LAMINATIONS SEMINAR

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Abstract. These notes are based upon the results by G. Levin from Hebrew University of Jerusalem and the author which appeared under the title **An inequality for laminations, Julia sets and 'growing trees'** in Erg. Th. and Dyn. Syst., **22** (2002), pp. 63-97 [BL3] (see also [BL1, BL2] for preliminary versions).

Introduction

Dear readers! These are the handouts for the Laminations Seminar at the University of Alabama at Birmingham. Please carefully read these introductory comments before you begin studying the notes.[1]

Motivation. The dynamical systems theory started by studying invertible maps. There is a well-developed theory of **diffeomorphisms** of two-dimensional manifolds. However this theory cannot be fully extended onto non-invertible maps (**endomorphisms**) because it heavily relies upon invertibility. One of the classes of endomorphisms onto which the theory of dynamical systems is being currently extended is the class of polynomials which has extra-analytical properties simplifying the study and to some extent replacing invertibility.

The main tool in working with polynomials with connected Julia set is related to the Riemann map between the basin of infinity A_∞ and the unit disk D. This map establishes the correspondence between the radii of D and their preimages in A_∞ called **rays**. If the Julia set J is locally connected then all rays land at points of the Julia set which allows one to introduce a map $p : S^1 \to J$ associating to every argument the landing point of the ray corresponding to this argument.

Say that two angles a and b are equivalent ($a \sim b$) if $p(a) = p(b)$. This equivalence relation on S^1 is called a **lamination** of the circle. It serves as a main combinatorial-topological tool in studying polynomials which justifies our interest in laminations.

Text. The results proven in these notes are very recent (in fact they have been only published in the beginning of 2002). In order to make these notes easier to read and

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more available to students I spent a lot of time rewriting the above mentioned paper. Essentially, as a source for studying at a seminar, this is yet a work in progress, and for that reason the material is far from being as polished as it is in usual textbooks devoted to the results obtained 100 years ago, studied by generations of students, and taught by generations of teachers. Therefore I would suggest that you apply the appropriate criteria judging the text. I would also appreciate your remarks concerning possible errors which will be promptly corrected.

Level of difficulty. The material studied in the handouts is complicated and difficult. Sometimes it makes clear explanation of the material hard, specially within limited time when the seminar meets. However, all efforts will be made to have the material worked out as thoroughly as possible. If you need extra time in class to discuss some results, you are encouraged to ask us to slow down and ask as many questions as you need. You are also encouraged to talk to me or to your advisor about difficulties you encounter. My aim as a teacher has always been to have all students understand the material, and I want to achieve this goal in this seminar too.

All the above inevitably means that significant attention will be paid to the details, the seminar might have to consider certain problems for longer than it was originally planned, etc. However this does not mean that the time is not used effectively, it means that effectiveness is measured not in terms of the number of theorems proved or pages of very dense material covered, rather it is measured in terms of your understanding and being able to use these methods in the future. Keep in mind that looking the notes through at home will help your understanding a lot!

Grading. All students who actively participate in the seminar and give at least one talk will get the grade A.

Aim of the notes. Our aim is to develop the necessary tools and then to prove a combinatorial version of Sullivan No Wandering Domain Theorem [Su] and several related results. The theme of wandering sets is present in a variety of different dynamical system. In my view, this interest can be explained as follows. An important question of the dynamical systems theory is that of the behavior of the majority of points. In the topological setting majority means a **massive** set, that is a countable intersections of open dense subsets. E.g., if a map has a dense orbit then actually there is a massive subset of points with dense orbits, so in this case the typical behavior is dense.

Similarly, the fact that there are no wandering Fatou domains of polynomials allows us to study the typical behavior on the filled Julia set: if a wandering domain existed we could not predict its behavior. However since such domains do not exist we know that the typical point of a Fatou set eventually maps inside a periodic Fatou domain, and in this case it is well-known that it converges to a periodic orbit or has the limit set which is a simple Jordan curve on which the map is acting like an irrational rotation.

All this justifies our interest in wandering sets and also to a question as to what sets cannot be wandering . Intuitively, big sets have less chance to be wandering, and to some extent this is confirmed by No Wandering Domain Theorem. “Next in size” after Fatou domains come continua, and so we prove in the notes that all continua in J are non-wandering. Yet next in size are vertices of J for which the problem is not solved yet! Thus our methods can serve as the basis for individual future research of the participants of the seminar.

1. Growing trees

A **tree** is a connected compact one-dimensional branched manifold T with no subsets homeomorphic to a circle. Let $a \in T$. If $T \setminus \{a\}$ has n connected components, then the **order** of T at a is $\mathrm{ord}_T(a) = n$. The point a is called an **endpoint** (of T) if $\mathrm{ord}_T(a) = 1$, an **inner point** (of T) if $\mathrm{ord}_T(a) = 2$ and a **vertex** (of T) if $\mathrm{ord}_T(a) \geq 3$. Clearly, a tree has finitely many vertices and endpoints. An **arc** (in T) is a subset of T homeomorphic to an interval. An **edge** (of T) is an arc whose endpoints are vertices or endpoints and whose other points are inner points of T. The absence in T of sets homeomorphic to circles makes the arc $[a, b]$ with endpoints $a, b \in T$ well-defined. The number of edges of T is finite.

Let X be a metric space, $T \subset X$ be a tree, $f : X \to X$ be a continuous map. Denote the sets $\bigcup_{i=0}^{n} f^i(T)$ by T_n and the set $\bigcup_{i=0}^{\infty} f^i(T)$ by T_∞. If (a) $f(T) \cap T \neq \emptyset$, (b) T_n is a tree for any n, and (c) there is a finite set of **critical** points $C_f = \{c_1, \dots, c_k\} \subset T_0$ with $f|T_\infty$ injective in some neighborhood of any $x \in T_\infty \setminus C_f$, then we call the sequence of sets $T_0 \subset T_1 \subset \cdots \subset T_\infty$ (or the set T_∞) a **growing tree**. Also, a point $x \in T_\infty$ is called a **vertex of** T_∞ if x is a vertex of some T_n.

For example, let $T = T_0$ be a letter $E \subset \mathbb{R}^2$ with horizontal segments $[(0, 1), (1, 1)]$, $[(0, 0), (1, 0)]$, $[(0, -1), (1, -1)]$. Let $f(x, y) = (x, 2y)$. Then $T_1 = T \cup f(T)$ consists of 5 horizontal and 1 vertical segments, $T_1 \setminus T_0$ consists of 2 semi-open arcs and, moreover, $T_{n+1} \setminus T_n$ consists of 2 semi-open arcs. This example is illustrated on Figure 1.

Lemma 1.1 shows how trees can grow; its proof of is left to the reader.

Lemma 1.1. *Let $T \subset T'$ be trees. Then the set $T' \setminus T$ has finitely many components $t_1, \dots, t_l$, all $\overline{t_i}$ are trees, $\bar{t}_i \cap T = \{x(t_i)\}$ is a point and, moreover, $\mathrm{ord}_{T'}(x(t_i)) \geq \mathrm{ord}_T(x(t_i)) + 1 \geq 2$ for any i.*

In the situation of Lemma 1.1 for a component t of $T' \setminus T$ we call the point $x(t)$ the **basepoint** (of t) and other endpoints of t **outer** endpoints of t (T'). Let the number of outer endpoints of t be $\mathrm{oen}(T, t)$ and the number of all outer endpoints of T' be $\mathrm{oen}(T, T')$. Then $\mathrm{oen}(T, T') = \sum_i \mathrm{oen}(T, t_i)$; e.g., if T' has the shape of the letter H and T is its "plank" then $T' \setminus T$ consists of 4 intervals $\{t_i\}_{i=1}^4, \mathrm{oen}(t_i) = 1$ and $\mathrm{oen}(T, T') = 4$. For a growing tree T_∞ Lemma 1.1 implies that $T_{n+1} \setminus T_n = \cup_{j=1}^{k_{n+1}} t_j^{n+1}$ where t_j^{n+1} are components of $T_{n+1} \setminus T_n$ with basepoints x_j^{n+1}.

Lemma 1.2. *Let $T_n \subset T_{n+1} \subset T_{n+2}$ come from a growing tree. Then $\mathrm{oen}(T_n, T_{n+1}) \geq \mathrm{oen}(T_{n+1}, T_{n+2})$ and any outer endpoint of T_{n+2} is the image of an outer endpoint of T_{n+1} (and all outer endpoints of any T_n are eventual images of outer endpoints of T_1).*

Proof. If a be an outer endpoint of T_{n+1} then $a = f(b)$ with $b \in T_{n+1} \setminus T_n$. Since f on a component of $T_{n+1} \setminus T_n$ is a homeomorphism then b is an outer endpoint of T_{n+1}. ■

On Figure 1 $\mathrm{oen}(T_0, T_1) = 2$, and actually $\mathrm{oen}(T_n, T_{n+1}) = 2$ for any $n \geq 0$.

By Lemma 1.2 $\mathrm{oen}(T_n, T_{n+1})$ is a non-increasing integer sequence, so $\mathrm{oen}(T_n, T_{n+1}) = \mathrm{oen}(T_\infty)$ for some $\mathrm{oen}(T_\infty)$ and big n (in the above example $\mathrm{oen}(T_\infty) = 2$). We assume that $\mathrm{oen}(T_n, T_{n+1}) = \mathrm{oen}(T_\infty)$.

The **(f)-orbit** of a set $A \subset X$ is $\cup_{n=0}^\infty f^n(A)$, and the **grand (f)-orbit** of A is the set of all points x so that there are $m, n \geq 0$ with $f^m(x) \in f^n(A)$.

Call the grand orbit of a point x **non-cyclic** if it contains no cycles; it is non-cyclic iff the orbit of x is infinite. For a tree $W \subset X$ or a growing tree $T_\infty \subset X$ vertices with infinite orbits are called **(W-)** or **(T_∞-) exceptional**. Call a growing tree **normal** if the images of endpoints of T_0 belong to T_0. Denote the *number of vertices (endpoints, edges)* of a tree T by $V(T)$ $(End(T), D(T))$ and the *set of vertices* of T by $\mathcal{V}(T)$.

Lemma 1.3. (1) *For any tree W, $V(W) + 1 = End(W) - 1 - \sum_{v \in \mathcal{V}(W)} (\mathrm{ord}_W(v) - 3)$ (equivalently, $1 + \sum_{v \in \mathcal{V}(W)} (\mathrm{ord}_W(v) - 2) = End(W) - 1$), and so $V(W) + 1 \leq End(W) - 1$.*

(2) $\sum_{j=1}^{k_{n+1}} (V(t_j^{n+1}) + 1) \leq \mathrm{oen}(T_\infty) \leq \mathrm{oen}(T, T_1)$.

Proof. (1) Induction over the number of edges.

(2) Sum up the inequality from (1) over the components of $T_{n+1} \setminus T_n$. ■

Consider a tree W and its vertices. Call a vertex $v \in W$ **quasi-last** if $f(v)$ is not a vertex of W. Denote the set of quasi-last vertices of W by $QL(W)$. We use this notion in Theorem 1.4 to estimate the number of T_∞-exceptional grand orbits.

Theorem 1.4. *Let $T_\infty = T_0 \subset T_1 \subset \dots$ be a normal growing tree. Then:*

(1) *The outer endpoints of T_m are f^m-images of critical points; thus,* $\text{oen}(T_\infty) \le k$.

(2) *Quasi-last vertices of T_m are critical points which are vertices of T_∞, or vertices of components of $T_m \setminus T_{m-1}$, or their basepoints which are not vertices of T_{m-1}. The images of quasi-last vertices of T_m of the second and third type are either vertices of components of $T_{m+1} \setminus T_m$, or basepoints of such components which are not vertices of T_m.*

(3) $|QL(T_m)| \le k + \text{oen}(T_\infty) \le k + \text{oen}(T_0, T_1)$ *and among points of $QL(T_m)$ there are at most* $\text{oen}(T_\infty)$ *vertices which are not critical points.*

(4) *The infinite orbit of a vertex x of T_m contains a unique $d_x \in QL(T_m)$ such that $f^i(d_x)$ is not a vertex of T_m for all $i > 0$. The number of T_∞-exceptional grand orbits is at most $k + \text{oen}(T_\infty)$ and the number of them containing no critical points is at most* $\text{oen}(T_\infty) \le k$.

Proof. (1) If $x \in T_m \setminus T_{m-1}$ then $x = f^m(y)$ for some $y \in T_0$ and there are no $j < m, z \in T_0$ such that $x = f^j(z)$. If y is not a critical point/endpoint of T_0 then x is not an endpoint of T_m. However, if y is an endpoint of T_0 then $f(y) \in T_0$, hence we can pick $j = m-1, z = f(y)$ which will satisfy the above conditions and therefore are not supposed to exist. Hence, y is a critical point as desired.

(2) A vertex a of T_{m-1} which is not a critical point is not a quasi-last vertex of T_m because $f(a)$ is a vertex of T_m. So, quasi-last vertices of T_m are either a) critical points which are vertices of T_∞ or b) vertices of T_m but not vertices of T_{m-1}, i.e. vertices of components of $T_m \setminus T_{m-1}$ or their basepoints which are not vertices of T_{m-1}.

Let v' be a quasi-last vertex of T_m but not a critical point. Consider two cases.

(i) $u = f(v') \notin T_m$. Then u is a vertex of a component of $T_{m+1} \setminus T_m$.

(ii) $u = f(v') \in T_m$. If u is not a basepoint of a component of $T_{m+1} \setminus T_m$ then there are no new branches which appear at u in T_{m+1} compare to T_m. In other words, u is a vertex of T_m which contradicts the assumption that v' is a quasi-last vertex of T_m.

(3) Immediately follows from (2) and Lemma 1.3(2).

(4) The former part of the claim is obvious; the latter follows from (3) because quasi-last vertices d_x corresponding to vertices from distinct grand orbits are distinct. ■

In the next theorem we apply a bit more sophisticated arguments to the situation of Theorem 1.4 to estimate from above the order of T_∞- exceptional vertices. The arguments below depend heavily on Theorem 1.4. We need another notion. Given a tree W and a point $a \in W$, consider arcs $[a, b] \subset W$ such that (a, b) contains no vertices/critical points of W. Call arcs $[a, b]$ and $[a, b']$ **equivalent** if $(a, b) \cap (a, b') \neq \emptyset$; clearly, equivalent arcs are ordered by inclusion. Classes of equivalence of arcs $[a, b]$ of W are called **germs** of W at a. One can say that a germ of a tree W at $a \in W$ is a pair (a, S), where S is an infinitesimal interval in W with one endpoint at a containing no vertices/critical points of W inside. Its image is defined as $f(a, S) = (f(a), f(S))$, so we may speak of the image of a germ contained in a tree.

Theorem 1.5. *If x is a T_∞-exceptional non-precritical vertex then* $\mathrm{ord}_{T_\infty}(x) \le 2k+1$.

Proof. Suppose that $\mathrm{ord}_{T_\infty}(x) > 2k+1$. Then for some big N we have that $\mathrm{ord}_{T_N}(x) > 2k+1$. Let us consider the orbit of x and study the behavior of germs of T_N at x along this orbit. Let us show that each germ except for at most 2 will have to leave T_N at some moment. Indeed, if 3 or more germs of T_N at x are kept in T_N then all the images of x are vertices of T_N. Hence x will be eventually mapped into a periodic point, a contradiction with x being exceptional.

Now, for each germ which leaves T_N (thus, for each germ of T_N at x with the possible exception of two) there is the first time when it does it. The idea of the proof is to study these times, sum up all the germs "cast" at these moments, and estimate this sum from above. More precisely, let i be the least number such that some germ (x, S) of T_N leaves T_N at this moment (we call this an *important event*). This can happen in two ways.

1. $f^{i+1}(x)$ is a basepoint of a component (or components) of $T_{N+1} \setminus T_N$, and germs which used to be germs of T_N at $f^i(x)$ are mapped onto basegerm(s) of those components.

2. $f^{i+1}(x)$ is outside T_N, and is therefore a vertex of a component of $T_{N+1} \setminus T_N$.

In any case, all but at most two germs of T_N are eventually mapped outside of T_N, and for each such germ there is the first moment when it escapes T_N. Let us put these moments in order. Since the important event of type 2. above implies that all germs of T_N are mapped outside T_N we see that in this sequence of important events it can only appear at the last moment. Therefore, the sequence of important events which we construct consists of several important events of type 1. and then the last important event which could be of type 1. or 2.

Suppose that the last important event takes place at the moment when f maps $f^j(x)$ into $f^{j+1}(x)$. Clearly, the number of germs of T_N at x equals the number of germs of T_N lost along the way to $f^{j+1}(x)$ plus the number of those which were not lost. All the germs of T_N lost before j, are basegerms of various components of $T_{N+1} \setminus T_N$ because all these important events are of type 1. Since x is not preperiodic, the point x cannot pass through the same basepoint of such a component twice. Hence the number of germs of T_N at x mapped along the way outside T_N is no more than the number of components of $T_{N+} \setminus T_N$, i.e. k.

Now, if the last important event is of type 1. then the germs of T_N mapped outside T_N at this moment together with the previously lost germs form a set of no more than k elements by the previous paragraph. On the other hand, at most two germs of T_N will not be mapped outside T_N even at this moment. Hence to begin with we had no more than $k+2$ germs of T_N at x.

However, if the last important event is of type 2. then all germs of T_N will be lost at this point and will become germs of some component of $T_{N+1} \setminus T_N$. The number of such germs is at most $k+1$ (the number of endpoints of any component of $T_{N+1} \setminus T_N$ is at most the number of all outer endpoints of T_{N+1} with respect to T_N plus one). Since before that at most k germs of T_N were mapped outside of T_N, we see that the overall number of germs of T_N at x was at most $k+k+1 = 2k+1$. ■

2. Laminations

Let us start with precise definitions. Consider an equivalence relation $\sim$ on the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (identified with $S^1 = \{z \in \mathbb{C} : |z| = 1\}$) with the following properties ([Do], [McM], cf. [Th]):

(E1) $\sim$ is **closed**: the graph of $\sim$ is a closed set in $\mathbb{T} \times \mathbb{T}$;

(E2) $\sim$ defines a **lamination**, i.e. it is **unlinked**: if $t_1 \sim t_2 \in S^1$ and $t_3 \sim t_4 \in S^1$, but $t_2 \not\sim t_3$, then the open intervals in $\mathbb{C}$ with the endpoints t_1, t_2 and t_3, t_4 are disjoint;

(E3) each class of equivalence $\sim$ is totally disconnected.

Call $\sim$ a **closed lamination**. We always assume that it is **non-degenerate**, i.e. has a class of more than one point. Equivalence classes of $\sim$ are called **($\sim$-)classes**; for $x \in S^1$ let $\mathrm{Cl}(x)$ be its class. A $\sim$-class that consists of exactly two (2) points is called a **leaf** while a $\sim$-class that consists of at least three (3) points is called a **gap** (cf. [Th]). Note that laminations in [Th] do not always arise from an equivalence relation on $\mathbb{T}$. Also, a gap in [Th] is defined as a component of $\mathbb{D} \setminus \{$*the union of convex hulls of leaves*$\}$. Our definitions are closer to [Do], [McM].

Fix an integer $d > 1$, denote by $\sigma_d = \sigma : \mathbb{T} \to \mathbb{T}$ the map $\sigma(t) = d \cdot t(\mod 1)$ and identify it with the map $z \mapsto z^d$ on S^1. Say that a subset of S^1 is *split* into classes if it contains a class of each its element. The relation $\sim$ is called **(σ-)invariant** iff:

(D1) $\sim$ is **forward invariant**: for a class g, the set $\sigma(g)$ is a class too

which implies that

(D2) $\sim$ is **backward invariant**: for a class g, its preimage $\sigma^{-1}(g) = \{x \in \mathbb{T} : \sigma(x) \in g\}$ is split into classes;

(D3) for any gap g, the map $\sigma : g \to \sigma(g)$ is a covering map with positive orientation.

Call a class g **critical** iff the map $\sigma : g \to \sigma(g)$ is not 1-to-1. Let $k_\sim$ be the maximal number of critical classes g such that $\sigma(g)$ is a single point with the infinite σ-orbit (i.e., $\sigma(g)$ is an irrational point of $\mathbb{T}$) and the orbits of g are pairwise disjoint.

Let $\mathbb{D}$ be the open disk bounded by S^1, $L_\sim = L$ be the union of **$\sim$-hulls**, i.e. convex hulls (in the Poincaré metric) of $\sim$-classes; by the definition $\sim$-hulls are contained in $\bar{\mathbb{D}}$ but not in $\mathbb{D}$. Define an extension $\simeq$ of $\sim$ onto $\bar{\mathbb{D}}$ as follows [Do]: a $\simeq$-class is a $\sim$-hull or a point of $\bar{\mathbb{D}} \setminus L$. Extend $\simeq$ onto $\mathbb{C}$ by declaring that a point in $\mathbb{C} \setminus \bar{\mathbb{D}}$ is equivalent only to itself. Call a connected component of the complement $\mathbb{D} \setminus L$ a **($\sim$-)component**. Given an open set Ω in $\mathbb{D}$, denote by $E(\Omega)$ the set $\bar{\Omega} \cap S^1$. Below Ω is usually bounded by geodesics intersecting each other only at their endpoints on the circle, and then $\bar{\Omega}$ is the convex hull of the set $E(\Omega)$.

However first we need to develop the language which allows us to new definitions. Let (x, y) be the Poincaré geodesic in $\mathbb{D}$ joining $x, y \in S^1$. Call (x, y) a **($\sim$-)geodesic** if $x \sim y$. We identify the $\sim$-geodesic (x, y) with the pair of points $\{x, y\}$ and speak of these two objects interchangeably. If (x, y) is a $\sim$-geodesic we say that σ **maps** (x, y)

onto (x', y') if $\sigma(x) = x'$ and $\sigma(y) = y'$. By $< x, y >$ we mean one of two arcs in S^1 with endpoints x, y.

Lemma 2.1. *Let Ω be a $\sim$-component. Then $E(\Omega) = E$ is a Cantor set and $\sigma(E) = E(\Omega')$, where Ω' is a $\sim$-component. Moreover, let $x_1, x_2 \in E$ be the endpoints of a component I of $S^1 \setminus E = E'$. Then $x_1 \sim x_2, \mathrm{Cl}(x_1) \subset \bar{I}$, and if $x_1' \in E$ is such that $\sigma(x_1') = \sigma(x_1)$ then one of the following cases holds:*

1. *$\sigma(x_1) \neq \sigma(x_2)$ and there is $x_2' \in E$ such that $\sigma(x_2') = \sigma(x_2)$ and x_1', x_2' are endpoints of another component of E';*
2. *$\sigma(x_1) = \sigma(x_2)$ and there is $x_2' \in E$ such that $\sigma(x_2') = \sigma(x_1)$ and x_1', x_2' are endpoints of a component of E';*
3. *$\sigma(x_1) = \sigma(x_2)$ and there is no $x_2' \in E$ such that x_1', x_2' are endpoints of a component of E'.*

Proof. For the sake of definiteness we assume that a point which runs within I from x_1 to x_2 has to run counterclockwise.

First we show that $x_1 \sim x_2$ and $\mathrm{Cl}(x_1) \subset \bar{I}$. Let l be a component of $\partial\Omega \setminus \{x_1, x_2\}$, which is disjoint with S^1. Any point $x \in l$ is then the limit of a sequence of points x_n so that each x_n lies in a boundary of a $\sim$ hull. Hence, $x_n \in l_n$ where l_n are pairwise disjoint $\sim$-geodesics. Consider two possibilities.

(i) The sequence $\{l_n\}$ is finite. Then x belongs to one of them, $l(x)$.

(ii) The sequence $\{l_n\}$ is infinite. Then x belongs to a geodesic $l(x)$ which is the limit of l_n.

Since the geodesics $l(x)$ for different $x \in l$ are either disjoint or coincide, we see that $l(x) = (x_1, x_2)$ for every $x \in l$. Thus $l = (x_1, x_2)$. Moreover, the endpoints of l_n are $\sim$ equivalent and the lamination is closed, therefore, $x_1 \sim x_2$. Also, Ω is disjoint with the $\simeq$ classes, therefore $\mathrm{Cl}(x_1) \subset \bar{I}$. Denote $\mathrm{Cl}(x_1)$ by K.

Let us show that E is a Cantor set. The fact that $\mathrm{Cl}(x_1) \subset \bar{I}$ implies that x_1 is not an isolated point in E. Indeed, otherwise there is another complementary to E arc $< z, x_1 >$ and by the above proven $z \sim x_1$, a contradiction to $\mathrm{Cl}(x_1) \subset \bar{I}$. Clearly, this means that there are no isolated points in E at all. To prove that E is a Cantor set it remains to prove that E contains no subintervals. This follows from the fact that some σ-iterate of any interval covers S^1.

Let $I' = S^1 \setminus I$. Let J' be the arc running clockwise from $\sigma(x_1)$ to $\sigma(x_2)$ and $J = S^1 \setminus J'$. Then J' contains σ-images of small semi-neighborhoods of x_1, x_2 non-disjoint from E. We show that J is disjoint from $\sigma(E)$. It is clear if $\sigma(x_1) = \sigma(x_2)$, so we assume that $\sigma(x_1) \neq \sigma(x_2)$. By (D3) for every class-preimage of $\sigma(K)$ we can find two points x_1', x_2' with $\sigma(x_1') = \sigma(x_1), \sigma(x_2') = \sigma(x_2)$ such that the closure of the arc T running counterclockwise from x_1' to x_2' contains $\mathrm{Cl}(x_1')$. Moreover, T is disjoint from E because there are points of E in a small counterclockwise semi-neighborhood of, say, x_1 and the geodesic (x_1', x_2') separates T from those points. Thus, the union A of all such arcs T is disjoint from E too. On the other hand by the construction A covers all preimages of J. Therefore, $\sigma(E)$ is disjoint from J as claimed which implies that $\sigma(E) \subset \bar{J}'$.

Let us show that $\sigma(K) \subset \overline{J}$. Let x_1'' be the counterclockwise closest to x_2 point such that $\sigma(x_1) = \sigma(x_1'')$ (i.e., x_2'' is of the form $x_1 + j/d$ for some j). Let R be the arc running counterclockwise from x_2 to x_1''. If $\sigma(K) \not\subset \overline{J}$ then inside R there must be points of a class K' such that $\sigma(K') = \sigma(K)$ which is impossible because A contains K' and is on the other hand disjoint from R.

Let us show that the alternative (1)-(3) follows. Assume that $x_1' \in E$ is such a point that $\sigma(x_1') = \sigma(x_1)$; let $\mathrm{Cl}(x_1') = K'$. If $\sigma(x_1) \neq \sigma(x_2)$ then by the proven above x_1' is an endpoint of a maximal arc $< x_1', x_2' >= A_j$ which is complementary to E. If $\sigma(x_1) = \sigma(x_2)$ then by (D3) we see that $u = \sigma(K)$ is a one-point set. Hence K' consists of a few points from $\sigma^{-1}(u)$. If $K' = \{x_1'\}$ then the case (3) holds. Otherwise by the above analysis the case (2) holds.

This completes the proof. ■

Let us introduce some maps and spaces. First, $K = \bar{\mathbb{D}}/\simeq$ is the quotient space, called the **pinched disc defined by** $\sim$ ([Do]). Denote the interior of K by F. The factor space $\mathbb{C}/\simeq$ is called the **pinched plane**; K is imbedded in $\mathbb{C}/\simeq$. Let $p : \mathbb{C} \to \mathbb{C}/\simeq$ be the factor map. Then $p : \mathbb{C} \setminus \bar{\mathbb{D}} \to (\mathbb{C}/\simeq) \setminus K$ and $p : \mathbb{D} \setminus L \to F$ are homeomorphisms. The set $J = p(S^1) = p(L)$ is the boundary of K in $\mathbb{C}/\simeq$. Also, call $A_\infty = (\mathbb{C}/\simeq) \setminus K$ the **basin of infinity** of a map f defined as follows. Since the map $\sigma(z) = z^d$ acts on S^1 and on $\mathbb{C} \setminus \bar{\mathbb{D}}$ and the relation $\sim$ is σ-invariant, we can introduce a map $f : J \to J$; also, since $p : \mathbb{C} \setminus \bar{\mathbb{D}} \to A_\infty$ is a homeomorphism, $f : J \to J$ extends to the map $f : A_\infty \to A_\infty$ as $f = p \circ \sigma \circ p^{-1}$. Observe, that K, J are compact, connected and locally connected because $p : \bar{\mathbb{D}} \to K$ is continuous. Finally, J and A_∞ are completely f-invariant, and $f|_{J \cup A_\infty}$ is continuous. We fix a metric on $\mathbb{C}/\simeq$ compatible with the topology which makes $\mathbb{C}/\simeq$ a Hausdorff metric space.

According to a theorem of Moore [Mo], the pinched plane $\mathbb{C}/\simeq$ is homeomorphic to the plane.

Proposition 2.2. *Let U be a connected component of the interior F of K. Then its closure is a topological disc. In particular, the boundary ∂U is a Jordan curve.*

Proof. $\bar{U}$ is the quotient of the closure of a $\sim$-component Ω by a closed equivalence relation on $\partial\Omega$ whose classes are points of S^1 and closed arcs in $\bar{\mathbb{D}}$ with the endpoints in S^1. Therefore, it is homeomorphic to $\bar{\mathbb{D}}$. ■

Our next aim is to extend f to $F \neq \emptyset$ (no extension is necessary if $F = \emptyset$).

Lemma 2.3. *If U is a component of F, then $f(\partial U)$ is a boundary of some component U' of F, the map $f : \partial U \to \partial U'$ preserves orientation and is an unbranched degree l covering map with $l \geq 1$ finite.*

Proof. Follows from Lemma 2.1. ■

Lemma 2.4. *If U is a component of F, such that $f^p(\partial U) = \partial U$, for some $p \geq 1$, then the map $f^p : \partial U \to \partial U$ is topologically conjugate either to*

(S) an irrational rotation on S^1, or to

(A) the map $z \mapsto z^l$, for some $l \geq 2$.

Moreover, if $x \in \partial U$ is such that the $\sim$-class $g = p^{-1}(x)$ is not a point, then g is either eventually mapped into a point (and thus precritical) or preperiodic; so if Ω is a $\sim$-component such that $p(\Omega) = U$ then $\sigma^p|\partial\Omega$ is not injective.

Proof. Denote by g the map $f^p : \partial U \to \partial U$. It is enough to show that g has no wandering intervals (i.e., non-trivial arcs $I \subset \partial U$ with $g^k(I) \cap g^n(I) = \emptyset, k \neq n$). Indeed, if g has no wandering intervals then by Lemma 2.3, $g : \partial U \to \partial U$ is conjugate to the rotation (if $l = 1$) or the map $z \mapsto z^l$ (if $l > 1$), see e.g. [MS]. Moreover, the rotation has to be irrational, because the map σ has finitely many periodic orbits of each period.

To prove that there are no wandering intervals we find a finite non-empty set $A \subset \partial U$ and a dense set $S \subset \partial U$ such that any point $x \in S$ eventually hits A (i.e., there exists $k \geq 0$ s.t. $g^k(x) \in A$). Let Ω be a $\sim$-component such that $p(\Omega) = U, S = \{p(l)\}$ where l runs over the geodesics in $\partial\Omega$. Also, let $A = \{p(l_b)\}$ where l_b runs over the family A' of geodesics in $\partial\Omega$ with the radial length (the length of the shortest arc of $S^1 \setminus l_b$) at least $1/(2d^p)$. By Lemma 2.1, the geodesics l are dense in $\partial\Omega$, hence S is dense in ∂U. Also, A is finite because the number of the geodesics l_b as above in the boundary of the same component Ω is at most $2d^p$. Finally, A is non-empty because any geodesic l on $\partial\Omega$ will be eventually mapped by σ^p onto a geodesic of radial length at least $1/(2d^p)$.

Note that if the case (A) holds then $\sigma^p|\Omega$ is not injective because $z \mapsto z^l, l \geq 2$ is not. Suppose that the case (S) holds. Then some geodesics in A' have to map into points since otherwise by the previous paragraph they will all be preperiodic, a contradiction with the case (S). So again $\sigma^p|\Omega$ is not injective which completes the proof. ■

We call a $\sim$-component U for which the condition of the lemma holds **periodic Siegel** iff (S) holds and **periodic attractive** iff (A) holds (cf. with rational maps [Mi]).

Proposition 2.5. *The following properties hold.*

(1) *Let $g \subset S^1$ be a $\sim$-class or the set $E(\Omega)$ for some $\sim$-component Ω. Then the number of such sets g with the additional property that $\sigma : g \to S^1$ is not injective, is finite. In particular, the number of components U of F such that $f : \partial U \to \partial U$ is an unbranched degree l covering map, $l \geq 2$, is finite.*
(2) *The number of all periodic components of F (Siegel and attractive) is finite.*

Proof. (1) Every g satisfying the assumptions, contains two points $x, y \in S^1$ with $\sigma(x) = \sigma(y)$, and so the radial distance between x, y equals to j/d for some $j = 0, 1, ..., [d/2]$. The geodesic (x, y) lies in the convex hull of g and these convex hulls are pairwise disjoint, thus these geodesics are pairwise disjoint too. However there may be only finitely many pairwise disjoint geodesics (x, y) such that the radial distance between x, y equals to j/d for some $j = 0, 1, ..., [d/2]$, hence there are finitely many sets g.

(2) Follows from (1) and the last claim of Lemma 2.4. ■

We strengthen Proposition 2.5 later in Proposition 2.8.

To extend f from $J = \partial K$ to components of F choose a component U of F and consider the grand orbit of U (the components U^n with the boundaries contained in $f^n(\partial U)$, $n = 0, \pm 1, \pm 2, \dots$).

Case A. ∂U is invariant under f^p for some p, and $f^p : \partial U \to \partial U$ is an unbranched degree l covering with $l \geq 1$. Let U^i be the component with the boundary $f^i(\partial U)$, $i = 0, 1, ..., p-1$. Keeping the dynamics on $f^i(\partial U)$ we extend it on all U^i in two steps.

(1) Extend $f^p : \partial U \to \partial U$ to $f_{p,U} : \bar{U} \to \bar{U}$ as follows. Using Lemma 2.4, consider a homeomorphism $H : \partial U \to S^1$ conjugating f^p to g_l (g_l is an irrational rotation if $l = 1$, and $g_l(z) = z^l$ otherwise): $g_l \circ H = H \circ f^p$ on ∂U. The map g_l is defined on $\bar{\mathbb{D}}$ and fixes zero. Extend H to a homeomorphism $\bar{H} : \bar{U} \to \bar{\mathbb{D}}$ and let $a_U = \bar{H}^{-1}(0) \in U$. The desired extension of f^p on $\bar{U}$ is $f_{p,U} = \bar{H}^{-1} \circ g_l \circ \bar{H}$. Note that $f_{p,U}|_{\partial U} = f^p$.

Define the set $\mathcal{G}_U = \mathcal{G}_0 = \{\Gamma_z\}_{z \in \partial U}$ of curves in $\bar{U}$ as $\Gamma_z = \bar{H}^{-1}(r_x)$ where $x = H(z) \in S^1$ and r_x is the radius in $\bar{\mathbb{D}}$ between 0 and $x \in S^1$. Then the system of curves $\mathcal{G}_U$ is invariant under $f_{p,U}$, each Γ_z joins $z \in \partial U$ with $a_U = a_0 = \bar{H}^{-1}(0)$, the curves of $\mathcal{G}_U$ form a **foliation** of $\bar{U} \setminus a_U$ (i.e., fill in this set and are pairwise disjoint), and $f_{p,U}(a_U) = a_U$.

(2) We set $U^0 = U^p = U, \mathcal{G}_p = \mathcal{G}_0, a_p = a_0$ and define maps $f_i : \bar{U}^i \to \bar{U}^{i+1}$ ($i = 1, ..., p-1$) so that $f_{p,U} = f_{p-1} \circ f_{p-2} \circ \cdots \circ f_0$. Simultaneously we define points $a_{U^i} = a_i \in U^i$ and foliations $\mathcal{G}_{U^i} = \mathcal{G}_i = \{\Gamma_z\}_{z \in \partial U^i}$ of $\bar{U}^i \setminus a_i$. We begin by defining maps $f_i, i = 1, \dots, p-1$ as follows:

(a) f_i is a continuous extension of $f : \partial U^i \to \partial U^{i+1}$;

(b) f_i is an unbranched degree l_i covering map with a unique branched point a_i such that $a_{i+1} = f_i(a_i)$ (here l_i is the degree of the map $f : \partial U^i \to \partial U^{i+1}$);

(c) $a_{i+1} = f_i(a_i)$ and $\mathcal{G}_{i+1} = f_i(\mathcal{G}_i)$ (that is, the foliation $\mathcal{G}_i$ is obtained as a pull-back of $\mathcal{G}_{i+1}$ under the map f_i which is possible because $f_i(a_i) = a_{i+1}$).

To begin with the foliation $\mathcal{G}_p = \mathcal{G}_0$ and the point a_p are defined. Let $f_{p-1} \circ \cdots \circ f_1 = h$ and $r = \prod_{i=1}^{p-1} l_i$. Then $h : U^1 \to U$ is of degree r. Define a map $f_0 = h^{-1} \circ f_{p,U}$ first along a curve $\Gamma_{z_0} \in \mathcal{G}_U$. As the point z moves along ∂U^0, extend the germ of f_0 over the curves Γ_z from the map $f : \partial U^0 \to \partial U^1$ to a well-defined map $f_0 : \bar{U}^0 \to \bar{U}^1$ so that $f_{p,U} = f_{p-1} \circ f_{p-2} \circ \cdots \circ f_0$ and properties (a)-(c) above are satisfied for $f_i, 1 \leq i \leq p-1$.

By the construction, the union of curves of families $\mathcal{G}_i, i = 0, ..., p-1$, is invariant under the map $\bar{f} : \cup_{i=0}^{p-1} U^i \to \cup_{i=0}^{p-1} U^i$ defined as $\bar{f}|U^i = f_i$. Each curve $\Gamma_z \in \mathcal{G}_i$ joins the point $z \in \partial U^i$ and the marked point a_i, and the curves of $\mathcal{G}_i$ form a foliation of $\bar{U} \setminus a_i$.

Case B. U is a preimage W^{-m} of a periodic component W, i.e. $f^m(\partial W^{-m}) = \partial W$. Consider all preimages $W^{-n}, n \geq 1$, other than iterates of W and introduce the dynamics on all W^{-n} inductively (first on all W^{-1}, then all W^{-2}, etc) as follows. We have done it on each periodic W. Assume we have already defined the map $f_{V'} : V' \to f_{V'}(V')$ on every component V' which is not an iterate of W such that $f^i(\partial V') = \partial W$ for some $0 \leq i \leq n-1$. If now $f^n(\partial V) = \partial W$ and $f : \partial V \to f(\partial V)$ is an l-cover ($l \geq 1$) we define f_V on

$\bar{V}$ in such a way, that $f_V|\partial V = f$, $f_V : \bar{V} \to f(\bar{V})$ is a covering map with a chosen point a_V (which is a unique branch point if $l > 1$) such that $f_{f^{n-1}(V)}(a_V) \circ \dots \circ f_{f(V)} \circ f_V = a_W$. Preimages of the curves of $\mathcal{G}_W$ inside components V form families of curves $\mathcal{G}_V$ which are in fact foliations of sets $V \setminus \{a_V\}$.

Case C. If U is a wandering domain ($f^k(\partial U) \cap f^r(\partial U) = \emptyset$, $k \neq r$), fix a high forward iterate V of U, so that maps $f^n : \partial V \to f^n(\partial V), n > 0$, are isomorphisms. Mark a point $a_V \in V$ and choose a foliation $\mathcal{G}_V = \{\Gamma_x\}_{z \in \partial V}$ of $V \setminus \{a_V\}$, where Γ_z is a curve joining a_V and $z \in \partial V$. Define f on all images of V so that it becomes a homeomorphic extension of f defined on their boundaries; for any such image $U = f^n(V)$ also define the point $a_U = f^n(a_V)$. Now define f on all preimages of all images of V as in Case B.

We get a continuous map $\bar{f} : \mathbb{C}/\simeq \to \mathbb{C}/\simeq$ of the pinched plane as follows (here we define some new notions mimicking [DH], [Do]). First, $\bar{f}$ coincides with f on $(\mathbb{C}/\simeq) \setminus F$ and with f_U on all components of F. Every component U of F has the marked point a_U called the **center** of U, and $a_{f(U)} = \bar{f}(a_U)$. Every set $\bar{U} \setminus a_U$ is foliated by the curves Γ_x joining a_U with points $x \in \partial U$; these curves, called **internal rays**, form the family $\mathcal{G}_U$. The union $\mathcal{G}(K)$ of $\mathcal{G}_U$ over all components U of F is $\bar{f}$-invariant. An arc l in K is called **legal** if for any component U of the interior F of K, the set $l \cap \bar{U}$ is contained in the union of two internal rays. Talking of an arc defined by a map $\gamma : [0, 1] \to K$ we often denote this arc (i.e. the set $\gamma([0, 1])$) by γ. Also, by a **loop** in K we mean a continuous map $\gamma : [0, 1] \to K$ such that $\gamma(t) \neq \gamma(\tau)$, for all $0 \leq t < \tau \leq 1$, except if $\gamma(0) = \gamma(1)$.

It is easy to see that the map $\bar{f}$ is a local homeomorphism at any point x of the pinched plane except for a finitely many (by Proposition 2.5) **critical** points $c_1, ..., c_m$ of the form: either $c_i = p(g) \in J$, where g is a critical $\sim$-class, or $c_i = a_U$, where a_U is the center of a component U of F and $f : \partial U \to f(\partial U)$ is an l-cover, $l \geq 2$ (note that each critical point of the latter type is preperiodic whenever U is preperiodic).

Indeed, sets J, F, and A_∞ are completely invariant under the map $\bar{f}$. Moreover, by the construction, for every point $x \in J$ there is a neighborhood U such that $\bar{f}$ is one-to-one on every component of $U \setminus J$. Therefore, it is enough to check that $\bar{f}|J = f$ is a local homeomorphism at any non-critical point. Let us check that f is actually an open map everywhere; we do this by way of contradiction. If f is not open at x then there is its neighborhood U and a sequence of classes x_n such that $f(x_n) \to f(x)$ while no class $f(x_n)$ has preimages in U. We can assume that $x_n \to y$ and then $f(y) = f(x)$. Then we can choose points $x'_n \in x_n$ which converge to a point $x' \in y$ so that $\sigma(x') \in f(y) = f(x)$. By the properties of laminations we can find a point $z' \in x$ such that $\sigma(z') = \sigma(x')$ which implies that there exists a sequence of points $z'_n \to z'$ such that $\sigma(z'_n) = \sigma(x'_n)$. Choosing a subsequence, we may assume that classes z_n of points z'_n converge in J, and then they can only converge to the class x. On the other hand, classes z_n from some time on belong to U which proves that classes $f(z_n) = f(x_n)$ belong to $f(U)$, contrary to our assumption. The verification of the fact that f is 1-to-1 at a non-critical point is just as elementary as is left to the reader as a useful exercise.

The **external ray** R_t of argument $t \in \mathbb{T}$ is the curve $p(\{r \exp(2\pi i t) : r > 1\})$, the external rays $R_t, t \in \mathbb{T}$ foliate the basin of infinity A_∞. If $r \to 1$ then the point

$p(r\exp(2\pi it))$ of R_t tends to the point $x = p(\exp(2\pi it))$ in J (R_t **lands** at x) and vice versa, every point $x = p(\exp(2\pi it)) \in J$ is a landing point of the external ray R_t.

Lemma 2.6. *The set K is arcwise connected and has the following properties:*

(1) *there is no loop γ in K which is the union of finitely many legal arcs;*
(2) *given points $x, y \in K$, there exists a unique legal arc in K with endpoints at x, y;*
(3) *if γ is a legal arc, then $\bar{f}(\gamma)$ is a finite union of the legal arcs containing no loops.*

Proof. K is arcwise connected because it is the image of $\bar{\mathbb{D}}$ under a continuous map p.

(1) If γ lies in a component U of F, the statement clearly holds. Otherwise fix points $a \neq b \in \gamma$ who split γ into two closed arcs γ_1, γ_2, so that $\gamma_1 \cap \gamma_2 = \{a, b\}$. Consider subsets $\tilde{\gamma} = p^{-1}(\gamma), \tilde{\gamma}_i = p^{-1}(\gamma_i), i = 1, 2$ of $\bar{\mathbb{D}}$. Since $p^{-1}(x)$ is a connected closed subset of the plane for any $x \in \mathbb{C}/\simeq$, the sets $\tilde{\gamma}_1, \tilde{\gamma}_2$ are compact connected subsets of $\bar{\mathbb{D}}$ while $\tilde{\gamma}_1 \cap \tilde{\gamma}_2 = p^{-1}(a) \cup p^{-1}(b)$ is not connected. Hence ([Ku]) $\tilde{\gamma}_1 \cup \tilde{\gamma}_2 = \tilde{\gamma}$ separates the plane. Let $\tilde{A}$ be a bounded component of $\mathbb{C} \setminus \tilde{\gamma}$. Since $\tilde{\gamma}$ consists of $\simeq$ classes, $\tilde{A}$ consists of $\simeq$ classes as well. Also, $\tilde{A}$ is open. Then $\tilde{A} \subset \mathbb{D}$ because $\tilde{\gamma} \subset \bar{\mathbb{D}}$ and so if $\tilde{A}$ hits $\mathbb{C} \setminus \mathbb{D}$ it must be unbounded. Hence, $\tilde{A}$ is disjoint from any $\sim$-class because otherwise it would contain points of $\bar{\mathbb{D}}$ (every $\sim$-class contains points of $\bar{\mathbb{D}}$ by definition), and so $\tilde{A}$ contains an interior point $\tilde{x}$ of a $\sim$-component Ω. Thus, the point $x = p(\tilde{x})$ lies in the component $U = p(\Omega)$ of F.

Now, $\tilde{A} \subset \mathbb{D}$ implies $\Omega \cap \tilde{\gamma} \neq \emptyset$. Hence $\gamma \cap U \neq \emptyset$ too. By the definition of a loop, $\gamma \cap U$ is a finite union of internal rays. Moreover, since γ has no points of self-intersection, $x \in \gamma \cap \bar{U} = \Gamma_{x_1} \cup \Gamma_{x_2}$ where $x_1, x_2 \in \partial U$ and $\Gamma_{x_1}, \Gamma_{x_2}$ are the corresponding internal rays. Let $A = p(\tilde{A})$; clearly, A is an open and connected subset of a pinched disk. Then one of two open arcs of $\partial U \setminus \{x_1, x_2\}$ lies in A, a contradiction with $A \subset U$.

(2) Let $\gamma : [0, 1] \to K$ be a curve, connecting $x = \gamma(0)$ with $y = \gamma(1)$. Then the set $\gamma \cap \bar{U}$ is closed for any component U of F. Let α_U, β_U be the least and the greatest numbers with $\gamma(\alpha) \in \bar{U}, \gamma(\beta) \in \bar{U}$. For every U, we can redefine γ on $[\alpha_U, \beta_U]$ so that γ maps the interval $[\alpha_U, \beta_U]$ onto $\Gamma_{\gamma(\alpha_U)} \cup \Gamma_{\gamma(\beta_U)}$. We proceed this way, applying the construction on every step to the current map γ.

It is easy to see that the sequence of maps γ (and corresponding curves) converges to a legal arc with endpoints x, y as desired. By (1) this arc is unique.

(3) Follows immediately from (1). ∎

In the next section we construct a growing tree in the quotient space of $\simeq$ and apply results of Section 1 in order to study the dynamics of Fatou domains in this quotient space. This will provide alternative proof of Sullivan's famous No Wandering Fatou Domain Theorem [Su] (which he proved in a much more general situation of rational functions). In fact in the case of laminations one can prove that there are no wandering continua, and an interesting question is in what way this combinatorial result can be extended onto rational functions. In other words, for what kinds of continua other than Fatou domains can one prove that there are wandering continua of this kind?

3. Combinatorial version of Sullivan's No Wandering Fatou Domain Theorem

A set A is called *wandering* if all its iterates are pairwise disjoint. The main aim of this section is to prove the following theorem.

Theorem 3.1. *Let $\sim$ be a closed invariant lamination $\sim$. Let Ω be a $\sim$-component. Then the set $E(\Omega) \subset \mathbb{T}$ is σ-preperiodic in the following sense: there exist $n \geq 0, m > 0$ with $\sigma^m(E(\Omega)) = \sigma^{m+n}(E(\Omega))$.*

To prove Theorem 3.1 we need several lemmas. Given $x, y \in K$, denote by $[x, y]$ a unique well-defined by Lemma 2.6 legal arc in K with ends at x, y. Now we step by step define a growing tree $T_0 \subset T_1 \subset \dots$ in K for the map $\bar{f}$. Let $\beta = p(0)$ ($0 \in S^1$ is a fixed point of the map $\sigma(z) = z^d$ of S^1). Then β is also a fixed point of $\bar{f}$. By (D2) from Section 2 any $\sim$-class in $\sigma^{-1}(\mathrm{Cl}(0))$ contains at least one point of $\sigma^{-1}(0)$. Hence there are no more than d preimages of β; denote them by $\{\gamma_i\}$ and then define the **initial tree** $T_0 = \cup_i[\gamma_i, \beta]$.

Let $T_n = \cup_{i=0}^n \bar{f}^i(T_0)$. By Lemma 2.6 all T_n are trees. Given $x \in J$, denote by $N(x)$ the number of the external rays landing at x (in other words, $N(x)$ is the number of elements in the $\sim$-class $p^{-1}(x)$). In the next proposition we study the trees T_n and the orbits of the points $x \in J$ with $N(x) \geq 2$. We say that two external rays R_{t_1} and R_{t_2} are **separated (by the tree T_0)** if t_1 and t_2 lie in different components of $\mathbb{T} \setminus \{0, 1/d, 2/d, ..., 1 - 1/d\}$. Denote by $k_\sim$ is the maximal number of critical $\sim$-classes g with infinite and pairwise disjoint orbits, and such that $\sigma(g)$ is a point.

Proposition 3.2. *The following properties hold.*

(1) *If separated external rays R_{t_1}, R_{t_2} land at the same point x then $x \in T_0$.*
(2) *All critical points of $\bar{f}$ belong to T_0 and $T_0 \subset T_1 \subset \dots$ is a growing tree.*
(3) *If $x \in J$ then $N(x) = |p^{-1}(x)|$.*
(4) *If $M \subset J$ is a continuum or $M = \{x\}$ with $N(x) \geq 2$ then there exists i with $f^i(M) \cap T_0 \neq \emptyset$. Moreover, the following holds:*
(a) *in the case of continuum there are infinitely many i such that $\bar{f}^i(M) \cap T_0 \neq \emptyset$ and the set of points eventually mapped into T_0 is dense in M.*
(b) *if $x \in J$ is not an $\bar{f}$-preimage of a critical point or of β then $N(x) \geq 2$ if and only if there are infinitely many i such that $\bar{f}^i(x) \in T_0$ is not an endpoint of the tree T_0.*
(c) *if $x \in J$ is not an $\bar{f}$-preimage of a critical point and $N(x) \geq 3$, then, for every finite $n \leq N(x)$, and for some i, m the point $\bar{f}^i(x)$ is a vertex of T_m with $\mathrm{ord}_{T_m}(f^i(x)) \geq n$.*
(5) *For every component U of F, some iterate $V = \bar{f}^i(U)$ intersects T_0; moreover, the center a_V of V lies in T_0, and $\bar{V} \cap T_0$ is homeomorphic to the n-od with the branching point at a_V.*
(6) $\mathrm{oen}(T_\infty) \leq k_\sim$.

Proof. (1) Since β and all its f-preimages γ_i belong to T_0 we see that $\sim$-classes corresponding to γ_i include points $0, 1/d, \dots, (d-1)/d$. Since $g = p^{-1}(x)$ is a $\sim$-class containing t_1, t_2 then the convex hull of g intersects a connected set $p^{-1}(T_0)$ and hence $x \in T_0$.

(2) Suppose that g is a critical class. If g contains i/d for some i then $p(g) \in T_0$ by the definition of T_0. Otherwise suppose that $p(g) \notin T_0$. Then by (1) all rays with arguments from g are not separated. On the other hand, since g is critical there must be two points x and $x + j/d, 1 \le j \le d-1$ in g, a contradiction.

Similarly, if c is a critical point which is the center of a Fatou domain U, then there are external rays landing on the boundary of U which are mapped into one ray. If these rays are of the form $i/d, j/d$ then the legal arc connecting i/d with j/d must cross U and thus must pass through c. On the other hand, this arc is contained in T_0. Hence in this case $c \in T_0$. If these rays are not of the form $i/d, j/d$ then they rays are separated. Hence there are two angles of the form $i/d, j/d$ such that the legal arc connecting i/d with j/d must cross U and the same argument implies that $c \in T_0$ again. Together with Lemma 2.6 this implies that indeed $T_0 \subset T_1 \subset \dots$ is a growing tree.

(3) Follows from the definition of the external rays.

(4) In our situation we can find two external rays R_{t_1}, R_{t_2} landing at points of M so that for some i either one of the rays $\bar{f}^i(R_{t_1})$, $\bar{f}^i(R_{t_2})$ has the argument j/d (and lands at a point of T_0) or the rays $\bar{f}^i(R_{t_1})$, $\bar{f}^i(R_{t_2})$ are separated in which case by connectivity the continuum $\bar{f}^i(M)$ must intersect T_0. This proves the main claim of (4).

To prove (a) observe that subcontinua of arbitrarily small diameters are dense in M and that the image of a continuum under a power of f is a continuum itself.

To prove (b) observe that if $x \in J$ is not an $\bar{f}$-preimage of a critical point or of β then the claim (4)(a) can be applied to x infinitely many times, so $\bar{f}^i(x) \in T_0$ for infinitely many i. Since x is not a preimage of β then $\bar{f}^i(x)$ is not an endpoint of T_0. Therefore, for infinitely many i we have that $\bar{f}^i(x) \in T_0$ is an inner point of T_0.

On the other hand, suppose that there are infinitely many i such that $\bar{f}^i(x) \in T_0$ is an inner point of T_0. Then for some k we have that $\bar{f}^k(x)$ is not an endpoint of T_0 and there are at least two external rays landing at $\bar{f}^k(x)$ which by the assumptions implies that there are at least two external rays landing at x and so $N(x) \ge 2$.

Consider claim (c). First we show that any non-critical $x \in J$ has a neighborhood U such that for any $y \in J \cap U$ the cyclic order on the set $p^{-1}(x) \cup p^{-1}(y) \cap S^1$ is preserved by σ. Indeed, $\simeq$ is a closed equivalence relation on the plane such that every equivalence class is closed, connected and non-separating. Hence, there is an arbitrarily small neighborhood $\tilde{U}$ of $p^{-1}(x)$ such that $\tilde{U}$ consists of $\simeq$-classes. We can set $U = p(\tilde{U})$, and by (D3) the property is satisfied.

Let $N(x) \ge 3$. Fix $n, 3 \le n \le N(x)$. Let $R_{t_i}, i = 0, \dots, n-1$ be external rays landing at x in the cyclic order of their arguments $t_0, ..., t_{n-1}$. By (D3) this order will not change under iterations of σ. For each $i = 0, ..., n-1$, find the minimal $r_i = r > 0$ so that $\bar{f}^r(R_{t_i}), \bar{f}^r(R_{t_{i+1}})$ are separated. By (D3) the arc $I_i =< \sigma^r(t_i), \sigma^r(t_{i+1}) >$ containing no σ^r-images of other t_j is well-defined. Also, by the first part of claim (4) $\bar{f}^r(x)$ is not an endpoint of T_0, so there are points $y \in T_0$ arbitrarily close to $\bar{f}^r(x)$ such that $p^{-1}(y \cap S^1) \subset I_i$.

Repeating this we find numbers $r_i, i = 0, \dots, n-1$. Let R be their maximum. Pick a small neighborhood U of x so that for all $y \in U$ all the iterates $\sigma^j, 0 \le j \le R$ preserve the cyclic order on $p^{-1}(x) \cup p^{-1}(y) \cap S^1$. Then choose points $y_i \in U$ so that $\sigma^{r_i}(p^{-1}(y_i)) \subset I_i$ and $\bar{f}^{r_i}(y_i) \in T_0$. Since σ^R preserves the cyclic order on $p^{-1}(x) \cap S^1$, the cyclic order of points $\{\sigma^R(t_i)\}$ is the same as that of points $\{t_i\}$. Thus the pairwise disjoint arcs $< \sigma^R(t_i), \sigma^R(t_{i+1}) >$ are well-defined.

By the choice of U each set $\sigma^R(p^{-1}(y_i))$ is contained in $< \sigma^R(t_i), \sigma^R(t_{i+1}) >$. Since $\bar{f}^R(y_i) = \bar{f}^{R-r_i}(\bar{f}^{r_i}(y_i)) \subset T_{R-r_i} \subset T_R$ we see that in fact all points $\bar{f}^R(x), \bar{f}^R(y_i)$ belong to T_R and that n rays $\bar{f}^R(R_{t_i}), i \le 0 \le n-1$ land at $\bar{f}^R(x)$ and divide the disk into n components containing distinct points $\bar{f}^R(y_i)$ and therefore non-disjoint from T_R. Thus, the number of components of $T_R \setminus \{f^R(x)\}$ is at least n as desired. Since x is not pre-critical we conclude that for any $m \ge R$ we have $\mathrm{ord}_{T_m}(\bar{f}^R(x)) \ge n$.

(5) Let Ω be the corresponding to U component of $\bar{\mathbb{D}} \setminus L$. Take any two $t_1, t_2 \in E(\Omega)$, which are non-precritical, non-preperiodic, and whose σ-images are not $\sim$-equivalent (it is possible since $E(\Omega)$ is a Cantor set). Then $\bar{f}^i$-iterates of external rays R_{t_1} and R_{t_2} land at distinct points of $\partial f^i(U)$ and are separated for some i. Let Ω' be the $\sim$-component with $E(\Omega') = \sigma^i(E(\Omega))$. Then $p^{-1}(T_0)$ intersects $\partial\Omega'$ at least at two points. Since T_0 consists of legal arcs and by Lemma 2.6 we conclude that $\bar{f}^i(\bar{U})$ is of the desired form.

(6) Clearly, $\mathrm{oen}(T_\infty) \le k''$ where k'' is the number of critical points c of $\bar{f} : T_\infty \to T_\infty$ such that for any m, $\bar{f}^m(c) \in T_m \setminus T_{m-1}$ (call such critical points *fast* and others *slow*). Let us show that a fast critical point c is such that $N(f^n(c)) = 1$ for every n. Indeed, if $N(f^n(c)) \ge 2$ then by (4) some forward image of $f^n(c)$ maps into T_0 and it is not fast. Also, if c is in the interior of Fatou component U then by the construction it coincides with a_U and maps into $a_{f^k(U)}$ by any f^k. By (5) there exists k such that $f^k(U)$ intersects T_0. At this point we would have $f^k(c) = a_{f^k(U)} \in T_0$, a contradiction with the fact that c is fast. Finally, all preperiodic critical points are slow.

We conclude that fast critical points are non-preperiodic, belong to J and such that $N(f^j(c)) = 1$ for every j. Let us denote the set of all such critical points by A. Then there exists the maximal number M such that if two critical points from A are mapped into the same point y by the same power of f then this power is less than M. Therefore the set of points $f^M(A) = B$ consists of points with disjoint grand orbits. Moreover, let us also assume that all other critical points which as we saw above are not fast are mapped into T_{m-1} by some f^m with $m < M$ (in other words, the fact that they are slow can be observed before the time M). Then the only outer endpoints of T_M are the points of B and so $\mathrm{oen}(T_\infty) \le |B|$. Since $|B| \le k_\sim$ (all points of B have disjoint grand orbits) we see that $\mathrm{oen}(T_\sim) \le k_\sim$ as desired. ■

Proof of Theorem 3.1. Assume to the contrary that for a component Ω, the set $E(\Omega)$ is wandering under σ. Replacing Ω by its sufficiently high iterate we may assume that no iterate of $U = p(\Omega)$ contains a critical point of $\bar{f}$. Moreover, by Proposition 2.5 there are only finitely many Fatou components U such that $\bar{f}|\partial U$ is not injective. Hence we may assume that $\bar{f}|\bar{f}^n(\bar{U})$ is injective for every $n \geq 0$.

Let us show that for any n we can find high iterate R of U whose boundary intersects T_R at least at n points. This is done similar to Proposition 3.2(4)(c). Choose $n \geq 3$ external rays $R_{t_i}, i = 0, \ldots, n-1$ landing at points $x_i \in \partial U = S$ in the cyclic order of their arguments $t_0, ..., t_{n-1}$. Since $\bar{U}$ is wandering by the assumption, these angles $t_0, \ldots, t_{n-1}$ are not preperiodic, in particular they are not eventual preimages of β. Since locally the cyclic order is not changed under σ because images of U contain no critical points and because all iterates of $\bar{f}$ on $\bar{U}$ are 1-to-1, we conclude that the cyclic order among $t_0, ..., t_{n-1}$ will not change under iterations of σ.

For each $i = 0, ..., n-1$, find the minimal $r_i = r > 0$ so that $\bar{f}^r(R_{t_i}), \bar{f}^r(R_{t_{i+1}})$ are separated. Then the arc $I_i =< \sigma^r(t_i), \sigma^r(t_{i+1}) >$ containing no σ^r-images of other t_j is well-defined and its p-image must intersect T_0. Hence the subarc $\tilde{I}_i$ of the boundary $\bar{f}^r(S)$ of the Fatou component $\bar{f}^i(U)$ corresponding to I_i (in fact $\tilde{I}_i \subset p(I_i)$) contains points of T_0. A useful point of view here is to look at all this downstairs, that after the quotient map p has been applied.

Let R be the maximum of numbers r_i. Then by the previous paragraph the set $\bar{f}^R(S)$ is divided into subarcs $p < \sigma^R(t_i), \sigma^R(t_{i+1}) >$, to each of these arcs we associate the appropriate subarc of the boundary of $\bar{f}^R(\bar{U})$, and each such piece of the boundary of $\bar{f}^R(\bar{U})$ intersects T_R. Hence $\bar{f}^R(\bar{U})$ intersects T_R over at least n points where n is chosen arbitrarily. By Proposition 3.2(5) and properties of T_∞ the center $a_{\bar{f}^N(U)}$ is a vertex of T_R with the order at least n. By Proposition 2.5 the number k of all critical points of $\bar{f}$ is finite, and if we choose $n > 2k+1$ then by Theorem 1.5 we see that $a_{\bar{f}^N(U)}$ must be a periodic point, a contradiction with the assumption that $\bar{U}$ is wandering.

Now, if U itself is non-wandering then it is preperiodic. Theoretically it is possible that U is a wandering Fatou domain while its boundary is not wandering. To exclude this situation assume by way of contradiction that ∂U is non-wandering. Then passing to the appropriate iterate of U and power of $\bar{f}$ we can assume that $\bar{f}(S) \cap S \neq \emptyset$.

This implies that actually for any n we have $\bar{f}^n(S) \cap \bar{f}^{n+1}(S) \neq \emptyset$. Let us show that this implies that for any n and i such that $i \geq 2$ we have $\bar{f}^n(S) \cap \bar{f}^{n+i}(S) \neq \emptyset$. We need the following definition: if any point of a continuum JK can be reached from the infinity by a ray then such plane continuum is called *unshielded*. Our Julia set J is indeed unshielded and we will rely upon this below. Let us make the following observation: two iterations of S cannot intersect at more than one point. Indeed, otherwise there will be points shielded from infinity, a contradiction.

Now, let us prove the above made claim. Suppose that contrary to what we want to show we have a chain of boundaries $\bar{f}^n(S), \bar{f}^{n+1}(S), \ldots, \bar{f}^{n+i}(S)$ such that every next boundary is non-disjoint from the previous one while the last one ($\bar{f}^{n+i}(S)$) intersects the first one ($\bar{f}^n(S)$). This can happen in two ways and we will now prove that either way it leads to a contradiction.

One of the ways is as follows: $\bar{f}^{n+2}(S), \bar{f}^{n+1}(S)$ and $\bar{f}^n(S)$ all intersect in one point. Let us show that this is impossible. Indeed, if so then the point a of their intersection is fixed for $\bar{f}$. Moreover, since the domains $\bar{f}^n(U), \bar{f}^{n+1}(U)$ and $\bar{f}^{n+2}(U)$ are disjoint we see that there are at least three rays landing at a. If so, then by Proposition 3.2(4) some iterate of a must belong to T_0, and since a is fixed then $a \in T_0$.

Therefore there are finitely many germs of T_0 at a. Now, the intersection of any set $\bar{f}^{n+i}(\bar{U})$ with T_0 is homeomorphic to an n-od with some n. Therefore a is an endpoint of such intersection for any $i \geq 0$, and the germs of T_0 at a are actually also germs of intersection $\bar{f}^{n+i}(\bar{U})$ at a. Since there are finitely many germs of T_0 at a, the intersections $\bar{f}^{n+i}(\bar{U})$ will have to be non-disjoint at some moment, a contradiction with the assumption that U is wandering.

It remains to consider the case when there are no three consecutive iterations of S having a common point. That is, we can assume that $\bar{f}^n(S) \cap \bar{f}^{n+1}(S) \neq \emptyset$ for any n and study this case with the above assumption. We want to prove that for any $i \geq 2$ we have then $\bar{f}^n(S) \cap \bar{f}^{n+i}(S) = \emptyset$. Indeed, otherwise we may assume that there are numbers n and $i \geq 2$ such that $\bar{f}^n(s), \dots, \bar{f}^{n+i-1}(S)$ are disjoint other than one-point intersections between consecutive iterations of S while $\bar{f}^{n+i}(S) \cap S \neq \emptyset$. In this case it is easy to see that there will be points shielded from infinity which is again a contradiction.

Let us show that $\bar{U}$ is wandering under the map $\bar{g} = \bar{f}^2$. Indeed, any two sets $\bar{g}^k(S)$ and $\bar{g}^l(S)$ with $k \neq l$ are disjoint by the previous paragraph, and U is wandering as we showed before, therefore the entire $\bar{U}$ is wandering. However, this contradicts the proven above because $\sim$ is a lamination invariant under a power of the original circle map σ and therefore all proven above applies. This contradiction completes the proof. ■

Next we prove a similar result which extends the No Wandering Domain Theorem: we prove that there are no wandering continua subsets of J. However before that let us motivate our interest to wandering sets. To do so let us point out that one of the central questions of the dynamical systems theory is that of the typical behavior of points. In other words, one wants to know what happens with the majority of points. In the topological setting majority means a **massive** set, that is a countable intersections of open dense subsets. As we know, if a map has a dense orbit then actually there is a massive subset of points with dense orbits, so in this case the typical behavior is dense.

Similarly, the absence of wandering Fatou domains of polynomials allows us to study the typical behavior on the filled-in Julia set K. Indeed, if a wandering domain existed we could not predict its behavior. However since such domains do not exist we know that the typical point of a Fatou set eventually maps inside a periodic Fatou domain, and in this case it is well-known that it converges to a periodic orbit or has the limit set which is a simple Jordan curve on which the map is acting like an irrational rotation.

All this justifies our interest to wandering sets and also to a question as to what sets cannot be wandering for the dynamical system which we study. Intuitively, big sets have less chance to be wandering, and to some extent this is confirmed by No Wandering Domain Theorem. "Next in size" after Fatou domains come continua, and so a question as to whether there are a wandering subcontinua of the Julia set is very natural. Below we prove that all continua in J are non-wandering, however before that we need to prove a useful lemma.

Lemma 3.3. *Suppose that $K \subset J$ is a continuum that meets the boundary of any Fatou domain in at most 1 point. Then K is uniquely arcwise connected, and moreover there is a unique subarc of J connecting any two points of K.*

Proof. Let $x \neq y \in K$. Since J is arcwise connected, there exists an arc $A \subset J$ joining x and y. Assume that $K \not\supset A$. Choose a point $z \in A \setminus K$ and let I be the component of $A \setminus K$ containing z. Then it is easy to see that $\bar{I} \cup K$ bounds a Fatou domain, a contradiction. So any arc connecting two points of K must be contained in K. Let us now show that K is uniquely arcwise connected. Indeed, by way of contradiction assume that A_1 and A_2 are arcs in K joining x and y. If $A_1 \neq A_2$ then arguing similar to the previous paragraph we can show that A_1 and A_2 bound a Fatou domain, a contradiction. ∎

Theorem 3.4. *There are no wandering continua $K \subset J$.*

Proof. Suppose that $K \subset J$ is wandering. We may assume that its orbit does not contain critical points (there are finitely many critical points of f and infinitely many pairwise disjoint images of K). To show that K cannot intersect the boundary of a Fatou domain over more than one point suppose otherwise and choose points $x, y \in K \cap \partial U$ where U is a Fatou domain. There are two disjoint arcs in ∂U which connect x and y. Let us first show that at least one of them is contained in K. Indeed, otherwise we may assume that there points of the complement of K in either arc. This easily implies that one of them must be shielded from infinity by the other arc and K, a contradiction.

We may assume that an arc $I \subset \partial U$ is contained in K (U is a Fatou domain). Since U is non-wandering, I is eventually mapped onto a subarc I' of the boundary S of a periodic Fatou domain V. By Lemma 2.4 $f^n|S$ is conjugate to an irrational rotation or the map $z \to z^d$ on S^1 for the appropriate n, and in either case I' (and I) is non-wandering. Hence the intersection of K with the boundary of a Fatou domain consists of no more than 1 point. By Lemma 3.3 this implies that K is uniquely arcwise connected.

Choose points $a, b \in K$ and consider the unique arc $I = [a, b] \subset J$; as we know I intersects boundaries of Fatou domains at no more than one points each, and I is wandering. Let us show that I contains a dense subset of vertices of J. To this end let us introduce the following notion: an arc $M \subset J$ such that M contains no vertices of J and intersects the boundary of any Fatou domain at no more than one point is said to be **interval-like**. We study properties of interval-like arcs in a series of claims.

Claim 3.5. *Let $L = [a, b]$ is interval-like. Choose the "endpoints" a', a'' of $p^{-1}(a)$ and the "endpoints" b', b'' of $p^{-1}(b)$ so that entire circle S^1 is divided by them into arcs $(a', a''), (a'', b'), (b', b''), (b'', a')$. Then $L \subset p([a'', b']) \cap p([a', , b''])$.*

Proof of Claim 3.5. Indeed, otherwise we may assume that $L \not\subset p([a'', b'])$. Then there is a subarc of L shielded from $[a'', b']$ which therefore must be a subarc of a Fatou domain, a contradiction with the assumption about L which completes the proof of the claim. ∎

Claim 3.6. *Let $M = [a, b]$ is interval-like. Choose the "endpoints" a', a'' of $p^{-1}(a)$ and the "endpoints" b', b'' of $p^{-1}(b)$ so that entire circle S^1 is divided by them into arcs $(a', a''), (a'', b'), (b', b''), (b'', a')$. Then $M = p([a'', b']) = p([a', , b''])$.*

Proof of Claim 3.6. By way of contradiction suppose that $M \not\supset p([a'', b'])$. Then there exists a class $g \subset (a'', b')$ such that $p(g) \notin I$. Let us denote the "endpoints" of g by c and d so that c is closer to a'' and d is closer to b'. Consider the p-images of arcs $[a'', c]$ and $[d, b']$ denoted by $S = p([a'', c])$ and $T = p([d, b'])$. These are paths in J which enter M at some moment. This must happen at the same point of M because otherwise the subinterval of M is shielded from $[a'', b']$ by S and T which implies that this subinterval must be contained in the boundary of a Fatou domain, a contradiction.

So, S and T enter M at the same point e. Consider the case when $e \neq a, b$. Since the path S enters M at e then there is a p-preimage of e inside $(a'', c]$ while the fact that T enters M at e means that there is a p-preimage of e inside $[d, b)$ (points a'', b' above are excluded because $e \neq a, b$). Finally, by Claim 3.5 there must be a p-preimage of e in (a', b''). All these p-preimages of e are different, so e is a vertex of J, a contradiction.

It remains to consider the case when $e = a$ (the case when $e = b$ can be considered similarly). In this case observe that since T enters M at a then there are p-preimages of a in $[d.b']$, a contradiction with the fact that a', a'' are the "endpoints" of the class $p^{-1}(a)$. Hence this is impossible either, and the claim is proved. ■

Let us go back to the proof of Theorem 3.4. We are trying to prove that a wandering arc $I \subset J$ (which as we know intersects any Fatou domain at no more than 1 point) must contain a dense subset of vertices of J. Indeed, otherwise I contains an interval-like subarc M, and by Claims 3.5 and 3.6 we can find an arc $[c, d]$ on the circle such that $p([c, d]) = M$. However, this implies that M is not wandering, a contradiction.

Now, we also know that all the forward iterates of I avoid critical points. Hence by Proposition 3.4(c) vertices of J contained in I are eventually mapped onto vertices of $T_l\infty$. On the other hand by Theorem 1.4(4) there are finitely many grand orbits of such orbits. Therefore there are two points $x \neq y \in I$ which belong to the same grand orbit.

Let show that this implies a contradiction. Indeed, since x, y belong to the same grand orbit there exist numbers n, m such that $f^n(x) = f^m(y) = z$, and without loss of generality we can assume that $n \leq m$. Now, if $n = m$ then $f^n|I$ is not 1-to-1, a contradiction. On the other hand, if $n < m$ then $z \in f^m(I)$ and also $z \in f^n(I)$ which implies that $f^m(I) \cap f^n(I) \neq \emptyset$, again a contradiction which finally completes the proof of the theorem. ■

Continuing to investigate the question of the "size" of a set which guarantees that the set if non-wandering one inevitably comes to the idea of measuring the "size" of a point x of $J = S^1/\sim$ by counting the number of rays landing at x, that is by $N(x)$. The question then is whether in cases when $N(x)$ is big we can guarantee that x is a non-wandering point (equivalently, that x is a pre-periodic point). To avoid unnecessary details we may well assume that x is not precritical. In this case Theorem 1.5 tells us in the growing trees setting that $N(x)$ must be at most $2k + 1$ - that is, if we know that $N(x) > 2k + 1$ then we are guaranteed that x is non-wandering.

It turns out that this kind of claim can greatly specified. Indeed, Thurston proves in his preprint about laminations that for quadratic laminations there are no wandering non-precritical vertices at all. Later Kiwi [Ki] proved that if the lamination is z^d-invariant then non-precritical wandering vertex x must have $N(x) \leq d$; Levin [Le] proved also that if there is one critical point then wandering vertices do not exist.

It turns out however that even more can be said. Namely, the main result of the paper on which these notes are based is that not only is $N(x)$ bounded for wandering vertices but also that the sum of $N(x_i)$ over points x_i from different grand orbits of wandering vertices is bounded which therefore gives an estimate upon the number of such grand orbits and specifies the situations when wandering vertices may exist much further.

As an example consider the cubic maps. For them the following claim follows from the above quoted results (recall that by a *dendrite* we mean a locally connected continuum containing no subsets homeomorphic to the circle).

Corollary 3.7. *Let $\sim$ be a cubic lamination which has wandering gaps which are not precritical. Then the following facts hold:*

(1) *J is a dendrite;*
(2) *$f|J$ has two wandering critical points c, d with distinct grand orbits such that no vertex ever maps into a critical point, $N(c) = N(d) = 2$ and all forward images of them are endpoints of J;*
(3) *wandering vertices x', x'' are not precritical, have the same grand orbits, and are such that $N(x') = N(x'') = 3$.*

Let us now end this section of the handouts by stating some of the aforementioned general results without proof and then showing how Corollary 3.7 follows from these results. To do so we need some new notation. Namely, let k_S be the number of periodic orbits of the Siegel discs (a Fatou component is called a *Siegel disc* if $\bar{f}$ restricted on its boundary is an irrational rotation). Also, let k_p be the number of all periodic orbits of Fatou components of a lamination $\sim$. Then the following theorem holds.

Theorem 3.8. *Let Γ be a non-empty collection of classes of $\sim$, such that:*

(a) *any $g \in \Gamma$ is non-preperiodic under the map σ (i.e., each $t \in g$ is irrational);*
(b) *the orbits of $g \in \Gamma$ are pairwise disjoint;*
(c) *$|g| \geq 3$ for every $g \in \Gamma$ (i.e., g is a gap);*
(d) *σ^n is injective on g for every $n = 1, 2, \ldots$ and every $g \in \Gamma$.*

Then $\sum_{g \in \Gamma}(|g| - 2) \leq k_\sim - k_S - 1 \leq d - 2 - k_p \leq d - 2$ so that the number of classes in Γ is at most $k_\sim - k_S - 1 \leq d - 2 - k_p \leq d - 2$ and for every $g \in \Gamma$ we have $|g| \leq k_\sim - k_S + 1 \leq d - k_p \leq d$.

Let us apply this theorem to the case of cubic laminations and show how in the cubic case Theorem 3.8 implies rather specific information about the lamination and its critical points described in Corollary 3.7.

Corollary 3.7. *Let $\sim$ be a cubic lamination which has wandering gaps which are not precritical. Then the following facts hold:*

(1) *J is a dendrite;*
(2) *$f|J$ has two wandering critical points c, d with distinct grand orbits such that no vertex ever maps into a critical point, $N(c) = N(d) = 2$ and all forward images of them are endpoints of J;*

(3) *wandering vertices x', x'' are not precritical, have the same grand orbits, and are such that $N(x') = N(x'') = 3$.*

Proof of Corollary 3.7. (1) If there are σ-components then by Theorem 3.1 they must be periodic so that $k_p \geq 1$. However then by Theorem 3.8 we would have that $d-2-k_p \leq 0$ and hence wandering classes cannot exist. So we conclude that there are no σ-components and therefore J is a dendrite.

(2) By Theorem 3.8, $k_\sim = 2$. Therefore, $f|J$ has two wandering critical points c, d with distinct grand orbits. Moreover, all forward images of them are endpoints of J. Observe that the order of J at critical points cannot be less than 2. Now, if the order of J at a critical point is greater than 2 then, since the image of this critical point must be an endpoint of J we see that f has to be at least 3-to-1 at this critical point which implies that $c = d$, a contradiction. Hence $N(c) = N(d) = 2$ which in turn implies that no vertex ever maps into a critical point.

(3) Translating the results of Theorem 3.8 into the language of $f|J$ we see that if there are wandering vertices x', x'' which are not precritical then they have the same grand orbit and also $N(x') = N(x'') = 3$. On the other hand, if x is a precritical vertex then the critical point which it maps onto under the least positive iteration of f is a vertex of J too, a contradiction to (2). Hence wandering vertices of J must be non-precritical. This completes the proof of Corollary 3.7. ∎

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