SPECTRAL DECOMPOSITION AND MISIURETWICZ CONJECTURE FOR GRAPH MAPS

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Abstract. We verify the conjecture of M.Misiurewicz and prove that for any graph X there exists a number L=L(X) such that any continuous self-mapping of X with cycles of periods 1,2,...,L has in fact cycles of all possible periods.

0. Introduction

Let us call one-dimensional branched manifolds *graphs*. We study properties of a set $P(f)$ of periods of cycles of a graph map f. One of the well-known and impressive results on this topic is Sharkovskii theorem [S1] about the co-existence of periods of cycles for maps of the real line. To formulate it let us introduce the following *Sharkovskii ordering* for positive integers:

$$(*) \qquad 3 \prec 5 \prec 7 \prec \cdots \prec 2\cdot 3 \prec 2\cdot 5 \prec 2\cdot 7 \prec \cdots \prec 8 \prec 4 \prec 2 \prec 1$$

Denote by $S(k)$ the set of all such integers m that $k \prec m$ or $k = m$ and by $S(2^\infty)$ the set $\{1, 2, 4, 8, \dots\}$.

Theorem[S1]. *Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous map. Then either $P(g) = \emptyset$ or there exists such $k \in \mathbb{N} \cup 2^\infty$ that $P(g) = S(k)$. Moreover for any such k there exists a map $g : [0,1] \longrightarrow [0,1]$ with $P(g) = S(k)$ and there exists a map $g_0 : \mathbb{R} \longrightarrow \mathbb{R}$ with $P(g_0) = \emptyset$.*

Other information about sets of periods of cycles for one-dimensional maps is contained in papers [AL,M] for maps of the circle, [ALM] for maps of the letter Y and [Ba] for maps of the n-od.

Sharkovskii theorem implies that if a map $f : \mathbb{R} :\longrightarrow \mathbb{R}$ has a cycle of period 3 then it has cycles of all possible periods. The following conjecture, which was formulated by M.Misiurewicz at the Problem Session at Czecho-Slovak Summer Mathematical School near Bratislava in 1990, seems to be closely related to the mentioned property of maps of the real line.

Misiurewicz Conjecture. *For a graph X there exists an integer $L = L(X)$ such that for a continuous map $f : X \longrightarrow X$ inclusion $P(f) \supset \{1, 2, \dots, n\}$ implies $P(f) = \mathbb{N}$.*

We verify Misiurewicz conjecture in Section 2. Furthermore, using *the spectral decomposition for graph maps* [B3] which is similar to that for interval maps [B1,B2] we then prove the following

Theorem 2.2. *Let $f : X \to X$ be a continuous graph map. Then the following statements are equivalent.*

1) The map f has a positive entropy.

2) There exists such n that $P(f) \supset nZ \equiv \{ni : i \in \mathbb{N}\}$.

Section 1 is devoted to the brief description of the spectral decomposition, in Section 2 we verify Misiurewicz conjecture and prove Theorem 2.

Notations

$\operatorname{int} Z$ is the interior of a set Z;
∂Z is the boundary of Z;
$\overline{Z}$ is the closure of Z;
f^n is the n-fold iterate of a map f;
$\operatorname{orb} x \equiv \{f^n x\}_{n=0}^{\infty}$ is the *orbit (trajectory)* of x;
$\omega(x)$ is the limit set of $\operatorname{orb} x$;
$\mathbb{N} \equiv \{1, 2, 3, \dots\}$ is the set of natural numbers;
$\operatorname{Per} f$ is the set of all periodic points of a map f;
$P(f)$ is the set of all periods of periodic points of a map f;
$h(f)$ is a topological entropy of a map f.

1. The Spectral Decomposition

In this section we briefly describe the spectral decomposition for one-dimensional maps (for the proofs see [B3]). Let us begin with some historical remarks.

A.N.Sharkovskii constructed the decomposition of the set $\omega(f) = \bigcup_{x\in I} \omega(x)$ for continuous interval maps $f : I \to I$ in [S2]. Then in [JR] Jonker and Rand constructed for unimodal maps the decomposition which is in fact close to that of Sharkovskii; however they used completely different methods based on symbolic dynamics. In [H] the decomposition for piecewise-monotone maps with discontinuities was constructed by Hofbauer and then Nitecki in [N] considered the decomposition for piecewise-monotone continuous maps from more geometrical point of view. The author's papers [B1,B2] were devoted to the case of arbitrary continuous interval maps; they contained the different approach to the problem in question which allowed us to obtain some new corollaries (e.g. describing generic properties of invariant measures for interval maps). The similar approach was used in [B3] to construct the decomposition for graph maps and now we pass to the desription of the results of the paper [B3].

Let X be a graph, $f : X \to X$ be a continuous map. We use terms *edge, vertex, endpoint* in the usual sense; the numbers of edges and endpoints of X are denoted by $\operatorname{Edg}(X)$, $\operatorname{End}(X)$. If necessary we add some "artificial" vertices to make all edges of a graph homeomorphic to an interval. We construct the decomposition of the set $\omega(f)$, which is defined similar to that for interval maps. First we need some definitions. A closed connected set $Y \subset X$ is called *subgraph*. A subgraph Y is called *periodic (of period k)* if $Y, fY, \dots, f^{k-1}Y$ are pairwise disjoint and $f^k Y = Y$; the union of all iterations of Y is denoted by $\operatorname{orb} Y$ and called *a cycle of subgraphs.* Let $Y_0 \supset Y_1 \supset \dots$ be periodic subgraphs of periods $m_0, m_1, \dots$; then m_{i+1} is divided by $m_i\,(\forall i)$. If $m_i \to \infty$ then the subgraphs Y_i, $i = 1, 2, \dots$ are said to be *generating.* We call any invariant closed set $S \subset Q = \cap(\operatorname{orb} Y_i)$ *a solenoidal set* and denote the solenoidal set $Q \cap \omega(f)$ by $S_\omega(Q)$ (note that $\omega(f)$ is closed for graph maps, see [B3]).

One can use a transitive shift in an Abelian zero-dimensional infinite group as a model for the map on a solenoidal set. Namely, let $D = \{n_i\}$ be a sequence of integers, n_{i+1} is divided by $n_i\,(\forall i)$ and $n_i \to \infty$. Let us consider a subgroup $H(D) \subset \mathbb{Z}_{n_0} \times \mathbb{Z}_{n_1} \times \dots$, defined in the following way:

$$H(D) \equiv \{(r_o, r_1, \dots) : r_{i+1} \equiv r_i \pmod{m_i}\,(\forall i)\}.$$

Denote by τ the minimal shift in $H(D)$ by the element $(1, 1, \dots)$.

Theorem 1.1[B3]. *Suppose that $\{Y_i\}$ are generating subgraphs and that they have periods $\{m_i\}$. Let $Q = \bigcap_{i\geq 0} \operatorname{orb} Y_i$. Then there exists a continuous surjective map $\varphi : Q \longrightarrow H(D)$ with the following properties:*

1) $\tau \circ \varphi = \varphi \circ f$ (i.e.φ semiconjugates $f|Q$ to τ);

2) there exists the unique set $S \subset Q \cap \overline{Per\, f}$ such that $\omega(x) = S$ for any $x \in Q$ and if $\omega(z) \cap Q \neq \emptyset$ then $S \subset \omega(z) \subset S_\omega$;

3) for any $\bar{r} \in H(D)$ the set $J = \varphi^{-1}(\bar{r})$ is a connected component of Q and $\varphi|S_\omega$ is at most 2-to-1;

4) $h(f|Q) = 0$.

Let us turn to another type of an infinite limit set. Let $\{Y_i\}_{i=1}^{l}$ be a collection of connected graphs, $K = \bigcup_{i=1}^{l} Y_i$. A continuous map $\psi : K \longrightarrow K$ which permutes these graphs cyclically is called *non-strictly periodic* or *non-strictly l-periodic*; for example if Y is a periodic subgraph then $f|\operatorname{orb} Y$ is non-strictly periodic. In what follows we will consider monotone semiconjugations between non-strictly periodic graph maps (a continuous map $g : X \longrightarrow Y$ is *monotone* provided $g^{-1}(Y)$ is connected for any $y \in Y$). We need the following

Lemma 1.1. *Let X be a graph. Then there exists a number $r = r(X)$ such that if $M \subset X$ is a cycle of subgraphs and $g : M \longrightarrow Y$ is monotone then the following property holds for any $y \in M$: card $\{\partial(g^{-1}(y))\} \leq r(X)\,(\forall y \in M)$.*

Lemma 1.1 makes natural the following definition. If $\varphi : K \longrightarrow M$ is continuous, monotone, semiconjugates a non-strictly periodic map $f : K \longrightarrow K$ to a non-strictly periodic map $g : M \longrightarrow M$ and there is a closed f-invariant set $F \subset K$ such that $\varphi(F) = M$ and $\varphi^{-1}(y) \cap F \subset \partial(\varphi^{-1}(y))\,(\forall y \in M)$ then we say that φ *almost conjugates* $f|F$ to g.

Let Y be an n-periodic subgraph, $\operatorname{orb} Y = M$. Denote by $E(M, f)$ the following set:

$$E(M, f) \equiv \{x \in M : \text{for any open } U \ni x, U \subset M \text{ we have } \overline{\operatorname{orb} U} = M\}$$

provided it is infinite. We call the set E(M,F) *a basic set* and denote it by $B(M, f)$ provided $\operatorname{Per}(f|M) \neq \emptyset$; otherwise we denote $E(M, f)$ by $C(M, f)$ and call it *a circle-like set.*

Theorem 1.2[B3]. *Let Y be an n-periodic subgraph, $M = \operatorname{orb} Y$ and $E(M, f) \neq \emptyset$. Then there exist a transitive non-strictly n-periodic map $g : K \longrightarrow K$ and a monotone continuous surjection $\varphi : M \longrightarrow K$ which almost conjugates $f|E(M, f)$ to g. Furthermore, the following properties hold:*

1) $E(M, f)$ is a perfect set;

2) $f|E(M, f)$ is transitive;

3) if $\omega(x) \supset E(M, f)$ then $\omega(x) = E(M, f)$;

4) if $E(M,f)=C(M,f)$ is a circle-like set then K is a union of n circles, g permutes them, g^n on any of them is an irrational rotation and $h(g)=h(f|E(M,f))=0$;

5) if $E(M,f)=B(M,f)$ is a basic set then $h(f|B(M,f))>0$, $B(M,f)\subset \overline{Per\, f}$ and there exist a number k and a closed subset $D\subset B(M,f)$ such that $\varphi(D)$ is connected,
sets $f^iD\cap f^jD$ and $\varphi(f^iD)\cap\varphi(f^jD)\,(0\le i<j<kn)$ are finite, $f^{kn}D=D$, $\bigcup_{i=0}^{kn-1} f^iD=B(M,f)$ and $f^{kn}|D$, $g^{kn}|\varphi D$ are topologically mixing.

A number kn from the statement 5) of Theorem 1.2 is called *a period* of $B(M,f)$.

To formulate the decomposition theorem denote by Z_f the set of all cycles maximal by inclusion among all limit sets of f.

Theorem 1.3[B3]. *Let $f:X\to X$ be a continuous graph map. Then there exist a finite number of circle-like sets $\{C(K_i,f)\}_{i=1}^k$, an at most countable family of basic sets $\{B(L_j,f)\}$ and a family of solenoidal sets $\{S_\omega(Q_\alpha)\}$ such that*

$$\omega(f)=Z_f\bigcup(\bigcup_{i=1}^k C(K_i))\bigcup(\bigcup_j B(L_j))\bigcup(\bigcup_\alpha(S_\omega(Q_\alpha)).$$

Moreover, there exist numbers $\gamma(X)$ and $\nu(X)$ such that $k\le\gamma(X)$, the only possible intersections in the decomposition are between basic sets and at most $\nu(X)$ basic sets can intersect.

Theorem 1.3 shows that one can consider mixing graph maps as models for graph maps on basic sets. The following theorem seems to be important in this connection; to formulate it we need the definition of maps with the specification property (see, for example, [DGS]).

Theorem 1.4[B3]. *Let $f:X\to X$ be a continuous mixing graph map. Then f has the specification property.*

It is well-known [DGS] that maps with the specification have nice properties concerning the set of invariant measures. Using them and Theorems 1.1 - 1.4 we can describe generic properties of invariant measures for graph maps. First we need some definitions. Let $T:X\to X$ be a map of a compact metric space into itself. The set of all T-invariant Borel normalized measures is denoted by D_T. A measure $\mu\in D_T$ with *supp* μ containing in one cycle is said to be a $CO-measure$. The set of all CO-measures concentrated on cycles with minimal period p is denoted by $P_T(p)$. Let $V(x)$ be the set of accumulation points of time-averages of iterations of the point x. A point $x\in X$ is said to have *maximal oscillation* if $V_T(x)=D_T$.

Theorem 1.5[B3]. *Let B be a basic set. Then:*

1) for any l the set $\bigcup_{p\ge l}P_{f|B}(p)$ is dense in $D_{f|B}$;

2) the set of all ergodic non-atomic invariant measures μ with supp $\mu=B$ is a residual subset of $D_{f|B}$;

3) if $V\subset D_{f|B}$ is a non-empty closed connected set then the set of all such points x that $V(x)=V$ is dense in X (in particular every measure $\mu\in D_{f|B}$ has a generic point);

4) points with maximal oscillation are residual in B.

Theorem 1.6[B3]. *Let μ be an invariant measure. Then the following properties of μ are equivalent:*

1) there exists such a point x that $\operatorname{supp}\mu \subset \omega(x)$;

2) μ has generic points;

3) μ is concentrated on a circle-like set or can be approximated by CO-measures.

In particular, CO-measures are dense in all ergodic neasures which are not concentrated on circle-like sets.

Let us denote by nZ the set $nZ \equiv \{in : i \geq 1\}$. Now we can formulate an easy property of maps with the specification which we need in Lemma 1.2.

Property 1.1. *If T is a map with the specification then $P(T)$ almost coincides with $\mathbb{N}$.*

Property 1.1 and Theorem 1.2 easily imply the following

Lemma 1.2. *Let $f : X \longrightarrow X$ be a graph map, B be a basic set of f, m be a period of B. Then both sets $P(f|B) \setminus mZ$ and $mZ \setminus P(f|B)$ are finite and so there exists such n that $P(f|B) \supset nZ$.*

2. Misiurewicz Conjecture

During the Problem Session at Czecho-Slovak Summer Mathematical School near Bratislava in 1990 M.Misiurewicz formulated the following

Conjecture. *For a graph X there exists an integer $L = L(X)$ such that for a continuous map $f : X \longrightarrow X$ inclusion $P(f) \supset \{1, 2, \dots, n\}$ implies $P(f) = \mathbb{N}$.*

We verify this conjecrure and give a sketch of the proof. First let us formulate the following

Lemma 2.1. *Let R be a positive integer. Then one can find such $N = N(R) > R$ that for any $M \geq N$ there exist positive integers $0 = a_0 < a_1 < a_2 < \dots < a_l = M$ with the following properties:*

1) $a_{i+1} - a_i \geq R\,(0 \leq i < l)$;

2) for any proper divisor s of M there exists $j, 1 \leq j < l$ such that a_j is divided by s.

Proof. Let $M = p_1^{b_1} \cdot \ldots p_k^{b_k}$, where $p_1, \dots, p_k$ are prime integers. Set $m_i = \dfrac{M}{p_i}$, $1 \leq i \leq k$. Clearly numbers $\{m_i\}$ have the required property 2). So it is sufficient to find numbers $a_0 = 1 < a_1 < \dots < a_l = M$ such that $a_{i+1} - a_i \geq R$, $0 \leq i < l$ and for any j there exists such i that a_i is divided by m_j. To this end suppose that $\{q_1 < q_2 < \dots < q_r\}$ is the set of all prime integers less that $R+1$ and set $\alpha = \min(\dfrac{1}{q_{i+1}} - \dfrac{1}{q_i})_{i=1}^{r-1}$, $N = \max(\dfrac{R}{\alpha}, 3q_r)$.

Now if $\dfrac{M}{p_k p_{k-1}} \geq R$ then $\dfrac{M}{p_i} - \dfrac{M}{p_{i+1}} \geq \dfrac{M}{p_k p_{k-1}} \geq R$. If $\dfrac{M}{p_k p_{k-1}} < R$ then $p_1, p_2, \dots < p_{k-2} \leq R$ and so

$$m_i - m_{i+1} = \frac{M}{} - \frac{M}{\phantom{p_{i+1}}} \quad \geq \alpha M \geq \alpha N \geq R\,(1 \leq i \leq k-2).$$

Thus it remains to consider the differences $\dfrac{M}{p_{k-1}} - \dfrac{M}{p_{k-2}}, \dfrac{M}{p_k} - \dfrac{M}{p_{k-1}}$ which is left to the reader. Clearly we may assume that $N(R)$ increases with R. $\square$

Let us call a subset of a graph *an interval* if it is homeomorphic to the interval [0,1]; we use for intervals standart notations $[a,b],[a,b),(a,b],(a,b)$. Let us fix for the rest of this section a graph X and a continuous map $f: X \to X$.

Lemma 2.2. *There exists a number $m = m(X)$ such that if $a \in X$ and $[a,b_1],[a,b_2],\dots,$ $[a,b_{m+1}]$ are intervals then one of them contains some of others.*

Proof. Left to the reader. $\square$

Suppose that there exist an edge $I = [a,b] \subset X$ and two periodic points, $P \in I$ of prime period $p > m(X)$ and $Q \in X$ of prime period $q > m(x)$, $p \neq q$ such that if $[P,Q] \subset I$ then $(P,Q) \cap (\operatorname{orb} Q \cup \operatorname{orb} P) = \emptyset$; fix them for Lemmas 2.3 - 2.7.

Lemma 2.3. *We have $f^{p(q-1)m(X)}[P,Q] \supset \operatorname{orb} Q$, $\quad f^{q(p-1)m(X)}[P,Q] \supset \operatorname{orb} P$ and so $f^t[P,Q] \supset \operatorname{orb} Q \cup \operatorname{orb} P$ for $t \geq pqm(X) - \min(p,q)\cdot m(X)$.*

Proof. Consider all the intervals of type $\{T_i = [P,c_i]\}_{i=1}^k$, where $c_i \in \operatorname{orb} Q$, containing no points of $\operatorname{orb} Q$ but c_i (some of points $\{c_i\}$ may coincide with each other). Then $k \leq m(X)$ and we may assume $Q = c_1, [P,Q] = T_1$. On the other hand for any i there exists $j = j(i)$ such that $f^p T_i \supset T_j$. Hence there exist such numbers l and n that $l + n \leq k$ and, say, $f^{pl} T_1 \supset T_2, f^{pn} T_2 \supset T_2$ which implies that $f^{pnj} T_2 \supset \{f^{pni} c_2\}_{i=0}^j$. But p,q are prime numbers and $n \leq m(X) < q$; thus $\{f^{ipn} c_2\}_{i=0}^{q-1} = \operatorname{orb} Q$ and $f^{pn(q-1)+lp}[P,Q] \supset \operatorname{orb} Q$ (recall that $T_1 = [P,Q]$). It implies that $f^{p(q-1)m(X)}[P,Q] \supset \operatorname{orb} Q$. Similarly $f^{q(p-1)m(X)}[P,Q] \supset \operatorname{orb} P$ and we are done. $\square$

Let us call subintervals of I with endpoints from $\operatorname{orb} Q$ or $\operatorname{orb} P$ *basical intervals* provided their interiors contain no points from $\operatorname{orb} P$ or $\operatorname{orb} Q$. In what follows basical interval will be called *P-interval, Q-interval* or *PQ-interval* depending on periodic orbits containing its endpoints. Furthermore, suppose that there are two intervals $G \subset X$ and $H \subset X$ and a continuous map $\varphi: X \to X$ such that $\varphi(G) \supset H$ and there is a subinterval $K \subset G$ such that $\varphi(K) = H$; then say that G $\varphi-$*covers* H. Note the following property: if G φ-covers H and H ψ-covers M then G $\psi \circ \varphi$-covers M.

Lemma 2.4. *Let $Z \subset X$ be an interval, $Y = [\alpha,\beta] \subset X$ be an edge and $g: X \to X$ be a continuous map; suppose that $\alpha,\beta \in g(Z)$. Then there are points $\gamma,\delta \in Y$ such that $g(Z) \cap Y = [\alpha,\gamma] \cup [\delta,\beta]$ and Z g-covers $[\alpha,\gamma]$ and $[\delta,\beta]$.*

Proof. Left to the reader. $\square$

Lemma 2.5. *Let A be a PQ-interval. Then for any $i \geq pqm(X)$ this interval f^i-covers all basical intervals except at most one.*

Proof. Follows from Lemmas 2.3 and 2.4. $\square$

Lemma 2.6. *Suppose that card$\,(\operatorname{orb} P \cap I) \geq 4$, card$\,(\operatorname{orb} Q \cap I) \geq 4$. Then the following statements are true.*

1) Either for any P-interval M there exists $i < p^2$ such that $f^i M$ contains a PQ-interval or there exist two P-intervals Y and Z such that each of them f^i-covers both of them for $i \geq (p-1)^2$

2) Either for any Q-interval N there exists $i < q^2$ such that f^iN contains a PQ-interval or there exist two Q-intervals Y' and Z' such that each of them f^i-covers both of them for $i \geq (p-1)^2$.

Proof. We will prove only statement 1). Consider a P-interval $[c,d]$ which has a neighbouring PQ-interval, say, $[d,e]$. Let the point c be closer to the point a than the point d (recall that $I = [a,b] \supset [c,d] \cup [d,e]$). Divide the proof by steps.

Step 1. *If $f^i[c,d]$ contains a PQ-interval then for any P-interval M there exists such $j \leq p-1+i$ that f^jM contains a PQ-interval.*

Indeed, for any P-interval M one can find such $m < p$ that either $f^mM \supset [c,d]$ or $f^mM \supset [d,e]$ which implies the required.

Step 2. *Suppose there exists such $i < (p-1)^2$ that $f^i[c,d]$ contains a PQ-interval. Then for any P-interval M there exists an integer $j < (p-1)^2+p$ such that f^jM contains a PQ-interval.*

Step 2 easily follows from Step 1.

Denote by x the closest to e point from $\operatorname{orb} P$ lying to the other side of e than d; clearly x may not exist.

Step 3. *Suppose that $f^i[c,d]$ does not contain PQ-intervals for $i < (p-1)^2$. Then for $i \geq (p-1)(p-2)$ the interval $[c,d]$ f^i-covers $[a,d]$ (and $[x,b]$ provided x exists).*

Let $l < p$ be such that $f^lc = d$. Then $f^l[c,d] \supset [c,d]$ and moreover $[c,d]$ f^l-covers $[c,d]$. But p is a prime integer which as in Lemma 2.3 implies that $f^i[c,d] \supset \operatorname{orb} P$ for every $i \geq l(p-2)$. Since $f^i[c,d]$ does not contain $[d,e]$ for $l(p-2) \leq i < l(p-1)$ we have by Lemma 2.4 that $[c,d]$ f^i-covers $[a,d]$ (and $[x,b]$ provided x exists). But $[c,d]$ f^i-covers $[c,d]$ which easily implies that for any $i \geq l(p-2)$ the interval $[c,d]$ f^l-covers $[a,d]$ (and $[x,b]$ provided x exists).

Step 4. *Suppose that $f^i[c,d]$ does not contain PQ-intervals for $i < (p-1)^2+p$. Then for any P-interval M and $i \geq (p-1)^2$ we have that M f^i-covers $[a,d]$ (and $[x,b]$ provided x exists).*

Clearly there exists $l < p$ such that either M f^l-covers $[c,d]$ or M f^l-covers $[d,e]$. Now by Step 3 $f^{(p-1)(p-2)}[c,d] \supset M$; so if M f^l-covers $[d,e]$ then $f^{(p-1)(p-2)+l}[c,d] \supset [d,e]$ which is a contradiction. Thus M f^l-covers $[c,d]$ and by Step 3 we get the required.

Now suppose there exists a P-interval M such that f^iM contains no PQ-intervals for $i < p^2$. Then by Step 1 $f^i[c,d]$ contains no PQ-intervals for $i < p^2-(p-1) = (p-1)^2+p$. Applying Step 4 and using simple geometrical arguments we may assert that there exist two P-intervals Y and Z such that $Y \cap Z = \emptyset$ and for any $i \geq (p-1)^2$ the interval Y f^i-covers intervals Y, Z and the interval Z f^i-covers intervals Y, Z which completes the proof of Lemma 2.6. $\square$

Lemma 2.7. *Suppose that $\operatorname{card}(\operatorname{orb} P \cap I) \geq 4$, $\operatorname{card}(\operatorname{orb} Q \cap I) \geq 4$. Let*

$$T \equiv T(p,q) \equiv N(pqm(X) - \min(p,q) \cdot m(X) + [\max(p,q)]^2)$$

(recall that function $N(x)$ was defined in Lemma 2.1). Then $P(f) \supset \{i : i \geq T\}$ and $h(f) > 0$.

Proof. Let us make use of Lemmas 2.1 and 2.6 and consider all possible cases.

Case A. *There exist such P-intervals Y and Z that each of them f^i-covers both of them for $i \geq (p-1)^2$.*

Let $k \geq N((p-1)^2)$ be an integer. By Lemma 2.1 one can easily see that there exist integers $1 = a_0 < a_1 < \cdots < a_l = k$, $a_{i+1} - a_i \geq (p-1)^2$ such that for any proper divisor s of k there exists a_i which is divided by s. Properties of f^i-covering imply that there exists an interval $K \subset Y$ such that $f^{a_i} K \subset Z$ for any $1 < i < l$ and $f^k K = Y$. Hence there exists a point $\zeta \in Y$ such that $f^{a_i} \zeta \in Z$ for $0 < i < l$ and $f^k \zeta = \zeta$; by the properties of the numbers $\{a_i\}$ it implies that k is the minimal period of the point ζ and so $P(f) \supset \{i : i \geq N((p-1)^2)\} \supset \{i : i \geq T\}$. Standart one-dimensional arguments show also that $h(f) > 0$ (see, for example, [BGMY]).

Case B. *There are such Q-intervals Y' and Z' that each of them f^i-covers both of them for $i \geq (q-1)^2$.*

Similarly to Case A we have $P(f) \supset \{i : i \geq N((q-1)^2)\} \supset \{i : i \geq T\}$ and $h(f) > 0$.

Case C. *For any basical interval M there exists a number $s = s(M) < [\max(p,q)]^2$ such that $f^s M$ contains a PQ-interval.*

Let for definitness $p > q$. Then similarly to Lemma 2.5 we can conclude by Lemmas 2.3 and 2.4 that any basical interval M f^i-covers all basical intervals except at most one of them for $i \geq H = pqm(X) - qm(X) + p^2$. Choose four basical intervals $\{M_j\}_{j=1}^4$ which are pairwise disjoint and show that for any $k \geq N(H)$ there exists a periodic point ζ of minimal period k.

Let $k \geq N(H)$. As in Case A choose integers $1 = a_0 < a_1 < \cdots < a_l = k$ with the properties from Lemma 2.1. Let $u = a_l - a_{l-1}$. Then it is easy to see that there exists such basical interval, say, M_1, that at least two other basical intervals, say M_2 and M_3, f^u-cover M_1. On the other hand one can easily show that there are two numbers $i, j \in \{2,3,4\}$ and two intervals $K_i \subset M_1$ and $K_j \subset M_1$ such that for any $1 \leq v \leq l-2$ we have $f^{a_v}(K_i) \subset M_{r(v)}$ and $f^{a_v}(K_j) \subset M_{t(v)}$ where $r(v), t(v) \in \{2,3,4\}$ are appropriate integers and moreover $f^{a_{l-1}} K_i = M_i$, $f^{a_{l-1}} K_j = M_j$. Clearly one of the numbers i, j belongs to the set $\{2,3\}$; let, say, $i = 2$. Then choosing correspondent subintervals and using simple properties of f-coverings one can easily find an interval $K \subset M_1$ such that $f^{a_v} K \cap M_1 = \emptyset$, $1 \leq v \leq l-1$, and $f^k K = M_1$. Thus f has a periodic point of minimal period k. Moreover, it is clear that $h(f) > 0$ which completes the proof. □

Theorem 2.1. *Let X be a graph, $s = Edg(X) + 1$ and $\{p_i\}_{i=1}^s$ be s ordered prime integers greater than $4Edg(X)$. Set $L = L(X) = T(p_s, p_{s-1})$. If a continuous map $f : X \to X$ is such that $P(f) \supset \{1, 2, \ldots, L\}$ then $P(f) = \mathbb{N}$ and $h(f) > 0$.*

Proof. Clearly in the situation of Theorem 2.1 one can find two periodic points with properties from Lemma 2.7. It completes the proof. □

Remark 1[B4]. *If X is a tree then one may set $L(X) = 2(p-1)End(X)$ where p is the least prime integer greater than $End(X)$.*

Theorem 2.2. *Let $f : X \to X$ be a continuous graph map. Then the following statements are equivalent.*

1) The map f has a positive entropy.

2) There exists such n that $P(f) \supset nZ \equiv \{ni : i \in \mathbb{N}\}$.

Proof. By the decomposition if $h(f) > 0$ then f has a basic set. By Lemma 1.2 it implies statement 2) of Theorem 2.1. On the other hand by Theorem 2.1 statement 2) of Theorem 2.2 implies that $h(f) > 0$ which completes the proof. □

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