DENSE SET OF NEGATIVE SCHWARZIAN MAPS WHOSE CRITICAL POINTS HAVE MINIMAL LIMIT SETS

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Abstract. We study C^2-structural stability of interval maps with negative Schwarzian. It turns out that for a dense set of maps critical points either have trajectories attracted to attracting periodic orbits or are persistently recurrent. It follows that for any structurally stable unimodal map the ω-limit set of the critical point is minimal.

0. Introduction

One of the main open questions in one-dimensional dynamics is to identify the C^r structurally stable interval maps for $r \geq 2$. For $r = 1$ this has been done by Jakobson in [J]. The reader can find a good account of this problem in [MS]. The purpose of this paper is to narrow down the possible set of structurally stable maps for the case of $r = 2$. The class of maps we find is dense in the space of all C^2 interval maps with negative Schwarzian. Thus, every structurally stable map with negative Schwarzian belongs to this class. The assumption of negative Schwarzian is not very restrictive, since the main examples of interval maps satisfy it (see e.g. [M]). Moreover, many important results proved first for such maps have been later generalized to all maps, so there is a good chance that results similar to ours hold for all C^2 maps.

Our dense class of maps consists of the C^2 interval maps with finitely many critical points, all of them nondegenerate, none of them being an endpoint of the interval, and each critical point being either attracted to an attracting periodic orbit or persistently recurrent. The definition of persistent recurrence is given in Section 4. For unimodal maps this implies that the ω-limit set of the critical point is minimal (in the dynamical sense) and thus nowhere dense.

As in our previous paper on the similar subject [BM], the perturbations we make are localized in a small neighborhood of the critical points. This approach works since we can make such perturbations that whenever the new trajectory returns to the neighborhood on which the map was modified, the perturbation is enhanced

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rather than damped. There are unimodal maps for which this is automatic, so called *long branched* maps. For these maps it was proved by Bruin [Br] that they are not structurally stable. However, in our case the perturbation has to be carefully chosen to have this property.

Local perturbation have been also used by Contreras [C], who proved that there exists a dense set of interval maps of class $C^{2+\alpha}$ for which there exists an ergodic invariant measure with Lyapunov exponent non-positive.

We also prove that if a map has all critical points non-recurrent then it is unstable in C^r for all r. In fact, we prove even more: this map can be arbitrarily well C^r approximated by maps with all critical points attracted to periodic sinks. This fact is not too difficult to prove. However, we could not find it in the literature and it is related to our main results.

The paper is organized as follows. In Section 1 we introduce notation and preliminary results. In Sections 2-4 we prove some technical lemmas that serve as "bricks" with which we build the proofs of the main results in Section 5.

1. Preliminaries

Let us explain some terminology that we will be using.

By the *limit set* of a point we will mean its ω-limit set.

When we speak of perturbations, our terminology will be perhaps not perfectly logical, but commonly used. Thus, a map g which is close to a map f (in some specified topology) will be called a *perturbation* of f. However, when we say that g is a *small perturbation* of f, we mean that $g - f$ (not g) is small. Moreover, when we say that a bump perturbation (see Section 2) g is *small*, we mean that both the (appropriate) norm and the support of $g - f$ are small.

Let f be a smooth interval map. A point x is called a *periodic sink* (from one side) if there exists $n > 0$ and a (one-sided) neighborhood U of x such that $f^n(x) = x$, $f^n(U) \subset U$ and the diameter of $f^k(U)$ tends to 0 as $k \to \infty$. The *basin of attraction* of x is then the set $\bigcup_{k=0}^{\infty} f^{-k}(U)$. In this situation the number $(f^n)'(x)$, called *multiplier*, has absolute value less than or equal to 1. If the multiplier has the absolute value strictly less than 1 then x is called an *attracting periodic point* and its orbit is also called *attracting*. Finally, if the multiplier at a periodic point x has the absolute value 1 then the point x and its entire orbit are called *neutral*.

Let us note the following well known fact (by a *(pre)periodic point* we mean a periodic or preperiodic point).

Lemma 1.1. *Any point from the boundary of the basin of attraction of a periodic sink is either a (pre)periodic point or an endpoint of the domain of f.*

A point whose limit set is an attracting periodic orbit will be called *sinking*. If its limit set is a neutral periodic orbit, it will be called *weakly sinking*. If its limit set is a repelling periodic orbit which belongs to the boundary of a periodic sink, it will be called *almost sinking*. Otherwise, it will be called *floating*.

There are further useful well known facts. We still assume that f is smooth.

Lemma 1.2. *Suppose that x has finite limit set P. Then P is a periodic orbit. Moreover, either $f^n(x) \in P$, so x is (pre)periodic, or the orbit P is attracting or neutral and x is in its basin of attraction, so x is sinking or weakly sinking.*

A point $c \in [0, 1]$ is *critical* for a smooth map $f : [0, 1] \to [0, 1]$ if $f'(c) = 0$. If there is $n \geq 1$ such that $f^n(x)$ is critical then x will be called *precritical*. A point contained in its limit set is called *recurrent*.

Normally, one defines *Schwarzian* (or *Schwarzian derivative*) of a function f of class C^3 as $Sf = f'''/f' - (3/2)(f''/f')^2$. It is defined at all non-critical points of f. Thus, usually *negative Schwarzian* means $Sf < 0$ at all non-critical points. As can be easily checked, this property implies strict convexity of the function $1/\sqrt{|f'|}$ on each component of the complement of the set of critical points. This requires only C^1 smoothness, and we will adopt it as a definition of negative Schwarzian. Thus, a function f is said to have negative Schwarzian if it is of class C^1 and the function $1/\sqrt{|f'|}$ is strictly convex on each component of the complement of the set of critical points. It is almost equivalent to the classical definition if f is C^3, and it is well known that it yields the same useful properties of interval maps.

For simplicity, we assume that our interval is $[0,1]$, but of course every closed interval would do. By $\mathcal{N}_r$ we denote the subspace of the space $C^r([0,1],[0,1])$ consisting of maps with finitely many critical points, all of them non-degenerate, and none of them 0 or 1. It is well known (see e.g. [MS], Chapter III) that the space of C^r maps with finitely many critical points, all of which are non-degenerate, is open and dense in $C^r([0,1],[0,1])$. The set of C^r maps for which 0 and 1 are not critical points is also open and dense, so $\mathcal{N}_r$ is open and dense in $C^r([0,1],[0,1])$.

We will write $[a;b]$ for the interval $[a,b]$ if $a < b$ and $[b,a]$ if $b < a$.

Next basic properties of limit sets that will be used in this paper are summarized in the following theorem. An interval J is called *wandering* if its images $f^n(J)$, $n \geq 0$, are pairwise disjoint and do not converge to a periodic orbit.

Theorem 1.3. *For $f \in \mathcal{N}_2$ the following properties hold.*

1. *There are no wandering intervals.*
2. *Points with finite limit sets are dense in* $[0,1]$.
3. *If there are no critical and no precritical points in (x,y) then the limit set of every $z \in [x,y]$ is a periodic orbit.*
4. *Any floating point is the limit from both sides (one side for* 0 *and* 1*) of both (pre)periodic and precritical points.*

Proof. For property (1), see e.g. [MS], Chapter IV, Theorem A. Property (2) follows immediately from (1) and Lemma 6.1 of [Bl]. Property (3) follows immediately from (1) and Lemma 3.1 of Chapter II of [MS].

To prove (4), assume that x is a floating point. Suppose that there is $y \neq x$ such that there are no precritical points in $(x;y)$. We may assume that there are also no critical points in $(x;y)$ (if there is a critical point in $(x;y)$ then we may replace y by the closest to x critical point in $(x;y)$). By (3) the limit set of x is a periodic orbit. Since x is floating, by Lemma 1.2 x is preperiodic.

Suppose now that there is $y \neq x$ such that there are no (pre)periodic points in $(x;y)$. By (2), there is $z \in (x;y)$ with a finite limit set. By Lemma 1.2, z is a in the basin of attraction of a periodic sink. By Lemma 1.1, the whole interval $(x;z)$ is contained in this basin of attraction. Thus, x cannot be floating, a contradiction. This completes the proof of (4). ■

2. Perturbations and negative Schwarzian

Before we start perturbing maps with negative Schwarzian derivative to get maps with some special behavior of the trajectories of the critical points, we have to make these critical points "manageable". To this end, we introduce the space $\widetilde{\mathcal{S}}$ of the maps from $\mathcal{N}_2$ with negative Schwarzian, that are C^3 in a neighborhood of critical points. Let $\mathcal{S}_2$ be the space of all C^2 maps from $[0,1]$ to itself with negative Schwarzian. We consider the spaces $\widetilde{\mathcal{S}}$ and $\mathcal{S}_2$ with C^2 topology.

Theorem 2.1. *The space $\widetilde{\mathcal{S}}$ is dense in $\mathcal{S}_2$.*

To prove this theorem, we need some auxiliary lemmas.

Lemma 2.2. *Let $\varphi : [0,1] \to \mathbb{R}$ be a continuous function and let $\varepsilon > 0$. Then there exists a set $A \subset [0,1]$ such that $\varphi(x) < \varepsilon$ for $x \in A$, $\varphi(x) > \varepsilon/2$ for $x \notin A$, and A is the union of finitely many closed intervals.*

Proof. Set $B = \{x \in [0,1] : \varphi(x) < \varepsilon\}$ and $C = \{x \in [0,1] : \varphi(x) \leq \varepsilon/2\}$. Then B is open, C is compact and $C \subset B$. Let E be the union of these connected components of B that contain elements of C. Clearly, $C \subset E \subset B$. We claim that E has finitely many components. Suppose that it has infinitely many components E_i. Choose one point $x_i \in C$ from each of them. The sequence (x_i) has an accumulation point $x \in C$. Since $C \subset B$, this point belongs to some component D of B. The set D is open, so infinitely many x_i's belong to it. However, different x_i's belong to different components of B, a contradiction. This proves the claim.

Now for each component E_i of E we choose a closed interval A_i such that $E_i \cap C \subset A_i \subset E_i$. Clearly, the set $A = \bigcup A_i$ satisfies the conditions of the lemma. ∎

We will sometimes replace our function f by a quadratic function on some interval. Namely, for a given C^2 function f and a point x such that $f''(x) \neq 0$ there is the unique quadratic function $\widetilde{f}$ such that $\widetilde{f}(x) = f(x)$, $\widetilde{f}'(x) = f'(x)$ and $\widetilde{f}''(x) = f''(x)$. We will denote this function by $\Phi_{f,x}$, its critical point by $\xi(f,x)$, and $\Phi_{f,x}(\xi(f,x))$ by $\zeta(f,x)$.

Lemma 2.3. *Let $f : [a,c] \to \mathbb{R}$ be a C^2 function such that $f'(x) \neq 0$ for $x < c$ but $f'(c) = 0$.*

(1) *If $f''(c) \neq 0$ then $\xi(f,x) > x$ for every point $x < c$ sufficiently close to c, $\lim_{x\to c} \xi(f,x) = c$, and $\lim_{x\to c} \zeta(f,x) = f(c)$.*
(2) *If $f''(c) = 0$ then there is a sequence of points $x_n \nearrow c$ such that $f''(x_n) \neq 0$, $\xi(f,x_n) > x_n$, $\lim_{n\to\infty} \xi(f,x_n) = c$, and $\lim_{n\to\infty} \zeta(f,x_n) = f(c)$.*

Proof. Elementary calculations show that

$$\xi(f,x) = x - \frac{f'(x)}{f''(x)}\,. \tag{2.1}$$

and

$$\zeta(f,x) = f(x) - \frac{(f'(x))^2}{2f''(x)}\,. \tag{2.2}$$

If $f''(c) \neq 0$ then close to c to the left of it, f' has the opposite sign to f'', so by (2.1) $\xi(f,x) > x$. Also by (2.1), $\xi(f,x)$ is a continuous function of x in a neighborhood of c, that attains value c at c. Moreover, by (2.2), $\zeta(f,x)$ is a continuous function of x in a neighborhood of c, that attains value $f(c)$ at c. This proves (1).

Assume now that $f''(c) = 0$. We can rewrite (2.1) as

$$\xi(f,x) = x - \frac{1}{(\ln|f'(x)|)'}\,. \tag{2.3}$$

and (2.2) as

$$\zeta(f,x) = f(x) - \frac{f'(x)}{2(\ln|f'(x)|)'} \tag{2.4}$$

respectively. Since $\lim_{x\to c} \ln|f'(x)| = -\infty$, there is a sequence $x_n \nearrow c$ such that

$$\lim_{n\to\infty} (\ln|f'(x)|)' = -\infty.$$

We have $f''(x_n) = f'(x)(\ln|f'(x)|)' \neq 0$. By (2.3), $\xi(f,x_n) > x_n$. Again by (2.3), $\lim_{n\to\infty} \xi(f,x_n) = c$, and by (2.4) $\lim_{n\to\infty} \zeta(f,x_n) = f(c)$. This proves (2). ■

Lemma 2.4. *Let $0 \le a' < b' \le 1$ and let $\varphi(a'), \varphi'(a'), \varphi''(a'), \varphi(b'), \varphi'(b'), \varphi''(b')$ be given, such that $\varphi(a') = \varphi(b')$, $\varphi'(a') = \varphi'(b') = 0$ and $0 < |\varphi''(a')| < \varepsilon$, $0 < |\varphi''(b')| < \varepsilon$. Then there exists a polynomial $f : [a',b'] \to \mathbb{R}$ with negative Schwarzian, such that f and φ agree up to second derivatives at a' and b' and the C^2 norm of $f - \varphi(a')$ is smaller than 60ε.*

Proof. Suppose first that the signs of $\varphi''(a')$ and $\varphi''(b')$ are the same. Then we set

$$f(x) = \alpha(x-a')^2(x-b')^2(x-d)^2 + \varphi(a')$$

for suitably chosen $\alpha \in \mathbb{R}$ and $d \in (a',b')$. We have

$$f'(x) = 2\alpha(x-a')(x-b')(x-d)[(x-a')(x-b') + (x-a')(x-d) + (x-b')(x-d)]$$

and

$$\begin{aligned} f''(x) = {} & 2\alpha[(x-a')(x-b') + (x-a')(x-d) + (x-b')(x-d)]^2 \\ & + 4\alpha(x-a')(x-b')(x-d)[(x-a') + (x-b') + (x-d)]. \end{aligned}$$

In particular,

$$f''(a') = 2\alpha(a'-b')^2(a'-d)^2; \quad f''(b') = 2\alpha(a'-b')^2(b'-d)^2.$$

From this we can determine d (by dividing the first equation by the second one and relying upon the assumption that $f''(a') = \varphi''(a')$, $f''(b') = \varphi(b')$), and then α.

For the sake of definiteness let us assume that d is closer to b' than to a'. Then $|a'-d| \ge (b'-a')/2$. Therefore $|\alpha| \le 2|f''(a')|/(b-a)^4 < 2\varepsilon/(b'-a')^4$. Thus for any $x \in [a',b']$ we get $|f(x) - \varphi(a')| < 2\varepsilon(b'-a')^2 \le 2\varepsilon$, $|f'(x)| < 4\varepsilon \cdot 3(b'-a') \le 12\varepsilon$ and $|f''(x)| < 4\varepsilon \cdot 3^2 + 8\varepsilon \cdot 3 = 60\varepsilon$.

Assume now that the signs of $\varphi''(a')$ and $\varphi''(b')$ are different. Then we set

$$f(x) = \alpha(x-a')^2(x-b')^2(x-d) + \varphi(a').$$

Similar computations as above yield

$$f''(a') = 2\alpha(a'-b')^2(a'-d); \quad f''(b') = 2\alpha(a'-b')^2(b'-d).$$

This allows us to find $d \in (a',b')$ and $\alpha \in \mathbb{R}$. Applying the same method as in the first case, we get $|f(x) - \varphi(a')| < \varepsilon$, $|f'(x)| < 5\varepsilon$ and $|f''(x)| < 20\varepsilon$ for $x \in [a',b']$.

It remains to check that the polynomials we used have negative Schwarzian. However, for each of them the number of critical points is by 1 less than the degree. Therefore their derivatives have all zeros real, and according to [S] or [M], they have negative Schwarzian. ∎

Proof of Theorem 2.1. We want to show that if $f \in \mathcal{S}_2$ then we can find its small (in C^2 topology) perturbation that belongs to $\widetilde{\mathcal{S}}$. We will "improve" the behavior of f step by step. Formally, after each step we get a different function, so we should use a different name for it. However, this would lead to cumbersome notation, so the function we are working with will be called f all the time.

Note that the adjustments can be made separately on various intervals. If f has negative Schwarzian on $[0, x]$ and on $[x, 1]$ (and is C^2 on the whole $[0, 1]$) then it has negative Schwarzian on the whole $[0, 1]$.

One of the elements of our construction will consist of moving around pieces of the graph of our map. Suppose that we want to make a perturbation smaller than ε. Then we may move the graph up or down by less than ε. We may also move the graph left or right by less than some δ, depending on ε and f (since the functions f, f', f'' are uniformly continuous). Moreover, we may apply affine transformations sufficiently close to the identity to the x and y axes (that is, expand or contract a little the graph in the horizontal or vertical directions). All these operations preserve negative Schwarzian.

The first thing we take care of are the endpoints of $[0, 1]$. Suppose 0 is a critical point of f. We extend f to the left of 0 to a C^2 function by a quadratic polynomial if $f''(0) \neq 0$ and by $x \mapsto x^4 + x^3$ if $f''(0) = 0$. In both cases the map we obtain has negative Schwarzian. Now we rescale our map from an interval slightly larger than $[0, 1]$ back to $[0, 1]$. We deal with $f''(1) = 0$ in a similar way.

Now we want to improve the behavior of f in the regions where the first and the second derivatives are small. Fix some $\varepsilon > 0$ and let $\varphi = \max(|f'|, |f''|)$. By Lemma 2.2, we can find a set A which is the union of finitely many closed intervals, and such that $\varphi < \varepsilon$ on A, while outside A the function φ is bounded away from 0. We may assume that $f' \neq 0$ on the boundary of A. Let $[a, b]$ be one of the intervals constituting A.

If f has no critical points in $[a, b]$ then we do not change f on $[a, b]$. Otherwise, let c be the leftmost critical point of f in $[a, b]$. By our construction, $c > a$. By Lemma 2.3, there is $x < c$ such that $\xi(f, x) > x$, the points x and $\xi(f, x)$ are as close to c as we want and $\zeta(f, x)$ is as close to $f(c)$ as we want. Then we replace $f|_{[x,c]}$ by $\Phi_{f,x}|_{[x,\xi(f,x)]}$. If $\xi(f, x) < c$, we set $a' = \xi(f, x)$; if $\xi(f, x) \geq c$, we move the graph of our function restricted to $[0, \xi(f, x)]$ to the left in order to make its domain disjoint from $[c, 1]$ and denote the new position of $\xi(f, x)$ by a'. The whole perturbation is as small as we want, independently of ε. Then we make an analogous construction to the right of the point c' which is the rightmost critical point of f in $[a, b]$, and denote the point analogous to a' by b'. To complete this step we need to fill in the gap between a' and b'.

Our gap is $[a', b']$, and the values of f, f', f'' at a', b' are given. We have $f'(a') = f'(b') = 0$ and $0 < |f''(a')| < \varepsilon$, $0 < |f''(b')| < \varepsilon$. Moreover, for our original function f we had $|f'| < \varepsilon$ on $[a, b]$, so $|f(y) - f(z)| < \varepsilon(b - a)$ for every $y, z \in [a, b]$. After the perturbation, this changed as little as we wanted, so we have $|f(a') - f(b')| < \varepsilon(b - a)$. To prepare for filling the gap we move the left or the right part of the graph of f up or down to get $f(a') = f(b')$. This perturbation is smaller than

$\varepsilon(b-a)$. Now we fill the gap $[a', b']$ with a polynomial, using Lemma 2.4. This perturbation has norm less than 61ε.

We repeat this construction for every component of A. In such a way we make some perturbations that are arbitrarily small (independently of ε), some perturbations smaller than 61ε that have disjoint supports, and some global perturbations (moving parts of the graph up or down), each of them smaller than ε times the length of the corresponding component of A. Thus the total size of the latter perturbations is smaller than ε. Hence, the total size of all perturbations we made is smaller than 63ε. During our construction we could change the size of the domain and the range of f, but not more than by ε.

The map we got has no points where both the first and the second derivative vanish. Therefore it has finitely many critical points, all of them nondegenerate. Now we will make the map C^3 (even quadratic) in a neighborhood of these points.

With the tools we developed, this is fairly easy. If c is a critical point then $f''(c) \neq 0$ and we choose x_1 and x_2 very close to c from the left and right respectively and we replace f on $[x_1, c]$ and $[c, x_2]$ by Φ_{f,x_1} on $[x_1, \xi(f, x_1)]$ and Φ_{f,x_2} on $[\xi(f, x_2), x_2]$ respectively. Then we move the right and the left parts of the graph a little (up or down and left or right) so that they fit together and the point at which the two parts of the graph are glued has the x-coordinate c. By Lemma 2.3, this is a small perturbation. The values of f and f' at c are the same from both sides. However, the second derivatives may not agree; we have $\Phi''_{f,x_1} \equiv f''(x_1)$ and $\Phi''_{f,x_2} \equiv f''(x_2)$. They are nevertheless very close to each other, so we can make them identical by affine rescaling of one of the parts of the graph in the vertical direction. This will be again as small perturbation as we want, provided x_1 and x_2 are sufficiently close to c. After all this, the function we get will be quadratic in a neighborhood of its critical point corresponding to c. We have to make only finitely many such adjustments, so the total perturbation is small (say, smaller than ε).

At this moment the function f we have, satisfies all conditions for belonging to $\mathcal{S}$, except that its domain and range may be slightly wrong (up to 2ε). Then a suitable horizontal and/or vertical rescaling takes care of it. Since ε is as small as we want, this rescaling is as small perturbation as we want, at least on the "old" parts of f. For the "new" parts of f we have to have some estimate, since how these parts look may depend on ε.

We want to know that small vertical and horizontal rescaling have small effect on the values of f, f' and f'' on these "new" parts. In the regions where $|f'|$ and $|f''|$ were small it follows precisely from this smallness. Close to the nondegenerate critical points, where we were gluing in pieces of parabolas, this follows from the fact that $|f''|$ is commonly bounded. This completes the proof. ∎

We can work with maps from $\widetilde{\mathcal{S}}$, but it is more convenient to impose more restrictions on them. Let us call the critical points of a map f and the endpoints of the interval $[0, 1]$ *exceptional points* of f. Consider the function $\rho : \widetilde{\mathcal{S}} \to \mathbb{N}$ such that $\rho(f)$ is the number of sinking exceptional points of f. Let $\mathcal{S}$ be the subspace of $\widetilde{\mathcal{S}}$ consisting of all maps f such that ρ has a local maximum at f and no critical point of f is mapped into 0 or 1.

Theorem 2.5. *The space $\mathcal{S}$ is open and dense in $\widetilde{\mathcal{S}}$.*

Proof. The set of these elements of $\widetilde{\mathcal{S}}$ for which no critical point is mapped to 0 or 1 is obviously open and dense.

If $f \in \widetilde{\mathcal{S}}$ then all critical points of f are non-degenerate and none of them is 0 or 1. Therefore for maps from some neighborhood of f the number of all exceptional points is the same as for f. Thus, ρ is bounded in this neighborhood, so the number $\limsup_{g\to f} \rho(g) = k$ is finite. In every neighborhood of f there are maps g for which $\rho(g) = k$, and if this neighborhood is sufficiently small then $g \in \mathcal{S}$. This proves that the set of those elements of $\widetilde{\mathcal{S}}$ at which ρ has a local maximum is dense.

Since sinking critical points remain sinking under small perturbations, we have $\rho(g) \geq \rho(f)$ for all g sufficiently close to f. Hence, if $f \in \mathcal{S}$ then ρ is constant in some neighborhood of f. This proves that the set of those elements of $\widetilde{\mathcal{S}}$ at which ρ has a local maximum is open. Hence, $\mathcal{S}$ is open and dense in $\widetilde{\mathcal{S}}$. ∎

The special property of elements of $\mathcal{S}$ that we will be using is given in the next proposition.

Proposition 2.6. *If $f \in \mathcal{S}$ then f has no neutral periodic points.*

To prove this proposition, we will have to perturb maps from $\mathcal{S}$ in neighborhoods of neutral periodic points. We will do these perturbations in two stages. Therefore we start with some definitions and a lemma.

We will say that f has *strongly negative Schwarzian* on an interval J if it is piecewise C^3 on J and there is $\varepsilon > 0$ such that $2f'''f' - 3(f'')^2 < -\varepsilon$ on J.

As we mentioned earlier, our definition of negative Schwarzian of f is strict convexity of $1/\sqrt{|f'|}$ on intervals without critical points of f. On any such interval we can think of the function $Tf = 1/\sqrt{|f'|}$ as a transform of f. The inverse transform is defined up to an additive constant. To specify a concrete transform, we have to add an initial condition (a value of f at one point).

A simple calculation shows that

$$(Tf)' = -\frac{f''}{2f'}Tf \tag{2.5}$$

and

$$(Tf)'' = -\frac{Tf}{4(f')^2}\left(2f'''f' - 3(f'')^2\right). \tag{2.6}$$

If f has strongly negative Schwarzian on J and s is sufficiently small in C^3 topology, the function $g = f + s$ has also strongly negative Schwarzian on J. By (2.6), strongly negative Schwarzian implies negative Schwarzian. Hence, our terminology is consistent.

Lemma 2.7. *Let x be an interior point of a closed interval J. Let $f : J \to \mathbb{R}$ be a function of class C^2 with negative Schwarzian and without critical points. Then for every $\varepsilon > 0$ and every side of x (left or right) there exists a point a on that side of x and a function $g : [x; a] \to \mathbb{R}$ such that*

(1) *$g(x) = f(x)$, $g(a) = f(a)$;*
(2) *$g'(x) = f'(x)$, $g'(a) = f'(a)$;*
(3) *$g''(x) = f''(x)$, $g''(a) = f''(a)$;*
(4) *g is of class C^3;*
(5) *$(Tg)'' > 0$;*
(6) *The C^2 norm of $g - f|_{[x;a]}$ is smaller than ε.*

Proof. We may assume without loss of generality that we want a to be to the right of x. We start by choosing any $a > x$ such that $a \in J$.

Consider the function $h = Tf$. By the assumption, it is strictly convex and positive. If a is sufficiently close to x then h is monotone on $[x, a]$. Draw the tangent lines to the graph of h at x and a. They intersect at some point P. Set $Q = (x, h(x))$ and $R = (a, h(a))$. Let h_0 be the function whose graph consists of the segments QP and PR and let h_1 be the function whose graph consists of the segment QR. Clearly, $h_0 < h < h_1$ on (x, a).

The function $|f'|$ is continuous and positive on a closed interval, so it is bounded away from zero. Hence, if g is a small C^2 perturbation of f then $k = Tg$ is a small C^1 perturbation of h. Moreover, the conditions (2)-(5) are satisfied if and only if the following conditions are satisfied:

(2') $k(x) = h(x),\ k(a) = h(a)$;
(3') $k'(x) = h'(x),\ k'(a) = h'(a)$;
(4') k is of class C^2;
(5') $k'' > 0$.

There exist functions satisfying (2')-(5') that are arbitrarily C^0 close to h_0 and h_1. Since $h_0 < h < h_1$ on (x, a), we have

$$\int_x^a (h_0(y))^{-2}\, dy > \int_x^a (h(y))^{-2}\, dy > \int_x^a (h_1(y))^{-2}\, dy.$$

Therefore there exist functions k_0 and k_1 satisfying (2')-(5') and such that

$$\int_x^a (k_0(y))^{-2}\, dy > \int_x^a (h(y))^{-2}\, dy > \int_x^a (k_1(y))^{-2}\, dy.$$

We can join them by a one-parameter family of functions satisfying (2')-(5'), for instance affinely: $k_t = (1-t)k_1 + tk_2$. Then there exists t such that for $k = k_t$ we have

$$\int_x^a (k(y))^{-2}\, dy = \int_x^a (h(y))^{-2}\, dy.$$

Then the corresponding function g for which $Tg = k$ and $g(x) = f(x)$, satisfies (1)-(5).

We get property (6) automatically if a is sufficiently close to x. Indeed, take $\delta > 0$. The values of k and $h|_{[x,a]}$ are between $h(x)$ and $h(a)$, and if a is sufficiently close to x then $|h(x) - h(a)| < \delta$. Thus k and $h|_{[x,a]}$ are at most δ-apart at any point of $[x, a]$. Similarly, the values of k' and $h'|_{[x,a]}$ are between $h'(x)$ and $h'(a)$, and if a is sufficiently close to x then $|h'(x) - h'(a)| < \delta$. Thus k' and $h'|_{[x,a]}$ are at most δ-apart at any point of $[x, a]$. We have

$$f' = \pm 1/h^2,\ g' = \pm 1/k^2,\ f'' = \pm 2h'/h^3,\ g'' = \pm 2k'/k^3$$

and h, k are bounded away from 0. Hence, if δ is sufficiently small, then $|f'(y) - g'(y)| < \varepsilon$ and $|f''(y) - g''(y)| < \varepsilon$ for every $y \in [x, a]$. Since $f(x) = g(x)$, if $a - x < 1$ then we get by integration also $|f(y) - g(y)| < \varepsilon$ for $y \in [x, a]$. This completes the proof. ■

Proof of Proposition 2.6. Suppose that a map $f \in \mathcal{S}$ has a neutral periodic point a. It is well known that then a is attracting (topologically) at least from one side and there exists an exceptional point c whose orbit is attracted by a. Without changing orbits of sinking exceptional points we can make a small perturbation of f in the left and right neighborhoods of a, as in Lemma 2.7. The new map has strongly negative Schwarzian (by property (5) of Lemma 2.7 and (2.6)) in a neighborhood of a. Now by a small C^3 perturbation in a neighborhood of a we can make a attracting. If these perturbations are sufficiently small (the first one in C^2 and the second one in C^3) then the resulting map g is in $\mathcal{S}$ and $\rho(g) > \rho(f)$. This contradicts the definition of $\mathcal{S}$. ∎

Now we address the problem of how to perturb maps from $\mathcal{S}$ in neighborhoods of critical points.

Lemma 2.8. *For any $\delta, \varepsilon > 0$ there exists an even function $s_{\delta,\varepsilon} = s : \mathbb{R} \to \mathbb{R}$ of class C^3 such that*

(1) *$s(x) = 0$ for any $x \notin [-\varepsilon, \varepsilon]$, while $s(0) = \varepsilon^2\delta/1000$,*
(2) *s is strictly increasing on $[-\varepsilon, 0]$ and strictly decreasing on $[0, \varepsilon]$,*
(3) *$|s'(x)| < \varepsilon\delta$, $|s''(x)| \le \delta$ and $|s'''(x)| \le \delta/\varepsilon$ for any x.*

If $\varepsilon < 1$ then the C^2-norm of s is smaller than or equal to δ.

Proof. Set $h(x) = (x^2 - 1)^4$ for $x \in [-1, 1]$ and $h(x) = 0$ otherwise. This function satisfies the conditions of the lemma with $\varepsilon = 1$ and $\delta = 1000$. Now the function $s(x) = \delta\varepsilon^2 h(x/\varepsilon)/1000$ has all the required properties. ∎

In the typical situation both ε and δ are small, which guarantees that s is small in C^2 topology (yet to see whether s is small in C^3 topology, we need to know also how δ and ε are related). Since our main focus is C^2 topology, this allows us to perturb our maps by adding or subtracting functions similar to s.

Let $c \in \mathbb{R}$; we call the function $s_c(x) = s_{\delta,\varepsilon}(x - c)$ the *(ε, δ)-bump function at c.* If both ε and δ are small then we say that the (ε, δ)-bump function is small too. If c is a critical point of f, we can consider a map $g = f + s_c$ or $g = f - s_c$ where s_c is an (ε, δ)-bump function at c. Moreover, if ε is smaller than half the minimal distance between critical points of f then intervals supporting functions s_c for different critical points c are pairwise disjoint. From now on we consider functions s_c only for ε smaller than half the minimal distance between critical points of f. An *(ε, δ)-bump perturbation* of f is the result of adding to or subtracting from f some of maps s_c corresponding to different critical points c (the maps $f + s_c$ and $f - s_c$ are called *(ε, δ)-bump perturbations at c*). Notice that by choosing small ε and δ we can get (ε, δ)-bump perturbations of f arbitrarily close to f in C^2 topology.

Observe that we can vary δ and then our bump perturbations depend continuously on it. We will use this fact in the proof of Lemma 4.2. Observe also that since for $f \in \mathcal{S}$ no critical point is mapped into $0, 1$ then all sufficiently small bump perturbations at any critical point map $[0, 1]$ to itself.

Any bump perturbation has the same set of critical points as the original map, provided that ε and δ are sufficiently small and all the critical points of the original map are non-degenerate. Moreover, in the circumstances we will be using it, it preserves negative Schwarzian. The next lemma shows it. It explains why we needed extra smoothness close to critical points.

Lemma 2.9. *Let $f : [-a, a] \to \mathbb{R}$ be a function of class C^3 with a non-degenerate critical point 0. Then there exist positive numbers ε and δ such that for any C^3*

function $s : [-a, a] \to \mathbb{R}$ *whose support is contained in* $[-\varepsilon, \varepsilon]$ *and such that*

$$|s(x)| \le \delta, \quad |s'(x)| \le \varepsilon\delta, \quad |s''(x)| \le \delta, \quad |s'''(x)| \le \frac{\delta}{\varepsilon} \tag{2.7}$$

for all x*, the function* $g = f + s$ *has strongly negative Schwarzian and a unique critical point in* $[-\varepsilon, \varepsilon]$*. In particular, this applies to any* (ε, δ)*-bump perturbation of* f *at* 0*. In this case, the critical point in* $[-\varepsilon, \varepsilon]$ *is* 0*.*

Proof. Set

$$b = |f''(0)|, \quad d = \max_{x \in [0,1]} |f'''(x)|. \tag{2.8}$$

Since 0 is a non-degenerate critical point, we have $b > 0$. Choose $\varepsilon \in (0, a)$ such that

$$|f'(x)| < \min\left(1.3\, b|x|, 0.2\, \frac{b^2}{d}\right), \quad |f''(x)| > 0.8\, b \tag{2.9}$$

for $x \in [-\varepsilon, \varepsilon]$. Then choose $\delta > 0$ such that

$$\delta < \min\left(0.1\, b, 0.2\, \frac{b^2}{d\varepsilon}\right) \tag{2.10}$$

for $x \in [-\varepsilon, \varepsilon]$.

Let us check that these numbers have the required property. Let s be a function with the properties from the statement of the lemma and consider the function $g = f + s$. For $x \in [-\varepsilon, \varepsilon]$ we have $|g''(x)| \ge |f''(x)| - |s''(x)| \ge 0.8b - 0.1b \ne 0$, so g has at most one critical point in $[-\varepsilon, \varepsilon]$. It has to have one, since $g'(-\varepsilon) = f'(-\varepsilon)$ and $g'(\varepsilon) = f'(\varepsilon)$ have opposite signs. If s is a bump function, it is even, so $s'(0) = 0$. Hence, $g'(0) = f'(0) + s'(0) = 0$ and 0 is the only critical point of $g|_{[-\varepsilon,\varepsilon]}$.

Now we will check whether $g|_{[-\varepsilon,\varepsilon]}$ has strongly negative Schwarzian. To this end we show that

$$2(f'''(x) + s'''(x))(f'(x) + s'(x)) < 3(f''(x) + s''(x))^2 - 0.39\, b^2 \tag{2.11}$$

for $x \in [-\varepsilon, \varepsilon]$. In order to do this let us estimate both parts of the inequality (2.11) step by step. We will use all the time inequalities (2.7)-(2.10). We get

$$3(f''(x) + s''(x))^2 > 3(0.8\, b - 0.1\, b)^2 = 1.47\, b^2, \tag{2.12}$$

$$|f'''(x)| \cdot |f'(x) + s'(x)| < d\left(0.2\, \frac{b^2}{d} + \varepsilon \cdot 0.2\, \frac{b^2}{d\varepsilon}\right) = 0.4\, b^2, \tag{2.13}$$

$$|s'''(x)| \cdot |f'(x) + s'(x)| < \frac{0.1\, b}{\varepsilon} \cdot (1.3\, b\varepsilon + \varepsilon \cdot 0.1\, b) = 0.14\, b^2. \tag{2.14}$$

From (2.12)-(2.14) we get

$$\begin{aligned} 2(f'''(x) + s'''(x))(f'(x) + s'(x)) &< 2(0.4\, b^2 + 0.14\, b^2) \\ &= 1.08\, b^2 = 1.47\, b^2 - 0.39\, b^2 < 3(f''(x) + s''(x))^2 - 0.39\, b^2 \end{aligned}$$

for $x \in [-\varepsilon, \varepsilon]$. This proves (2.11) and completes the proof. ∎

Note that due to Lemma 2.9, a map which is C^3 in a neighborhood of a non-degenerate critical point has strongly negative Schwarzian in a sufficiently small neighborhood of this point.

One more tool that will be useful in our considerations is so called Koebe Lemma for negative Schwarzian maps (known also as Koebe Principle, etc.). It is not only called, but also stated differently in different places. It has a part about the distortion and a part about lengths of intervals (that follows from the first one). We need only the second one. Therefore we state the second part in the form most useful for us and the first one in the form good for the proof of the second part. The first part in essentially the same form can be found for instance in [Br]. Moreover, it can be obtained easily from the convexity of the function $1/\sqrt{|h'|}$ by way of integration. We supply a proof of the second part, since we could not find it stated anywhere in this form.

Koebe Lemma. *Let $h : [a, b] \to \mathbb{R}$ be a function with negative Schwarzian and such that $h' \neq 0$ on (a, b).*

(1) *Let $a < a' < b' < b$. Assume that $|h(a') - h(a)| \geq \delta |h(b') - h(a')|$ and $|h(b) - h(b')| \geq \delta |h(a') - h(b')|$. Then for every $x, y \in [a', b']$ we have $|h'(x)|/|h'(y)| \leq ((1 + \delta)/\delta)^2$.*

(2) *Let $a < a'' < b'' < b$. Assume that $|h(b'') - h(a'')| \leq \omega |h(a'') - h(a)|$ and $|h(b'') - h(a'')| \leq \omega |h(b) - h(b'')|$. Then $b'' - a'' < 2\omega(3 + 2\omega)^2(a'' - a)$ and $b'' - a'' < 2\omega(3 + 2\omega)^2(b - b'')$. In particular, if $\omega \leq 1$, then $b'' - a'' < 50\omega(a'' - a)$ and $b'' - a'' < 50\omega(b - b'')$.*

Proof of (2). Without loss of generality, we may assume that h is increasing. Set $\widetilde{a} = h(a'') - (h(b'') - h(a''))/(2\omega)$ and $\widetilde{b} = h(b'') + (h(b'') - h(a''))/(2\omega)$, and then $a' = h^{-1}(\widetilde{a})$, $b' = h^{-1}(\widetilde{b})$. Then the assumptions of (1) are satisfied with $\delta = 1/(2 + 2\omega)$. By (1) and Mean Value Theorem we get

$$\frac{(h(a'') - h(a'))/(a'' - a')}{(h(b'') - h(a''))/(b'' - a'')} \leq \left(\frac{1 + \delta}{\delta}\right)^2 = (3 + 2\omega)^2.$$

Since $h(b'') - h(a'') = 2\omega(h(a'') - h(a'))$ and $a'' - a' < a'' - a$, we get $b'' - a'' < 2\omega(3 + 2\omega)^2(a'' - a)$. Similarly, $b'' - a'' < 2\omega(3 + 2\omega)^2(b - b'')$. If $\omega \leq 1$ then $3 + 2\omega \leq 5$, and we get the required estimates. ∎

3. Non-recurrent critical points

In this and the next section we will describe the small steps we need to make our map "better and better". We start with the case when a critical point c of our map f is non-recurrent. We will assume that $f \in \mathcal{N}_r$ and the perturbations will be C^r small ($r \geq 2$).

The simplest non-recurrent critical points are those with finite limit set. Then this limit set is a periodic orbit. We should exclude the case when our critical point is periodic itself, since then it is recurrent. However, since then no action is necessary (this is a very good critical point), it does not matter whether formally we assign this case to this or to the next section.

"Improvements" that we will try to make in this section are of two types. If possible, we will try to make c sinking. This is a good step in the direction of making our map hyperbolic. The second best choice is to make c precritical. If the orbit of c falls into c itself, then c is sinking. Otherwise, it falls into another critical point d. Then we can continue to try to improve the behavior of the orbit of d. If we succeed, the orbit of c will be as good as the orbit of d.

The first lemma will be applied when dealing with a weakly sinking critical point. It is trivial, but we state it in order to unify the proof of Theorem 5.1. In fact, we used it already in the proof of Proposition 2.6.

Lemma 3.1. *Let x be a neutral periodic point of $f \in \mathcal{N}_r$. Then for every neighborhood U of f in $\mathcal{N}_r$ and every $\varepsilon > 0$ there exists $g \in U$ such that the support of $g - f$ is contained in the ε-neighborhood of x and x is an attracting periodic point for g. Moreover, if x is topologically attracting for f from one or both sides and $[x, x+\varepsilon]$ or/and $[x-\varepsilon, x]$ is contained in the basin of attraction, then the same holds for g (and the basin of attraction of x from that side remains the same).*

The next lemma deals with the case of an almost sinking critical point and is equally trivial. A small bump perturbation at c will push the orbit of c into the basin of attraction of a periodic sink. If this sink is attracting, we get c sinking. If it is neutral, we get c weakly sinking.

Lemma 3.2. *Let c be an almost sinking critical point of $f \in \mathcal{N}_r$. Then for any sufficiently small C^r bump perturbation g of f at c in the right direction (up or down), c is either sinking or weakly sinking.*

The third possibility we have to take into account is when c is floating. Since we assumed that the limit set of c is a periodic orbit, this means that this periodic orbit is repelling. Moreover, by Lemma 1.2, the orbit of c falls into this orbit.

Lemma 3.3. *Let c be a floating critical point of $f \in \mathcal{N}_r$ with finite limit set. Then there are arbitrarily small C^r bump perturbations g of f at c for which c is precritical.*

Proof. Let x be a point from the limit set of c. Then x is a periodic repelling point. Denote the period of x by n. We claim that $f^j(c) = x$ for some j. To see that, let us fix a small interval J containing x, such that $f^n(J)$ contains J, but no point from the orbit of x, other than x itself, belongs to $f^n(J)$. If the orbit of c does not contain x then it passes through J infinitely many times, each time escaping from it after a while. Then it passes infinitely many times through $f^n(J) \setminus J$, so there is a point from the limit set of c there, a contradiction. This proves our claim.

By Theorem 1.3 (4), there are (pre)periodic points arbitrarily close to x. There is a small neighborhood U of x such that a suitable branch of $h = (f|_U)^{-n}$ is a contraction and x is its fixed point. Choose a precritical point $y \in U$. Then $\lim_{k\to\infty} h^k(y) = x$. There is an arbitrarily small bump perturbation g of f at c such that $g^j(c) = h^k(y)$ for some k. If it is sufficiently small, the orbits of $g^j(c)$ under f and g coincide until they get to a critical point. Therefore c is precritical for g. ∎

At last we have to consider the case when c is non-recurrent, but its limit set is infinite. Then we just make a step that reduces this case to the previous ones.

Lemma 3.4. *Let c be a non-recurrent critical point of $f \in \mathcal{N}_r$ with infinite limit set. Then there are arbitrarily small C^r bump perturbations g of f at c for which c is precritical.*

Proof. By Theorem 1.3 (3), we can find a precritical point y so close to c that $y' = \tau_c(y)$ is well defined and the orbit of $f(c)$ is disjoint from $[y; y']$. There is a neighborhood of U of c, contained in $[y; y']$, and such that the orbit of y misses U before it gets to a critical point.

Let V be a small neighborhood of $f(c)$. We want to prove that there is a precritical point $z \in V$ whose orbit misses U before it hits a critical point. If the union of images of V is disjoint from $[y; y']$ then this follows from Theorem 1.3 (3). If the union of images of V is not disjoint from $[y; y']$ then there is the smallest k such that $f^k(V)$ intersects $[y; y']$. We have $f^k(f(c)) \notin [y; y']$, so there is $z \in V$ such that either $f^k(z) = y$ or $f^k(z) = y'$. Then z is precritical and $f^i(z) \notin [y; y']$ for $i < k$. Thus, z is the point we were looking for.

Now, for every bump perturbation g of f at c such that the support of $g - f$ is contained in U, the point z found above is also precritical for g. Since V is arbitrarily small, we can choose g so that $g(c) = z$. This completes the proof. ∎

4. Reluctantly recurrent critical points

In this section we will show what can be done if a critical point is recurrent. This is a much more difficult case than in the preceding section, so we will need some additional assumptions. Namely, we assume that $f \in \mathcal{S}$. We consider this space with C^2 topology.

Moreover, we do not know how to deal with all recurrent critical points, but only with so called reluctantly recurrent ones. To define them, we have to introduce new notation.

Let $f : [0, 1] \to [0, 1]$ be a piecewise monotone map. For $x \in [0, 1]$ let us denote by $H_n(x)$ the maximal interval containing x on which f^n is monotone and let $f^n(H_n(x)) = M_n(x)$. Let $r_n(x)$ be the minimal distance between $f^n(x)$ and the endpoints of $M_n(x)$. If f^n has a local extremum at x, there is an ambiguity in the choice of $H_n(x)$ and $M_n(x)$, but $r_n(x) = 0$ independently of this choice. Moreover, in that case $r_m(x) = 0$ for all $m \geq n$. Also if $x = 0$ or 1, then $r_n(x) = 0$ for all n. Thus either for some m we have $r_m(x) = 0$ (and then $r_n(x) = 0$ for all $n \geq m$) or $r_n(x) \neq 0$ for any n, in which case x is neither a precritical point nor $0, 1$. In fact, we will be interested mainly in the asymptotic behavior of $r_n(x)$, i.e. in whether it converges to 0 or not. Following Yoccoz we call a recurrent critical point c *reluctantly recurrent* if $r_n(f(c)) \not\to 0$ and *persistently recurrent* otherwise. Note that since $f \in \mathcal{S}$ then $f(c) \neq 0, 1$.

We start by proving a technical lemma. Apart from topological considerations, it relies upon Koebe Lemma for negative Schwarzian maps (see Section 2).

Lemma 4.1. *Let c be a reluctantly recurrent critical point of a map $f \in \mathcal{S}$. Then c has infinite limit set and for any $\beta, \varepsilon_0 > 0$ there exist points a (close to c), b (close to $f(c)$), and a number n such that:*

(1) *if f has local maximum at c then $f(a) < f(c) < b$, whereas if f has local minimum at c then $f(a) > f(c) > b$;*
(2) $|a - c| < \varepsilon_0$ *and* $|\tau_c(a) - c| < \varepsilon_0$*;*
(3) $|b - f(c)| < \beta |f(c) - f(a)|$*;*
(4) $f^n|_{[f(a);b]}$ *is monotone;*
(5) *the orbit of $f^n(b)$ misses the interval $[a; \tau_c(a)]$;*
(6) *if $f^i(b) \in [a; \tau_c(a)]$ for some i then $f^i(f(c))$ lies on the same side of c as $f^i(b)$, but farther away from c.*

Proof. Since c is recurrent, it either has infinite limit set or is periodic. If it is periodic then it is precritical, so $r_n(f(c)) = 0$ for all sufficiently large n. This contradicts the assumption that c is reluctantly recurrent. Therefore the limit set of c is infinite.

For the sake of definiteness we may assume that f has local maximum at c. Reluctant recurrence of c means that there is a number $\varepsilon > 0$ and a sequence (n_i) such that $r_{n_i}(f(c)) > \varepsilon$ for any i. Set $H_i = H_{n_i}(f(c))$ and $M_i = M_{n_i}(f(c))$. We may assume that the maps $f^{n_i}|_{H_i}$ are either all increasing or all decreasing, $f^{n_i}(f(c)) \to y$ for some y and $[y - \varepsilon/2, y + \varepsilon/2] \subset M_i$ for every i (replace the sequence (n_i) by its subsequence if necessary).

Denote by H_i^- the part of H_i lying to the left of $f(c)$ and set $M_i^- = f^{n_i}(H_i^-)$. The reason why we are interested in H_i^- is that this set is the image of some neighborhood of c (since f has a local maximum at c).

All sets M_i^- lie on the same side of $f^{n_i}(f(c))$ and the points $f^{n_i}(f(c))$ approach y. Therefore every point z on one side of y does not belong to M_i^- for sufficiently large i (if the sets M_i^- lie to the left of $f^{n_i}(f(c))$ then we look at z to the right of y and vice versa). Let $\beta, \varepsilon_0 > 0$ be given (we may assume $\beta < 50$; in fact, β is small). Let us choose z as above, such that

$$|z - y| < \beta\varepsilon/400 \tag{4.1}$$

and such that its limit set is finite. This is possible by Theorem 1.3 (2). Since $\beta\varepsilon/400 < \varepsilon/8 < \varepsilon/2$, we get $z \in [y - \varepsilon/2, y + \varepsilon/2] \subset M_i$ for every i.

Since the limit set of c is infinite, so is the limit set of $f(c)$. Thus, $f(c)$ is bounded away from the limit set of z. Moreover, by Theorem 1.3 (4), the length of H_i goes to 0 as $i \to \infty$. Thus, there exists j such that H_j is disjoint from the orbit of z, the component of the f-preimage of H_j containing c is shorter than ε_0, and

$$|f^{n_j}(f(c)) - y| < \beta\varepsilon/400. \tag{4.2}$$

The component of the f-preimage of H_j containing c is of the form $[a; \tau_c(a)]$ for some a and then $H_j^- = [f(a), f(c)]$. Since this preimage is shorter than ε_0, (2) is satisfied. Let $n = n_j$ and let b be the $f^n|_{H_j}$-preimage of z. Then $[f(a); b] \subset H_j$, so (4) holds. Since M_j^- and z lie on the opposite sides of $f^n(f(c))$ and $f^n|_{H_j}$ is monotone, $f(a)$ and b lie on the opposite sides of $f(c)$. This proves (1). Since H_j is disjoint from the orbit of z, (5) holds.

Since $[y - \varepsilon/2, y + \varepsilon/2] \subset M_j$ and $\beta < 100$, we get from (4.1) and (4.2) that each component of $M_j \setminus [f^n(f(c)); z]$ has length at least $\varepsilon/4$. On the other hand, from (4.1) and (4.2) we get $|f^n(f(c)) - z| < \beta\varepsilon/200$. Since $\beta < 50$, from Koebe Lemma we get (3).

Suppose that for some i we have $f^i(b) \in [a, \tau_c(a)]$. By (5), since $f^n(b) = z$, we get $i < n$. By (4) and (1), the points $f^i(f(a))$, $f^i(f(c))$ and $f^i(b)$ lie on the same side of c in this or reverse order. If $f^i(f(a))$ is the closest one to c among them, then either $[a; c]$ or $[\tau_c(a); c]$ is mapped into itself by f^{i+1} in a monotone way. Then the orbit of c is attracted to a periodic orbit, a contradiction. Thus, $f^i(b)$ is closer to c than $f^i(f(c))$, so (6) holds. ■

Lemma 4.2. (1) *Let c be a critical point of $f \in \mathcal{N}_2$. Assume that c has an infinite limit set. Let g be an (ε, δ)-bump perturbation of f at c. Suppose that there exist* *a number m and a point z such that the f-orbit of z misses $(c-\varepsilon, c+\varepsilon)$, while $f^m(c) \neq g^m(c)$ lie on the opposite sides of z (non-strictly). Then the f-itinerary of c and the g-itinerary of c are distinct.*

(2) *Let g be an (ε, δ)-bump perturbation of f at a critical point c, such that the f-itinerary of c and the g-itinerary of c are distinct. Then there is $\delta' \leq \delta$ and an (ε, δ')-perturbation h of f at c such that c is precritical for h.*

Proof. (1) Suppose that the f-itinerary of c and the g-itinerary of c coincide. Then $f^k(c)$ and $g^k(c)$ always belong to the same lap (the laps of f and g coincide, so we may talk just about *the laps*). Let $f^m(c) = x$ and $g^m(c) = y$. Then by the assumption $x \neq y$.

We claim that for every $i \geq 0$ we have $f^i(z) \in [f^i(x); g^i(y)]$ and $[f^i(x); g^i(y)]$ is non-degenerate. We prove it by induction. If $i = 0$, this is simply the assumption of the lemma. Suppose that $f^j(z) \in [f^j(x); g^j(y)]$ and $[f^j(x); g^j(y)]$ is non-degenerate. Then the points $f^j(z), f^j(x), g^j(y)$ belong to the same lap on which either both f, g are increasing or they are both decreasing. Thus, application of f to the points $f^j(x)$ and $f^j(z)$ will change (or not) their order in the same way as application of g to the points $f^j(z) = g^j(z)$ and $g^j(y)$. On the other hand, $f(f^j(z)) = g(f^j(z))$, which together with the previous remark shows that $f^{j+1}(z) \in [f^{i+j}(x); g^{i+j}(y)]$. Moreover, at least one of the intervals $[f^j(x); f^j(z)]$, $[f^j(z); g^j(y)]$ is non-degenerate, so at least one of the intervals $[f^{j+1}(x); f^{j+1}(z)]$, $[f^{j+1}(z); g^{j+1}(y)]$ is also non-degenerate. Therefore $[f^{i+j}(x); g^{i+j}(y)]$ is non-degenerate. This completes the induction step and proves the claim.

Thus, $f^i|_{[x,z]}$ is monotone for all i, which by Theorem 1.3 (4) implies that x is not floating. Hence, the f-limit set of $x = f^m(c)$ (and thus the f-limit set of c) is a periodic orbit, a contradiction. This completes the proof of (1).

(2) It is enough to consider a one-parameter family of (ε, δ')-bump perturbations of f, where δ' varies from 0 to δ. Then (2) follows immediately from the definitions by continuity arguments. ■

Now we can prove the main result of this section. The space we consider here is $\mathcal{S}$. Thus, g is a small bump perturbation of f at c if $g - f$ is C^2 small and has a small support.

Lemma 4.3. *Let c be a reluctantly recurrent critical point of a map $f \in \mathcal{S}$. Then there is an arbitrarily small bump perturbation of f at c for which c is precritical.*

Proof. By Lemmas 4.2 (2) and 2.8, it is enough to show that for every $\varepsilon_0, \delta_0 > 0$ there are $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$ and an (ε, δ)-bump perturbation g of f at c such that the itineraries of c for f and g are distinct.

To do this, we fix $\varepsilon_0, \delta_0 > 0$ and use Lemma 4.1 with $\beta = \delta_0/(500\gamma)$, where $\gamma = \sup_{x\in[0,1]} |f''(x)|$. Set $\varepsilon = \min(|a-c|, |\tau_c(a) - c|)$ and $U = (a; \tau_c(a))$. Then

$$(c - \varepsilon, c + \varepsilon) \subset U \tag{4.3}$$

and by Lemma 4.1 (2) we have $\varepsilon < \varepsilon_0$. Choose δ such that an (ε, δ)-bump perturbation g of f at c maps c to b. By (4.3), the support of $g - f$ is contained in U.

By Lemma 2.1 (1) we have $|b - f(c)| = s_{\varepsilon,\delta}(0) = \varepsilon^2\delta/1000$. Therefore, by Lemma 4.1 (3),

$$\delta = \frac{1000|b - f(c)|}{\varepsilon^2} < \frac{1000\beta|f(c) - f(a)|}{\varepsilon^2} = \frac{2\delta_0|f(c) - f(a)|}{\gamma\varepsilon^2}. \tag{4.4}$$

Since $f'(c) = 0$, we have $|f'(x)| \le \gamma|x - c|$ for any x, so $|f(c) - f(a)| \le \gamma\varepsilon^2/2$. Together with (4.4) this gives us $\delta < \delta_0$.

Suppose that the itineraries of c for f and g coincide. Set $z = f^n(b)$ and $m = n+1$, where n is the number from Lemma 4.1. We want to show that the assumptions of Lemma 4.2 (1) are satisfied. By Lemma 4.1 (5) and (4.3), the f-orbit of z misses $(c-\varepsilon, c+\varepsilon)$. The limit set of c is infinite by Lemma 4.1. Thus, it remains to check whether $f^m(c)$ and $g^m(c)$ lie on the opposite sides of z (non-strictly). Note that this would imply that they are distinct, since $b \ne f(c)$ and by Lemma 4.1 (1) and (4) we have $z = f^n(b) \ne f^n(f(c)) = f^m(c)$.

Let us prove by induction that for any $j \le n$ the point $f^j(b)$ lies (non-strictly) between the points $f^j(f(c))$ and $g^j(b) = g^{j+1}(c)$. If $j = 0$ then $f^j(b) = g^j(b) = b$, so we have the induction base. Assume now that

$$f^i(b) \in [f^i(f(c)); g^i(b)] \tag{4.5}$$

for some $i < n$. Since we assumed that the itineraries of c for f and g coincide, the interval $[f^i(f(c)); g^i(b)]$ belongs to one lap (the laps of f and g are the same). We have to prove that

$$f^{i+1}(b)) \in [f^{i+1}(f(c)); g^{i+1}(b)]. \tag{4.6}$$

We may assume that $f^i(f(c)) < g^i(b)$; the other case differs only by the direction of inequalities.

If $g^i(b) \notin U$ then $g^{i+1}(b) = f(g^i(b))$, and (4.6) follows from (4.5) and monotonicity of f on $[f^i(f(c)), g^i(b)]$. Assume now that $g^i(b) \in U$. Then by Lemma 4.1 (6), c lies on the same side of $f^i(f(c))$ as $f^i(b)$. By (4.5), we get

$$f^i(f(c)) < f^i(b) \le g^i(b) \le c. \tag{4.7}$$

Since f is monotone on $[f^i(f(c)), g^i(b)]$ and on U, it is monotone on the whole $[f^i(f(c)), c]$. If it is increasing, then f has a local maximum at c, so $g \ge f$ by Lemma 4.1 (1). Therefore from (4.7) we get $f^{i+1}(f(c)) < f^{i+1}(b) \le f(g^i(b)) \le g^{i+1}(b)$, and (4.6) follows. If it is decreasing, then f has a local minimum at c, so $g \le f$. Therefore from (4.7) we get $f^{i+1}(f(c)) > f^{i+1}(b) \ge f(g^i(b)) \ge g^{i+1}(b)$, and (4.6) also follows. This completes the induction step.

Thus, by Lemma 4.2 (1), we get a contradiction. This shows that the itineraries of c for f and g are distinct. This completes the proof. ■

5. Main results

Let us start with a result on C^r maps, announced in Introduction.

Theorem 5.1. *Assume that a map $f \in \mathcal{N}_r$ ($r \ge 2$) has all critical points non-recurrent. Then it can be approximated arbitrarily well in C^r topology by a map from $\mathcal{N}_r$ with all critical points sinking.*

Proof. We will make several small perturbations of the map. In particular, each perturbation will be such that the set of the critical points of the map will not change. Although after each step we get a different map, in order to keep the notation simple we will always call it f.

Denote by A the set of critical points of f that are neither sinking nor precritical, and by B the set of those elements of A that are almost sinking. Again, although A and B may change after each step, we will not change their names.

Our objective is to make A empty. If we accomplish it, every critical point of f will be either sinking or precritical. If some image of a critical point c is a sinking critical point then c is also sinking. If some image of c is c itself, then c is sinking, too. Thus, all critical points will be sinking.

Each step either will make A smaller, or will make B smaller without enlarging A. Moreover, after each step all critical points will remain non-recurrent. Thus, after finite number of steps, A will become empty.

To perform a step, we choose $c \in A$ such that its limit set is not a proper subset of the limit set of any other element of A. Since the critical points of f are non-recurrent, no element of A has c in its limit set.

If $c \in B$, we apply Lemma 3.2 and with a small bump perturbation at c make this point sinking or weakly sinking, so it is no longer in B. If the bump is sufficiently small (remember that this means also that its support is small), this can only affect limit sets of c and of critical points with c in the limit set. Thus, the limit sets of the elements of $A \setminus \{c\}$ remain the same. Moreover, the critical points that are not in A stay sinking or precritical. Thus, B gets smaller, A does not get bigger, and all critical points remain non-recurrent.

If c does not belong to B then it is either weakly sinking or floating. In either case it is not precritical.

If it is weakly sinking, we apply Lemma 3.1 to a point x from the neutral periodic orbit P which is the limit set of c. If ε from that lemma is sufficiently small, after this perturbation all critical points that were not in A stay sinking or preperiodic. The critical points whose limit set is P (including c) become sinking with the same limit set (the only way we can have P as the limit set for some y is that either the orbit of y hits x or x is attracting from one or both sides and the orbit of y falls into the basin of attraction). Therefore after the perturbation A gets smaller. All other critical points from A retain their limit sets. Indeed, if the limit set of an element of A contains x then by the choice of c, this limit set is equal to P. Thus, all critical points remain non-recurrent.

If c is floating, we apply Lemma 3.3 or 3.4. A small bump perturbation at c makes it precritical. As above, the critical points that are not in A stay sinking or precritical, and the limit sets of the points of A other than c are not affected. Thus, A becomes smaller and all critical points stay non-recurrent.

Hence, in all cases the step works like we prescribed. This completes the proof. ∎

Now we derive the main results of this paper. We start with a more technical, but stronger version.

Theorem 5.2. *The set of those elements of $\mathcal{S}$ for which all critical points are either sinking or persistently recurrent, is dense in $\mathcal{S}$.*

Proof. The idea of the proof is the same as for the preceding theorem, except that we do not have to care about the behavior of yet unmodified orbits of the critical points. In particular, we do not need any special ordering of critical points.

Again, we make perturbations so small that the set of critical points will not change. Our perturbations will be exclusively bump perturbations at critical points. They will be of class C^3 (but small only in C^2 topology, see Lemmas 2.8 and 2.9). Therefore we will not leave the space $\widetilde{\mathcal{S}}$. Moreover, by Theorem 2.5, $\mathcal{S}$ is open in $\widetilde{\mathcal{S}}$, so we will not leave $\mathcal{S}$. Therefore by Proposition 2.6 the maps we get have no neutral points. Hence, their critical points cannot be weakly sinking. In particular, an application of Lemma 3.2 makes the critical point sinking.

We apply to the critical points, one by one, Lemmas 3.2, 3.3, 3.4 and 4.3. These perturbations can be made sufficiently small, so the critical points that are already sinking or precritical, stay like that. We leave alone sinking, precritical and persistently recurrent points. However, a persistently recurrent point that is not precritical may change its status while we are making a bump perturbation at another point. Then we go back to it. In such a way at the end we arrive at a map g that is a small C^2 perturbation of f, belongs to $\mathcal{S}$ and has all critical points sinking, precritical or persistently recurrent. Similarly as in the proof of the preceding theorem, it follows that all critical points of g are either sinking or persistently recurrent. $\blacksquare$

Remark 5.3. By the definition, each precritical recurrent point is persistently recurrent. In the above theorem (and in Corollary 5.4) these persistently recurrent points have a stronger property that all their images are also persistently recurrent. $\square$

From Theorems 5.2, 2.1 and 2.5 we get the following corollary.

Corollary 5.4. *Every C^2 interval map with negative Schwarzian can be approximated arbitrarily close in C^2 topology by C^2 maps with negative Schwarzian, finitely many critical points, all of them nondegenerate and either sinking or persistently recurrent.*

It is known that for a unimodal map with negative Schwarzian, persistent recurrence of the unique critical point c implies that the limit set of c is nowhere dense and minimal (in the dynamical sense) (see e.g. [BL]). Combining our results with well known facts from the theory of unimodal maps, we get the following corollary. Here $\mathcal{U}_2$ is the space of all unimodal C^2 maps with negative Schwarzian, endowed with C^2 topology.

Corollary 5.5. *The set of those elements of $\mathcal{U}_2$ for which the limit set of the critical point is minimal and nowhere dense, is open and dense in $\mathcal{U}_2$.*

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