Special Topics in Mathematics, MA 793
Attractors

1 Basic Sets 1

We begin by developing some tools necessary for studying attractors both on the interval and the circle. In what follows $f : R \to R$ is a map of one of the following topological spaces: R is either a circle, or a collection of intervals (in particular, R may be an interval). The arguments apply to both classes of spaces, and actually to an even more general class of all *graphs*, i.e. *one-dimensional branched manifolds*, however we will only work with the above listed two cases. Also, as a general remark let us observe that even though we only deal with compact spaces some of our more general results and definitions hold for non-compact spaces as well. Finally, occasionally we will refer the reader to the spring course of *Topological Dynamics* as TD; it can be located on the Math Dept web site and easily printed (or the referred results can be considered as useful exercises whose solution can be discussed with the instructor anytime).

Given a map f of a compact space X into itself and a point $\in X$, the sequence $\{x, f(x), \dots\}$ is said to be the *trajectory* (or *orbit*) of x (under f) and is denoted $\operatorname{orb} x$ (similarly one can define the *trajectory* or *orbit* of a set S, denoted $\operatorname{orb} S$). Speaking of *positive orbit* we consider iterations of x (S) starting at $f(x)$ $(f(S))$. Now, define the set $E(f, R)$ of points $x \in R$ such that there exists at least one side A of x in R with the following property: any A-sided semi-neighborhood of x has a dense orbit in R (we call such side A and corresponding semi-neighborhoods of x *dense*).

EXAMPLE 1.1. Let $R = [0, 1]$ and let $f : R \to R$ be the identity map. Then $E(f, R)$ is empty; moreover, for any subinterval $[a, b]$ the set $E(f, [a, b])$ is empty. $\square$

To describe the next example we need the following definitions. The set $\omega(x)$ of all limit points of the trajectory of x is said to be the (ω-)*limit* set of x. If there exists a point x such that $\omega(x) = X$ then the map f is said to be *transitive*. It is known that f is transitive if and only if any open subset of X has a dense orbit in X (TD).

EXAMPLE 1.2. If $R = [0, 1]$ and $f : R \to R$ is transitive then $E(f, R) = R$. $\square$

To state our first lemma concerning the sets $E(f, R)$ we need the following definition: a set B is said to be (f-)*invariant* if $f(B) \subset B$. Also, a set D is said to be *wandering* if sets $D, f(D), \dots$ are pairwise disjoint.

LEMMA 1.3. *The set $E(f, R)$ is invariant.*

Proof. We consider the case of $R = [0, 1]$ and leave all the necessary modifications to the other cases to the reader. Denote $E(f, R)$ by E. By way of contradiction suppose that $y \in E$ while $f(y) \notin E$. Then there are small closed non-degenerate semi-neighborhoods U, V of $f(y)$ from either side such that their orbits are not dense in R; clearly this implies that the same holds for any subset of U and any subset of V. On the other hand, the orbit of any neighborhood of $f(y)$ (in particular of the set $U \cup V$) is dense in R because it contains the image of a small dense semi-neighborhood of y (verification of this statement is a good exercise!). Let us study the situation in detail in order to get a contradiction.

We claim that for some $n > 0$ we have $f^n(U \cup V) \cap (U \cup V) \neq \emptyset$. Indeed, suppose otherwise. Then we can shrink $U \cup V$ to a much smaller neighborhood W of $f(y)$, and by the assumption the positive orbit of W will never intersect the set $(U \cup V) \setminus W$, a contradiction with density of its orbit. Let us show that in fact at least one of the sets U, V is not wandering. To this end observe that if an interval $[z, f(y)]$ is wandering then choosing a point z' much closer to $f(y)$ than z we can assume that all intervals $f^k([z', f(y)])$ are very small (verification of this claim is also a good exercise!). The same applies to intervals chosen on the other side of $f(y)$. Hence we can choose a very small neighborhood W of $f(y)$ whose all images are very small too.

Let us show that together with the fact that the orbit of W is dense it implies that $f(y)$ has a dense orbit in R. Indeed, otherwise there is an open interval $D \subset R$ whose points are ε-distant from the entire orbit of $f(y)$ with $\varepsilon > 0$. Choose an open interval W around y so small that all intervals $f^k(W)$ are less than $\varepsilon/10$. Since $f^{k+1}(y) \in f^k(W)$ then every point $z \in \operatorname{orb} W$ is at most $\varepsilon/10$-distant from $\operatorname{orb} f(y)$. Thus $f^k(W)$ is disjoint from D and so $\operatorname{orb} W$ is not dense, a contradiction. Thus, $f(y)$ must have a dense orbit in R which contradicts the assumption that neither U nor V have a dense orbit.

We may assume now that U is not wandering, i.e. that there exist numbers $k \geq 0$ and $l > 0$ such that $f^{k+l}(U) \cap f^k(U) \neq \emptyset$. It is not very hard to show [TD] that then the orbit B of U is the union of finitely many intervals of some type (verification of this claim is a good exercise!). Also, B is invariant. Since B is not dense in R we conclude that there exists a small dense semi-neighborhood S of y which is disjoint from B (otherwise B would cover a small dense semi-neighborhood of y and will therefore be dense itself, a contradiction with non-density of B). For the sake of definiteness assume that $S = (z, y)$. Let us show that then the orbit of V must contain a left semi-neighborhood of y. Indeed, images of V are "attached" to intervals from the orbit of U. If there is no interval $f^k(V)$ containing y and its small left semi-neighborhood then whenever images of V "reach out" to points of S, they must do it from the left. This and the fact that B is disjoint from S implies that the orbit of V covers some left semi-neighborhood of y and hence V has a dense orbit, a contradiction. Hence $f(y) \in E$ and E is invariant. ∎

2 Basic Sets 2

Some of the facts established in the proof of Lemma 1.3 are useful by themselves, so without proving them again we will simply state them below as corollaries.

COROLLARY 2.1. *If $x \in E(f, R)$ then for any dense semi-neighborhood V of x the following holds:*

1. *there exists a number n such that $f^n(V) \cap V \neq \emptyset$;*
2. *the set $R \setminus \operatorname{orb} V$ is finite.*

Let us continue studying properties of sets $E(f, R)$. First we need a technical result dealing with continuous interval maps.

LEMMA 2.2. *Let $x \in f(I)$ where I is a closed interval. Suppose that S is a side of x such that a small S-semi-neighborhood of x is contained in $f(I)$. Then there exists a point $y \in I$ and its side T such that any T-semi-neighborhood of x has the image which contains an S-semi-neighborhood of y.*

Proof. The preimage $f^{-1}(x) \cap I$ of x in I is non-empty. Let us show that there exists a point $y \in f^{-1}(x) \cap I$ such that any neighborhood of y in I has the image containing an S-semi-neighborhood of x. Indeed, by way of contradiction suppose that this is not true. Then every point of $f^{-1}(x) \cap I$ can be covered by a small neighborhood whose image does not contain some S-semi-neighborhood of x. Since $f^{-1}(x) \cap I$ is compact, we can choose a finite subcover; the union of all its elements is an open set $U \supset f^{-1}(x) \cap I$ such that $f(U)$ does not contain some small S-semi-neighborhood of x. On the other hand, the set $I \setminus U$ is a compact set whose image is disjoint from x, so $f(I \setminus U)$ does not contain a small S-semi-neighborhood of x either. Hence, $f(I)$ does not contain a small S-semi-neighborhood of x, a contradiction with the assumption.

By way of contradiction suppose now that for either side of y there exists a corresponding semi-neighborhood whose image does not contain an S-semi-neighborhood of x. Denote these semi-neighborhoods by U, V. Then $U \cup V$ is a neighborhood of y whose image does not contain an S-semi-neighborhood of x, a contradiction. Hence there exists a side T of y such that any T-semi-neighborhood of y has the image which contains an S-semi-neighborhood of x. ■

COROLLARY 2.3. *Let $x \in f(I)$ where I is a closed interval. Suppose that S is a dense side of x such that a small S-semi-neighborhood of x is contained in $f(I)$. Then I contains a point of $E(f, R)$ mapped into x by f.*

Proof. Apply Lemma 2.2 to S and x; this shows that there is a point $y \in J$ and its side T such that any T-semi-neighborhood W of y has the image which contains an S-semi-neighborhood of x. Thus the orbit W is dense and hence T is a dense side of y and $y \in E(f, R)$ as desired. ∎

LEMMA 2.4. *If $E(f, R)$ is infinite then it has no isolated points.*

Proof. We prove that if $x \in E(f, R)$ and S is a dense side of x, then yx is not isolated in $E(f, R)$ from the side S. Indeed, for the sake of definiteness let S be the right side. By way of contradiction assume that there exists an interval $I = [x, z]$ such that $(x, z]$ contains no points of $E(f, R)$. First consider the case when there is a number n such that $\cup_{i=0}^{n} f^i(I) = R$. Then there exists a number i such that $f^i(I)$ contains infinitely many points of $E(f, R)$. By Lemma 2.2 there are points $t \in E(f, R) \cap (x, z]$, a contradiction.

Now, assume that $\cup_{i=0}^{n} f^i(I) \neq R$ for any n. Consider different possibilities for the images of I. By Corollary 2.1 there exists k such that $f^k(I) \cap I \neq \emptyset$. This implies that $f^{i+k}(I) \cap f^i(I) \neq \emptyset$ for any i. Consider the sets $A_m = \cup_{i=0}^{\infty} f^{m+ik}(I), 0 \leq m < k$. By Corollary 2.1 the union of A_i's covers all but finitely many points of R, and it is easy to see that these points can only be the endpoints of the closures of the sets A_i.

Consider the case when x is preperiodic. Since $E(f, R)$ is infinite, then there exists a point $y \in E(f, R)$ which belongs neither to the set of endpoints of sets A_i nor to the orbit of x. Then y must belong to the interior of some A_i which implies that there is a small dense semi-neighborhood of y, covered by some iteration of I. By Corollary 2.3 it implies that there is a point $z \in E(f, R) \cap I$ mapped into y by the appropriate power of f. Since $z \neq x$ (remember that y does not belong to the orbit of x) we get a contradiction again.

Consider the case when x is not preperiodic and has an infinite orbit. Let us show that then if $i \neq j$ then $f^j(x) \notin \operatorname{int} f^i(I)$; indeed, otherwise $f^j(x)$ belongs to $f^i(I)$ with a small dense semi-neighborhood, so by Corollary 2.3 there are points of $E(f, R)$ in I mapped into $f^j(x)$ and since x is not preperiodic they are distinct from x, a contradiction. In particular, the orbit of x never enters (x, z). Also, $f^r([x, z])$ does not contain a right semi-neighborhood of x for any r because otherwise by Corollary 2.3 there exists an f^r-preimage y of x in $I \cap E(f, R)$, and $y \neq x$ because x is not periodic, a contradiction.

It is clear now, that $f^i(I) \cap I \neq \emptyset$ for infinitely many i. Indeed, otherwise by the previous paragraph a small right semi-neighborhood of x is disjoint from $\operatorname{orb} I$, a contradiction. Hence we can choose numbers $i < j$ such that $f^i(I)$ and $f^j(I)$ are non-disjoint from $(x, z]$ and do not contain a right semi-neighborhood of x. Of the points $f^i(x), f^j(x)$ one is closer to x from the right than the other; assume that $f^j(x)$ is closer. Then $f^i(I)$ contains $f^j(x)$ in its interior, a contradiction. ∎

3 Basic Sets 3

Let us continue our study of sets $E(f, R)$.

LEMMA 3.1. *The map f restricted onto $E(f, R)$ is onto.*

Proof. It is enough to consider the case when $E = E(f, R)$ is non-empty. Observe that in this case $f|R$ is onto (otherwise E is empty). Consider a point $x \in E$; we need to show that there exists a point $y \in E$ such that $f(y) = x$. Indeed, x belongs to the interior of $f(R) = R$ with a dense semi-neighborhood, hence by Corollary 2.3 there exists a point $y \in E(f, R) \cap R = E(f, R)$. ∎

LEMMA 3.2. *The set $E(f, R)$ is closed.*

Proof. If a sequence of points $x_j \in E(f, R)$ converges to a point x then taking a subsequence we may assume that points converge to x from the same side. Clearly, any semi-neighborhood of x from this side has a dense orbit in R and so $x \in E(f, R)$. ∎

Now we are finally ready to introduce the central notion of the first lectures of the course: a *basic set* is a set $E(f, R)$ provided it is infinite.

COROLLARY 3.3. *A basic set is perfect; more precisely, it is either a Cantor set, or coincides with R.*

Proof. By Lemma 3.2 a basic set is closed, and by Lemma 2.4 it has no isolated points. Hence it is perfect. Now suppose that $E(f, R)$ is not a Cantor set. Then there exists at least one non-degenerate component K of $E(f, R)$. Pick a point $x \in \operatorname{int} K$; then there exists a small dense semi-neighborhood U of x contained in K. By Corollary 2.1 it follows that for some $n > 0$ we have $f^n(U) \cap U \neq \emptyset$; hence the same applies to K, and so $f^n(K) \cap K \neq \emptyset$. Now, since K is a component of $E(f, R)$ and $E(f, R)$ is forward invariant by Lemma 1.3 we see that $f^n(K) \subset K$. On the other hand, K contains U and so the orbit of K must be dense in R. Since $\operatorname{orb} K = \cup_{i=0}^{n-1} f^i(K)$ is closed, we see that $\operatorname{orb} K = R$ as desired. ∎

LEMMA 3.4. *If $E(f, R)$ is basic then there exists a monotone map $h : R \to R$ which semi-conjugates $f|R$ and a transitive map $g : R \to R$.*

Proof. Consider the case when $E(f, R) = R$. In this case it is enough to show that f is transitive. To do so observe that every open set U contains at least one point with its dense semi-neighborhood, and hence every open set U has a dense orbit. It is known [TD] that this implies the transitivity of the map f (i.e. the existence of a point whose orbit is dense).

Suppose now that $E(f,R)$ is a Cantor set. Consider an interval $U=(a,b)$ complementary to $E(f,R)$. Let us show that then $\overline{U}$ is mapped into $\overline{V}$ where V is another complementary to $E(f,R)$ interval. Indeed, otherwise there are points of $E(f,R)$ which belong to the interior of $f(U)$. By Corollary 2.3 this implies that there are points of $E(f,R)$ in U, a contradiction.

Consider a monotone map $\varphi: R\to R$ which collapses all intervals complementary to $E(f,R)$ into points. We claim that this map φ semi-conjugates $f|R$ with another continuous map $g|R$. The fact that g is well-defined follows from the preceding paragraph. Indeed, given a point y of R-range we see that $\varphi^{-1}(y)$ is a complementary to $E(f,R)$ interval I or a point x. In any case we define $g(y)$ as $\varphi(f(\varphi^{-1}(y)))$. In general, this expression defines a set, and to show that $g(y)$ is well-defined we need to show that this set is a singleton. If $\varphi^{-1}(x)$ is a point, the claim is obvious. If however $\varphi^{-1}(x)=I$ is an interval, then by the construction I is complementary to $E(f,R)$, and by the proven in the preceding paragraph $f(I)$ is contained in the closure of another interval, complementary to $E(f,R)$. Clearly this implies that $\varphi(f(I))$ is a singleton as desired.

Now we need to show that g is continuous and transitive. The fact that it is continuous follows from the upper semi-continuity of $\varphi^{-1}(y)$ a function of y. Indeed, if $y_n\to y$ then $\varphi^{-1}(y_n)$ converge either to a point or to two endpoints of an interval $\varphi^{-1}(y)$. Either way, the f-image of this one or those two points belongs to the φ-preimage of the same point in R-range, namely of the point $g(y)$. By continuity of f we see that $g(y_n)\to g(y)$ as desired.

It remains to show that g is transitive. Indeed, take an open set U in R-range. Then its preimage V is an open set in R-domain which must contain some points with their dense semi-neighborhoods. Hence the orbit of V is dense, and so is the orbit of U which proves that f is transitive. ■

We have not considered examples for a while, so it is s good time to do it now. To begin with observe that if an interval map is transitive then the interval R is a unique basic set of our map. Now, consider a full tent map which is transitive, and do the following procedure. First, insert an interval I instead of the point 0, then instead of its preimage 1, then instead of its second preimage 1/2, etc. Now, define our map on new intervals in such a way that the map is continuous and the intervals are mapped one onto another as prescribed by the original map. This can be done in such a way that no new turning points are created anywhere inside the inserted intervals except for the interval I on which we can define the map almost as we please (the only exception is the right endpoint of I which must be fixed to guarantee continuity). Then the former interval $[0,1]$ becomes a Cantor basic set of the new map. The described process is the inverse of the process of collapsing of complementary to basic sets intervals described in Lemma 3.4.

4 Circle maps without periodic points 1

The first applications of the tools we have developed is to the circle maps without periodic points. In fact, similar results hold for graph maps without periodic points, but this requires the construction of basic sets for graph maps which strictly speaking has not been done. We will also need some tools very close to those developed in [TD]. From now on let us assume that $f : S^1 \to S^1$ is a continuous circle map such that $\mathrm{Per}(f) = \emptyset$ where $\mathrm{Per}(f)$ is the set of all periodic points of f.

LEMMA 4.1. *The set $E(f, R)$ is infinite.*

Proof. Let us show that it is enough to prove that $E(f, R)$ is non-empty. Indeed, by Lemma 1.3 the set $E(f, R)$ is invariant. Therefore, if it is finite then there exist periodic points, a contradiction.

Now, consider any point x and its limit set $\omega(x)$. Choose a point $y \in \omega(x)$ and show that $y \in E(f, R)$. In order to do so we may assume that there exists a sequence $\{n_k\}$ of iterations of f such that along this sequence x approaches y "clockwise" (i.e. the points $f^{n_k}(x)$ are very close to x and such that to get from a point in this sequence to the next point one needs to move clockwise). This means that the points $f^{n_k}(x)$ approach x from the "counter-clockwise" side which we denote C. Take a C-semi-neighborhood U of y and consider its orbit.

As follows from the choices made above, there exists n such that $f^n(U) \cap U \neq \emptyset$. Then as we have seen before a number of times, the set $A = \cup_{i=0}^{\infty} f^{ni}(U)$ is a connected set. We want to show that $\overline{A} = S^1$. By way of contradiction assume that this is not so. Then $\overline{A}$ is a closed interval, and since $f^n(A) \subset A$ (follows easily from the definition of A), we see that $f^n(\overline{A}) \subset \overline{A}$, and hence there are periodic points, a contradiction. Thus, any C-semi-neighborhood of y has a dense orbit and $y \in E(f, R)$ which shows that $E(f, R) \neq \emptyset$ and proves the lemma. ∎

Now we can use Lemma 3.4 according to which $f|S^1$ is monotonically semiconjugate to a transitive map $g : S^1 \to S^1$. In fact, in our situation we can say a little bit more.

LEMMA 4.2. *The map $f : S^1 \to S^1$ is monotonically semi-conjugate to a transitive map $g : S^1 \to S^1$ such that g has no periodic points.*

Proof. By Lemma 3.4 the semi-conjugacy φ of f to some transitive circle map g exists and it is enough to show that g has no periodic points. Indeed suppose it does. Assume that x is such a point that $g^n(x) = x$ for some n. The set $I = \varphi^{-1}(x)$ is a closed interval because φ is monotone, hence the fact that $g^n(x) = x$ implies that $f^n(I) \subset I$ and therefore that f has periodic points, a contradiction. ∎

Now we need to study transitive circle maps without periodic points. Our first aim is to show that these maps are all homeomorphisms. To this end we will need some general tools close to those developed in [TD]; we will cover them here in full for the sake of completeness. First we need the following definition: a map $f : X \to X$ of a compact metric space into itself is said to be *ε-sensitively dependent on initial conditions* at a point x if and only if for any open U containing x there exists $n \geq 0$ such that $\operatorname{diam}(f^n(U)) \geq \varepsilon$; in this case the point x is called *ε-sensitive*.

Denote the set of all ε-sensitive points by S_ε. Also, a point is *sensitive* if it is ε-sensitive for some $\varepsilon > 0$; the set $\cup_{\varepsilon>0} S_\varepsilon$ of all sensitive points is denoted by S. If there exists $\varepsilon > 0$ such that $S_\varepsilon = X$ then we say that the map f is *sensitive*.

LEMMA 4.3. *The set S_ε is invariant and closed. The set S is invariant.*

Proof. Let $x \in S_\varepsilon$ and prove that $f(x) \in S_\varepsilon$. Suppose that $f(x) \notin S_\varepsilon$. Then there exists a neighborhood U of $f(x)$ such that $\operatorname{diam}(f^n(U)) < \varepsilon$ for any $n \geq 0$. We can choose a ball W of radius $\delta < \varepsilon$ centered at x so that $f(W) \subset U$. This implies that $\operatorname{diam}(f^n(W)) < \varepsilon$ for any $n \geq 0$, a contradiction with the fact that $x \in S_\varepsilon$.

Let us prove that S_ε is closed. Indeed, if $x \in \overline{S}_\varepsilon$ then any neighborhood of x is a neighborhood of a point of S_ε, hence some image of this neighborhood will have the diameter at least ε and therefore $x \in S_\varepsilon$ by the definition. ∎

Consider some examples. Let us recall that a *saw* interval map is a map $f : [0, 1] \to [0, 1]$ such that for some $n > 1$ the map has n intervals of monotonicity each of which is mapped onto the entire interval linearly, and $n - 1$ turning points at $1/n, 2/n, \dots, (n - 1)/n$. For example, the unimodal saw map is the full tent map, and other examples are just as easy to come up with as this one.

LEMMA 4.4. *A saw map is sensitive.*

Proof. Assume that f is a saw map with $n > 1$ intervals of monotonicity. Let us prove that f is topologically exact. Indeed, any interval I which does not contain a turning point of f has the property that $|f(I)| = n|I|$. Hence any interval after a while contains a fixed point which is an endpoint of the interval $[0, 1]$. For the sake of definiteness assume that $0 \in f^n(I)$. Consider a small subinterval $J = [0, n^{-m}] \subset f^n(I)$; clearly, m such that this holds can be chosen. Then under $f, f^2, \dots$ the interval J grows until its f^{m-1}-image covers $[0, 1/n]$ which is the leftmost interval of monotonicity of f. Clearly, the next image of J (that is, $f^m(J)$) covers the entire $[0, 1]$ which proves that f is topologically exact because we have found $n + m$ with the property that $f^{n+m}(J) = [0, 1]$. Hence, f is sensitive. ∎

5 Circle maps without periodic points 2

Let us introduce another map whose study is very important for some dynamical systems, in particular for complex dynamical systems. Namely, let S^1 be a circle whose circumference equals 1. Define $f : S^1 \to S^1$ in angle coordinates as $f(\alpha) = 2\alpha$. This is a so-called *doubling* map which as we will see in the following lemma is sensitive as well.

LEMMA 5.1. *The doubling map f is sensitive.*

Proof. Denote the length of an arc J by $|J|$. Clearly, any small arc doubles its length under f. Moreover, the definition implies that if I is an arc such that $f(I) \neq S^1$ then $|f(I)| = 2|I|$. This implies that f is topologically exact and so f is sensitive. ∎

Our list of one-dimensional sensitive maps is concluded by topologically expanding interval maps. A continuous map $f : [0, 1] \to [0, 1]$ is called *topologically expanding* if there exists $\gamma > 1$ such that for any interval I on which f is monotone we have $|f(I)| \geq \gamma|I|$. In this case γ is called a *constant of expansion* or *expansive constant.* Obviously, γ is not uniquely defined because if γ is a constant of expansion then so is any $\gamma' < \gamma$. For example, a saw map is topologically expanding; more precisely, the saw map with d intervals of monotonicity is topologically expansive with the constant of expansion d. The next lemma shows that Lemmas 4.4 and 5.1 could be deduced from a more general result.

LEMMA 5.2. *A topologically expanding interval/circle map f is sensitive.*

Proof. First of all observe that for any k the map f^k is also a topologically expanding map with the expansive constant γ^k. Indeed, on the one hand it is easy to see that f^k is piecewise-monotone. On the other hand, if $f^k|I$ is monotone then $f|_I, f|_{f(I)}, \dots, f|_{f^{k-1}(I)}$ is monotone. Hence by the properties of topologically expanding maps we conclude that $|f(I)| \geq \gamma|I|, |f^2(I)| = |f(f(I))| \geq \gamma(\gamma|I|) = \gamma^2|I|, \dots, |f^k(I)| \geq \gamma^k|I|$ as desired.

Now, pick k so that $\gamma^k| \geq 4$ and consider the map $g = f^k$. Suppose that the length of the shortest interval of monotonicity of g is ε. Let us prove that for any interval J there exists m such that $|g^m(J)| \geq \varepsilon$. To this end we prove that for any interval I if $|I| \leq \varepsilon$ then $|g(I)| \geq 2|I|$. Indeed, if $|I| \leq \varepsilon$ then I cannot contain two turning points of g (the minimal distance between two turning points of g is ε). Therefore it either contains one or no turning points of g which implies that at least half of I is contained in some interval of monotonicity of g. The length of the g-image of this half of J is at least $4(|I|)/2 = 2|I|$ as desired. Clearly, this implies our claim and thus completes the proof of the lemma. ∎

Now that we have received some experience in dealing with sensitive maps, let us continue our study of transitive circle maps without periodic points. First consider a general transitive map $f : X \to X$ of a compact metric space X into itself. The map f can be either sensitive or not. The next lemma considers non-sensitive maps and connects them with so-called recurrent maps (let us recall that a map $g : X \to X$ is said to be *recurrent* if there exists a sequence of powers g^{n_k} such that $g^{n_k} \to id_X$, see [TD], Lecture 17).

LEMMA 5.3. *If f is not-sensitive then it is recurrent.*

Proof. Let x be a point such that $\omega(x) = X$. If x is ε-sensitive for some $\varepsilon > 0$ then $S_\varepsilon = X$. Indeed, S_ε is closed and invariant by Lemma 4.3, so it has to contain $\omega(x) = X$ as desired. However by the assumption f is not sensitive, hence x is not sensitive either. By the definition it implies that for every ε there exists $\delta > 0$ such that for any point y with $d(x, y) < \delta$ we have that for any $n \geq 0$ the distance $d(f^n(x), f^n(y))$ is less than ε. Now, since x has a dense orbit there exists m such that $d(f^m(x), x) < \delta$. If we let $f^m(x)$ play the role of the point y above we will have to conclude that $d(f^{m+n}(x), f^n(x)) < \varepsilon$ for any non-negative n. Since the orbit of x is dense, this implies that in fact for any $z \in X$ we have $d(f^m(z), z) \leq \varepsilon$. Clearly this implies that f is recurrent. ■

There are some properties of recurrent maps proven in [TD], Lecture 17. For the sake of completeness we will reprove the following one.

LEMMA 5.4. *Recurrent maps are homeomorphisms.*

Proof. Let a map f be such that for some sequence $\{n_k\}$ we have $f^{n_k} \to \mathrm{id}_X$. Let us show now that must be a homeomorphism. Indeed, if f is not onto then for $z \notin f(X)$ the convergence $f^{n_k}(z) \to z$ is impossible, a contradiction. On the other hand, if $f(y) = f(z) = u$ then $f^{n_k - 1}(u)$ must converge to both y and z, which is impossible. So f is a continuous bijection and therefore any recurrent map is a homeomorphism. ■

So, either a transitive map is recurrent (and therefore a homeomorphism) or sensitive. We will now concentrate upon sensitive transitive maps of the circle; our aim is to show that they must have periodic points.

LEMMA 5.5. *If a map $f : S^1 \to S^1$ is sensitive then it has periodic points.*

Proof. Let f be ε-sensitive. Partition S^1 into k adjacent arcs $I_1, \dots, I_k$ so that each arc is of diameter less than $\varepsilon/4$. Then for each arc I_r there exists a number n_r such that $\mathrm{diam}(f^{n_r}(I_r) > \varepsilon$. At this moment by the choice of the sizes of the arcs we see that f^{n_r} contains completely at least one other arc. Construct an oriented graph with k vertices such that its vertex s is connected with an arrow to the vertex t if and only if $f^{n_s}(I_s) \supset I_t$. This graph G is such that there is at least one arrow coming out of every vertex. It is easy to see then that there are loops in the graph. The existence of such loops immediately implies that there are associated with them periodic points (this sort of argument was applied a number of times in [TD]) and completes the proof. ■

It remains to study transitive recurrent circle homeomorphisms without periodic points.

6 The rest of the course

In the rest of the course we use hand outs based on fragments of various sources. First, we cover the remaining questions about circle maps without periodic points relying upon the first chapter of the book “Differentiable dynamics; an introduction to the orbit structure of diffeomorphisms” by Z. Nitecki, published by M.I.T. Press Cambridge, Mass., in 1971. Then we studied interval maps with the negative Schwarzian derivative. The introductory part of this topic was covered by the Section 1.11 from the textbook “An Introduction into Chaotic Dynamical Systems”, 2nd edition, by B. Devaney, published by Addison-Wesley in 1989. Then we studied the first 3 sections of the paper “Attractors of Maps of the Interval” by A. Blokh and M. Lyubich published in Banach Center Publications of the Semester on Dynamical Systems held in Warsaw, **23** (1989), pp. 427–442. The students gave talks devoted to the claims proven in this paper, sometimes rewriting them in a more appropriate for a graduate course style, clarifying certain claims etc etc. The writing of their own proofs of important claims from this paper was actually one of the requirements of the course, and in fact the files with their proofs are available. However, having thought about this for a while I decided not to post their proofs here because they all continue and rely upon a number of initial statements of the above quoted paper which is only available to me in the form of manuscript and does not exist in the tex-format.