be a complete continuum. A *point* $a \in X$ is *critical with respect to* $X$ if in a neighborhood $U$ of $a$ in $X$ the map $f|_U$ is not one-to-one. An *atom* $A \subset X$ is *critical with respect to* $X$ if it is either a critical point with respect to $X$ or a Fatou atom $A \subset X$ of degree greater than 1.

**Lemma 3.11.** *Let $I \subset X$ be two non-degenerate continua in $J$ such that $X$ is complete and invariant. Suppose that $X$ is not a Siegel atom. Then $f^n(I)$ contains a critical atom of $f|_X$ for some $n$. Thus, (pre)critical atoms are condense in $J$ (every continuum in $J$ intersects a (pre)critical atom).*

*Proof.* If $I$ contains more than one point of the boundary $\mathrm{Bd}(U)$ of a Fatou domain $U$, then it contains an arc $K \subset \mathrm{Bd}(U)$. As all Fatou domains are (pre)periodic, $K$ maps eventually to a subarc $K'$ of the boundary $\mathrm{Bd}(V)$ of a periodic Fatou domain $V$. If $V$ is of degree greater than 1, then eventually $K'$ covers $\mathrm{Bd}(V)$; since in this case $\mathrm{Bd}(V)$ is a critical atom of $f|_X$, we are done. If $V$ is Siegel, then every point of $\mathrm{Bd}(V)$ is eventually covered by $K'$. Since we assume that $X$ is not a Siegel atom, it follows that a critical point of $f|_X$ belongs to $\mathrm{Bd}(V)$, and again we are done.

By the preceding paragraph, from now on we may assume that non-empty intersections of $I$ with boundaries of Fatou domains are single points. By [BL02a], $I$ is not wandering; we may assume that $I \cap f(I) \neq \varnothing$. Set $L = \bigcup_{k=0}^{\infty} f^k(I)$. Then $L$ is connected, and $f(L) \subset L$. If $\overline{L}$ contains a Jordan curve $Q$, then $Q$ is the boundary of an invariant Fatou domain. If $L \cap Q = \varnothing$, then $L$ is in a single component of $J \setminus Q$, hence $\overline{L} \cap Q$ is at most one point, a contradiction. Choose the smallest $k$ with $f^k(I) \cap Q \neq \varnothing$; by the assumption $f^k(I) \cap Q = \{q\}$ is a singleton. Since the orbit of $I$ cannot be contained in the union of components of $J \setminus \{q\}$ disjoint from $Q$, there exists $m$ with $f^m(I) \cap Q$ not being a singleton, a contradiction.

Hence $\overline{L}$ is an invariant dendrite, and all cutpoints of $\overline{L}$ belong to images of $I$. Suppose that no cutpoint of $\overline{L}$ is critical. Then $f|_{\overline{L}}$ is a homeomorphism (if two points of $\overline{L}$ map to one point, there must exist a critical point in the open arc connecting them). However, it is proven in [BFMOT10] that if $D \subset J$ is an invariant dendrite, then it contains infinitely many periodic cutpoints. Hence we can choose two points $x, y \in \overline{L}$ and a number $r$ such that $f^r(x) = x, f^r(y) = y$. Then the arc $I' \subset \overline{L}$ connecting $x$ and $y$ is invariant under the map $f^r$ which is one-to-one on $I'$. Clearly, this is impossible inside $J$ (it is easy to see that a self-homeomorphism of an interval with fixed endpoints and finitely many fixed points overall must have a fixed point attracting from at least one side which is impossible in $J$). $\square$

We need new notation. Let $\mathrm{CrA}(X)$ be the set of critical atoms of $f|_X$; set $\mathrm{CrA} = \mathrm{CrA}(J)$. Also, denote by $\mathrm{PC}(X)$ the union of all periodic cut-atoms of $X$ and by $\mathrm{PC}_1(X)$ the set of all periodic cut-atoms of $X$ of degree 1.

**Lemma 3.12.** *Let $X \subset J$ be an invariant complete continuum which is not a Siegel atom. Then the following facts hold.*

(1) *For every cutpoint $x$ of $X$, there exists an integer $r \geq 0$ such that $f^r(x)$ either* (a) *belongs to a set from* $\mathrm{CrA}(X)$, *or* (b) *separates two sets of* $\mathrm{CrA}(X)$, *or* (c) *separates a set of* $\mathrm{CrA}(X)$ *from its image. In any case $f^r(x) \in \mathrm{IC}(\mathrm{CrA}(X))$, and in cases* (b) *and* (c) *$f^r(x)$ is a cutpoint of* $\mathrm{IC}(\mathrm{CrA}(X))$. *In particular, the dynamical span of all cut-atoms of $X$ is contained in* $\mathrm{IC}(\mathrm{CrA}(X))$.
(2) $\mathrm{PC}(X) \subset \mathrm{IC}(\mathrm{CrA}(X))$. *In particular, if $X = \mathrm{IC}(\mathrm{PC}(X))$ then* $X = \mathrm{IC}(\mathrm{CrA}(X))$.

*Proof.* (1) Suppose that $x$ does not eventually map to $\mathrm{CrA}(X)$. Then all points in the forward orbit of $x$ are cutpoints of $X$ (in particular, there are at least two $X$-legs at any such point).

If $f^r(x)$ has more than one critical $X$-leg for some $r \geq 0$, then $f^r(x)$ separates two sets of $\mathrm{CrA}(X)$. Assume that $f^k(x)$ has one critical $X$-leg for every $k \geq 0$. By Lemma 3.10, each non-critical $X$-leg $L$ of $f^k(x)$ maps in a one-to-one fashion to some $X$-leg $M$ of $f^{k+1}(x)$. There is a connected neighborhood $U_k$ of $f^k(x)$ in $X$ so that $f|_{U_k}$ is one-to-one. We may assume that $U_k$ contains all of the non-critical legs at $f^k(x)$. Hence there exists a bijection $\varphi_k$ between components of $U_k \setminus f^k(x)$ and components of $f(U_k) \setminus f^{k+1}(x)$ showing how small pieces (*germs*) of components of $X \setminus \{f^k(x)\}$, containing $f^k(x)$ in their closures, map to small pieces (*germs*) of components of $X \setminus \{f^{k+1}(x)\}$, containing $f^{k+1}(x)$ in their closures.

By Lemma 3.11 choose $r > 0$ so that a non-critical $X$-leg of $f^{r-1}(x)$ maps to the critical $X$-leg of $f^r(x)$. Then the bijection $\varphi_{r-1}$ sends the germ of the critical $X$-leg $A$ of $f^{r-1}(x)$ to a non-critical $X$-leg $B$ of $f^r(x)$. Let $C \subset A$ be the connected component of $X \setminus (\mathrm{CrA}(X) \cup \{f^{r-1}(x)\})$ with $f^{r-1}(x) \in \overline{C}$; let $R$ be the union of $\overline{C}$ and all the sets from $\mathrm{CrA}(X)$ non-disjoint from $\overline{C}$. Then $f(R) \subset \overline{B}$ while all the critical atoms are contained in the critical leg $D \neq B$ of $f^r(x)$. It follows that $f^r(x)$ separates these critical atoms from their images. By definition of IC(CrA($X$)) this implies that either $f^r(x)$ belongs to a set from CrA($X$), or $f^r(x)$ is a cutpoint of IC(CrA($X$)). In either case $f^r(x) \in \mathrm{IC}(\mathrm{CrA}(X))$. The rest of (1) easily follows.

(2) If $x$ is a periodic cutpoint of $X$ then, choosing $r$ as above, we see that $f^r(x) \in \mathrm{IC}(\mathrm{CrA}(X))$ that implies that $x \in \mathrm{IC}(\mathrm{CrA}(X))$ (because $x$ is an iterated image of $f^r(x)$ and IC(CrA($X$)) is invariant). Thus, all periodic cutpoints of $X$ belong to IC(CrA($X$)). Now, take a periodic Fatou atom $Y$. If $Y$ is of degree greater than 1, then it has an image $f^k(Y)$ which is a critical atom of $f|_X$. Thus, $Y \subset \mathrm{IC}(\mathrm{CrA}(X))$. Otherwise for some $k$ the set $f^k(Y)$ is a periodic Siegel atom with critical points on its boundary. Since $X$ is not a Siegel atom itself, $f^n(Y)$ contains a critical point of $f|_X$. Hence the entire $Y$ is contained in the limit set of this critical point and again $Y \subset \mathrm{IC}(\mathrm{CrA}(X))$. Hence each periodic Fatou atom in $X$ is contained in IC(CrA($X$)). Thus, $\mathrm{PC}(X) \subset \mathrm{IC}(\mathrm{CrA}(X))$ as desired. □

We can now relate various dynamical cores to the critical atoms contained in these cores. First first let us consider the following heuristic example. Suppose that the lamination $\sim$ of sufficiently high degree has an invariant Fatou gap $V$ of degree 2 and, disjoint from it, a super-gap $U$ of degree 3. The super-gap $U$ is subdivided ("tuned") by an invariant quadratic gap $W \subset U$ with a critical leaf on its boundary (or a finite critical gap sharing an edge with its boundary as in Figure 2) so that $W$ concatenated with its appropriate pullbacks fills up $U$ from within.

Also assume that the strip between $U$ and $V$ is enclosed by two circle arcs and two edges, a fixed edge $\ell_u$ of $U$ and a prefixed edge $\ell_v$ of $V$ (that is, $\sigma_d(\ell_v)$ is a fixed edge of $V$). Moreover, suppose that $U$ and $V$ have only two periodic edges , namely, $\ell_u$ and $\sigma_d(\ell_v)$, so that all other edges of $U$ and $V$ are preimages of $\ell_u$ and $\sigma_d(\ell_v)$. All other periodic gaps and leaves of $\sim$ are located in the component $A$ of $\mathbb{D}\setminus\sigma_d(\ell_v)$ which does not contain $U$ and $V$. In Figure 2 an example of this construction is shown for $d = 6$.

It follows from Theorem 1.2 that in this case $\mathrm{COR}_1$ includes a continuum $K \subset p(A \cup V)$ united with a connector-continuum $L$ connecting $K$ and $p(\ell_u)$. Moreover, $p(\mathrm{Bd}(V)) \subset K$. Basically, all the points of $L$ except for $p(\ell_u)$ are "sucked into" $K$ while being repelled away from $p(\ell_u)$. Clearly, in this case even though $p(\ell_u) \in \mathrm{COR}_1$, still $p(\ell_u)$ does not belong to the set $\mathrm{IC}(\mathrm{CrA}(\mathrm{COR}_1))$ because $p(\mathrm{Bd}(W))$, while being a critical atom, is not contained in $\mathrm{COR}_1$. This shows that some

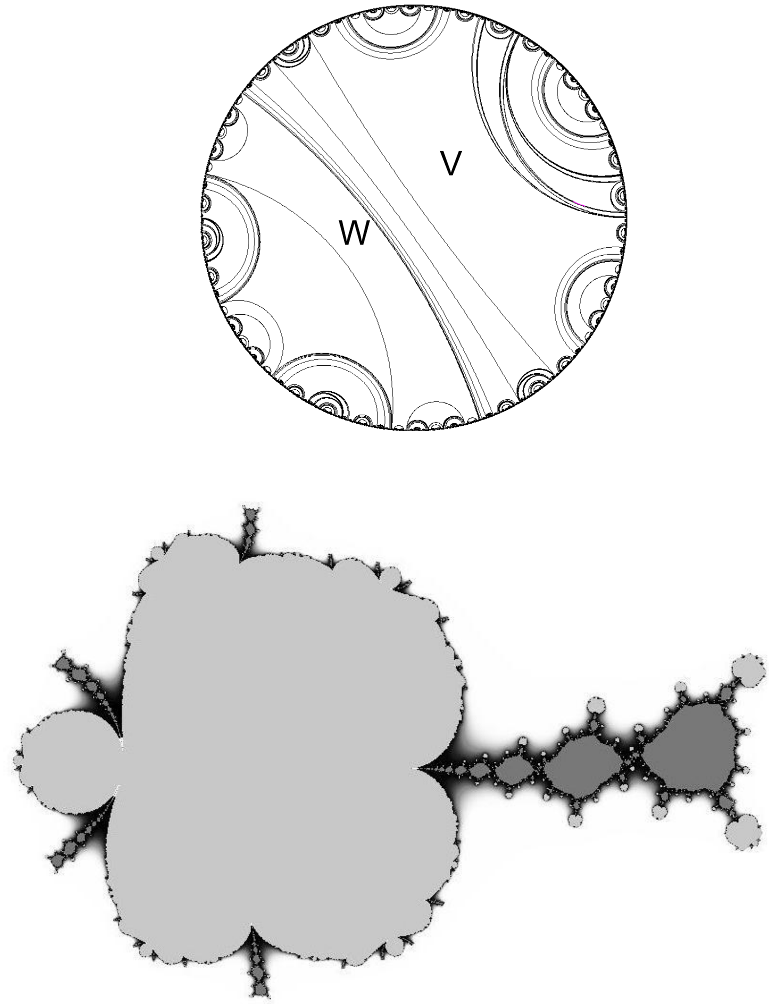


FIGURE 2. Example of a lamination and the corresponding Julia set, for which $\mathrm{COR}_1 \neq \mathrm{IC}(\mathrm{CrA}(\mathrm{COR}_1))$. The picture corresponds to a degree 6 polynomial. The invariant quadratic gap $V$ corresponds to the largest dark grey region on the right. The invariant gap $W$ corresponds to the large light grey "cauliflower" on the left. This is an invariant parabolic domain that contains a critical point on its boundary.

points of $\mathrm{COR}_1$ may be located outside $\mathrm{IC}(\mathrm{CrA}(\mathrm{COR}_1))$ and also that there might exist non-degenerate critical atoms intersecting $\mathrm{COR}_1$ over a point (and hence not contained in $\mathrm{COR}_1$). This example shows that the last claim of Lemma 3.12 cannot be established for $X = \mathrm{COR}_1$.

Also, let us consider the case when COR is a Siegel atom $Z$. Then by definition there are no critical points or atoms of $f|_{\mathrm{COR}}$, so in this case CrA(COR) is empty. However this is the only exception.

**Theorem 3.13.** *Suppose that* COR *is not just a Siegel atom. Then* $\mathrm{COR} = \mathrm{IC}(\mathrm{CrA}(\mathrm{COR}))$, $\mathrm{COR}_{rot} = \mathrm{IC}(\mathrm{CrA}(\mathrm{COR}_{rot}))$, *and* $\mathrm{COR}_s = \mathrm{IC}(\mathrm{CrA}(\mathrm{COR}_s))$.

*Proof.* By Theorem 1.2 we have $\mathrm{COR} = \mathrm{IC}(\mathrm{PC})$. Let us show that in fact COR is the dynamical span of its periodic cutpoints and its periodic Fatou atoms. It suffices to show that any periodic cutpoint of $J$ either belongs to a Fatou atom or is a cutpoint of COR. Indeed, suppose that $x$ is a periodic cutpoint of $J$ which does not belong to a Fatou atom. Then by Theorem 2.15 applied to different components of $J \setminus \{x\}$ we see that $x$ separates two periodic elements of PC. Hence $x$ is a cutpoint of COR as desired. By Lemma 3.12 we have $\mathrm{COR} = \mathrm{IC}(\mathrm{CrA}(\mathrm{COR}))$. The remaining claims can be proven similarly. □

There is a bit more universal way of stating a similar result. Namely, instead of considering critical atoms of $f|_{\mathrm{COR}}$ we can consider critical atoms of $f$ *contained in* COR. Then the appropriately modified claim of Theorem 3.13 holds without exception. Indeed, it holds trivially in the case when COR is an invariant Siegel atom. Otherwise it follows from Theorem 3.13 and the fact that the family of critical atoms of $f|_{\mathrm{COR}}$ is a subset of the family of all critical atoms of $f$ contained in COR. We prefer the statement of Theorem 3.13 to a more universal one because it allows us not to include “unnecessary” critical points of $f$ which happen to be endpoints of COR; clearly, the results of Theorem 3.13 hold without such critical points. Notice, that the explanations given in this paragraph equally relate to $\mathrm{COR}_1$ and $\mathrm{COR}_{rot}$.

In the dendritic case the following corollary holds.

**Corollary 3.14.** *If $J$ is a dendrite, then the following holds:*

$$\mathrm{COR} = \mathrm{IC}(\mathrm{PC}_{rot}) = \mathrm{IC}(\mathrm{CrA}(\mathrm{COR})).$$

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