Laminational models for some spaces of polynomials of any degree

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Abstract

The so-called “pinched disk” model of the Mandelbrot set is due to A. Douady, J. H. Hubbard and W. Thurston. It can be described in the language of geodesic laminations. The combinatorial model is the quotient space of the unit disk under an equivalence relation that, loosely speaking, “pinches” the disk in the plane (whence the name of the model). The significance of the model lies in particular in the fact that this quotient is planar and therefore can be easily visualized. The conjecture that the Mandelbrot set is actually homeomorphic to this model is equivalent to the celebrated MLC conjecture stating that the Mandelbrot set is locally connected.

For parameter spaces of higher degree polynomials no combinatorial model is known. One possible reason may be that the higher degree analog of the MLC conjecture is known to be false. We investigate to which extent a geodesic lamination is determined by the location of its critical sets and when different choices of critical sets lead to essentially the same lamination. This yields models of various parameter spaces of laminations similar to the “pinched disk” model of the Mandelbrot set.

Received by the editor December 22, 2016; revised April 27, 2017 and May 12, 2017.

2010 *Mathematics Subject Classification.* Primary 37F20; Secondary 37F10, 37F50.

Key words and phrases. Complex dynamics; laminations; Mandelbrot set; Julia set.

The first and the third named authors were partially supported by NSF grant DMS–1201450.

The study has been funded by the Russian Academic Excellence Project ’5-100’.

CHAPTER 1

Introduction

The **parameter space** of complex degree d polynomials is by definition the space of affine conjugacy classes of these polynomials. An important subset of the parameter space is the *connectedness locus* $\mathcal{M}_d$ consisting of classes of all degree d polynomials P, whose Julia sets $J(P)$ are connected. General properties of the connectedness locus $\mathcal{M}_d$ of degree d polynomials have been studied for quite some time. For instance, it is known that $\mathcal{M}_d$ is a compact cellular set in the parameter space of complex degree d polynomials (this was proven in [**BrHu88**] in the cubic case and in [**Lav89**] for higher degrees, see also [**Bra86**]; by definition, going back to Morton Brown [**Bro60**], a subset X of a Euclidean space $\mathbb{R}^n$ is *cellular* if there exists a sequence Q_n of topological n-cells such that $Q_{n+1} \subset \mathrm{Int}(Q_n)$ and $X = \cap Q_n$).

For $d = 2$, the connectedness locus is the famous *Mandelbrot set* $\mathcal{M}_2$, which can be identified with the set of complex numbers c such that 0 does not escape to infinity under the iterations of the polynomial $P_c(z) = z^2 + c$. The identification is based on the fact that every quadratic polynomial is affinely conjugate to P_c for some $c \in \mathbb{C}$ as well as a classical theorem of Fatou and Julia. The Mandelbrot set $\mathcal{M}_2$ has a complicated self-similar structure (for instance, homeomorphic copies of the Mandelbrot set are dense in the Mandelbrot set itself). A crucial role in understanding its structure is played by the "pinched disk" model by Adrien Douady, John Hamal Hubbard and William Thurston [**DH82, DH8485, Thu85**]. This model can be described as a *geodesic lamination* (see the index in the back for the definitions of non-standard terms).

In this paper, we will partially generalize these results to the higher degree case. We replace the notion of non-disjoint minors by linked or essentially equal critical quadrilaterals and show that in certain cases two linked or essentially equal laminations must coincide. We apply these results to construct models of some spaces of laminations.

In what follows we assume basic knowledge of complex dynamics (a good reference is John Milnor's book [**Mil00**]). Important developments can be found in Curtis McMullen's book [**McM94b**]. We use standard notation. However, we describe in detail less well known facts concerning, e.g., combinatorial concepts (such as *geodesic laminations* developed by Thurston in [**Thu85**], or *laminational equivalence relations*) that will serve as important tools for us.

1.1. Laminations

Laminations were introduced by Thurston in his paper [**Thu85**] and have been used as a major tool in complex dynamics ever since.

We will write $\mathbb{C}$ for the plane of complex numbers, $\widehat{\mathbb{C}}$ for the Riemann sphere, and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for the open unit disk. A *laminational equivalence relation* is a closed equivalence relation $\sim$ on the unit circle $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$, whose classes are finite sets, such that the convex hulls of distinct classes are disjoint. A laminational equivalence relation is (σ_d-)*invariant* if the map $\sigma_d : \mathbb{S} \to \mathbb{S}$, defined by $\sigma_d(z) = z^d$, takes classes to classes, and the restriction of σ_d to every class $\mathfrak{g}$ can be extended to an orientation preserving covering map τ of the circle of some degree $k \leqslant d$ so that $\mathfrak{g}$ is a full preimage of $\tau(\mathfrak{g})$.

If a polynomial P has a connected filled Julia set K, then, by the Riemann mapping theorem, there exists a conformal map $\varphi : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K$ so that $\varphi \circ \sigma_d = P \circ \varphi$. The image of the radial segment $\{re^{i\theta} \mid r > 1\}$ under φ is called an *external* ray of K with argument θ. If, in addition, $J = \mathrm{Bd}(K)$ is locally connected, then φ extends over $\mathbb{S}$. In this case, there is a laminational equivalence relation $\sim_P$ on $\mathbb{S}$ which identifies pairs of angles if the corresponding external rays land at the same point in J. The quotient $J_{\sim_P} = \mathbb{S}/{\sim_P}$ is homeomorphic to J, and the self-mapping $f_{\sim_P}$ of $J_{\sim_P}$ induced by σ_d is topologically conjugate to $P|_{J_P}$; the map $f_{\sim_P}$ and the set $J_{\sim_P}$ are called a *topological polynomial* and a *topological Julia set*, respectively.

Laminational equivalence relations can play a significant role even for some polynomials whose connected Julia sets are not locally connected. For these polynomials $\sim_P$ still can be defined, although $P|_{J_P}$ and $f_{\sim_P}|_{J_{\sim_P}}$ are no longer conjugate. However, they are semiconjugate by a *monotone map* (a continuous map, whose fibers are continua). A topological polynomial and topological Julia set can be defined for every σ_d-invariant laminational equivalence relation even if it does not correspond (in the above sense) to a complex polynomial.

With every laminational equivalence relation $\sim$, it is useful to associate geometric objects defined below. We identify $\mathbb{S}$ with $\mathbb{R}/\mathbb{Z}$. For a pair of points $a, b \in \mathbb{S}$, we will write $\overline{ab}$ for the *chord* (a straight line segment in $\mathbb{C}$) connecting a and b (in particular, a chord is always contained in the closed unit disk $\overline{\mathbb{D}}$). If G is the convex hull $\mathrm{CH}(G')$ of some closed set $G' \subset \mathbb{S}$, then we write $\sigma_d(G)$ for the set $\mathrm{CH}(\sigma_d(G'))$. The boundary of G will be denoted by $\mathrm{Bd}(G)$. If A is a $\sim$-class, then we call a chord $\overline{ab}$ in $\mathrm{Bd}(\mathrm{CH}(A))$ a *leaf* of $\sim$. All points of $\mathbb{S}$ are also called (*degenerate*) *leaves*. The family $\mathcal{L}_\sim$ of all leaves of $\sim$ is called the (σ_d-)*invariant geodesic lamination generated by the relation* $\sim$.

Let us explain the terminology. The set $\mathcal{L}_\sim$ is called *invariant* for two reasons: for every non-degenerate leaf $\overline{xy} \in \mathcal{L}_\sim$ we have $\sigma_d(\overline{xy}) \in \mathcal{L}_\sim$, and, on the other hand, there exist d disjoint leaves in $\mathcal{L}_\sim$ such that their σ_d-images equal $\overline{xy}$. The set $\mathcal{L}_\sim$ is called *geodesic* because the standard visual interpretation of chords of $\mathcal{L}_\sim$ uses *geodesics* in the unit disk with respect to the Euclidean (or, equivalently, Poincaré) metric (see Figure 1 for an illustration). Denote by $\mathcal{L}^+_\sim$ the union of the unit circle and all the leaves in $\mathcal{L}_\sim$. Then $\mathcal{L}^+_\sim$ is a subcontinuum of the closed unit disk $\overline{\mathbb{D}}$. In general, collections of leaves with properties similar to those of collections $\mathcal{L}_\sim$ are also called *invariant geodesic laminations*. In fact, it is these collections that Thurston introduced and studied in [**Thu85**].

Let $\mathcal{L}$ be an invariant geodesic lamination (for instance, we may have $\mathcal{L} = \mathcal{L}_\sim$ for some invariant lamination $\sim$). The closure in $\mathbb{C}$ of a non-empty component of $\mathbb{D} \setminus \mathcal{L}^+$ is called a *gap* of $\mathcal{L}$. *Edges of a gap* G are defined as leaves of $\mathcal{L}$ on the boundary of G and we call $G \cap \mathbb{S}$ the *basis* of the gap G. A gap is said to be *finite* (*infinite*) if its basis is finite (infinite). Gaps of $\mathcal{L}$ with uncountable basis are called *Fatou gaps*.

The first application of geodesic laminations was in the quadratic case [**Thu85**]. Let us discuss it in more detail.

1.2. "Pinched disk" model of the Mandelbrot set

The "pinched disk" model for $\mathcal{M}_2$ is constructed as follows, cf. [**Dou93, Thu85**]. We will identify $\mathbb{S}$ with $\mathbb{R}/\mathbb{Z}$ by means of the mapping taking an **angle** $\theta \in \mathbb{R}/\mathbb{Z}$ to the point $e^{2\pi i\theta} \in \mathbb{S}$. Under this identification, we have $\sigma_2(\theta) = 2\theta$.

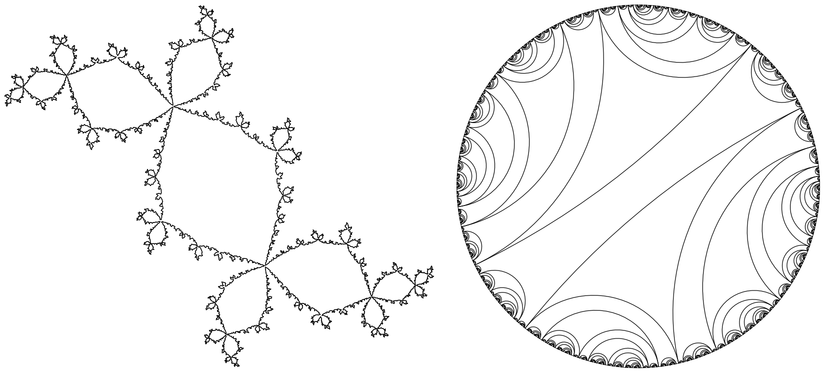

FIGURE 1. Left: Douady rabbit, the Julia set of the polynomial $f(z) = z^2 - 0.12.. + 0.74..i$ Right: its geodesic lamination.

If the Julia set $J(P_c)$ is locally connected, then, as was explained above in Subsection 1.1, Thurston associates to the polynomial P_c (and hence to the parameter value c) a laminational equivalence relation $\sim_{P_c}$ and then the corresponding σ_2-invariant geodesic lamination $\mathcal{L}_{\sim_{P_c}} = \mathcal{L}_c$. The dynamics of $\sigma_2 : \mathbb{S} \to \mathbb{S}$ descends to the quotient space, and the induced dynamics $f_{\sim_{P_c}} : \mathbb{S}/\sim_{P_c} \to \mathbb{S}/\sim_{P_c}$ is topologically conjugate to $P_c|_{J(P_c)} : J(P_c) \to J(P_c)$. A σ_2-invariant geodesic lamination is also called a *quadratic invariant geodesic lamination*. The lamination $\mathcal{L}_c$ is also called the *(quadratic invariant) geodesic lamination* of P_c. In what follows when talking about quadratic invariant geodesic laminations we often omit "invariant" or "geodesic" (indeed, we only deal with **geodesic** laminations, and "quadratic" already assumes "invariant").

Thurston's geodesic laminations model the topological dynamics of quadratic polynomials with locally connected Julia sets. So far, this construction only provides topological models for individual quadratic polynomials, and not even for all of them, since there are polynomials P_c such that $J(P_c)$ is connected but not locally connected; however, we need to model the space of **all** polynomials P_c with connected Julia sets. Metaphorically speaking, there are two parallel worlds: the "analytic" world of complex polynomials and the "combinatorial" world of geodesic laminations. Both worlds often come close to each other: whenever we have a polynomial P_c with locally connected $J(P_c)$, then we have the corresponding invariant geodesic lamination $\mathcal{L}_c$. On the other hand, sometimes the two worlds diverge. Still, a model for $\mathcal{M}_2$ can be built within the combinatorial world.

Since the space $C(\overline{\mathbb{D}})$ of all subcontinua of the closed unit disk with the Hausdorff metric is a continuum, it makes sense to consider the closure $\mathbb{L}_2$ of the family of all quadratic geodesic laminations $\mathcal{L}_c$ in $C(\overline{\mathbb{D}})$, where $J(P_c)$ is locally connected. Limit points of this family (called *quadratic geodesic limit laminations*) do not immediately correspond to polynomials with connected Julia sets. However, one can extend the correspondence between polynomials and geodesic laminations to all polynomials. More precisely, one can associate to each polynomial P all Hausdorff limits of geodesic laminations obtained by approximating P by polynomials with locally connected Julia sets.

The main property of the leaves of an invariant geodesic lamination is that they are not *linked*, that is, they do not cross in $\mathbb{D}$. Thurston gave a simple parameterization of a quotient of $\mathbb{L}_2$. The idea is to take one particular leaf from every quadratic limit geodesic lamination $\mathcal{L}$, namely, the leaf, called the *minor* of $\mathcal{L}$, whose endpoints are the σ_2-images of the endpoints of a longest leaf of $\mathcal{L}$ called a *major* of $\mathcal{L}$ (it is easy to see that a quadratic invariant geodesic lamination can have at most two longest leaves, each of which is the rotation of the other leaf by one half of the full angle around the center of $\mathbb{D}$).

One of the main results of [**Thu85**] is that the minors of all quadratic limit geodesic laminations are **pairwise unlinked** and hence form a geodesic lamination called the *quadratic minor lamination* QML (observe that QML is not invariant). The geodesic lamination QML generates a laminational equivalence relation $\approx_{\mathrm{QML}}$ where two points x, y of the unit circle are declared to be $\approx_{\mathrm{QML}}$-equivalent if there exists a finite chain of minors connecting x and y (the fact that $\sim_{\mathrm{QML}}$ indeed is a laminational equivalence relation follows from [**Thu85**]). This gives a conjectural model for the Mandelbrot set, in the sense that the boundary $\mathrm{Bd}(\mathcal{M}_2)$ of $\mathcal{M}_2$ is conjecturally homeomorphic to $\mathbb{S}/\approx_{\mathrm{QML}}$ (it is known that there exists a monotone map from $\mathrm{Bd}(\mathcal{M}_2)$ to $\mathbb{S}/\approx_{\mathrm{QML}}$ and, hence, $\mathbb{S}/\approx_{\mathrm{QML}}$ is at least a monotone model of $\mathrm{Bd}(\mathcal{M}_2)$).

The leaves of QML can be described without referring to quadratic geodesic limit laminations. To this end, let us denote by $|x-y|$, x, $y \in \mathbb{S} = \mathbb{R}/\mathbb{Z}$, the length of the shortest circle arc with endpoints x and y. Hence the length of a diameter is $\frac{1}{2}$. Denote by $\overline{ab}$ the chord with endpoints $a \in \mathbb{S}$ and $b \in \mathbb{S}$. Consider the chord $\overline{ab}$ assuming that $\lambda = |a-b| \leqslant 1/3$. Let Q be the convex hull of the set $\sigma_2^{-1}(\{a,b\})$ in the plane. **Assume that all four sides of Q are unlinked with all images $\overline{\sigma_2^n(a)\sigma_2^n(b)}$ ($n = 0, 1, 2 \dots$) of $\overline{ab}$** (this holds automatically if $\overline{ab}$ is a minor of a quadratic invariant geodesic lamination).

The set Q is called a *critical quadrilateral* (Q is a quadrilateral that maps onto its image $\overline{ab}$ two-to-one). The set Q has two pairs of sides of equal length opposite to each other. Clearly, two opposite sides are of lengths $\lambda/2 \leqslant 1/6$ and the other two are of length $1/2 - \lambda/2 \geqslant 1/3$. Denote by ℓ_1 and ℓ_2 the two longer sides of Q (so that the circle arcs "behind" ℓ_1 and ℓ_2 are of length $1/2 - \lambda/2 \geqslant 1/3$). Then the strip S, the part of the unit disk $\mathbb{D}$ located between ℓ_1 and ℓ_2, is called the *critical strip* (of ℓ_1 or ℓ_2).

Comparing the lengths of various chords involved in the described picture, we see that the points a and b do not belong to S; indeed otherwise we would have had either that $\lambda < \lambda/2$ or that $\lambda > 1/2 - \lambda/2 > 1/3$, a contradiction. In other words, $\sigma_2(\ell_1) = \sigma_2(\ell_2) = \overline{ab}$ is disjoint from S (it can be contained in the boundary of S if $a = \frac{1}{3}$ and $b = \frac{2}{3}$). Similar considerations involving critical strips play an important role in [**Thu85**] and, in particular, lead to the so-called Central Strip Lemma (see Section 3.3.1). This lemma yields that minors of quadratic invariant geodesic laminations are pairwise unlinked. In the paper [**CHMMO15**] the Central Strip Lemma is studied and extensions of this lemma to the case of degree greater than two are obtained, however the conclusions of these extensions are weaker than the conclusion of the original Central Strip Lemma.

By [**Thu85**], if, for every positive integer n, the chord $\sigma_2^n(\ell_1) = \sigma_2^n(\ell_2)$ is disjoint from the interior of S, then Thurston's **pullback construction** yields a quadratic invariant geodesic lamination with the majors ℓ_1 and ℓ_2. Therefore, the condition that for every positive integer n, the chord $\sigma_2^n(\ell_1) = \sigma_2^n(\ell_2)$ is disjoint from the interior of S, describes all chords ℓ_1 and ℓ_2 that are majors of quadratic geodesic laminations. Clearly, this description does not depend on quadratic invariant geodesic laminations.

Observe that if $a = b$, then Q is a diameter of $\mathbb{S}$. In this case, Q is trivially a major. However, if $a \neq b$, then the conditions from the previous paragraph (that $\sigma_2^n(\overline{ab})$ is disjoint from the interior of S) are non-trivial. An alternative — and more straightforward — way of defining QML is by saying that QML is formed by minors of all quadratic invariant geodesic laminations, that is by chords $\sigma_2(\ell)$ taken for all majors ℓ of all quadratic invariant geodesic laminations.

As was mentioned above, one of the main results of Thurston's from [**Thu85**] is that QML is in fact a geodesic lamination itself (it is not at all obvious that minors described above are pairwise unlinked). Moreover, Thurston shows in [**Thu85**] that leaves of QML can be broken into single ("stand alone") leaves and finite collections of leaves with each such collection being the boundary of a geodesic polygon in $\mathbb{D}$. One can then collapse all such leaves and geodesic polygons to points thus defining the quotient space $\overline{\mathbb{D}}/\approx_{\mathrm{QML}}$, which serves as a combinatorial model for $\mathcal{M}_2$, see [**Dou93, Sch09**]. Denote the corresponding quotient map by $\pi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}/\approx_{\mathrm{QML}}$. Also, denote $\overline{\mathbb{D}}/\approx_{\mathrm{QML}}$ by $\mathcal{M}_2^{comb}$ reflecting the combinatorial nature of the "pinched disk" model of $\mathcal{M}_2$.

The importance of results of [**Thu85**] lies, in particular, in the fact that thanks to the minors being unlinked one can visualize QML and, hence, a quotient of the space $\mathbb{L}_2$ (distinct quadratic invariant geodesic laminations may have the same minor and our parameterization identifies such laminations). This in turn allows for a visualization of $\overline{\mathbb{D}}/\approx_{\mathrm{QML}}$ as the result of "planar pinching" of the unit disk which collapses all the above described geodesic polygons formed by minors. More precisely, by [**Dav86**], there exists a homotopy $\gamma : \mathbb{C} \times [0, 1] \to \mathbb{C}$ such that, for each $t \in [0, 1)$, the map $\gamma_t : z \to \gamma(z, t)$ is an orientation preserving homeomorphism that shrinks every geodesic polygon formed by minors more and more (as t approaches 1) so that, for $t = 1$, we do not have a homeomorphism, rather a "pinching" map $\gamma_1 : z \to \gamma(z, 1)$ representing the quotient map of $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}/\approx_{\mathrm{QML}}$.

Recall that a continuous mapping from one continuum to another continuum is *monotone* if the *fibers* (that is, preimages of points) are connected. It is known [**Sch09**] that there exists a monotone map $\pi : \mathcal{M}_2 \to \mathcal{M}_2^{Comb} = \mathbb{S}/QML$. The set $\mathcal{M}_2$ is locally connected if and only if the fibers of π are points, hence, π gives the desired homeomorphism between $\mathrm{Bd}(\mathcal{M}_2)$ and $\mathbb{S}/\approx_{\mathrm{QML}}$ provided that the MLC conjecture holds. In other words, the conjecture that the boundary of $\mathcal{M}_2$ is homeomorphic to $\mathbb{S}/\approx_{\mathrm{QML}}$ is equivalent to the celebrated **MLC conjecture** claiming that the Mandelbrot set is locally connected.

1.3. Previous work

The structure of the cubic connectedness locus $\mathcal{M}_3$ (or some parts of it) has been studied by many authors. There are several approaches. In some papers, higher degree connectedness loci are considered too. In the rest of this subsection we briefly describe some relevant results. Let us emphasize that our short overview is far from being complete.

Branner and Hubbard [**BrHu88**] initiated the study of $\mathcal{M}_3$, and investigated the complement of $\mathcal{M}_3$ in the full parameter space of cubic polynomials. The complement is foliated by so-called *stretching rays* that are in a sense analogous to external rays of the Mandelbrot set. The combinatorics of $\mathcal{M}_3$ is closely related to landing patterns of stretching rays. However, we do not explore this connection here. A significant complication is caused by the fact that there are non-landing stretching rays. Landing properties of stretching rays in the parameter space of real polynomials have been studied by Komori and Nakane [**KN04**]. Of special interest is a certain subset of the complement of $\mathcal{M}_3$ in the parameter space called the **shift locus** (see, for example, [**BrHu88, BrHu92, deM12, DP11**]).

Another approach to understanding parameter spaces of polynomials is based on a new notion, due to Thurston, of core entropy (entropy on the Hubbard tree of a polynomial) studied, for example, in [**Thu14, Tio15, Tio14, DS14**].

Lavaurs [**Lav89**] proved that $\mathcal{M}_3$ is not locally connected. Epstein and Yampolsky [**EY99**] showed that the bifurcation locus in the space of real cubic polynomials is not locally connected either. This makes the problem of defining a combinatorial model of

$\mathcal{M}_3$ that would admit visual interpretation very delicate. Buff and Henriksen [**BH01**] presented copies of quadratic Julia sets, including not locally connected Julia sets, in slices of $\mathcal{M}_3$. By McMullen [**McM07**], slices of $\mathcal{M}_3$ contain lots of copies of $\mathcal{M}_2$. In addition, Gauthier [**Gau14**] has shown that $\mathcal{M}_3$ contains copies of $\mathcal{M}_2 \times \mathcal{M}_2$. In fact, the last two papers contain more general results than what we mention here; we now confine ourselves to the cubic case. Various spaces of cubic polynomials are studied in [**Zak99, BKM10**].

In his thesis, D. Faught [**Fau92**] considered the slice $\mathcal{A}$ of $\mathcal{M}_3$ consisting of polynomials with a fixed critical point and showed that $\mathcal{A}$ contains countably many homeomorphic copies of $\mathcal{M}_2$ and is locally connected everywhere else. P. Roesch [**Roe06**] filled some gaps in Faught's arguments and generalized Faught's results to higher degrees. Milnor [**Mil09**] gave a classification of hyperbolic components in $\mathcal{M}_d$; however, this description does not involve combinatorial tags. Schleicher [**Sch04**] constructed a geodesic lamination modeling the space of *unicritical* cubic polynomials, that is, polynomials with a unique multiple critical point. We have also heard of an unpublished old work of D. Ahmadi and M. Rees, in which cubic geodesic laminations were studied, however we have not seen it. Finally, a paper by J. Kiwi [**Kiw05**] studies the parameter space of all central monic polynomials of arbitrary degree, focusing upon the intersection of the connectedness locus and the closure of the *shift locus* (i.e. the set of all polynomials so that *all* critical points escape). However, [**Kiw05**] does not deal with the combinatorial structure of the connectedness locus.

1.4. Overview of the method

We now sketch the main tools developed in the present paper. The need for them is justified by the fact that Thurston's tools used in the construction of QML do not generalize to the cubic case. His tools are based on the Central Strip Lemma stated in Section 3.3.1, and include the No Wandering Triangles Theorem (also stated in Section 3.3.1). A straightforward extension of the Central Strip Lemma as well as that of the No Wandering Triangles Theorem to the cubic case fail (see a recent paper [**CHMMO15**] with possible extensions of the Central Strip Lemma). As a consequence, cubic invariant geodesic laminations may have wandering triangles (see [**BO08**]). Thus, one needs a different set of combinatorial tools. Such tools are developed in the present paper and are based upon a principle which we call **smart criticality**. Smart criticality works for geodesic laminations of any degree.

Recall, that given a geodesic lamination $\mathcal{L}$ in $\mathbb{D}$, one defines *gaps* of $\mathcal{L}$ as closures of components of $\overline{\mathbb{D}} \setminus \mathcal{L}^+$ where $\mathcal{L}^+ \subset \overline{\mathbb{D}}$ is the union of all leaves of $\mathcal{L}$. The statement about quadratic geodesic laminations we are trying to generalize is the following: if the minors of two quadratic geodesic laminations intersect in $\mathbb{D}$, then they coincide. However, although minors can also be defined for higher degree laminations, they are not the right objects to consider because they do not define geodesic laminations in a unique way. The sets that essentially determine a given invariant geodesic lamination are in fact its critical sets rather than their images. Thus, for the purpose of characterizing invariant geodesic laminations we propose different objects.

For a quadratic invariant geodesic lamination $\mathcal{L}$, instead of its non-degenerate minor $\overline{m}$, we can consider the quadrilateral, whose vertices are the four σ_2-preimages of the endpoints of $\overline{m}$. Such a quadrilateral Q is called a *critical quadrilateral*. Note that Q is not necessarily a gap of $\mathcal{L}$. Thus, Q lies in some critical gap of $\mathcal{L}$ or, if $\overline{m}$ is a point, coincides with the critical leaf of $\mathcal{L}$ which we see as a generalized critical quadrilateral. Similarly, for a degree d invariant geodesic lamination $\mathcal{L}$, we can define critical quadrilaterals as (possibly degenerate) quadrilaterals lying in gaps or leaves of $\mathcal{L}$ (opposite vertices of these quadrilaterals must have the same σ_d-images). These critical quadrilaterals will play the role of minors and will be used to tag higher degree geodesic laminations.

The method of smart criticality helps to verify that, under suitable assumptions, two linked leaves ℓ_1, ℓ_2 (i.e., leaves such that $\ell_1 \cap \ell_2 \cap \mathbb{D} \neq \varnothing$) of **different** geodesic laminations have linked images $\sigma_d^n(\ell_1)$, $\sigma_d^n(\ell_2)$, for **all** n. One possible reason, for which $\sigma_d(\ell_1)$, $\sigma_d(\ell_2)$ may be linked if ℓ_1, ℓ_2 are linked, is the following: ℓ_1 and ℓ_2 are contained in a part of the unit disk bounded by several circle arcs and such that these circle arcs map forward under σ_d so that the circular order among their points is (non-strictly) preserved.

A typical reason for that phenomenon is that ℓ_1 and ℓ_2 are disjoint from a *full* collection of critical chords (here a *σ_d-critical chord* is a chord of $\mathbb{D}$, whose endpoints map to the same point under σ_d, and a *full collection of critical chords* is a collection of $d-1$ critical chords without loops). Implementing this idea, we prove that $\sigma_d^n(\ell_1)$, $\sigma_d^n(\ell_2)$ are linked for all n by choosing, for every n, a different and thus depending on n full collection of critical chords — this is the meaning of "smart" as in "smart criticality" above (alternatively, one could call this "adjustable criticality").

Smart criticality can be implemented in the following situation. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two invariant geodesic laminations. Suppose that we can choose *full* collections of critical quadrilaterals in $\mathcal{L}_1$ and $\mathcal{L}_2$ (i.e., such collections that on the boundaries of components of their complement the map σ_d is one-to-one except perhaps for boundary critical chords); then we say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are *quadratically critical*. Critical quadrilaterals of a quadratically critical σ_d-invariant geodesic lamination $\mathcal{L}$ can be ordered; if we fix that order we call the corresponding $(d-1)$-tuple of critical quadrilaterals a *quadratically critical portrait* of $\mathcal{L}$, and $\mathcal{L}$ is said to be *marked*.

Suppose that two quadratically critical portraits $\mathrm{QCP}_1, \mathrm{QCP}_2$ are such that equally numbered critical quadrilaterals in them either have alternating vertices, or share a diagonal; then we say that $\mathrm{QCP}_1, \mathrm{QCP}_2$ are *linked* (if at least one pair of corresponding critical quadrilaterals with alternating vertices exists) or *essentially equal* (if all pairs of corresponding quadrilaterals share a common diagonal). Two marked invariant quadratically critical geodesic laminations are said to be *linked (essentially equal)* if their quadratically critical portraits are linked (essentially equal).

In fact, being linked or essentially equal is slightly more general than the property just stated; the precise statements can be found in Definition 3.10 and in Definition 3.75. The main result of the paper is that in a lot of cases linked or essentially equal invariant geodesic laminations must coincide, or at least they must share a significant common portion. This fact can be viewed as a version of rigidity of critical data of invariant geodesic laminations. It serves as a basis for the applications discussed in Subsection 1.5.

To be more specific, we need to introduce a few notions. Suppose that $\mathcal{L}$ is a σ_d-invariant geodesic lamination. Then there are two types of leaves of $\mathcal{L}$. First, there are leaves ℓ of $\mathcal{L}$ such that in any neighborhood of ℓ there are uncountably many leaves of $\mathcal{L}$ (that is, if $\ell = \overline{ab}$, then, for every $\varepsilon > 0$, there are uncountably many leaves $\overline{xy} \in \mathcal{L}$ such that $\mathrm{d}(x, a) < \varepsilon$ and $\mathrm{d}(y, b) < \varepsilon$ where $\mathrm{d}(\cdot, \cdot)$ is a distance between points on the unit circle $\mathbb{S}$). The union of all such leaves (perhaps only consisting of the unit circle, but normally much more significant), is itself an invariant geodesic lamination denoted by $\mathcal{L}^p$. Since every leaf of $\mathcal{L}^p$ is a limit of other leaves of $\mathcal{L}^p$ we call $\mathcal{L}^p$ the *perfect part* of $\mathcal{L}$.

Another important part of $\mathcal{L}$ is related to so-called periodic Siegel gaps. Namely, an n-periodic Fatou gap U of an invariant geodesic lamination $\mathcal{L}$ is said to be a periodic *Siegel* gap if $\sigma_d^n : \mathrm{Bd}(U) \to \mathrm{Bd}(U)$ is a degree one map monotonically semiconjugate to an irrational rotation of the unit circle. It is easy to see that edges of periodic Siegel gaps are isolated in $\mathcal{L}$. The closure of the union of the grand orbits of all periodic Siegel gaps of an invariant geodesic lamination $\mathcal{L}$ is denoted by $\mathcal{L}^{Sie}$ and is called the *Siegel part* of $\mathcal{L}$. It is not hard to see that the union of the perfect part and the Siegel part of an invariant geodesic lamination is itself an invariant geodesic lamination.

We can also consider pullbacks of periodic Fatou gaps U. If there is an eventual non-periodic pullback W of U that maps forward by σ_d in a k-to-1 fashion with $k > 1$, then U is said to be of *capture* type. The terminology, due to Milnor [**Mil93, Mil09**], comes from the fact that in the case of complex polynomials the periodic Fatou domain corresponding to U **captures** a critical point that belongs to the appropriate non-periodic pullback Fatou domain.

Our two main rigidity theorems show that the fact that two invariant geodesic laminations are linked or essentially equal implies that the laminations themselves are "almost" equal. Thus, we obtain a tool allowing us to conclude that certain distinct geodesic laminations cannot be linked/essentially equal. As the linkage/essential equality of geodesic laminations is related to the mutual location of their critical sets, out of this we choose appropriate tags of the critical sets and draw conclusions about those tags being pairwise disjoint. This in the end yields parameterization of the corresponding space of geodesic laminations similar to Thurston's QML.

FIRST RIGIDITY THEOREM. *If two marked invariant quadratically critical geodesic laminations are linked or essentially equal, then the unions of their perfect parts and their Siegel parts are the same.*

To state the Second Rigidity Theorem we need the following definition. Let $\sim$ be an invariant laminational equivalence relation and $\mathcal{L}_\sim$ be the geodesic lamination generated by $\sim$. Let U be an n-periodic Fatou gap of $\mathcal{L}_\sim$ such that $\sigma_d^n : \mathrm{Bd}(U) \to \mathrm{Bd}(U)$ has degree two. Then we call U a *quadratic Fatou gap*. If U is quadratic, then there is a unique edge M (possibly degenerate) of $\mathrm{Bd}(U)$ of period n; we call M a *refixed* edge of U. Let us also denote by M^* the unique edge of U distinct from M but with the same image as M.

The convex hull of $M \cup M^*$ is said to be a *legal critical quadrilateral*. There may exist a finite gap G sharing the leaf M with U and, accordingly, a finite gap G^* sharing the leaf M^* with U such that $\sigma_d(G) = \sigma_d(G^*)$. Then *in some cases* one can erase M and its entire grand orbit from $\mathcal{L}_\sim$ and, possibly, replace it by a different leaf contained in G and its entire grand so that the new geodesic lamination generates the same laminational equivalence relation. In these cases one can insert in $U \cup G \cup G^*$ a critical quadrilateral Q with edges in G and G^* so that leaves from the forward orbit of Q do not cross each other. Thurston's pullback construction implies that we can pull Q back inside the grand orbit of U and add the thus constructed grand orbit of Q to $\mathcal{L}_\sim$. We will call such quadrilaterals *legal* too. Also, if a critical set G of $\mathcal{L}_\sim$ is finite, then any critical quadrilateral inserted in G and such that sets from its forward orbit do not cross is called *legal*.

Finally, suppose that an invariant geodesic lamination $\mathcal{L}_\sim$ is such that all critical sets of $\mathcal{L}^p_{\sim_i} \cup \mathcal{L}^{Sie}_{\sim_i}$ are either finite sets or periodic quadratic Fatou gaps. Then we say that $\sim$ (and $\mathcal{L}_\sim$) are *quadratically almost perfect-Siegel non-capture*. A full ordered collection of legal critical quadrilaterals inserted in critical sets of a quadratically almost perfect-Siegel non-capture geodesic laminations is said to be a *legal quadratically critical portrait* of $\sim$ if chords from the forward orbits of these quadrilaterals are not linked (these forward orbits are "dynamically consistent"). If such a portrait is chosen for $\mathcal{L}$, then $\mathcal{L}$ is said to be *marked*. Two marked quadratically almost perfect-Siegel non-capture laminational equivalence relations are said to be *linked* (*essentially equal*) if their legal quadratically critical portraits are *linked* (*essentially equal*).

SECOND RIGIDITY THEOREM. *If two marked invariant geodesic quadratically almost perfect-Siegel non-capture laminations $\mathcal{L}_{\sim_1}$ and $\mathcal{L}_{\sim_2}$ are linked or essentially equal, then they coincide (that is, $\sim_1 = \sim_2$ and $\mathcal{L}_{\sim_1} = \mathcal{L}_{\sim_2}$).*

1.5. Main applications

Questions concerning the existence of combinatorial models of the connectedness loci $\mathcal{M}_d$ of degree d polynomials arose soon after Thurston's construction of a combinatorial model for $\mathcal{M}_2$ (see, for example, [**Thu85, McM94**]). The main aim of the present paper is to generalize the "pinched disk" model onto some classes of invariant geodesic laminations as well as polynomials. Inevitably, the increase in the degree makes the problem more difficult. Thurston's work was based on his Central Strip Lemma [**Thu85**, Lemma II.5.1], which implied his No Wandering Triangle Theorem and the transitivity of the first return map of finite periodic polygons. However, the Central Strip Lemma fails in degrees higher than two. Moreover, it is known that in the cubic case wandering triangles exist [**BO04, BO08, BCO12, BCO13**] and that the first return map on a finite periodic polygon is not necessarily transitive in higher degree cases [**Kiw02**]. This shows that a new approach is necessary.

Furthermore, the connectedness locus $\mathcal{M}_3$ in the parameter space of complex cubic polynomials is a four-dimensional set, which is known to be non-locally connected [**Lav89**]. Thus, it is hopeless to look for a precise topological model of the boundary of $\mathcal{M}_3$ as a quotient of a locally connected space (any quotient space of a locally connected space is locally connected!). Yet another indication of the fact that a new approach is needed is the fact that in the cubic case the so-called combinatorial rigidity fails as shown by Henriksen in [**Hen03**].

In the present paper we concentrate on the part of Thurston's work (see [**Thu85**]) where it is shown that the family of quadratic invariant geodesic lamination can be tagged by their minors, which, by [**Thu85**], are pairwise unlinked. This yields the "pinched disk" model QML for the Mandelbrot set. We prove similar results, which allow us to describe various spaces of invariant geodesic laminations.

The first application can be found in Subsection 4.1. Consider the space of all polynomials with connected Julia sets such that all their periodic points are repelling. Such polynomials exhibit rich dynamics and have been actively studied before. In particular, there is a nice association, due to Jan Kiwi [**Kiw04**], between these polynomials and a certain class of invariant geodesic laminations of the same degree. These invariant geodesic laminations $\mathcal{L} = \mathcal{L}_\sim$ are generated by invariant laminational equivalence relations $\sim$ that have the following property: the associated topological Julia set $J_\sim$ is a *dendrite* (that is, a locally connected one-dimensional continuum that contains no Jordan curves); equivalently, all gaps of $\mathcal{L}_\sim$ must be finite. Then the corresponding invariant geodesic lamination and the corresponding invariant laminational equivalence are called *dendritic*.

Kiwi proves in [**Kiw04**] that in this case for a given polynomial P of degree d there exists an invariant laminational equivalence relation $\sim_P$ such that the filled Julia set $J(P)$ of the polynomial P can be **monotonically** (recall that this means that point-preimages are connected) mapped onto $J_{\sim_P}$. Moreover, the monotone map $\psi_P : J(P) \to J_{\sim_P}$ in question semiconjugates $P|_{J(P)}$ and the associated topological polynomial $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ induced by σ_d on the topological Julia set $J_{\sim_P} = \mathbb{S}/\sim_P$. Denote by φ_P the quotient map $\varphi_P : \mathbb{S} \to \mathbb{S}/\sim_P$.

Take a point $z \in J(P)$, project it by the map ψ_P to a point $\psi_P(z)$ of the topological Julia set $J_{\sim_P}$, lift the point $\psi_P(z)$ to the corresponding φ_P-fiber $\varphi_P^{-1}(\psi_P(z))$, and then to its convex hull $\mathrm{CH}(\varphi_P^{-1}(\psi_P(z)))$ denoted by G_z. Clearly, G_z is a gap or (possibly degenerate) leaf of $\mathcal{L}_{\sim_P}$; loosely, G_z is the laminational counterpart of the point z. This geometric association is important for a combinatorial interpretation of the dynamics of P. In particular, each critical point c of P is associated with the critical gap or leaf G_c of $\mathcal{L}_{\sim_P}$ (all dendritic invariant geodesic laminations have finite critical sets).

We call polynomials with connected Julia sets, all of whose cycles are repelling, *dendritic*. Let us emphasize that we do not mean that the Julia sets of dendritic polynomials are dendrites themselves; rather our terminology is justified because by [**Kiw04**] the Julia sets of dendritic polynomials can be mapped to a non-trivial dendrite under a monotone map. In particular, the Julia set of a dendritic polynomial may be non-locally connected and, hence, not a dendrite.

Dealing with polynomials, we specify the order of their critical points and talk about (*critically*) *marked polynomials*. In that we follow Milnor [**Mil12**]. More precisely, a (*critically*) *marked polynomial* is a polynomial P with an ordered collection $C(P)$ of its critical points, so that every multiple critical point is repeated several times according to its multiplicity (thus, $C(P)$ is a $(d-1)$-tuple, where d is the degree of P). Marked polynomials do not have to be dendritic (in fact, the notion is used by Milnor and Poirier for hyperbolic polynomials, that is, in the situation diametrically opposite to that of dendritic polynomials). However, we consider only dendritic marked polynomials. Thus, speaking of a marked polynomial, we mean a pair $(P, C(P))$.

In what follows, $C(P)$ is called an *ordered critical collection* of P; normally we use the notation $C(P) = (c_1, \dots, c_{d-1})$, where a multiple critical point c is repeated in $C(P)$ according to its multiplicity. Since we want to reflect convergence of polynomials, we allow for the same critical point of multiplicity k to be repeated $k-1$ times not in a row. For example, let a polynomial P of degree 5 have two critical points c and d of multiplicity 3 each. Then we can mark P with any ordered collection of points c and d as long as each point is repeated twice, such as (c, d, c, d), or (d, d, c, c), or (c, c, d, d) etc.

Endow the family of all critically marked polynomials with the natural topology that takes into account the order among critical points so that marked polynomials $(P_i, C(P_i))$ converge to a marked polynomial $(P, C(P))$ if and only if $P_i \to P$ and $C(P_i) \to C(P)$. Our aim is to provide local laminational models for some dendritic polynomials of arbitrary degree d. In other words, we suggest a class of marked dendritic polynomials $(P, C(P))$ with the following property. There exists a neighborhood $\mathcal{U}$ of $(P, C(P))$ and a continuous map from $\mathcal{U}$ to a special laminational parameter space. The definition of this map is based upon information on laminational equivalence relations $\sim_Q$ defined by dendritic polynomials $Q \in \mathcal{U}$. This approach is close to Thurston's original approach which led to the proof of the existence of a monotone map from the entire quadratic Mandelbrot set onto its laminational counterpart, the "pinched disk" model $\mathcal{M}_2^{comb}$. We implement this approach on open subsets of the space of marked polynomials of arbitrary degree d.

As polynomials P, we choose dendritic polynomials with the following additional property: the invariant dendritic geodesic lamination $\mathcal{L}_{\sim_P}$ has $d-1$ pairwise distinct critical sets. We will call such polynomials *simple dendritic*. If $(P, C(P))$ is a critically marked simple dendritic polynomial, then all critical points in $C(P)$ must be distinct. However, the mere fact that P is dendritic and has $d-1$ distinct critical points is not sufficient to conclude that $(P, C(P))$ is a simple dendritic polynomial. This is because distinct critical points of P may belong to the same fiber of ψ_P resulting in some critical sets of $\mathcal{L}_{\sim_P}$ being of multiplicity greater than two. One can show that the space of simple dendritic critically marked polynomials is open in the space of all critically marked dendritic polynomials.

Denote the space of all degree d simple critically marked dendritic polynomials by $\mathcal{CMD}_d^{sim}$. Consider the *ordered postcritical collection*$(P(c_1), \dots, P(c_{d-1}))$. The sets $G_{c_i}, 1 \leqslant i \leqslant d-1$ are critical sets of the invariant geodesic lamination $\mathcal{L}_{\sim_P}$ and the sets $G_{P(c_i)}, 1 \leqslant i \leqslant d-1$ are their σ_d-images. Define the following two maps from $\mathcal{CMD}_d^{sim}$ to the space of compact subsets of $\overline{\mathbb{D}}^{d-1}$. First, it is the map $\widehat{\Psi}_d$ defined as follows:

$$\widehat{\Psi}_d(P) = G_{c_1} \times G_{c_2} \times \dots \times G_{c_{d-1}}.$$

Second, it is the map Ψ_d, defined as follows:

$$\Psi_d(P) = G_{P(c_1)} \times G_{P(c_2)} \times \dots \times G_{P(c_{d-1})}.$$

These maps associate to any marked simple dendritic polynomial a compact subset of $\overline{\mathbb{D}}^{d-1}$ (moreover, this subset itself is the product of convex hulls of certain $\sim_P$-classes). Notice that each set G_{c_j} maps onto its image two-to-one. We call the set $\Psi_d(P)$ the *postcritical tag* of the critically marked polynomial $(P, C(P))$.

THEOREM ON LOCAL CHARTS FOR DENDRITIC POLYNOMIALS. *Suppose that $(P, C(P))$ is a marked simple dendritic polynomial of degree d. Then there is a neighborhood $\mathcal{U}$ of $(P, C(P))$ in $\mathcal{CMD}_d^{sim}$ such that for any two polynomials $(Q, C(Q)), (R, C(R)) \in \mathcal{U}$ with $\widehat{\Psi}(Q) \neq \widehat{\Psi}(R)$ we have that their Ψ_d-images $\Psi_d(Q)$ and $\Psi_d(R)$ are disjoint.*

The Theorem on Local Charts for Dendritic Polynomials implies the following corollary, in which the notation of the Theorem is used.

LOCAL PINCHED POLYDISK MODEL FOR DENDRITIC POLYNOMIALS. *Consider the union of all postcritical tags of polynomials in $\mathcal{U}$ in $\overline{\mathbb{D}}^{d-1}$ and its quotient space obtained by collapsing these tags to points. The constructed space is separable and metric. Moreover, the map Ψ_d viewed as a map from $\mathcal{U}$ to this space is continuous.*

The second application extends the results of [**BOPT15a**] and can be found in Subsection 4.2. In [**BOPT15a**], we studied the space $\mathbb{LP}_3^{np}(\overline{ab})$ of all cubic invariant geodesic laminations generated by cubic invariant laminational equivalence relations $\sim$ such that for some fixed critical leaf $D = \overline{ab}$ with non-periodic endpoints we have $a \sim b$, and there are no gaps of capture type. The main result of [**BOPT15a**] is that this family of cubic invariant geodesic laminations is modeled by a lamination. This result resembles the description of the combinatorial Mandelbrot set.

More specifically, to each cubic invariant geodesic lamination $\mathcal{L}$ from $\mathbb{LP}_3^{np}(D)$ we associate its critical set C whose criticality "manifests" itself inside the circle arc (b, a) of length $\frac{2}{3}$. We show that either C is finite, or C is a periodic Fatou gap of degree two and period k. Now, if $\mathcal{L} \in \mathbb{LP}_3^{np}(D)$ then a pair of sets $\mathrm{QCP} = (Q, D)$ is called a *quadratically critical portrait privileged for* $\mathcal{L}$ if and only if $Q \subset C$ is a critical leaf or a collapsing quadrilateral (by a *collapsing* quadrilateral we mean a quadrilateral whose boundary maps two-to-one to a chord). In the case when C is a critical periodic Fatou gap of period k, we require that Q be a collapsing quadrilateral obtained as the convex hull of a (possibly degenerate) edge ℓ of C of period k and another edge $\hat{\ell}$ of C such that $\sigma_3(\ell) = \sigma_3(\hat{\ell})$.

In [**BOPT15**], we show that for each $\mathcal{L} \in \mathbb{LP}_3^{np}(D)$ there are only finitely many privileged quadratically critical portraits. Let $\mathcal{S}_D$ denote the collection of all privileged for $\mathcal{L}$ quadratically critical portraits (Q, D). To each such (Q, D) we associate its *minor* (a chord or a point) $\sigma_3(Q) \subset \overline{\mathbb{D}}$. For each such chord we identify its endpoints, extend this identification by transitivity and define the corresponding equivalence relation $\simeq_D$ on $\mathbb{S}$. The main result of [**BOPT15**] is that $\simeq_D$ is itself a laminational equivalence (non-invariant!) whose quotient is a parameterization of $\mathbb{LP}_3^{np}(D)$.

In Subsection 4.2 of the present paper, we generalize the results of the paper [**BOPT15**] onto the degree d case. In order to do so we introduce the appropriate space analogous to $\mathbb{LP}_3^{np}(D)$. Namely, fix a collection $\mathcal{Y}$ of $d-2$ pairwise disjoint critical chords of σ_d with non-periodic endpoints. Let $\mathbb{L}(\mathcal{Y})$ be the space of all invariant geodesic laminations generated by laminational equivalence relations that are *compatible* with this collection in the sense that $\mathcal{L}_\sim$ belongs to $\mathbb{L}(\mathcal{Y})$ if and only if the endpoints of each critical leaf from $\mathcal{Y}$ are $\sim$-equivalent. Moreover, similar to the case of $\mathbb{LP}_3^{np}(D)$ we also require that $\mathcal{L}_\sim$ has no gaps of capture type. We prove in Corollary 4.24 that $\mathbb{L}(\mathcal{Y})$ is non-empty.

Let $\mathcal{Y}^+$ be the union of all critical leaves from $\mathcal{Y}$. There exists a unique component $A(\mathcal{Y}) = A$ of $\overline{\mathbb{D}} \setminus \mathcal{Y}^+$ on whose boundary the map σ_d is two-to-one except for its critical boundary edges (σ_d is one-to-one in the same sense on all other components of $\overline{\mathbb{D}} \setminus \mathcal{Y}^+$). Moreover, in Lemma 4.26 we show that for each $\mathcal{L} \in \mathbb{L}(\mathcal{Y})$, there exists a unique critical set C that contains a critical chord $\overline{c} \subset \overline{A}$. We then use the set C to define the *minor set* of $\mathcal{L}$. Namely, it is shown that C is either finite, or a periodic (of period, say, n) Fatou gap such that $\sigma_d^n : \mathrm{Bd}(U) \to \mathrm{Bd}(U)$ is two-to-one. In the former case, set $m(\mathcal{L}) = \sigma_d(C)$. In the latter case, choose a maximal finite gap whose vertices are fixed under σ_d^n and share an edge with U, and let $m(\mathcal{L})$ be its σ_d-image. The main result of Subsection 4.2 is the following theorem.

THEOREM ON CRITICALLY DEFINED SLICES OF LAMINATIONS. *There exists a non-invariant laminational equivalence relation $\sim_\mathcal{Y}$ such that minor sets of invariant geodesic laminations from $\mathbb{L}(\mathcal{Y})$ are convex hulls of classes of equivalence of $\sim_\mathcal{Y}$; this gives rise to the quotient space $\mathbb{S}/\sim_\mathcal{Y} = \mathcal{M}_\mathcal{Y}$ that parameterizes $\mathbb{L}(\mathcal{Y})$.*

Yet another application of the results of this paper will be contained in a forthcoming paper by the authors where we construct a higher dimensional lamination of a subset of $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$ whose quotient space is a combinatorial model for the space of all marked cubic polynomials with connected Julia set that only have repelling cycles [**BOPT16, BOPT17**].

1.6. Organization of the paper

In Section 2.1, we introduce invariant geodesic laminations. In Section 2.2, we discuss laminational equivalence relations in detail. General properties of invariant geodesic laminations are considered in Section 2.3. In Section 3.1, we introduce and study our major tool, quadratically critical portraits, for invariant geodesic laminations. The most useful results, based upon quadratically critical portraits, can be obtained for some special types of invariant geodesic laminations investigated in Section 3.2. In Section 3.3, we introduce another major tool, so-called *accordions*, which are basically sets of linked leaves of distinct invariant geodesic laminations. We first study accordions by postulating certain properties of them related to the orientation of leaves comprising these accordions. In Section 3.4, we develop the principle of smart criticality and show that owing to this principle we can apply the results of Section 3.3 to accordions of two linked or essentially equal invariant geodesic laminations. Arguments based upon smart criticality yield that accordions of linked or essentially equal geodesic laminations behave much like gaps of a single invariant geodesic lamination. This is established in Section 3.5, where the method of smart criticality is developed. Finally, in Section 4, we will prove the Main Theorems.

1.7. Acknowledgments

The authors are indebted to the referee for thoroughly reading this text and making a number of very useful and thoughtful suggestions leading to significant improvements in the paper.

CHAPTER 2

Invariant laminations: general properties

2.1. Invariant geodesic laminations

In this section, we give basic definitions, list some known results concerning (invariant) geodesic laminations, and establish some less known facts about them.

2.1.1. Basic definitions. We begin with simple geometry.

Definition 2.1 (Chords)**.** A *chord* is a closed segment connecting two points of the unit circle, not necessarily distinct. If the two points coincide, then the chord connecting them is said to be *degenerate*.

Let us now consider collections of chords.

Definition 2.2 (Solids of chord collections)**.** Let $\mathcal{R}$ be a collection of chords. Then we set $\bigcup \mathcal{R} = \mathcal{R}^+$ and call $\mathcal{R}^+$ the *solid of* $\mathcal{R}$.

We are mostly interested in collections of chords with specific properties.

Definition 2.3 (Geodesic laminations)**.** A *geodesic lamination* is a collection $\mathcal{L}$ of (perhaps degenerate) chords called *leaves* such that the leaves are pairwise disjoint in $\mathbb{D}$ (that is, in the open unit disk), $\mathcal{L}^+$ is closed, and all points of $\mathbb{S}$ are elements of $\mathcal{L}$. *Gaps* of $\mathcal{L}$ are defined as the closures of the components of $\mathbb{D} \setminus \mathcal{L}^+$. The solid $\mathcal{L}^+$ is called the *solid of the geodesic lamination* $\mathcal{L}$.

The notion of geodesic lamination is **static** in the sense that no map is even considered with resect to $\mathcal{L}$. In order to relate it to the dynamics of the map σ_d, it is useful to extend σ_d as described below.

Definition 2.4 (Extensions of σ_d)**.** Extend σ_d over leaves of $\mathcal{L}$ so that the restriction of the extended σ_d to every leaf is an affine map. This extension is continuous on $\mathcal{L}^+$ and well-defined (provided that $\mathcal{L}$ is given). By Thurston [**Thu85**], define a canonical *barycentric* extension of the map σ_d to the entire closed disk $\overline{\mathbb{D}}$. Namely, after σ_d is extended affinely over all leaves of an invariant geodesic lamination $\mathcal{L}$, extend it piecewise affinely over the interiors of all gaps of $\mathcal{L}$, using the barycentric subdivision. We will use the same notation for both σ_d and all its extensions.

Observe that while the extensions of σ_d can be defined for any geodesic lamination, they are really sensible only in the case of σ_d-invariant geodesic laminations considered below; thus, when talking about σ_d on $\overline{\mathbb{D}}$, we always have some invariant geodesic lamination in mind and we extend σ_d using Thurston's barycentric extension (see [**Thu85**] for details).

2.1.2. Sibling invariant geodesic laminations. Let us introduce the notion of a (sibling) *σ_d-invariant geodesic lamination*, which is a slight modification of the notion of an invariant geodesic lamination introduced by Thurston [**Thu85**]; in the case when d is fixed, we will often write "invariant" instead of "σ_d-invariant" without causing ambiguity.

Definition 2.5 (Invariant geodesic laminations [**BMOV13**])**.** A geodesic lamination $\mathcal{L}$ is (sibling) *(σ_d)-invariant* provided that:

(1) for each $\ell \in \mathcal{L}$, we have $\sigma_d(\ell) \in \mathcal{L}$,
(2) for each $\ell \in \mathcal{L}$ there exists $\ell^* \in \mathcal{L}$ so that $\sigma_d(\ell^*) = \ell$.
(3) for each $\ell \in \mathcal{L}$ such that $\sigma_d(\ell)$ is a non-degenerate leaf, there exist d **pairwise disjoint** leaves $\ell_1, \dots, \ell_d$ in $\mathcal{L}$ such that $\ell_1 = \ell$ and $\sigma_d(\ell_i) = \sigma_d(\ell)$ for all $i = 2, \dots, d$.

Observe that since leaves are chords, and chords are closed segments, pairwise disjoint leaves in part (3) of the above definition cannot intersect even on the unit circle (that is, they cannot even have common endpoints). Notice also, that Definition 2.5 can be given without condition (2); in that case we will talk about *forward (sibling) invariant geodesic lamination*. In particular, forward (sibling) invariant geodesic laminations may well contain finitely many non-degenerate leaves (in that case we will call it *finite*).

We call the leaf ℓ^* in (2) a *pullback* of ℓ and the leaves $\ell_2, \dots, \ell_d$ in (3) *sibling leaves* or just *siblings* of $\ell = \ell_1$. In a broad sense, a *sibling of* ℓ is a leaf with the same image but distinct from ℓ. Definition 2.5 is slightly more restrictive than Thurston's definition of an invariant geodesic lamination, which we give below. In what follows, given a set A, we let $\mathrm{CH}(A)$ denote the convex hull of A.

Definition 2.6 (Invariant geodesic laminations in the sense of Thurston)**.** A geodesic lamination is said to be *invariant* (*in the sense of Thurston*) if the following holds:

(1) for each non-degenerate $\ell \in \mathcal{L}$, we have $\sigma_d(\ell) \in \mathcal{L}$;
(2) $\mathcal{L}$ is *gap invariant*: if G is a gap of $\mathcal{L}$ and $H = \mathrm{CH}(\sigma_d(G \cap \mathbb{S}))$ is the convex hull of $\sigma_d(G \cap \mathbb{S})$, then H is a point, a leaf of $\mathcal{L}$, or a gap of $\mathcal{L}$, and, in the latter case, the map $\sigma_d|_{\mathrm{Bd}(G)} : \mathrm{Bd}(G) \to \mathrm{Bd}(H)$ of the boundary of G onto the boundary of H is a positively oriented composition of a monotone map and a covering map (in fact the set H as above will be called the σ_d-image of G and will be denoted by $\sigma_d(G)$ in what follows).
(3) there are d **pairwise disjoint** leaves $\ell^* \in \mathcal{L}$ such that $\sigma_d(\ell^*) = \ell$.

If $\mathcal{L}$ satisfies conditions (1) and (2) only, then $\mathcal{L}$ is called *forward invariant* (*in the sense of Thurston*).

The above quoted result of [**BMOV13**] claims that if $\mathcal{L}$ is sibling σ_d-invariant, then it is σ_d-invariant in the sense of Thurston. From now on, by $(\sigma_d$-$)$invariant geodesic laminations, we mean *sibling* σ_d-invariant geodesic laminations and consider **only** such invariant geodesic laminations.

The next definition is crucial for our investigation and shows in what ways different chords can coexist.

Definition 2.7 (Linked chords)**.** Two **distinct** chords of $\mathbb{D}$ are *linked* if they intersect inside $\mathbb{D}$ (we will also sometimes say that these chords *cross each other*). Otherwise two chords are said to be *unlinked*.

Definition 2.8 deals with gaps of geodesic laminations and their edges.

Definition 2.8 (Gaps and their edges)**.** A gap G is said to be *infinite* (*finite, uncountable*) if $G \cap \mathbb{S}$ is infinite (finite, uncountable). Uncountable gaps are also called *Fatou* gaps. For a closed convex set $H \subset \mathbb{C}$, straight segments from $\mathrm{Bd}(H)$ are called *edges* of H.

The *degree* of a gap or leaf G is defined as follows.

Definition 2.9 (Degree of a gap or leaf)**.** Let G be a gap or a leaf. If $\sigma_d(G)$ is degenerate (that is, if $\sigma_d(G)$ is a singleton), then the *degree of* G is the cardinality of $G \cap \mathbb{S}$. Suppose now that $\sigma_d(G)$ is not a singleton. Consider $\sigma_d|_{\mathrm{Bd}(G)}$. Then the *degree of* G equals the number of components in the preimage of a point $z \in \sigma_d(\mathrm{Bd}(G))$ under the map $\sigma_d|_{\mathrm{Bd}(G)}$.

Note that we talk about the number of components rather than the number of points since, say, an entire critical leaf is mapped to a single point, thus the full preimage of this point is infinite.

We say that ℓ is a *chord of a geodesic lamination* $\mathcal{L}$ if ℓ is a chord of $\overline{\mathbb{D}}$ unlinked with all leaves of $\mathcal{L}$.

Definition 2.10 (Critical sets)**.** A *critical chord* (*leaf*) $\overline{ab}$ of $\mathcal{L}$ is a chord (leaf) of $\mathcal{L}$ such that $\sigma_d(a) = \sigma_d(b)$. A gap is *all-critical* if all its edges are critical. An all-critical gap or a critical leaf (of $\mathcal{L}$) is called an *all-critical set* (of $\mathcal{L}$). A gap G is said to be *critical* if the degree of G is greater than one. A *critical set* is either a critical leaf or a critical gap.

Observe that a gap G may be such that $\sigma_d|_{\mathrm{Bd}(G)}$ is not one-to-one, yet G is not critical in the above sense. More precisely, a gap may have critical edges while not being critical. Indeed, let G be a triangle with one critical edge and two non-critical edges. Let $G \cap \mathbb{S} = \{x, y, z\}$ where $\overline{xy}$ is critical. Then $\sigma_d(G) = \overline{\sigma_d(y)\sigma_d(z)}$ is a leaf of $\mathcal{L}$ and $\sigma_d|_{\mathrm{Bd}(G)}$ is not one-to-one, but G is not critical because the degree of G is one.

Finally, we need to define a metric on the set of geodesic laminations. Heuristically two laminations should be close if for every leaf in one lamination there is a leaf in the other lamination that is close to it.

We will use the Hausdorff metric H to define the required metric. Given a compact metric space X with metric (distance function) ρ, let 2^X denote the set of all non-empty closed subsets of X. Let $\mathrm{Ball}_\rho(A, \varepsilon)$ denote the set of all points $x \in X$ so that $\rho(x, A) < \varepsilon$. Given A, $B \in 2^X$, the metric

$$H_\rho(A, B) = \inf\{\varepsilon > 0 \mid A \subset \mathrm{Ball}_\rho(B, \varepsilon) \text{ and } B \subset \mathrm{Ball}_\rho(A, \varepsilon)\}$$

is called the *Hausdorff metric*. It is well known that with this metric 2^X is a compact metric space. Since every point of $\mathbb{S}$ is a degenerate leaf of a geodesic lamination $\mathcal{L}$, the solid $\mathcal{L}^+$ is a compact (and connected) subset of $\overline{\mathbb{D}}$. It is tempting to define the distance between $\mathcal{L}_1$ and $\mathcal{L}_2$ as $H_\rho(\mathcal{L}_1^+, \mathcal{L}_2^+)$, where ρ is the usual Euclidean metric on the closed unit disk. Unfortunately, if $d > 2$, there exist distinct geodesic laminations $\mathcal{L}_1$, $\mathcal{L}_2$ such that $\mathcal{L}_1^+ = \mathcal{L}_2^+ = \overline{\mathbb{D}}$ and, hence, $H_\rho(\mathcal{L}_1, \mathcal{L}_2) = 0$. For example, in the cubic case, the two laminations consisting of all vertical and of all horizontal chords are two such laminations. Hence we need two refine the choice of the metric.

Clearly, every element ℓ of $\mathcal{L}$ is a compact set and, hence, a point in $2^{\overline{\mathbb{D}}}$. Thus each geodesic lamination is a closed subset of $2^{\overline{\mathbb{D}}}$. Let H_H denote the Hausdorff metric on $2^{\overline{\mathbb{D}}}$. Then the required distance on the set of geodesic laminations is $H_H(\mathcal{L}_1, \mathcal{L}_2)$. With this metric, the set of geodesic laminations is a compact metric space.

Theorem 2.11 (Theorem 3.21 [**BMOV13**])**.** *The family of sets $\mathcal{L}^+$ of all invariant geodesic laminations $\mathcal{L}$ is closed in the Hausdorff metric H_H. In particular, this family is compact.*

Theorem 2.11 allows us to give the following definition.

Definition 2.12. Suppose that a sequence $\mathcal{L}_i^+$ of solids of invariant geodesic laminations converges to a compact set T. Then by Theorem 2.11 there exists an invariant geodesic lamination $\mathcal{L}$ such that $T = \mathcal{L}^+$ is its solid. In this case we say that geodesic laminations

$\mathcal{L}_i$ *converge to* $\mathcal{L}$. Thus, from now on we will write $\mathcal{L}_i \to \mathcal{L}$ if $\mathcal{L}_i^+ \to \mathcal{L}^+$ in the Hausdorff metric H_H..

Clearly, $\mathcal{L}_i^+ \to \mathcal{L}^+$ implies that the collections of chords $\mathcal{L}_i$ converge to the collection of chords $\mathcal{L}$ (that is, each leaf of $\mathcal{L}$ is the limit of a sequence of leaves from $\mathcal{L}_i$, and each converging sequence of leaves of $\mathcal{L}_i$ converges to a leaf of $\mathcal{L}$).

2.2. Laminational equivalence relations

In this section, we discuss (invariant) laminational equivalence relations and (invariant) geodesic laminations generated by them. The relation between certain polynomials with connected Julia sets and laminational equivalence relations is also discussed. Finally, we introduce a few useful concepts, which we will rely upon in the rest of the paper.

2.2.1. Laminational equivalence relations and their relations to complex polynomials. A lot of geodesic laminations naturally appear in the context of invariant equivalence relations on $\mathbb{S}$ satisfying special conditions. We will call such equivalence relations *laminational*.

Definition 2.13 (Laminational equivalence relations)**.** An equivalence relation $\sim$ on the unit circle $\mathbb{S}$ is said to be *laminational* if either $\mathbb{S}$ is one $\sim$-equivalence class (such laminational equivalence relations are called *degenerate*), or the following holds:
(E1) the graph of $\sim$ is a closed subset of $\mathbb{S} \times \mathbb{S}$;
(E2) the convex hulls of distinct equivalence classes are disjoint;
(E3) each equivalence class of $\sim$ is finite.

As with geodesic laminations, the above definition is static. However for us the most interesting case is the dynamical case described below.

Definition 2.14 (Laminational equivalence relations and dynamics)**.** A laminational equivalence relation $\sim$ is called (σ_d-)*invariant* if:
(D1) $\sim$ is *forward invariant*: for a $\sim$-equivalence class $\mathfrak{g}$, the set $\sigma_d(\mathfrak{g})$ is a $\sim$-equivalence class;
(D2) for any $\sim$-equivalence class $\mathfrak{g}$, the map $\sigma_d : \mathfrak{g} \to \sigma_d(\mathfrak{g})$ extends to $\mathbb{S}$ as an orientation preserving covering map such that $\mathfrak{g}$ is the full preimage of $\sigma_d(\mathfrak{g})$ under this covering map.

For an invariant laminational equivalence relation $\sim$ consider the *topological Julia set* $\mathbb{S}/\sim = J_\sim$ and the *topological polynomial* $f_\sim : J_\sim \to J_\sim$ induced by σ_d. The quotient map $\pi_\sim : \mathbb{S} \to \mathbb{S}/\sim = J_\sim$ semi-conjugates σ_d with $f_\sim|_{J_\sim}$. A laminational equivalence relation $\sim$ **canonically extends** over $\mathbb{C}$: non-trivial classes of the extension are convex hulls of classes of $\sim$. By Moore's Theorem, the quotient space $\mathbb{C}/\sim$ is homeomorphic to $\mathbb{C}$.

The quotient map $\pi_\sim : \mathbb{S} \to \mathbb{S}/\sim$ extends to the plane with the only non-trivial point-preimages (*fibers*) being the convex hulls of non-degenerate $\sim$-equivalence classes. With any fixed identification between $\mathbb{C}/\sim$ and $\mathbb{C}$, one extends $f_\sim$ to a branched-covering map $f_\sim : \mathbb{C} \to \mathbb{C}$ of degree d called a *topological polynomial* too. The complement $K_\sim$ of the unique unbounded component $U_\infty(J_\sim)$ of $\mathbb{C} \setminus J_\sim$ is called the *filled topological Julia set*. The *(canonical) geodesic lamination* $\mathcal{L}_\sim$ *generated by* $\sim$ is the collection of edges of convex hulls of all $\sim$-equivalence classes and all points of $\mathbb{S}$.

Lemma 2.15 (Theorem 3.21 [**BMOV13**])**.** *Geodesic laminations $\mathcal{L}_\sim$ generated by σ_d-invariant laminational equivalence relations are sibling invariant. If a sequence of sets $\mathcal{L}^+_{\sim_i}$ converges to a compact set T, then there exists a sibling invariant geodesic lamination $\mathcal{L}$ such that $T = \mathcal{L}^+$.*

We would like to motivate the usage of laminational equivalence relations by showing in what way they are related to polynomials. Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geqslant 2$, let A_∞ be the basin of attraction of infinity, and let $J(P) = \mathrm{Bd}(A_\infty)$ be the Julia set of P. When $J(P)$ is connected, A_∞ is simply connected and conformally isomorphic to $\mathbb{C} \setminus \overline{\mathbb{D}}$ by a unique isomorphism $\phi : \mathbb{C} \setminus \overline{\mathbb{D}} \to A_\infty$ asymptotic to the identity at ∞. By a theorem of Böttcher (see, e.g., [**Mil00**, Theorem 9.1]), the map ϕ conjugates $P|_{A_\infty}$ with $z^d|_{\mathbb{C}\setminus\overline{\mathbb{D}}}$. If $J(P)$ is locally connected, then ϕ extends continuously to a semiconjugacy $\overline{\phi}$ between $\sigma_d = z \mapsto z^d|_{\mathbb{S}}$ and $P|_{J(P)}$:

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{\sigma_d|_{\mathbb{S}}} & \mathbb{S} \\ {\scriptstyle\overline{\phi}}\big\downarrow & & {\scriptstyle\overline{\phi}}\big\downarrow \\ J(P) & \xrightarrow{P|_{J(P)}} & J(P) \end{array} \tag{2.2.1}$$

The *laminational equivalence generated by* P is the equivalence relation $\sim_P$ on $\mathbb{S}$ whose classes are $\overline{\phi}$*-fibers*, i.e. point-preimages under $\overline{\phi}$. Call $J_{\sim_P} = \mathbb{S}/\sim_P$ the *topological Julia set associated with the polynomial* P. The map $f_{\sim_P}$, induced on $J_{\sim_P}$ by σ_d, will be called the *topological polynomial associated to the polynomial* P. Evidently $P|_{J(P)}$ and $f_{\sim_P}|_{J_{\sim_P}}$ are topologically conjugate. The collection $\mathcal{L}_P$ of chords of $\overline{\mathbb{D}}$ that are edges of convex hulls of $\sim_P$ classes is called the *geodesic lamination generated by the polynomial* P.

In fact, this connection between polynomials and appropriately chosen topological polynomials can be extended onto a wider class of polynomials with connected Julia sets. The first steps in this direction were made in a nice paper by Jan Kiwi [**Kiw04**].

Definition 2.16 (Irrationally indifferent periodic points)**.** Let x be a periodic point of a polynomial P of period n. Then x is said to be *irrationally indifferent* if the multiplier $(P^n)'(x)$ of P at x is of the form $e^{2\pi i\theta}$ for some irrational θ. If there exists an open P^n-invariant neighborhood of x, on which P^n is conjugate to an irrational rotation of an open unit disk, then x is said to be a periodic *Siegel* point . If such a neighborhood of x does not exist, then x is said to be a *periodic Cremer point*.

We also need to introduce a few topological concepts. Observe that in our definition of a laminational equivalence relation we require that classes of equivalence be finite. However the definitions may be given without this requirement; in these cases we will talk about laminational equivalence relations *possibly with infinite classes*.

Definition 2.17 (Dendrites and dendritic laminations)**.** A locally connected continuum is said to be a *dendrite* if it contains no subsets homeomorphic to the unit circle. If $\sim$ is a laminational equivalence relation on the unit circle $\mathbb{S}$ such that the quotient space $\mathbb{S}/\sim$ is a dendrite, then we call $\sim$ and the corresponding geodesic lamination $\mathcal{L}_\sim$ *dendritic*. Observe that, in this case, every $\sim$-class is finite, and hence every point x of the quotient space $\mathbb{S}/\sim$ is such that $\mathbb{S}/\sim\setminus\{x\}$ consists of finitely many components (in that case x is said to be of *finite order*). If, however, $\sim$ is a laminational equivalence relation on the unit circle $\mathbb{S}$ possibly with infinite classes such that $\mathbb{S}/\sim$ is a dendrite, then we call $\sim$ and $\mathcal{L}_\sim$ *dendritic possibly with infinite classes*.

In what follows when we talk about a *preperiodic* object (point, set etc) we mean that it is not periodic but maps to a periodic object after some (positive) number of iterations of the map. On the other hand when we talk about a (*pre*)*periodic* object, we mean that it is either preperiodic or periodic. In particular, when talking about (pre)periodic points we mean points that have finite forward orbits. Now we can state one of the important results proven in [**Kiw04**].

Theorem 2.18. *Suppose that a polynomial P with connected Julia set $J = J(P)$ has no Siegel or Cremer periodic points. Then there exist a laminational equivalence relation $\sim_P$, the corresponding topological polynomial $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ restricted to the topological Julia set, and a monotone semiconjugacy $\varphi_P : J \to J_{\sim_P}$. The semiconjugacy φ_P is one-to-one on all (pre)periodic points of P belonging to the Julia set. If all periodic points of P are repelling, then $J_{\sim_P}$ is a dendrite.*

In what follows, denote by $\mathcal{D}$ the space of all polynomials with connected Julia sets and only repelling periodic points. Let $\mathcal{D}_d$ be the space of all such polynomials of degree d.

Theorem 2.18 was extended [**BCO11**] onto all polynomials with connected Julia sets. Call a monotone map φ_P of a connected polynomial Julia set $J(P) = J$ onto a locally connected continuum L the *finest monotone map of $J(P)$ onto a locally connected continuum* if, for any monotone $\psi : J \to J'$ with J' locally connected, there is a monotone map h with $\psi = h \circ \varphi_P$. Then it is proven in [**BCO11**] that the finest monotone map on a connected polynomial Julia set semiconjugates $P|_{J(P)}$ to the corresponding topological polynomial $f_{\sim_P}$ on its topological Julia set $J_{\sim_P}$ generated by the laminational equivalence relation possibly with infinite classes $\sim_P$. It follows that the following diagram is commutative (recall that by $\pi_{\sim_P}$ we denote the quotient map corresponding to the lamination $\sim_P$).

$$\begin{tikzcd}
J(P) \arrow[rr, "P|_{J(P)}"] \arrow[drr, "\varphi_P"] & & J(P) \arrow[drr, "\varphi_P"] & & \mathbb{S}^1 \arrow[rr, "\sigma_d"] \arrow[dll, "\pi_{\sim_P}"'] & & \mathbb{S}^1 \arrow[dll, "\pi_{\sim_P}"] \\
& & J_{\sim_P} \arrow[rr, "f_{\sim_P}|_{J_{\sim_P}}"'] & & J_{\sim_P} & &
\end{tikzcd}$$

2.2.2. Other useful notions. Considering objects related to geodesic laminations, we do not have to fix these geodesic laminations. Recall that, given two points $a,\, b \in \mathbb{S}$, we write $\overline{ab}$ for the chord connecting a with b.

Definition 2.19. By a *periodic gap or leaf*, we mean a gap or a leaf G, for which there exists the least number n (called the *period* of G) such that $\sigma_d^n(G) = G$. Then we call the map $\sigma_d^n : G \to G$ the *remap*. An edge (vertex) of G, on which the remap is the identity, is said to be *refixed*.

Given points $a,\, b \in \mathbb{S}$, denote by (a, b) the *positively oriented* open arc from a to b (that is, moving from a to be b within (a, b) takes place in the counterclockwise direction). For a closed set $G' \subset \mathbb{S}$, we call components of $\mathbb{S} \setminus G'$ *holes* (of G' or of the convex hull $G = \mathrm{CH}(G')$ of G'). If $\ell = \overline{ab}$ is an edge of the convex hull $G = \mathrm{CH}(G')$ of G', then we let $H_G(\ell)$ denote the component of $\mathbb{S} \setminus \{a, b\}$ disjoint from G' and call it the hole of G *behind* ℓ (it is only unique if G' contains at least three points). The *relative interior* of a gap is its interior in the plane; the *relative interior* of a segment is the segment minus its endpoints.

Definition 2.20. If $A \subset \mathbb{S}$ is a closed set and all the sets $\mathrm{CH}(\sigma_d^i(A))$ are pairwise disjoint, then A is called *wandering*. If there exists $n \geqslant 1$ such that the sets $\mathrm{CH}(\sigma_d^i(A))$ with

$i = 0, \ldots, n-1$ have pairwise disjoint relative interiors while $\sigma_d^n(A) = A$, then A is called *periodic* of period n. If there exists a minimal $m > 0$ such that all $\mathrm{CH}(\sigma_d^i(A))$ with $0 \leqslant i \leqslant m+n-1$ have pairwise disjoint relative interiors and $\sigma_d^m(A)$ is periodic of period n, then we call A *preperiodic* of period n and preperiod m. A set is called *(pre)periodic* if it is periodic or preperiodic. If A is wandering, periodic or preperiodic, and, for every $i \geqslant 0$ and every hole (a, b) of $\sigma_d^i(A)$, either $\sigma_d(a) = \sigma_d(b)$, or the positively oriented arc $(\sigma_d(a), \sigma_d(b))$ is a hole of $\sigma_d^{i+1}(A)$, then we call A (and $\mathrm{CH}(A)$) a (σ_d)*-laminational set*. We call $\mathrm{CH}(A)$ *finite* if A is finite. A $(\sigma_d$-)*stand alone gap* is defined as a laminational set with non-empty interior in the plane.

Recall that when talking about a Jordan curve K that encloses a simply connected domain W in the plane, by the *positive direction* on K one means the counterclockwise direction with respect to W, i.e., the direction of a particle moving along K so that W remains on its left. When considering a Jordan curve K in the plane we **always** do so with positive direction on it. In particulary, we consider the boundary of a gap with positive direction on it. Accordingly, denote by $<$ the **positive** (counterclockwise) circular order on $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ induced by the usual order of $\mathbb{R}$. Note that this order is only meaningful for sets of cardinality at least three. For example, we say that $x < y < z$ provided that moving from x in the positive direction along $\mathbb{S}$ we meet y before meeting z.

Definition 2.21 (Order preserving maps of the circle)**.** Let $X \subset \mathbb{S}$ be a set with at least three points. We call σ_d *order preserving on* X if $\sigma_d|_X$ is one-to-one and, for every triple $x, y, z \in X$ with $x < y < z$, we have $\sigma_d(x) < \sigma_d(y) < \sigma_d(z)$.

Finally, we discuss in this section *proper invariant geodesic laminations*.

Definition 2.22 (Proper invariant geodesic lamination)**.** Two leaves with a common endpoint v and the same image are said to form a *critical wedge* (the point v the is said to be its vertex). An invariant geodesic lamination $\mathcal{L}$ is *proper* if it contains no critical leaf with a periodic endpoint and no critical wedge with periodic vertex.

Given an invariant geodesic lamination $\mathcal{L}$, define an equivalence relation $\approx_{\mathcal{L}}$ by declaring that $x \approx_{\mathcal{L}} z$ if and only if there exists a *finite* concatenation of leaves of $\mathcal{L}$ connecting x and z.

Theorem 2.23 (Theorem 4.9 [**BMOV13**])**.** *Let $\mathcal{L}$ be a proper Thurston invariant lamination. Then $\approx_{\mathcal{L}}$ is a nonempty invariant laminational equivalence relation.*

2.3. General properties of invariant geodesic laminations

Some results of this section are taken from [**BMOV13**].

Lemma 2.24 (Lemma 3.7 [**BMOV13**])**.** *If $\overline{ab}$ and $\overline{ac}$ are two leaves of an invariant geodesic lamination $\mathcal{L}$ such that $\sigma_d(a), \sigma_d(b)$ and $\sigma_d(c)$ are all distinct points, then the order among points a, b, c is preserved under σ_d.*

We prove a few corollaries of Lemma 2.24

Lemma 2.25. *If $\mathcal{L}$ is an invariant geodesic lamination, $\ell = \overline{ab}$ is a leaf of $\mathcal{L}$, and a is periodic of period n, then b is (pre)periodic of period n.*

PROOF. Assume that, while the point a is of period n, the point b is not σ_d^n-fixed. Then, by Lemma 2.24, either the circular order among the points $b_i = \sigma_d^{ni}(b)$ is the same as the order of subscripts or $b_i = b_{i+1}$ for some i. In the former case b_i converge to some limit point, a contradiction with the expansion property of σ_d^n. Hence for some (minimal)

i we have $b_i = b_{i+1}$. It follows that the period m of b_i cannot be less than n as otherwise we can consider σ_d^m which fixes b_i and does not fix a yielding the same contradiction with Lemma 2.24. $\square$

We will need the following elementary lemma. The notion of a (*pre*)*critical* object is similar to the notion of a (pre)periodic object; thus, a (pre)critical point is either a precritical point, or a critical point.

Lemma 2.26. *If $x \in \mathbb{S}$, and the chords $\overline{\sigma_d^i(x)\sigma_d^{i+1}(x)}$, $i = 0, 1, \ldots$ are pairwise unlinked, then the point x, and therefore the leaf $\overline{x\sigma_d(x)} = \ell$, are* (*pre*)*periodic.*

PROOF. The sequence of leaves from the lemma is the σ_d-orbit of ℓ, in which consecutive images are concatenated and no two leaves are linked. If, for some i, the leaf $\overline{\sigma_d^i(x)\sigma_d^{i+1}(x)} = \sigma_d^i(\ell)$ is critical, then $\sigma_d^{i+1}(\ell) = \{\sigma_d^{i+1}(x)\}$ is a σ_d-fixed point, which proves the claim in this case. Assume now that the leaf ℓ is not (pre)critical. If the point x is not (pre)periodic, then, by topological considerations, leaves $\sigma_d^n(\ell)$ must converge to a limit leaf or a limit point. Clearly, this limit set is σ_d-invariant. However, the map σ_d is expanding, a contradiction. $\square$

Lemma 2.26 easily implies Lemma 2.27.

Lemma 2.27. *Let $\mathcal{L}$ be a geodesic lamination. Then the following holds.*

(1) *If ℓ is a leaf of $\mathcal{L}$ and, for some $n > 0$, the leaf $\sigma_d^n(\ell)$ is concatenated to ℓ, then ℓ is* (*pre*)*periodic.*
(2) *If ℓ has a* (*pre*)*periodic endpoint, then ℓ is* (*pre*)*periodic.*
(3) *If two leaves ℓ_1, ℓ_2 from geodesic laminations $\mathcal{L}_1$, $\mathcal{L}_2$ share the same* (*pre*)*periodic endpoint, then they are* (*pre*)*periodic with the same eventual period of their endpoints.*

PROOF. Let $\ell = \overline{uv}$. First, assume that $\sigma_d^n(u) = u$. Then the statement (1) follows from Lemma 2.25. Second, assume that $\sigma_d^n(u) = v$. Then the statement (1) follows from Lemma 2.26. The statements (2) and (3) follow from (1) and Lemma 2.25. $\square$

A similar conclusion can be made for edges of periodic gaps.

Lemma 2.28. *Suppose that G is a gap of a geodesic lamination. Then, for every edge ℓ of G, there exists an integer k such that the length of the hole $H_{\sigma_d^k(G)}(\sigma_d^k(\ell))$ exceeds $\frac{1}{d+1}$. Moreover, suppose that a gap G is periodic of period m. Then, for every edge ℓ of G which is not* (*pre*)*critical, there exists an edge ℓ_* of G from the orbit of ℓ such that the length of the hole $H_G(\ell_*)$ exceeds $\frac{1}{d^m+1}$. In particular, any edge of a periodic gap is* (*pre*)*periodic or* (*pre*)*critical, and any periodic gap can have at most finitely many non-degenerate periodic edges.*

PROOF. To prove the first statement of the lemma, observe that the length s_n of the hole $H_{\sigma_d^n(G)}(\sigma_d^n(\ell))$ of $\sigma_d^n(G)$ behind the leaf $\sigma_d^n(\ell)$ grows with n as long as s_n stays sufficiently small. In fact, it is easy to see that the correct bound on s_n is that $s_n < \frac{1}{d+1}$. Indeed, suppose that $s_n < \frac{1}{d+1}$. Then the restriction $\sigma_d|_{H_{\sigma_d^n(G)}(\sigma_d^n(\ell))}$ is one-to-one and the hole $\sigma_d(H_{\sigma_d^n(G)}(\sigma_d^n(\ell)))$ is of length $ds_n > s_n$. Clearly, this implies that for some k the length of the hole $H_{\sigma_d^k(G)}(\sigma_d^k(ell))$ will exceed $\frac{1}{d+1}$ as desired.

Now, suppose that G is periodic of period m and ℓ is not (pre)critical. Then G is σ_d^m-invariant, and the second claim of the lemma follows from the first one. Observe that for any edge $\hat{\ell}$ of G such that $|H_G(\hat{\ell})| = s$ it is impossible that $\frac{1}{d^m+1} \leqslant s < \frac{1}{d^m}$ as in that case the arc T complementary to the arc $\sigma_d^m(H_G(\hat{\ell}))$ is of length $1-d^m s<s$, a contradiction (all the vertices of G must belong to $\overline{T}$ and hence $\overline{T}$ must contain $H_G(\hat{\ell})$, a contradiction). The remaining claims of the lemma now easily follow. $\square$

Given $v\in\mathbb{S}$, let $E(v)$ be the closure of the set $\{u\mid \overline{uv}\in\mathcal{L}\}$.

Lemma 2.29. *If v is not (pre)periodic, then $E(v)$ is at most finite. If v is (pre)periodic, then $E(v)$ is at most countable.*

PROOF. The first claim is proven in [**BMOV13**, Lemma 4.7]. The second claim follows from Lemma 2.27 as by that lemma both vertices of any leaf with an endpoint v must be preperiodic. $\square$

Properties of individual wandering polygons were studied in [**Kiw02**]; properties of collections of wandering polygons were studied in [**BL02**]; their existence was established in [**BO08**]. The most detailed results on wandering polygons and their collections are due to Childers [**Chi07**].

Let us describe the entire σ_d-orbit of a finite periodic laminational set.

Proposition 2.30. *Let T be a σ_d-periodic finite laminational set and X be the union of the forward images of T. Then, for every connected component R of X, there is an m-tuple of points $a_0<a_1<\dots<a_{m-1}<a_m=a_0$ in $\mathbb{S}$ such that R consists of eventual images of T containing $\overline{a_ia_{i+1}}$ for $i=0,\dots,m-1$. If $m>1$, then the remap of R is a transitive combinatorial rotation on the collection of all images of T in R.*

Loosely speaking, one can say that, under the appropriate power of σ_d, the set T "rotates" around the convex hull of $\{a_0,\dots,a_{m-1}\}$. Note that the case $m=1$ is possible. In this case, R consists of several images of T sharing a common vertex a_0, there is a natural cyclic order among the images of T, and the remap of R is a cyclic permutation of these images, not necessarily a combinatorial rotation. In particular, it may happen that there is a unique image of T containing a_0; in this case we deal with more standard dynamics where the sets $T,\sigma_d(T),\dots,\sigma_d^{k-1}(T)$ are pairwise disjoint while $\sigma_d^k(T)=T$.

PROOF. Set $T_k=\sigma_d^k(T)$. Let k be the smallest positive integer such that T_k intersects T_0; we may suppose that $T_k\neq T_0$. There is a vertex a_0 of T_0 such that $a_1=\sigma_d^k(a_0)$ is also a vertex of T_0. Clearly, both a_1 and $a_2=\sigma_d^k(a_1)$ are vertices of T_k. Set $a_i=\sigma_d^{ki}(a_0)$. Then we have $a_m=a_0$ for some minimal $m>0$. Let Q be the convex hull of the points $a_0,\dots,a_{m-1}$. Then Q is a convex polygon, or a chord, or a point. If $m>1$, then a_i and a_{i+1} are the endpoints of the same edge of Q (otherwise some edges of the polygons T_{ki} would cross in $\mathbb{D}$). Set $R=\cup_{i=0}^{m-1}T_{ki}$. If $m=1$, then the sets T_{ki} share the vertex a_0. If $m=2$ then Q is a leaf that flips under the action of σ_d^k. Finally, if $m>2$, then it follows from the fact that the boundary of Q is a simple closed curve that every chord $\overline{a_ia_{i+1}}$ is an edge of T_{ki}, $i=0,\dots,m-1$ shared with Q and the sets T_{ki} are disjoint from the interior of Q.

Since the case $m=2$ is straightforward, let us assume now that $m>2$. Notice that, by the construction, the map σ_d^k sends each set T_{ki} to the set $T_{k(i+1)}$ adjacent to T_{ki}, and $\sigma_d^k(\overline{a_ia_{i+1}})=\overline{a_{i+1}a_{i+2}}$. Now, let s be the least integer such that $\sigma_d^s(\overline{a_0a_1})=\overline{a_ja_{j+1}}$ is an edge of Q for some j. Evidently, the number s does not have to be equal to k. Still, it follows that $\sigma_d^s(\overline{a_1a_2})=\overline{a_{j+1}a_{j+2}}$, $\dots$, $\sigma_d^s(\overline{a_{m-1}a_0})=\overline{a_{j-1}a_j}$. Thus, the map $\sigma_d^s|_R$ is a combinatorial rotation. Moreover, the choice of s and the fact that $\overline{a_1a_2}=\sigma_d^k(\overline{a_0a_1})$ imply that the σ_d^s-orbit of $\overline{a_0a_1}$ is the collection of all edges of Q, i.e. that σ_d^s is transitive on the collection of all images of T forming R.

It remains to prove that R is disjoint from $R_j = \sigma_d^j(R)$ for $j < s$. By way of contradiction suppose that R_j intersects some T_{si}. Note that the "shape" of the set R_j mimics that of R: the set R_j consists of m sets that are σ_d^j-images of sets $T_{ki}, i = 0, \dots, m-1$ adjacent to a convex polygon $\sigma_d^j(Q)$ in the same way the sets $T_{ki}, i = 0, \dots, m-1$ are adjacent to Q.

Let us show that sets Q, $\sigma_d(Q)$, $\dots$, $\sigma_d^{s-1}(Q)$ have at most a vertex in common and that each set R_j is contained in one component of $\overline{\mathbb{D}} \setminus Q$. Indeed, consider the set $\sigma_d^j(Q)$. If all images of T adjacent to $\sigma_d^j(Q)$ are distinct from the images of T adjacent to Q, then all the images of T adjacent to $\sigma_d^j(Q)$ are contained in the same component of $\overline{\mathbb{D}} \setminus Q$. Hence $\sigma_d^j(Q)$ can have at most a common vertex with Q. Moreover, suppose that a set $\sigma_d^j(T)$, adjacent to $\sigma_d^j(Q), j < s$, in fact coincides with set $\sigma_d^{is}(T)$ adjacent to Q. Then by the choice of s it follows that the edge $\sigma_d^j(\overline{a_0 a_1})$ cannot coincide with the edge of $\sigma_d^j(T) = \sigma_d^{is}(T)$ shared by this set and Q. Since $\sigma_d^j(Q)$ is adjacent to $\sigma_d^j(T)$ along the edge $\sigma_d^j(\overline{a_0 a_1})$, it follows again that $\sigma_d^j(Q)$ and Q cannot have more than one vertex in common, and that R_j is contained in the corresponding component of $\overline{\mathbb{D}} \setminus Q$.

Now, suppose that $x \in R_j \cap R$. Assume that x belongs to a component A of $\overline{\mathbb{D}} \setminus Q$. Then on the one hand σ_d^s sends x to a component B of $\overline{\mathbb{D}} \setminus Q$ distinct from A. On the other hand, by the previous paragraph the entire R_j must be contained in A which implies that $\sigma_d^s(x)$ must belong to A, a contradiction. □

It is well-known (see [**Kiw02**]) that any infinite gap G of an invariant geodesic lamination $\mathcal{L}$ is (pre)periodic. By a *vertex* of a gap or leaf G we mean any point of $G \cap \mathbb{S}$.

Lemma 2.31. *Let G be a periodic gap of period n and set $K = \mathrm{Bd}(G)$. Then $\sigma_d^n|_K$ is the composition of a covering map and a monotone map of K. If $\sigma_d^n|_K$ is of degree one, then either statement* (1) *or statement* (2) *below holds.*

(1) *The gap G has at most countably many vertices, only finitely many of which are periodic, and no edge is preperiodic. All non-periodic edges of G are* (*pre*)*critical and isolated.*
(2) *The map $\sigma_d^n|_K$ is monotonically semiconjugate to an irrational circle rotation so that each fiber of this semiconjugacy is a finite concatenation of* (*pre*)*critical edges of G.*

PROOF. The first claim of the lemma holds since by [**BMOV13**] sibling invariant geodesic laminations are invariant in the sense of Thurston (see the beginning of Subsection 2.1.2). Consider now the case when G is of degree one. Then it follows that no edge of G can have two preimages under σ_d^n. In particular, G has no preperiodic edges. All other claims in the statement (1) of the lemma follow from Lemma 2.28. Observe that a critical edge ℓ of G must be isolated because by definition of an invariant geodesic lamination (more precisely, because geodesic invariant laminations are gap-invariant) there is another gap of $\mathcal{L}$ sharing the edge ℓ with $\mathcal{L}$ and mapping onto $\sigma_d(G)$ under σ_d.

In the statement (2) of the lemma we will prove only the very last claim. Denote by φ the semiconjugacy from (2). Let $T \subset K$ be a fiber of φ. By Lemma 2.28 all edges of G are (pre)critical. Hence if T contains infinitely many edges, then the forward images of T will hit critical leaves of σ_d^n infinitely many times as T cannot collapse under a finite power of σ_d^n. This would imply that an irrational circle rotation has periodic points, a contradiction that completes the proof. □

We can now recall the notion of a *periodic Siegel gap*; we will also introduce a useful notion of the *skeleton* of an infinite gap.

Definition 2.32. Let G be an infinite gap of a geodesic lamination $\mathcal{L}$. If $G \cap \mathbb{S}$ is at most countable, then we say that the *skeleton* of G is empty. Otherwise the *skeleton* of G is defined as the convex hull of the maximal Cantor subset of $G \cap \mathbb{S}$. Periodic infinite gaps G of geodesic laminations such that the remap on the boundary of G is monotonically semiconjugate to an irrational rotation are said to be (*periodic*) *Siegel gaps*.

Observe that the skeleton of a periodic Siegel gap is non-empty. Observe also that edges of the skeleton of a periodic Siegel gap G do not have to be edges of G itself.

By [**BL02**], if $\sim$ is an invariant laminational equivalence relation possibly with infinite classes then there are no countable infinite gaps of $\sim$, and the skeleton of a Siegel gap G coincides with G. In other words, in this case infinite gaps of $\sim$ are either periodic Siegel gaps or periodic Fatou gaps of degree greater than one, or their preimages.

It is known that periodic Siegel gaps must have critical edges that are isolated. Therefore, both countable and Siegel gaps must have isolated edges. Let us investigate other properties of periodic Siegel gaps.

Lemma 2.33. *Suppose that $\mathcal{L}$ is a geodesic lamination. Let G be a periodic Siegel gap of $\mathcal{L}$ of period n. Let H be the skeleton of G and $\ell = \overline{ab}$ be an edge of H. Consider the union of* all *finite concatenations of leaves of $\mathcal{L}$ coming out of a or b and let L be the convex hull of all such leaves. Then L is a finite polyhedron so that every two vertices can be connected by a chain of leaves from $\mathcal{L}$ and ℓ is an edge of L. Moreover, for a minimal $m > 0$, the σ_d^m-image of L is a singleton in H that is a limit from both sides of points of H. The semiconjugacy φ between σ_d^n and the corresponding irrational rotation can be extended onto the union of G and all sets L by collapsing each set L to the point $\varphi(a) = \varphi(b)$. Moreover, at each edge $\ell_* = \overline{a_* b_*}$ of L, there is an infinite gap of $\mathcal{L}$ that has on its boundary a finite concatenation of leaves of $\mathcal{L}$ connecting a_* and b_* and that maps onto G under σ_d^m.*

PROOF. It follows from Lemma 2.31 that for every edge $\ell = \overline{ab}$ of H there exists m such that $\sigma_d^m(a) = \sigma_d^m(b) = x$ is a point of the Cantor set $H \cap \mathbb{S}$ that is a limit point of $H \cap \mathbb{S}$ from both sides. This implies that there are no leaves of $\mathcal{L}$ coming out of x. Therefore, any finite concatenation of leaves of $\mathcal{L}$ coming out of a or b maps to x under σ_d^m and must be contained in the appropriate finite polygon mapped to x under σ_d^m. This implies the first two claims of the lemma. The existence of infinite gaps at edges of L follows now from the definition of an invariant geodesic lamination. $\square$

Observe that, by Lemma 2.33, the σ_d^{m-1}-image of L is an all-critical gap. We will need Lemma 2.33 in what follows, in particular, when we study Siegel gaps of two linked geodesic laminations.

Definition 2.34. In what follows sets L defined in Lemma 2.33 are said to be *decorations* of G. The union of G with all its decorations is said to be the *extension of* G.

In particular, Lemma 2.33 shows that the semiconjugacy φ can be defined on the extension of the corresponding Siegel gap. Then the fibers of φ (i.e., point-preimages under φ) are either decorations of G or single points of the set $G \cap \mathbb{S}$ that are limit points of $G \cap \mathbb{S}$ from both sides.

Lemma 2.31 implies Corollary 2.35.

Corollary 2.35. *Suppose that G is a periodic gap of an invariant geodesic lamination $\mathcal{L}$, whose remap has degree one. Then at most countably many pairwise unlinked leaves of other invariant geodesic laminations can be located inside G.*

We say that a chord is located *inside* G if it is a subset of G and intersects the interior of G.

PROOF. Any chord located inside G has its endpoints at vertices of G. Since in case (1) of Lemma 2.31 there are countably many vertices of G, we may assume that case (2) of Lemma 2.31 holds. Applying the semiconjugacy φ from this lemma, we see that if a leaf ℓ is located in G and its endpoints do not map to the same point by φ, then an iterated image of ℓ will eventually cross ℓ. If there are uncountably many leaves of geodesic laminations inside G, then among them there must exist a leaf ℓ with endpoints in distinct fibers of φ. By the above, some forward images of ℓ cross each other, a contradiction. $\square$

CHAPTER 3

Special types of invariant laminations

3.1. Invariant geodesic laminations with quadratically critical portraits

Here we define invariant geodesic laminations with quadratically critical portraits and discuss linked or essentially equal invariant geodesic laminations with quadratically critical portraits. First we motivate our approach and study families of collections of quadratic quadrilaterals with certain natural properties. Then we discuss properties of families of invariant geodesic laminations, for which the corresponding collections of critical quadrilaterals can be defined.

3.1.1. Collections of critical quadrilaterals and their properties. Thurston defines the *minor* m of a σ_2-invariant geodesic lamination $\mathcal{L}$ as the image of a longest leaf M of $\mathcal{L}$. Any longest leaf of $\mathcal{L}$ is said to be a *major* of $\mathcal{L}$. If m is non-degenerate, then $\mathcal{L}$ has two disjoint majors, both mapping to m; if m is degenerate, then $\mathcal{L}$ has a unique major that is a critical leaf. In the quadratic case, the majors are uniquely determined by the minor. Thus, a quadratic invariant geodesic lamination is essentially defined by its minor. Even though, in the cubic case, one could define majors and minors similarly, unlike in the quadratic case, these "minors" do not uniquely determine the corresponding majors.

The simplest way to see that is to consider distinct pairs of critical leaves with the same images. More precisely, choose an all-critical triangle Δ_1 with non-periodic vertices so that the common image x_1 of the vertices of Δ_1 is periodic (alternatively, has a dense orbit in $\mathbb{S}$). Choose a different all-critical triangle Δ_2 with similar properties. Now, choose an edge $\overline{c}$ of Δ_1. Clearly, there is a unique edge $\overline{d}$ of Δ_2 disjoint from $\overline{c}$. Under the assumptions made about Δ_1 and Δ_2 it is easy to see that the two critical leaves $\overline{c}$ and $\overline{d}$ have so-called **aperiodic kneadings** as defined by Kiwi in [**Kiw04**]. Therefore, by [**Kiw04**], these critical leaves generate the corresponding cubic invariant geodesic lamination. Any other similar choice of critical edges of Δ_1 and Δ_2 gives rise to a cubic invariant geodesic lamination too; clearly, these two invariant geodesic laminations are very different even though they have the same images of their critical leaves, that is, the same minors (see Figure 1). Thus, in the cubic case we should be concerned with critical sets, not only their images.

We study how ordered collections of critical sets of invariant geodesic laminations are located with respect to each other. The fact that critical sets may have different degrees complicates such study. So, it is natural to adjust our invariant geodesic laminations to make sure that the associated critical sets of two invariant geodesic laminations are of the same type. As associated critical sets we choose (generalized) *critical quadrilaterals*.

Definition 3.1. A (generalized) *critical quadrilateral* Q is the circularly ordered 4-tuple $[a_0, a_1, a_2, a_3]$ of marked points $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant a_0$ in $\mathbb{S}$ so that $\overline{a_0a_2}$ and $\overline{a_1a_3}$ are critical chords (called *spikes*); here critical quadrilaterals $[a_0, a_1, a_2, a_3]$, $[a_1, a_2, a_3, a_0]$, $[a_2, a_3, a_0, a_1]$ and $[a_3, a_0, a_1, a_2]$ are viewed as equal.

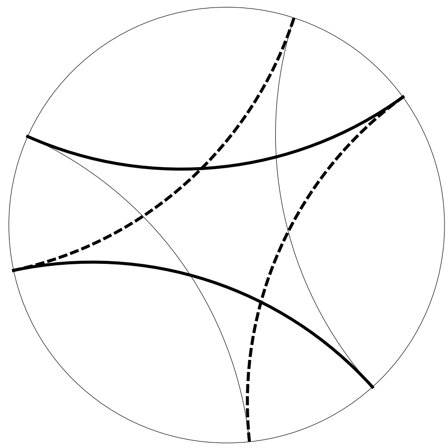

FIGURE 1. Two all-critical triangles in the cubic case. Taking similar outlined edges (thick and thick, thin and thin, or dashed and dashed), one from each of the two triangles, generates three different laminations with the same pair of minors.

We want to comment upon our notation. By $(X_1, \dots, X_k)$, we always mean a k-tuple, that is, an **ordered** collection of elements $X_1, \dots, X_k$. On the other hand, by $\{X_1, \dots, X_k\}$ we mean a collection of elements $X_1, \dots, X_k$ with no fixed order. Since, for critical quadrilaterals, we need to emphasize the **circular** order among its vertices, we choose the notation $[a_0, a_1, a_2, a_3]$ distinct from either of the two just described notations.

For brevity, we will often use the expression "critical quadrilateral" when talking about the convex hull of a critical quadrilateral. Clearly, if all vertices of a critical quadrilateral are distinct or if its convex hull is a critical leaf, then the quadrilateral is uniquely defined by its convex hull. However, if the convex hull of a critical quadrilateral is a triangle, this is no longer true. Indeed, let $T = \mathrm{CH}(a, b, c)$ be an all-critical triangle. Then $[a, a, b, c]$ is a critical quadrilateral, but so are $[a, b, b, c]$ and $[a, b, c, c]$.

A *collapsing quadrilateral* is a critical quadrilateral, whose σ_d-image is a leaf. A critical quadrilateral Q has two intersecting spikes and is either a collapsing quadrilateral, or a critical leaf, or an all-critical triangle, or an all-critical quadrilateral. If all its vertices are pairwise distinct, then we call Q *non-degenerate*, otherwise Q is called *degenerate*. Vertices a_0 and a_2 (a_1 and a_3) are called *opposite*. Considering invariant geodesic laminations, all of whose critical sets are critical quadrilaterals, is not very restrictive: we can "tune" a given invariant geodesic lamination by inserting new leaves into its critical sets in order to construct a new invariant geodesic lamination with all critical sets being critical quadrilaterals.

Lemma 3.2. *The family of all critical quadrilaterals is closed. The family of all critical quadrilaterals that are critical sets of invariant geodesic laminations is closed too.*

PROOF. The first claim follows easily because if a sequence of critical quadrilaterals converges then the limit is again a critical quadrilateral. The second one follows from Theorem 2.11 and the fact that if $\mathcal{L}_i \to \mathcal{L}$, then the critical quadrilaterals of invariant geodesic laminations $\mathcal{L}_i$ converge to critical quadrilaterals that are critical sets of $\mathcal{L}$. $\square$

In the quadratic case, we have less variety of critical quadrilaterals: only collapsing quadrilaterals and critical leaves. As mentioned above, each quadratic invariant geodesic lamination $\mathcal{L}$ either already has a critical quadrilateral, or can be tuned to have one. The latter can be done in several ways if $\mathcal{L}$ has a finite critical set (on which σ_2 acts in a two-to-one fashion). If, however, $\mathcal{L}$ does not have a finite critical set, then its critical set must be a periodic Fatou gap U of degree two. It follows from [**Thu85**] that it has a unique refixed edge M; then one can tune $\mathcal{L}$ by inserting into U the quadrilateral that is the convex hull of M and its sibling leaf.

Thurston's parameterization [**Thu85**] can be viewed as associating to every quadratic invariant geodesic lamination $\mathcal{L}$ with critical quadrilateral Q its minor m. It is easy to see that m is the σ_2-image of Q and that $Q = \sigma_2^{-1}(m)$ is the full σ_2-preimage of m. We would like to translate some crucial results of Thurston's into the language of critical quadrilaterals of quadratic invariant geodesic laminations.

To this end, observe, that, by the above, two minors cross if and only if their full pull-backs (which are collapsing quadrilaterals coinciding with convex hulls of pairs of majors) have a rather specific mutual location: their vertices alternate on the circle. A major result of Thurston's from [**Thu85**] is that **minors of different quadratic invariant geodesic laminations are unlinked**; in the language of critical quadrilaterals this can be restated as follows: **critical quadrilaterals of distinct quadratic invariant geodesic laminations cannot have vertices that alternate on the circle**. All this motivates Definition 3.3.

Definition 3.3 (cf with [**Thu85**]). Let A and B be two quadrilaterals. Say that A and B are *strongly linked* if the vertices of A and B can be numbered so that the following holds:

$$a_0 \leqslant b_0 \leqslant a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant a_3 \leqslant b_3 \leqslant a_0$$

where a_i, $0 \leqslant i \leqslant 3$, are vertices of A and b_i, $0 \leqslant i \leqslant 3$ are vertices of B. Equivalently, A and B are *strongly linked* if no hole of either quadrilateral contains more than one vertex of the other one.

Strong linkage is a closed condition: if two variable critical quadrilaterals are strongly linked and converge, then they must converge to two strongly linked critical quadrilaterals. An obvious case of strong linkage is between two non-degenerate critical quadrilaterals, whose vertices alternate on the circle so that all the inequalities in Definition 3.3 are strict. Yet even if both critical quadrilaterals are non-degenerate, some inequalities may be non-strict, which means that some vertices of both quadrilaterals may coincide.

For example, two coinciding critical leaves can be viewed as strongly linked critical quadrilaterals. Otherwise, an all-critical triangle A with vertices x, y, z and its edge $B = \overline{yz}$ can be viewed as strongly linked quadrilaterals if the vertices are chosen as follows: $a_0 = x, a_1 = a_2 = y, a_3 = z$ and $b_0 = b_1 = y, b_2 = b_3 = z$. Observe that if a critical quadrilateral Q is a critical leaf or has all vertices distinct, then Q as a critical quadrilateral has a well-defined assignment of vertices; the only ambiguous case is when Q is an all-critical triangle.

If an ordered collection of a few chords can be concatenated to form a Jordan curve, or if there are two identical chords, then we say that they form a *loop*. In particular, one chord does not form a loop while two equal chords do. If an ordered collection of chords $(\ell_1, \dots, \ell_k)$ contains no chords forming a loop, then we call it a *no loop collection*.

Lemma 3.4. *The family of no loop collections of critical chords is closed.*

PROOF. Suppose that there is a sequence of no loop collections of critical chords $\mathcal{N}^i = (\ell_1^i, \dots, \ell_s^i)$ such that $\mathcal{N}^i \to \mathcal{N} = (\ell_1, \dots, \ell_s)$. Clearly, all chords ℓ_i are critical.

We need to show that $\mathcal{N}$ is a no loop collection. By way of contradiction assume that, say, chords $\ell_1 = \overline{a_1a_2}, \dots, \ell_k = \overline{a_ka_1}$ form a loop $\widehat{\mathcal{N}}$, in which the order of points $a_1, \dots, a_k$ is positive. We claim that $\widehat{\mathcal{N}}$ cannot be the limit of no loop collections of critical chords, contradicting the convergence assumption that $\mathcal{N}^i \to \mathcal{N}$. This follows from the fact that if $G' \subset \mathbb{S}$ is a union of finitely many sufficiently *small* circle arcs such that all *straight* edges in the boundary of the convex hull $G = \mathrm{CH}(G')$ are critical, then in fact all circle arcs in G' are degenerate, so that G is a finite polygon.

A more formal proof follows. Consider chords $\ell_1^i = \overline{b_1^id_1^i}, \dots, \ell_k^i = \overline{b_k^id_k^i}$ such that points b_j^i converge to a_j and points d_j^i converge to a_{j+1} ($j+1$ is understood here and in the rest of the argument modulo k) as $i \to \infty$. Then for a well-defined collection of integers $m_1, \dots, m_k$ we have that $a_{j+1} = a_j + m_j \cdot \frac{1}{d}$ and $d_j^i = b_j^i + m_j \cdot \frac{1}{d}$. Moreover, since $\widehat{\mathcal{N}}$ is a loop, then $m_1 + \dots + m_k = d$. Now, since $\mathcal{N}^i$ is a no loop collection and all leaves in $\mathcal{N}^i$ are unlinked, for each $1 \leqslant j \leqslant k$ we have $d_j^i = b_j^i + m_j \cdot \frac{1}{d} \leqslant b_{j+1}^i$ and there exists at least one $1 \leqslant j \leqslant k$ such that $d_j^i < b_{j+1}^i$. Since by the above $m_1 + \dots + m_k = d$ it follows that after we follow k chords $\ell_1^i = \overline{b_1^id_1^i}, \dots, \ell_k^i = \overline{b_k^id_k^i}$ along the circle considering b_j^i as the initial point of ℓ_j^i and d_j^i as the terminal point of ℓ_j^i we see that the terminal point d_k^i of ℓ_k^i is located slightly beyond the initial point b_1^i of ℓ_1^i which implies that ℓ_k^i crosses ℓ_1^i, a contradiction. □

We will need the following definition.

Definition 3.5 (Full collections and complete samples of spikes)**.** Call a no loop collection of $d-1$ pairwise unlinked critical chords a *full collection (of critical chords)*. Given a collection $\mathcal{Q}$ of $d-1$ distinct critical quadrilaterals of an invariant geodesic lamination $\mathcal{L}$, we choose one spike in each of them and call this collection of $d-1$ critical chords a *complete sample of spikes (of $\mathcal{Q}$)*.

Now we are ready to investigate invariant geodesic laminations for which the appropriate collections of critical quadrilaterals can be defined.

3.1.2. Quadratically critical invariant geodesic laminations. Suppose that $\mathcal{L}$ is the invariant geodesic lamination generated by a laminational equivalence relation all of whose critical sets are critical quadrilaterals. Then any complete sample of spikes is a full collection because in this case distinct critical sets are disjoint. Observe that, by Lemma 3.4, full collections of critical chords form a closed family. It follows that the fact that complete samples of spikes form a full collection survives limit transition (unlike pairwise disjointness). This inspires another definition.

Definition 3.6 (Quadratic criticality)**.** Let $(\mathcal{L}, \mathrm{QCP})$ be an invariant geodesic lamination with a $(d-1)$-tuple QCP of critical quadrilaterals that are gaps or leaves of $\mathcal{L}$ such that any complete sample of spikes is a full collection. Then QCP is called a *quadratically critical portrait for $\mathcal{L}$* while the pair $(\mathcal{L}, \mathrm{QCP})$ is called an *invariant geodesic lamination with quadratically critical portrait* (if the appropriate invariant geodesic lamination $\mathcal{L}$ for QCP exists but is not emphasized we simply call QCP a *quadratically critical portrait*). The space of all quadratically critical portraits is denoted by $\mathcal{QCP}_d$. The family of all invariant geodesic laminations with quadratically critical portraits is denoted by $\mathbb{L}\mathcal{QCP}_d$.

Observe that any full collection of critical chords is a quadratically critical portrait. Notice also, that if C is a complementary component of a full collection of critical chords in $\mathbb{D}$, then σ_d is one-to-one on the boundary of C except for critical chords contained in the boundary of C. Therefore, if $\mathcal{L}$ admits a quadratically critical portrait then there are no gaps of $\mathcal{L}$ of degree greater than one that are different from critical quadrilaterals from this quadratically critical portrait. In particular, $\mathcal{L}$ cannot have infinite gaps of degree greater than one.

Corollary 3.7. *The spaces $\mathcal{QCP}_d$ and $\mathbb{L}\mathcal{QCP}_d$ are compact.*

PROOF. Let $(\mathcal{L}^i, \mathrm{QCP}^i) \to (\mathcal{L}, \mathcal{C})$, where convergence as always is understood in the Hausdorff (H_H) sense. By Theorem 2.11 and Lemma 3.2, here in the limit we have an invariant geodesic lamination $\mathcal{L}$ and an ordered collection $\mathcal{C}$ of $d-1$ critical quadrilaterals. Let $\mathcal{C} = (C_j)_{j=1}^{d-1}$ be these limit critical quadrilaterals. Choose a collection of spikes $\ell_j, j = 1, \dots, d-1$ of quadrilaterals of $\mathcal{C}$. Suppose that there is a loop formed by some of these spikes. By construction, there exist collections of spikes from quadratically critical portraits QCP^i converging to $(\ell_1, \dots, \ell_{d-1})$. Since by definition these are full collections of critical chords, this contradicts Lemma 3.4. Hence $(\ell_1, \dots, \ell_{d-1})$ is a full collection of critical chords too. That implies that $\mathcal{C}$ is a quadratically critical portrait for $\mathcal{L}$ and proves that $\mathcal{QCP}_d$ and $\mathbb{L}\mathcal{LP}_d$ are compact spaces. □

The following lemma describes invariant geodesic laminations admitting a quadratically critical portrait. Recall that by a *collapsing quadrilateral* we mean a critical quadrilateral that maps to a non-degenerate leaf.

Lemma 3.8. *An invariant geodesic lamination $\mathcal{L}$ has a quadratically critical portrait if and only if all its critical sets are collapsing quadrilaterals or all-critical sets.*

PROOF. If $\mathcal{L}$ has a quadratically critical portrait, then the claim of the lemma follows by definition. Assume that the critical sets of $\mathcal{L}$ are collapsing quadrilaterals and all-critical sets. Then $\mathcal{L}$ may have several critical leaves (some of them are edges of all-critical gaps, some are edges of other gaps, some are not edges of any gaps at all). Choose a no loop collection of critical leaves of $\mathcal{L}$ which is maximal by cardinality. Add to them the collapsing quadrilaterals of $\mathcal{L}$. Include all selected sets in the family of pairwise distinct sets $\mathcal{C} = (C_1, \dots, C_m)$ consisting of critical leaves and collapsing quadrilaterals.

We claim that $\mathcal{C}$ is a quadratically critical portrait. To this end we need to show that $m = d-1$ and that any collection $\mathcal{N}$ of spikes of sets from $\mathcal{C}$ is a no loop collection. First of all, let us show that any such collection $\mathcal{N}$ contains no loops. Indeed, suppose that $\mathcal{N}$ contains a loop $\ell_1 \in C_1, \dots, \ell_r \in C_r$. By construction there must be a collapsing quadrilateral among sets $C_1, \dots, C_r$. We may assume that, say, $C_1 = [a, x, b, y]$ is a collapsing quadrilateral and $\ell_1 = \overline{ab}$ is contained in the interior of C_1 except for points a and b. The spikes $\ell_2, \dots, \ell_r$ form a chain of concatenated critical chords which has, say, b as its initial point and a as its terminal point. Since these spikes come from sets $C_2, \dots, C_r$ distinct from C_1, they have to pass through either x or y as a vertex, a contradiction with C_1 being collapsing. Thus, $\mathcal{N}$ contains no loops, which implies that the number m of chords in $\mathcal{N}$ is at most $d-1$.

Assume now that $m < d-1$ and show that this leads to a contradiction. Indeed, if $m < d-1$, then we can find a component U of $\mathbb{D} \setminus \mathcal{N}^+$ with boundary including some circle arcs such that σ_d on the boundary of U is k-to-1 or higher with $k > 1$ (images of critical edges of U may have more than k preimages). We claim that there exists a critical chord ℓ of $\mathcal{L}$ inside U that connects points in $\mathrm{Bd}(U)$ not connected by a chain of critical edges in $\mathrm{Bd}(U)$. Observe that an arc on $\mathrm{Bd}(U)$ may include several critical chords from the collection $\mathcal{N}$. Consider all open arcs $A \subset \mathrm{Bd}(U)$ such that σ_d is non-monotone on A, and the endpoints of A are connected by a leaf of $\mathcal{L}$. Call such open arcs and the corresponding closed arcs *non-monotone*. Non-monotone arcs exist; indeed, by the assumptions there exist leaves ℓ of $\mathcal{L}$ inside U, and at least one of the two arcs in the boundary of U that connect the endpoints of ℓ must be non-monotone.

The intersection of a decreasing sequence of non-monotone arcs is a closed arc A_0 with endpoints connected with a leaf $\ell_0 \in \mathcal{L}$ such that either ℓ_0 is the desired critical leaf of $\mathcal{L}$ (the leaf ℓ_0 cannot connect two points otherwise connected by a chain of critical edges from $\mathrm{Bd}(U)$ as this would contradict the fact that arcs approaching A_0 are non-monotone), or A_0 is still non-monotone. Thus, it will be enough to show that if A_0 is a closed non-monotone arc which is minimal by inclusion, then there exists the desired critical chord of $\mathcal{L}$.

Clearly, $A_0 \cup \ell_0$ is a Jordan curve enclosing a Jordan disk T, and A_0 is not a union of spikes. If ℓ_0 is not critical, then, by the assumption of minimality of A_0, the leaf ℓ_0 cannot be approached by leaves of $\mathcal{L}$ from within T, thus ℓ_0 is an edge of a gap $G \subset T$. Take a component W of $T \setminus G$ that shares an edge $\overline{m}$ with G. Then, by minimality of A_0, either $\mathrm{Bd}(W)$ collapses to a point or $\mathrm{Bd}(W)$ maps in a monotone fashion to the hole of $\sigma_d(G)$ located "behind" $\sigma_d(\overline{m})$ united with $\sigma_d(\overline{m})$. This implies that G is critical as otherwise the quoted properties of components W of $T \setminus G$ and the fact that σ_d maps G onto $\sigma_d(G)$ in a one-to-one fashion show that $\sigma_d|_{A_0}$ is (non-strictly) monotone, a contradiction. The gap G cannot be all-critical, since ℓ_0 is an edge of G. Therefore, G is a collapsing quadrilateral, which contradicts our choice of $\mathcal{C}$. $\square$

Observe that there might exist several quadratically critical portraits for an invariant geodesic lamination $\mathcal{L}$ from Lemma 3.8. For example, consider a σ_4-invariant geodesic lamination $\mathcal{L}$ with two all-critical triangles $\Delta_1 = \mathrm{CH}(a, b, c), \Delta_2 = \mathrm{CH}(a, c, d)$ sharing an edge $\ell = \overline{ac}$. The proof of Lemma 3.8 leads to a quadratically critical portrait consisting of any three edges of Δ_1, Δ_2 not equal to ℓ in some order (recall that for each critical leaf its structure as a quadrilateral is unique). However it is easy to check that the collection $([a, b, b, c]$, $[a, a, c, c]$, $[a, c, d, d])$ is a quadratically critical portrait too. Notice that, in the definition of a complete sample of spikes, we do not allow to use more than one spike from each critical set, hence the pair of coinciding spikes in $[a, a, c, c]$ does not form a loop of spikes.

Given a quadratically critical portrait QCP, any complete sample of spikes is a full collection of critical chords. If QCP includes sets that are not leaves, then there are several complete samples of spikes as the choice of spikes is ambiguous. This is important for Section 3.4, where we introduce and study the so-called **smart criticality** and its applications to **linked invariant geodesic laminations with quadratically critical portraits** introduced below. First we need a technical definition.

Definition 3.9. A *critical cluster* of $\mathcal{L}$ is a convex subset of $\overline{\mathbb{D}}$ which is maximal by inclusion, whose boundary is a union of critical leaves of $\mathcal{L}$.

A critical leaf disjoint from all other leaves is itself a critical cluster. Consider also the example discussed after Lemma 3.8. There, a σ_4-invariant geodesic lamination $\mathcal{L}$ has two all-critical triangles sharing a critical edge; the union of these triangles is a critical cluster of $\mathcal{L}$.

Definition 3.10 (Linked invariant geodesic laminations)**.** Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be geolaminations with quadratically critical portraits $\mathrm{QCP}_1 = (C_1^i)_{i=1}^{d-1}$ and $\mathrm{QCP}_2 = (C_2^i)_{i=1}^{d-1}$ and a number $0 \leqslant k \leqslant d-1$ such that:

(1) for each $j > k$ the sets C_1^j and C_2^j are contained in a common critical cluster of $\mathcal{L}_1$ and $\mathcal{L}_2$ (in what follows these clusters will be called *special critical clusters* and leaves contained in them will be called *special critical leaves*).

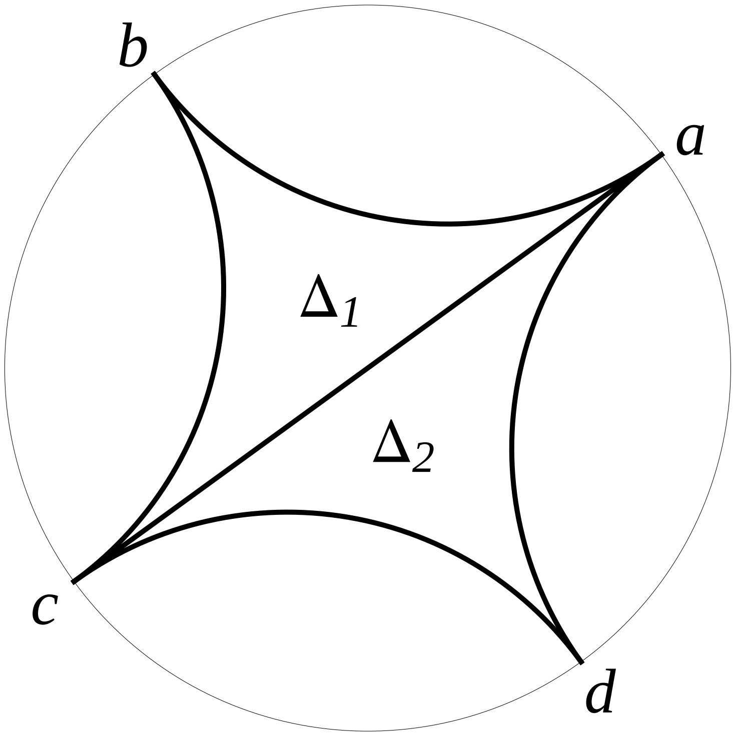

FIGURE 2. This figure illustrates the case with two all-critical triangles Δ_1 and Δ_2 described in the text.

(2) for every i with $1 \leqslant i \leqslant k$, the sets C_1^i and C_2^i are either strongly linked critical quadrilaterals or share a spike.

Then we use the following terminology:

(a) if in (1) for every i with $1 \leqslant i \leqslant k$, the quadrilaterals C_1^i and C_2^i share a spike, we say that QCP_1 and QCP_2, (as well as $(\mathcal{L}_1, \mathrm{QCP}_1)$ and $(\mathcal{L}_2, \mathrm{QCP}_2)$) are *essentially equal*),

(b) if in (1) there exists i with $1 \leqslant i \leqslant k$ such that the quadrilaterals C_1^i and C_2^i are strongly linked and do not share a spike, we say that QCP_1 and QCP_2 (as well as $(\mathcal{L}_1, \mathrm{QCP}_1)$ and $(\mathcal{L}_2, \mathrm{QCP}_2)$) are *linked*.

The critical sets C_1^i and C_2^i, $1 \leqslant i \leqslant d-1$ are called *associated* $($*critical sets of invariant geodesic laminations with quadratically critical portraits* $(\mathcal{L}_1, \mathrm{QCP}_1)$ *and* $(\mathcal{L}_2, \mathrm{QCP}_2))$.

3.2. Some special types of invariant geodesic laminations

Below, we discuss perfect invariant geodesic laminations and dendritic invariant geodesic laminations.

3.2.1. Perfect invariant geodesic laminations. The following is a natural basic definition.

Definition 3.11. An invariant geodesic lamination $\mathcal{L}$ is said to be *perfect* if no leaf of $\mathcal{L}$ is isolated. Given **any** invariant geodesic lamination $\mathcal{L}$, we can consider it with the Hausdorff metric; clearly this makes $\mathcal{L}$ a compact metric space. Define the *perfect part* of $\mathcal{L}$ as the maximal perfect subset $\mathcal{L}^p$ of $\mathcal{L}$.

Since points of $\mathbb{S}$, considered as degenerate leaves, belong to $\mathcal{L}$, it follows from Definition 3.11 that $\mathcal{L}^p$ must contain all singletons of $\mathbb{S}$. Lemma 3.12 easily follows from the definitions. Recall that by a *collapsing polygon* we mean a critical polygon that maps onto a non-degenerate leaf. In other words, if G is a collapsing polygon, then all its edges map to the image leaf.

Lemma 3.12. *The collection $\mathcal{L}^p$ is an invariant perfect geodesic lamination. For every $\ell \in \mathcal{L}^p$ and every neighborhood U of ℓ, there exist uncountably many leaves of $\mathcal{L}^p$ in U.*

PROOF. The fact that no leaf of $\mathcal{L}^p$ is isolated follows immediately. To see that $\mathcal{L}^p$ is invariant, notice that by definition only edges of collapsing quadrilaterals or their sibling leaves can have ambiguous collections of d pairwise disjoint sibling leaves. Indeed, if a leaf ℓ has more than $d-1$ sibling leaves, then two sibling leaves of ℓ must have a common vertex. This implies the claimed. It follows that there are at most finitely many leaves for which the choice of a collection of pairwise disjoint sibling leaves is ambiguous.

Now, let $\ell = \overline{xy}$ be a non-critical leaf of $\mathcal{L}^p$. Choose a sequence of leaves $\ell_i = \overline{x_i y_i}$ of $\mathcal{L}^p$ such that $x_i \to x, y_i \to y$ and for every i there are exactly d leaves in $\mathcal{L}$ (including ℓ_i) with the σ_d-image $\overline{\sigma_d(x_i)\sigma_d(y_i)}$. Moreover, their images are of length close to that of $\sigma_d(\ell)$ and therefore are bounded away from zero. This easily implies that the distance between any two endpoints of any two leaves from the collection of all sibling leaves of ℓ_i is bounded away from zero too. Therefore, the limits of these leaves form a collection of d leaves of $\mathcal{L}^p$ (including ℓ) with the same image $\sigma_d(\ell)$. By definition, this shows that $\mathcal{L}^p$ is invariant, as desired. □

While the existence of the perfect part $\mathcal{L}^p$ of $\mathcal{L}$ is thus established, the actual construction of it in the dynamical case is not obvious at all. The process of finding $\mathcal{L}^p$ was described in detail in [**BOPT14**]. In what follows we will need a few facts and concepts established in [**BOPT14**].

Definition 3.13 (Supergaps [**BOPT14**])**.** Consider a (periodic) infinite gap U of $\mathcal{L}^p$. Then U is said to be a (periodic) *super-gap* of $\mathcal{L}$.

Clearly, the part of the invariant geodesic lamination $\mathcal{L}$ contained in the orbit of U consists of no more than countably many non-degenerate leaves.

Suppose that $\sim$ is an invariant laminational equivalence relation and $\mathcal{L}_\sim$ is the corresponding invariant geodesic lamination. Let $\mathcal{L}^0_\sim = \mathcal{L}_\sim$ and define $\mathcal{L}^k_\sim$ inductively by removing all isolated leaves from $\mathcal{L}^{k-1}_\sim$ (one may call each step in this process a *countable cleaning*). It is proven in [**BOPT14**] that after finitely many countable cleaning steps we will obtain the invariant perfect geodesic lamination $\mathcal{L}^p_\sim$. This result is then used in proving the following lemma.

Lemma 3.14 (Lemma 3.2 [**BOPT14**])**.** *If $\sim$ is a laminational equivalence relation, then the following holds.*

(1) *Every leaf of $\mathcal{L}_\sim$ inside a super-gap G of $\sim$ is (pre)periodic or (pre)critical; every edge of a super-gap is (pre)periodic.*
(2) *Every edge of any gap H of $\mathcal{L}^p_\sim$ is not isolated in $\mathcal{L}^p_\sim$ from outside of H; all gaps of $\mathcal{L}^p_\sim$ are pairwise disjoint. Moreover, gaps of $\mathcal{L}^p_\sim$ are disjoint from leaves that are not their edges.*

(3) *There are no infinite concatenations of leaves in* $\mathcal{L}^p_\sim$. *Moreover, the invariant geodesic lamination* $\mathcal{L}^p_\sim$ *is generated by a laminational equivalence relation* $\sim^p$ *except that there may be the following leaves of* $\mathcal{L}_{\sim^p}$ *which by definition should not be included in* $\mathcal{L}^p_\sim$*: it is possible that one edge of certain finite gap of* $\sim^p$ *is a leaf passing inside an infinite gap of* $\mathcal{L}^p_\sim$.
(4) *Any periodic Siegel gap is a proper subset of its super-gap.*

The next lemma specifies some properties of any perfect invariant geodesic lamination $\mathcal{L}^p$.

Lemma 3.15. *If* $\mathcal{L}^p$ *is a perfect invariant geodesic lamination, then at most two leaves of* $\mathcal{L}^p$ *share an endpoint. Moreover, any leaf of* $\mathcal{L}^p$ *is a limit of uncountably many leaves of* $\mathcal{L}^p$ *disjoint from* ℓ. *If a leaf* ℓ *is critical, then* $\sigma_d(\ell)$ *is a point separated from the rest of the circle by images of those leaves converging to* ℓ *so that* ℓ *is either disjoint from all other leaves or gaps of* $\mathcal{L}^p$ *or is an edge of an all-critical gap of* $\mathcal{L}^p$ *disjoint from all other leaves or gaps of* $\mathcal{L}^p$.

PROOF. Suppose that there are more than two leaves of $\mathcal{L}^p$ coming out of the same point. Then, since, by Lemma 2.29, there are at most countably many leaves of $\mathcal{L}^p$ sharing an endpoint, $\mathcal{L}^p$ has isolated leaves, a contradiction. This implies that any leaf of $\mathcal{L}^p$ is a limit of an uncountably many leaves of $\mathcal{L}^p$ disjoint from ℓ. The rest of the lemma easily follows. □

Clearly, Lemma 3.15 implies the following corollary.

Corollary 3.16. *Let* $\mathcal{L}$ *be a perfect invariant geodesic lamination. Then the critical sets of* $\mathcal{L}$ *are pairwise disjoint and are either all-critical sets, or critical sets mapping exactly* k*-to-*1, $k > 1$, *onto their images.*

In this paper, we study, in particular, perfect invariant geodesic laminations and dendritic invariant geodesic laminations, which are a particular case of perfect ones. By Corollary 3.16, all such geodesic laminations have critical sets with certain natural properties. To avoid unnecessary complications, we will consider invariant geodesic laminations with similar properties even if we do not necessarily assume that they are perfect.

Definition 3.17. Let $\mathcal{L}$ be an invariant geodesic lamination. Suppose that their critical sets are pairwise disjoint except for the case when a critical leaf is a boundary edge of an all-critical set. Then we say that $\mathcal{L}$ is *regular*.

If an invariant geodesic lamination $\mathcal{L}$ is regular, then all its critical leaves are boundary leaves of all-critical sets. In particular, if C is a critical set of $\mathcal{L}$ which is not an all-critical set then it maps onto its image in exactly k-to-1 fashion. By Corollary 3.16, perfect invariant geodesic laminations are regular. However, it is easy to give examples of regular invariant geodesic laminations that are not perfect. Indeed, all quadratic invariant geodesic laminations corresponding to parabolic quadratic polynomials from the main cardioid are regular while it is well-known that they have only countably many non-degenerate leaves. Therefore, these laminations are not perfect, and the perfect part in any such lamination is the set of all points of $\mathbb{S}$.

We will use quadratically critical portraits to parameterize ("tag") certain classes of regular invariant geodesic laminations. An obstacle to this is the fact that an invariant geodesic lamination $\mathcal{L}$ with a k-to-1 critical set such that $k > 2$ does not admit a quadratically critical portrait. However, using Lemma 3.8, it is easy to see that in this case one can insert critical quadrilaterals in critical sets of higher degree in order to "tune" $\mathcal{L}$ into an invariant geodesic lamination with a quadratically critical portrait. This motivates the following definition.

Definition 3.18. Let $\mathcal{L}$ be a regular invariant geodesic lamination with pairwise disjoint critical sets (gaps or leaves) $D_1, \dots, D_k$. Let $\mathcal{L} \subset \mathcal{L}_1$ and $\mathrm{QCP} = (E_1, \dots, E_{d-1})$ be a quadratically critical portrait for $\mathcal{L}_1$. Clearly, there is a unique $(d-1)$-tuple $\mathcal{Z} = (C_1, \dots, C_{d-1})$ such that for every $1 \leqslant i \leqslant d-1$ we have $E_i \subset C_i$ and there is $1 \leqslant j(i) \leqslant k$ with $C_i = D_{j(i)}$. Then $\mathcal{Z}$ is called the *critical pattern of* QCP *in* $\mathcal{L}$; we will also say that QCP *generates* $\mathcal{Z}$. Observe that each $D_{j(i)}$ is repeated in $\mathcal{Z}$ exactly $m_{j(i)} - 1$ times, where $m_{j(i)}$ is the degree of $D_{j(i)}$.

In general, given a regular invariant geodesic lamination $\mathcal{L}$ with (pairwise disjoint) critical sets $D_1, \dots, D_k$, by an *invariant geodesic lamination with critical pattern* we mean a pair $(\mathcal{L}, \mathcal{Z})$, where $\mathcal{Z} = (C_1, \dots, C_{d-1})$ is a $(d-1)$-tuple of sets such that every C_i coincides with some D_j, and every D_j is repeated in $\mathcal{Z}$ exactly $m_j - 1$ times, where m_j is the degree of D_j. Then $\mathcal{Z}$ is called a *critical pattern for* $\mathcal{L}$.

Let us show that critical patterns from the second part of Definition 3.18 are always generated by quadratically critical portraits.

Lemma 3.19. *Given a regular invariant geodesic lamination with a critical pattern* $(\mathcal{L}, \mathcal{Z})$*, where* $\mathcal{L}$ *has (pairwise disjoint) critical sets* $D_1, \dots, D_k$ *of degrees* $m_1, \dots, m_k$ *respectively, there exists a full collection of critical chords of sets* $D_1, \dots, D_k$ *that generates* $\mathcal{Z}$.

Recall that a full collection of critical chords is a collection of $d-1$ pairwise unlinked critical chords with no loops. As was noticed before, a full collection can be viewed as a quadratically critical portrait. Observe also that $\sum_{i=1}^{k} m_i - 1 = d - 1$.

PROOF. If D_i is an all-critical set, then we can simply choose any $m_i - 1$ of its (critical) edges. If the critical set D_i is not all-critical, then we can still choose m_i points of $D_i \cap \mathbb{S}$ with the same image, take the convex hull of this collection of points, and, finally, choose $m_i - 1$ edges of this convex hull. Putting together the collections of critical chords just constructed, we will create a desired full collection of critical chords. It is easy to see now that one can order them so that they generate $\mathcal{Z}$ as desired. It remains to apply Thurston's pullback construction and this way construct the geodesic lamination with the critical sets just chosen as required. □

Observe that the choice of a full collection that generates a given critical pattern as explained above is far from unique. Notice also that by changing the order of the critical sets in which they show in a critical pattern, various critical patterns for the same invariant geodesic lamination can be obtained.

3.2.2. Dendritic invariant geodesic laminations with critical patterns. Below we introduce a useful notation. Recall that in Definition 2.17 we define dendritic laminations (in which case we do not allow for infinite classes of the corresponding laminational equivalence relations) and dendritic laminations possibly with infinite classes (in which case we do allow for infinite classes). Observe that by Kiwi [**Kiw02**] an infinite class of a lamination possibly with infinite classes must be (pre)periodic.

Definition 3.20. The family of all dendritic invariant geodesic laminations is denoted by $\mathbb{L}\mathcal{D}_d$. The family of all dendritic invariant geodesic laminations possibly with infinite classes is denoted by $\mathbb{L}\mathcal{D}_d^\infty$. The space of all dendritic invariant geodesic laminations with critical patterns is denoted by $\mathbb{LCPD}_d$. The space of all dendritic invariant geodesic laminations possibly with infinite classes and critical patterns is denoted by $\mathbb{LCPD}_d^\infty$.

Observe that if $\mathcal{L} = \mathcal{L}_\sim$ is a dendritic invariant geodesic lamination then all its gaps are finite (by definition) and correspond to $\sim$-equivalence classes. The situation is a little more complicated if $\mathcal{L} = \mathcal{L}_\sim$ is a dendritic invariant geodesic lamination possibly with infinite classes. Lemma 3.21 deals with these cases.

Lemma 3.21. *Dendritic invariant geodesic laminations $\mathcal{L}$ possibly with infinite classes are perfect. On the other hand, every perfect geodesic lamination can be viewed as a dendritic invariant lamination possibly with infinite classes.*

PROOF. Let ℓ be a leaf of $\mathcal{L}$. By way of contradiction, suppose that ℓ is isolated. Then ℓ is a common edge of two gaps. Denote these gaps by G and H. Since $\mathcal{L}$ is dendritic, it follows that in fact $\mathcal{L} = \mathcal{L}_\sim$ is generated by a dendritic laminational equivalence relation $\sim$ possibly with infinite classes. On the other hand, both G and H must be $\sim$-classes because $\sim$ is dendritic. This shows that G and H must be forming one $\sim$-class and therefore, by definition of the geodesic lamination generated by a laminational equivalence relation, we see that ℓ cannot be a leaf of $\mathcal{L}_\sim$, a contradiction.

Now, suppose that $\mathcal{L}$ is a perfect invariant geodesic lamination. Then, by Lemma 3.15, gaps of $\mathcal{L}$ are pairwise disjoint. Hence the set $\mathcal{L}^+$ can be partitioned into pairwise disjoint leaves or gaps. Declaring these sets as classes of equivalence of $\sim$ we see that $\mathcal{L}$ is generated by $\sim$ in the usual sense. Moreover, the fact that the corresponding quotient map collapses all gaps of $\mathcal{L}$ to points implies that the corresponding quotient space is a dendrite. This completes the proof. $\square$

Since Siegel gaps and countable gaps have isolated edges, it follows that the only gaps of a perfect geodesic lamination $\mathcal{L}$ are either finite gaps, or periodic Fatou gaps of degree greater than one, or their preimages. As we noticed in the proof of Lemma 3.21, the fact that $\mathcal{L}$ is perfect implies that no two gaps of $\mathcal{L}$ can intersect. In particular, all Fatou gaps of $\mathcal{L}$ are disjoint from other gaps, both finite and Fatou. Moreover, Fatou gaps of a perfect geodesic lamination have no critical edges as by the properties of invariant geodesic laminations such edges would be isolated.

Thus, by Lemma 2.28, if U is a periodic Fatou gap of an invariant perfect geodesic lamination $\mathcal{L}$, then there are finitely many periodic edges of U and all other edges are their preimages. Observe that if a perfect geodesic lamination $\mathcal{L}$ has some Fatou gaps, then $\mathcal{L}$ can be generated by several laminational equivalence relations depending on whether the corresponding quotient map collapses certain grand orbits of Fatou gaps. In particular, all these gaps must be collapsed under the quotient map in the case when the corresponding quotient space is a dendrite.

Strong conclusions about the topology of the Julia sets of non-renormalizable polynomials $P \in \mathcal{D}$ follow from [**KvS06**]. Building upon earlier results by Jeremy Kahn and Misha Lyubich [**KL09a, KL09b**] and by Oleg Kozlovskii, Weixiao Shen and Sebastian van Strien [**KSvS07a, KSvS07b**], Kozlovskii and van Strien generalized results of Artur Avila, Kahn, Lyubich and Shen [**AKLS09**] and proved in [**KvS06**] that if all periodic points of P are repelling, and P is non-renormalizable, then $J(P)$ is locally connected; moreover, by [**KvS06**], two such polynomials that are topologically conjugate are in fact quasi-conformally conjugate. Thus, in this case $f_{\sim_P}|_{J_{\sim_P}}$ is a precise model of $P|_{J(P)}$. Finally, for a given dendritic laminational equivalence relation $\sim$, it follows from another result of Jan Kiwi [**Kiw05**] that there exists a polynomial P with $\sim = \sim_P$. This polynomial does not have to have a locally connected Julia set. Thus, by [**Kiw05**] associating polynomials from $\mathcal{D}$ with their laminational equivalence relations $\sim_P$ and invariant geodesic laminations $\mathcal{L}_P = \mathcal{L}_{\sim_P}$, one maps polynomials from $\mathcal{D}_d$ **onto** $\mathbb{L}\mathcal{D}_d$.

To study the association of polynomials with their invariant geodesic laminations, we need Lemma 3.22 (it is stated as a lemma in [**GM93**] but goes back to Douady and Hubbard [**DH8485**]).

Lemma 3.22 ([**GM93, DH8485**])**.** *Let P be a polynomial of degree $d > 1$, and let R be an external ray of P landing at an iterated preimage y of a repelling periodic point x. We write n for the minimal non-negative integer such that $P^n(y) = x$. Then, for every polynomial P^* of degree d that is sufficiently close to P, the external ray R^* of P^* with the same argument as R lands at a point y^* that is close to y and such that $x^* = P^{*n}(y)$ is a repelling periodic point of P^* close to x.*

In what follows we need a result from [**BL02**] that deals with subcontinua of topological Julia sets.

Theorem 3.23 (No Wandering Continua [**BL02**])**.** *Let $\sim$ be a laminational equivalence relation possibly with infinite classes and $f_\sim : J_\sim \to J_\sim$ be the corresponding topological polynomial. Then for any non-degenerate continuum $K \subset J_\sim$ there exist $0 \leqslant l < m$ such that $f^l_\sim(K) \cap f^m_\sim(K) \neq \varnothing$.*

We will also need the following result, which combines Theorem 7.2.6 of [**BFMOT10**] and a part of Theorem 7.2.7 of [**BFMOT10**].

Theorem 3.24 (Theorems 7.2.6 and 7.2.7 of [**BFMOT10**])**.** *Let $D \subset J_\sim$ be a dendrite such that $f_\sim(D) \subset D$. Then $f_\sim$ has infinitely many periodic cutpoints in D.*

We are ready to prove the following lemma. Similar to the previously introduced terminology, by a precritical point we mean a non-critical point that eventually maps to a critical point while by a (pre)critical point we mean a critical or precritical point.

Lemma 3.25. *Suppose that $\sim$ is a dendritic laminational equivalence relation possibly with infinite classes. Then the following holds.*

1. *Each subcontinuum of $J_\sim$ contains a (pre)periodic non-(pre)critical point.*
2. *Each subcontinuum of $J_\sim$ contains a (pre)critical point.*
3. *Each leaf of $\mathcal{L}_\sim$ can be approximated by (pre)periodic leaves that will never map to a critical set of $\mathcal{L}_\sim$.*

PROOF. (1) Consider the topological polynomial $f_\sim$. Choose a continuum $I \subset J_\sim$. Assume that the sets I and $f^k_\sim(I)$ are non-disjoint. Consider the union T of all iterated $f^k_\sim$-images of I (this union is connected) and take its closure $\overline{T}$. Then $\overline{T} \subset J_\sim$ is an $f^k_\sim$-invariant dendrite. By Theorem 3.24, there are infinitely many periodic cutpoints in $\overline{T}$. Since $f^k_\sim$ has only finitely many critical points, there are infinitely many periodic non-(pre)critical cutpoints in $\overline{T}$. Since T is connected and dense in $\overline{T}$, it follows that T contains periodic non-(pre)critical points. Hence I contains (pre)periodic non-(pre)critical points as desired.

Now, suppose that $K \subset J_\sim$ is any subcontinuum of $J_\sim$. Then, by Theorem 3.23, there exist an eventual image I of K such that, for some $k > 0$, the sets I and $f^k_\sim(I)$ are non-disjoint. By the previous paragraph, it follows that I, and therefore K, contains (pre)periodic points as desired.

(2) The arguments are similar to those in the proof of the statement (1). Suppose that a continuum $I \subset J_\sim$ does not contain (pre)critical points of $f_\sim$. Then its forward images

do not contain critical points. By Theorem 3.23 we may assume that I and $f_\sim^k(I)$ are non-disjoint. Consider the union T of all $f_\sim^k$-images of I (this union is connected) and take its closure K. Then $K \subset J_\sim$ is an $f_\sim^k$-invariant dendrite. Since by the construction K and its images can only contain critical points of $f_\sim$ as its endpoints, it follows that $f_\sim^k|_K$ is one-to-one. By Theorem 3.24 there are infinitely many periodic cutpoints of K. Hence we can find two periodic points $x, y \in K$ such that $f_\sim^n(x) = x, f_\sim^n(y) = y$ and $f_\sim^n|_[x, y]$ is one-to-one where by $[x, y]$ we denote the unique arc inside $J_\sim$ with endpoints x and y. Moreover, we may assume that there are no $f_\sim^n$-fixed points in (x, y). This implies that either x or y attracts points of $[x, y]$ close to it, a contradiction.

(3) Let ℓ be a leaf of $\mathcal{L}_\sim$. Since by Lemma 3.21 the invariant geodesic lamination $\mathcal{L}_\sim$ is perfect, we can find a side of ℓ from which this leaf is non-isolated. Applying the quotient map, we can find an arc in $J_\sim$ and its preimage under the quotient map that is a connected union of a family of pairwise disjoint leaves and gaps of $\mathcal{L}_\sim$ (they are convex hulls of $\sim$-equivalence classes) including ℓ. These convex hulls of $\sim$-equivalence classes are approaching ℓ from the side from which ℓ is not isolated. By (1) there are (pre)periodic leaves or gaps in this family; moreover, we can choose them so that they never map to a critical set of $\mathcal{L}_\sim$. Since by the previous results all edges of a (pre)periodic gap of $\mathcal{L}$ are (pre)periodic themselves, it follows that ℓ can be approximated (from the above chosen side) by (pre)periodic leaves that never map to a critical set of $\mathcal{L}_\sim$ as desired. $\square$

In the dendritic case, the connection between critical patterns and invariant geodesic laminations can be studied using results of Jan Kiwi [**Kiw04**]. One of the results that can be easily deduced from [**Kiw04**] is the following theorem. We provide a sketch of an alternative geometric proof here.

Theorem 3.26 (cf [**Kiw04**]). *If $\mathcal{L}$ is a dendritic invariant geodesic lamination and $\mathcal{L}'$ is an invariant geodesic lamination such that $\mathcal{L}$ and $\mathcal{L}'$ share a collection of $d-1$ critical chords with no loops among them, then $\mathcal{L}' \supset \mathcal{L}$ and $\mathcal{L}' \setminus \mathcal{L}$ consists of at most countably many leaves inserted in certain gaps of $\mathcal{L}$.*

PROOF. Denote by $\sim$ a laminational equivalence relation generating $\mathcal{L}$. The critical chords shared by $\mathcal{L}$ and $\mathcal{L}'$ define $d-1$ complementary components to the closed unit disk $\overline{\mathbb{D}}$. Clearly, the closure $\overline{A}$ of each such component A intersected with $\mathbb{S}$ maps (under σ_d) onto the entire circle $\mathbb{S}$ in a one-to-one order preserving fashion (except for the endpoints). The boundary of $\overline{A}$ consists of circle arcs and concatenations of critical chords.

This allows one to consider pullbacks of chords into each such set $\overline{A}$. Indeed, given a chord ℓ and a set $\overline{A}$ as above, we can consider a set of all points in $\overline{A} \cap \mathbb{S}$ that map to the endpoints of ℓ. Generically, either endpoint will have exactly one preimage there. However if exactly one endpoint of ℓ equals the image of a boundary critical chord of $\overline{A}$ (or of a concatenation of boundary chords of A) then ℓ will have two preimages in $\overline{A}$. Finally, if both endpoints of ℓ are images of boundary concatenations of critical chords of $\overline{A}$, then we choose two preimages of ℓ that are disjoint (it is easy to see that such choice is unique).

The fact that the critical chords are shared by $\mathcal{L}$ and $\mathcal{L}'$ and the definition of a invariant geodesic lamination imply that all pullback chords constructed like that, except possibly for finitely many chords, are shared by $\mathcal{L}'$ and $\mathcal{L}$. Therefore, limits of these pullback chords are leaves of both $\mathcal{L}$ and $\mathcal{L}'$. As follows from [**Thu85**], these limits form an invariant geodesic lamination $\mathcal{L}''$. Moreover, by Lemma 3.25, each subcontinuum of $J_\sim$ contains (pre)critical points, which implies that $\mathcal{L}'' = \mathcal{L}$. Since all gaps of $\mathcal{L}$ are finite, it follows that $\mathcal{L}' \setminus \mathcal{L}$ consists of at most countably many leaves inserted in certain gaps of $\mathcal{L}$. $\square$

Critical patterns were introduced in Definition 3.18. We are ready to consider critical patterns of quadratically critical portraits in dendritic geodesic laminations. This notion is closely related to that of **critically marked (dendritic) polynomial**, which was introduced in the Introduction as we discussed there the Theorem on Local Charts for Dendritic Polynomials (in that we follow Milnor [**Mil93, Mil09**]). Recall that the space of all dendritic invariant geodesic laminations with critical patterns is denoted by $\mathbb{L}\mathcal{CPD}_d$.

Definition 3.27. To each marked dendritic polynomial $(P, C(P))$ of degree d we associate the corresponding dendritic invariant geodesic lamination with critical pattern $(\mathcal{L}_{\sim_P}, \mathcal{Z}(P, C(P)))$ by defining $\mathcal{Z}(P, C(P)) = \mathcal{Z}$ as the ordered collection of convex hulls of $\sim_P$-classes associated to critical points of P in the order they appear in $C(P)$; in the notation from the Introduction $C(P) = (c_1, \dots, c_{d-1})$ and $\mathcal{Z} = (G_{c_1}, \dots, G_{c_{d-1}})$. Also, define the map $\widehat{\Psi}_d$ so that $\widehat{\Psi}_d(P, C(P)) = G_{c_1} \times \dots \times G_{c_{d-1}}$.

Suppose that a sequence of regular invariant geodesic laminations with critical patterns $(\mathcal{L}^i, \mathcal{Z}^i)$ converges in the Hausdorff sense. Then, by Theorem 2.11, the limit $\mathcal{L}^\infty$ of σ_d-invariant geodesic laminations $\mathcal{L}^i$ is itself a σ_d-invariant geodesic lamination. Moreover, then critical patterns $\mathcal{Z}^i$ converge to the limit collection of $d-1$ critical sets of $\mathcal{L}^\infty$. We are interested in the case when the σ_d-invariant geodesic lamination $\mathcal{L}^\infty$ is in a sense compatible with a dendritic σ_d-invariant geodesic lamination.

Lemma 3.28. *Suppose that a sequence of regular invariant geodesic laminations with critical patterns $(\mathcal{L}^i, \mathcal{Z}^i)$ converges in the sense of the Hausdorff metric H_H to an invariant geodesic lamination $\mathcal{L}^\infty$ with a collection of limit critical sets $C_1, \dots, C_{d-1}$ and there exists a dendritic invariant geodesic lamination $\mathcal{L}$ with a critical pattern $\mathcal{Z} = (Z_1, \dots, Z_{d-1})$ such that $C_i \subset Z_i, 1 \leqslant i \leqslant d-1$. Then $\mathcal{L}^\infty \supset \mathcal{L}$.*

PROOF. By Lemma 3.19, for every i, we can choose a full collection $F^i = (\overline{c}^i_1, \dots, \overline{c}^i_d)$ of critical chords that generates $\mathcal{Z}^i$. By Lemma 3.4, we may assume that these full collections converge to a full collection $F = (\overline{c}_1, \dots, \overline{c}_d)$ as i tends to infinity. Clearly, elements of F are critical chords compatible with $\mathcal{L}^\infty$. On the other hand, by the assumptions they are compatible with dendritic invariant geodesic lamination $\mathcal{L}$. Therefore by Theorem 3.26 $\mathcal{L}^\infty \supset \mathcal{L}$ as desired. □

For an integer $m > 0$, we use a partial order by inclusion among m-tuples: $(A_1, \dots, A_m) \succ (B_1, \dots, B_m)$ (or $(B_1, \dots, B_m) \prec (A_1, \dots, A_m)$) if and only if $A_i \supset B_i$ for all $i = 1, \dots, m$. Thus m-tuples and k-tuples with $m \neq k$ are always incomparable. Lemma 3.28 says that if critical patterns of regular invariant geodesic laminations converge **into** a critical pattern of a dendritic invariant geodesic lamination $\mathcal{L}$, then the corresponding regular invariant geodesic laminations themselves converge **over** $\mathcal{L}$.

Definition 3.29. Let F be a map from a topological space A to the space 2^B of compact subsets of a compactum B. Then F is said to be *upper semicontinuous* if $x_i \to x$ in A implies that the limit of every convergent subsequence $y_{i_k} \in F(x_{i_k})$ belongs to $F(x)$. Equivalently, for any neighborhood $\mathcal{U}$ of $F(x)$ there exists a neighborhood V of x such that $F(y) \subset \mathcal{U}$ if $y \in V$.

The fact that F is upper semicontinuous does not necessarily mean that sets $F(x_i)$ must converge in the Hausdorff sense whenever $x_i \to x$. However all existing Hausdorff limits of subsequences of the sets $F(x_i)$ are contained in the set $F(x)$ as long as $x_i \to x$.

Corollary 3.30 easily follows from Lemmas 3.22, 3.25 and 3.28.

Corollary 3.30. *Suppose that a sequence $(P_i, C(P_i))$ of critically marked dendritic polynomials converges to a critically marked dendritic polynomial $(P, C(P))$. Consider invariant geodesic laminations with critical patterns $(\mathcal{L}_{\sim_{P_i}}, \mathcal{Z}(P_i, C(P_i)))$ and $(\mathcal{L}_{\sim_P}, \mathcal{Z}(P, C(P)))$. If $(\mathcal{L}_{\sim_{P_i}}, \mathcal{Z}(P_i, C(P_i)))$ converge in the sense of the Hausdorff metric to $(\mathcal{L}^\infty, \mathcal{Z}_\infty)$, then $\mathcal{L}^\infty \supset \mathcal{L}_{\sim_P}$ and $\mathcal{Z}_\infty \prec \mathcal{Z}(P, C(P))$. In particular, the map $\widehat{\Psi}_d$ is upper semicontinuous.*

By Corollary 3.30, critical sets of dendritic invariant geodesic laminations $\mathcal{L}_{\sim_P}$ associated with polynomials $P \in \mathcal{D}_d$ cannot explode under perturbation of P (they may implode though). Provided that a geometric (visual) way to parameterize $\mathbb{L}\mathcal{CPD}_d$ is given, the map $\widehat{\Psi}_d$ may yield the corresponding parameterization of the space of all dendritic critically marked polynomials and gives an important application of our tools. This justifies the introduction and studying dendritic geodesic laminations with critical patterns, which are natural counterparts of critically marked dendritic polynomials [**Mil12**].

3.3. Accordions of invariant geodesic laminations

In the Introduction, we mentioned that some of Thurston's tools from [**Thu85**] fail in the cubic case. This motivates us to develop new tools (so-called *accordions*), which basically track linked leaves from different invariant geodesic laminations. In this section, we study accordions in detail. In Sections 3.3 - 3.5, we assume that $\mathcal{L}_1$, $\mathcal{L}_2$ are σ_d-invariant geodesic laminations, and ℓ_1, ℓ_2 are leaves of $\mathcal{L}_1$, $\mathcal{L}_2$, respectively.

3.3.1. Motivation. For a quadratic invariant geodesic lamination $\mathcal{L}$ and a leaf ℓ of $\mathcal{L}$ that is not a diameter, let ℓ' be the sibling of ℓ. Denote by $C(\ell)$ the open strip of $\overline{\mathbb{D}}$ between ℓ and ℓ' and by $L(\ell)$ the length of the shorter component of $\mathbb{S} \setminus \ell$. Suppose that $\frac{1}{3} \leqslant L(\ell)$ and notice that by the assumptions we always have $L(\ell) < \frac{1}{2}$. Denote by k the smallest number such that $\sigma_2^k(\ell) \subset C(\ell)$ except perhaps for the endpoints (depending on the dynamics of ℓ, the number k is not necessarily defined, so the forthcoming conclusions should be understood conditionally). The Central Strip Lemma (Lemma II.5.1 of [**Thu85**]) claims that provided the number k **is** defined, we have $\sigma_2^k(\ell)$ separates ℓ and ℓ'. In particular, if $\ell = M$ is a *major*, that is, a longest leaf of some quadratic invariant geodesic lamination, then an eventual image of M cannot enter $C(M)$.

Let us list Thurston's results for which the Central Strip Lemma is crucial. A σ_2-*wandering triangle* is a triangle with vertices a, b, c on $\mathbb{S}$ such that the convex hull T_n of $\sigma_2^n(a)$, $\sigma_2^n(b)$, $\sigma_2^n(c)$ is a non-degenerate triangle for every $n = 0, 1, \dots$, and all these triangles are pairwise disjoint.

Theorem 3.31 (No Wandering Triangle Theorem [**Thu85**]). *Wandering triangles for σ_2 do not exist.*

Theorem 3.32 stated below follows from the Central Strip Lemma and is due to Thurston [**Thu85**] for $d = 2$. For arbitrary d, it is due to Jan Kiwi [**Kiw02**] who used different tools. Observe that by definition a gap must have at least three vertices (this trivial observation is important for the last claim of the theorem dealing with the quadratic case).

Theorem 3.32 ([**Thu85, Kiw02**]). *If A is a finite σ_d-periodic gap of period k, then either A is a d-gon, and σ_d^k fixes all vertices of A, or there are at most $d-1$ orbits of vertices of A under σ_d^k. Thus, for $d = 2$, the remap is transitive on the vertices of any finite periodic gap.*

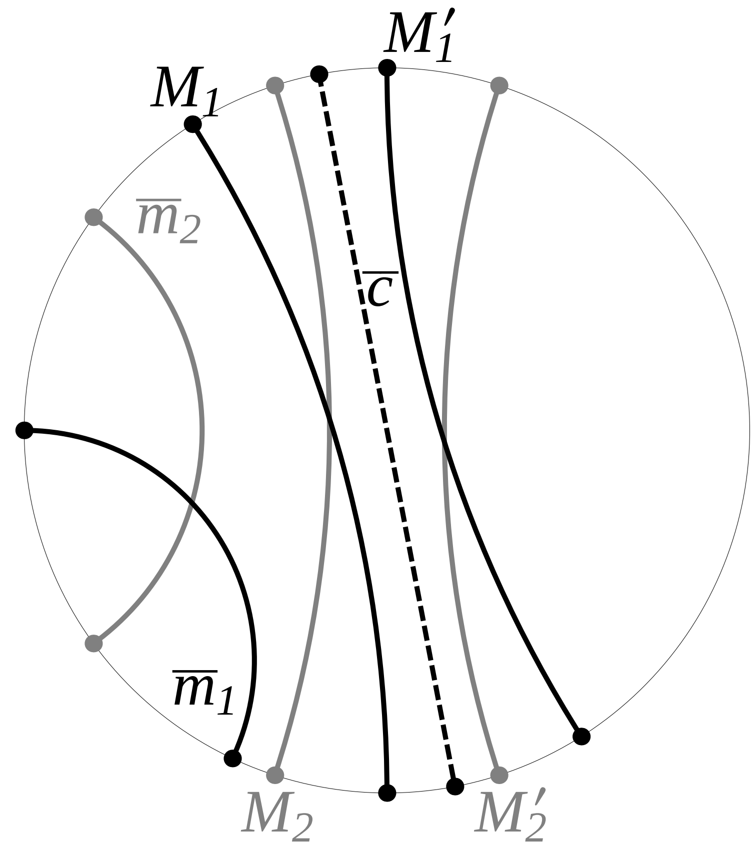

FIGURE 3. This figure illustrates Thurston's proof that quadratic minors are unlinked. The Central Strip Lemma forces orbits of both minors to not cross $\overline{c}$.

Another crucial result of Thurston is that minors of distinct quadratic invariant geodesic laminations are disjoint in $\mathbb{D}$. A sketch of the argument follows. Let $\overline{m}_1$ and $\overline{m}_2$ be the minors of two invariant geodesic laminations $\mathcal{L}_1 \neq \mathcal{L}_2$ that cross in $\mathbb{D}$. Let M_1, M_1' and M_2, M_2' be the two pairs of corresponding majors. We may assume that M_1, M_2 cross in $\mathbb{D}$ and M_1', M_2' cross in $\mathbb{D}$, but $(M_1 \cup M_2) \cap (M_1' \cup M_2') = \varnothing$ (see Figure 3) so that there is a diameter $\overline{c}$ with strictly preperiodic endpoints separating $M_1 \cup M_2$ from $M_1' \cup M_2'$. Thurston shows that there is a unique invariant geodesic lamination $\mathcal{L}$, with only finite gaps, whose major is $\overline{c}$. By the Central Strip Lemma, forward images of $\overline{m}_1$, $\overline{m}_2$ do not intersect $\overline{c}$. Hence $\overline{m}_1 \cup \overline{m}_2$ is contained in a finite gap G of $\mathcal{L}$. By the No Wandering Triangle Theorem, G is eventually periodic. By Theorem 3.32, some images of $\overline{m}_1$ intersect inside $\mathbb{D}$, a contradiction.

Examples indicate that statements analogous to the Central Strip Lemma fail in the cubic case. Indeed, Figure 4 shows a leaf $M = \overline{\frac{342}{728}\frac{579}{728}}$ of period 6 under σ_3 and its σ_3-orbit together with the leaf M' (which has the same image as M forming together with M a narrower "critical strip" S_n) and the leaf N' (which has the same image as $N = (\sigma_3)^4(M)$ forming together with N a wider "critical strip" S_w). Observe that $\sigma_3(M) \subset S_w$, which shows that the Central Strip Lemma does not hold in the cubic case (orbits of periodic

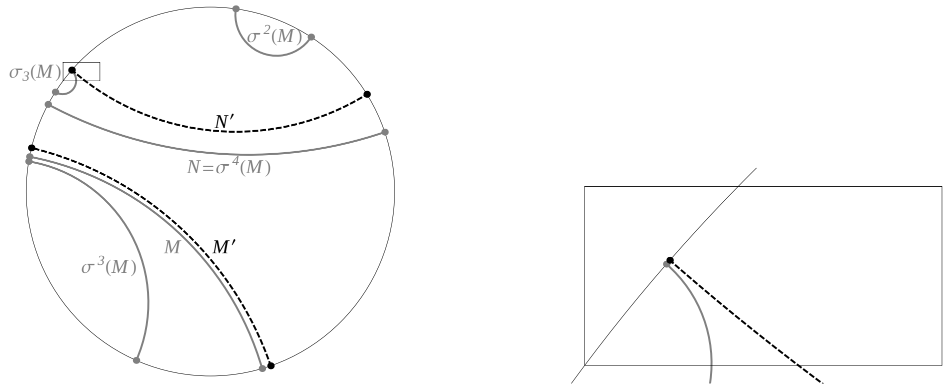

FIGURE 4. This figure shows that the Central Strip Lemma fails in the cubic case. Its left part has a fragment in which two endpoints of leaves are located very close to each other. Its right part is the zoomed-in version of the fragment indicating that the periodic points do not coincide.

leaves may give rise to "critical strips" containing some elements of these orbits of leaves). This apparently makes a direct extension of the arguments from the previous paragraph impossible leaving the issue of whether and how minors of cubic invariant geodesic laminations can be linked unresolved.

Another consequence of the failure of the Central Strip Lemma in the cubic case is the failure of the No Wandering Triangle Theorem (a counterexample was given in [**BO08**]; in fact, it was shown in [**BCO12, BCO13**] that there exists a large set of dendritic invariant geodesic laminations with wandering triangles). Properties of wandering polygons were studied in [**Kiw02, BL02, Chi07**].

3.3.2. Properties of accordions. We now give a definition of accordions.

Definition 3.33. Let $A_{\mathcal{L}_2}(\ell_1)$ be the collection of leaves of $\mathcal{L}_2$ linked with ℓ_1, together with ℓ_1. Let $A_{\ell_2}(\ell_1)$ be the collection of leaves from the forward orbit of ℓ_2 that are linked with ℓ_1, together with ℓ_1. The sets defined above are called *accordions* $($*of* $\ell_1)$ while ℓ_1 is called the *axis* (of the appropriate accordion). Sometimes we will also use $A_{\mathcal{L}_2}(\ell_1)$ and $A_{\ell_2}(\ell_1)$ to mean the union of the leaves constituting these accordions.

In general, accordions do not behave nicely under σ_d as linked leaves may have unlinked images. To avoid these problems, for the rest of this section, we will impose the following conditions on accordions.

Definition 3.34. A leaf ℓ_1 is said to *have order preserving accordions with respect to* $\mathcal{L}_2$ $($*respectively, to a leaf* $\ell_2)$ if $A_{\mathcal{L}_2}(\ell_1) \neq \{\ell_1\}$ (respectively, $A_{\ell_2}(\ell_1) \neq \{\ell_1\}$), and, for each $k \geqslant 0$, the map σ_d restricted to $A_{\mathcal{L}_2}(\sigma_d^k(\ell_1)) \cap \mathbb{S}$ (respectively, to $A_{\ell_2}(\sigma_d^k(\ell_1)) \cap \mathbb{S}$) is order preserving (in particular, it is one-to-one). Say that ℓ_1 and ℓ_2 have *mutually order preserving accordions* if ℓ_1 has order preserving accordions with respect to ℓ_2, and vice versa (in particular, ℓ_1 and ℓ_2 are not precritical).

Though fairly strong, these conditions naturally arise in the study of linked or essentially equal invariant geodesic laminations. In Section 3.5, we show that they are often satisfied by pairs of linked leaves of linked or essentially equal invariant geodesic laminations (Lemma 3.51) so that there are at most countably many pairs of linked leaves that do not have mutually order preserving accordions. If invariant geodesic laminations are perfect, this will imply that every accordion consisting of more than one leaf contains a pair of leaves with mutually order preserving accordions. Understanding the rigid dynamics of such pairs is crucial to our main results.

Proposition 3.35. *If σ_d is order preserving on an accordion A with axis ℓ_1 and $\ell \in A$, $\ell \neq \ell_1$, then $\sigma_d(\ell)$ and $\sigma_d(\ell_1)$ are linked. In particular, if ℓ_1 has order preserving accordions with respect to ℓ_2 then $\sigma_d^k(\ell) \in A_{\ell_2}(\sigma_d^k(\ell_1))$ for every $\ell \in A_{\ell_2}(\ell_1)$, $\ell \neq \ell_1$, and every $k \geqslant 0$.*

PROOF. The proof of Proposition 3.35 immediately follows from the definitions and is left to the reader. □

We now explore more closely the orbits of leaves from Definition 3.34.

Proposition 3.36. *Suppose that ℓ_1 and ℓ_2 are linked, ℓ_1 has order preserving accordions with respect to ℓ_2, and $\sigma_d^k(\ell_2) \in A_{\ell_2}(\ell_1)$ for some $k > 0$. In this case, if $\ell_2 = \overline{xy}$, then either ℓ_1 separates x from $\sigma_d^k(x)$ and y from $\sigma_d^k(y)$, or ℓ_2 has σ_d^k-fixed endpoints.*

PROOF. Suppose that ℓ_2 is not σ_d^k-fixed. Denote by $x_0 = x, y_0 = y$ the endpoints of ℓ_2; set $x_i = \sigma_d^{ik}(x_0)$, $y_i = \sigma_d^{ik}(y_0)$ and $A_t = A_{\ell_2}(\sigma_d^t(\ell_1))$, where $t = 0, 1, \dots$. If ℓ_1 does not separate x_0 and x_1, then either $x_0 \leqslant x_1 < y_1 \leqslant y_0 < x_0$ or $x_0 < y_0 \leqslant y_1 < x_1 \leqslant x_0$. We may assume the latter (cf. Figure 5).

Since σ_d^k is order preserving on $A_0 \cap \mathbb{S}$, then $x_0 < y_0 \leqslant y_1 \leqslant y_2 < x_2 \leqslant x_1 \leqslant x_0$ while the leaves $\overline{x_1y_1}$ and $\overline{x_2y_2}$ belong to the accordion A_k so that the above inequalities can be iterated. Inductively we see that

$$x_0 < y_0 \leqslant \dots \leqslant y_{m-1} \leqslant y_m < x_m \leqslant x_{m-1} \leqslant \dots \leqslant x_0.$$

All leaves $\overline{x_iy_i}$ are pairwise distinct as otherwise there exists n such that $\overline{x_{n-1}y_{n-1}} \neq \overline{x_ny_n} = \overline{x_{n+1}y_{n+1}}$ contradicting σ_d^k being order preserving on $A_{k(n-1)}$. Hence the leaves $\overline{x_iy_i}$ converge to a σ_d^k-fixed point or leaf, contradicting the expansion property of σ_d^k. □

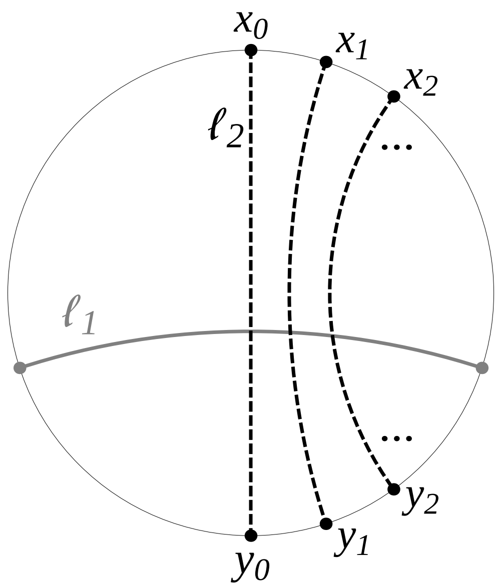

FIGURE 5. This figure illustrates Proposition 3.36. Although in the figure $\overline{x_2 y_2}$ is linked with ℓ_1, the argument does not assume this. In this and forthcoming figures, leaves marked in the same fashion belong to the same grand orbits of leaves.

In what follows, we often use one of the endpoints of a leaf as the subscript in the notation for this leaf.

Lemma 3.37. *If $\ell_a = \overline{ab}$ and $\ell_x = \overline{xy}$, where $a < x < b < y$, are linked leaves with mutually order preserving accordions, and a, b are of period k, then x, y are also of period k.*

PROOF. By the order preservation, $\sigma_d^k(x)$ is not separated from x by ℓ_a. It follows from Proposition 3.36 that $x = \sigma_d^k(x)$, $y = \sigma_d^k(y)$. Since, by Lemma 2.25, the points x and y have the same period (say, m), then m divides k. Similarly, k divides m. Hence $k = m$. □

We will mostly use the following corollary of the above results.

Corollary 3.38. *Suppose that $\ell_a = \overline{ab}$ and $\ell_x = \overline{xy}$ with $x < a < y < b$ are linked leaves. If ℓ_a and ℓ_x have mutually order preserving accordions, then there are the following possibilities for $A = A_{\ell_x}(\ell_a)$.*

(1) *$A = \{\ell_a, \ell_x\}$ and no forward image of ℓ_x crosses ℓ_a.*
(2) *$A = \{\ell_a, \ell_x\}$, the points a, b, x, y are of period $2j$ for some j, $\sigma_d^j(x) = y, \sigma_d^j(y) = x$, and either $\sigma_d^j(a) = b$, $\sigma_d^j(b) = a$, or $\sigma_d^j(\ell_a) \neq \ell_a$, and ℓ_x separates the points a, $\sigma_d^j(b)$ from the points b, $\sigma_d^j(a)$.*
(3) *$A = \{\ell_a, \ell_x\}$, the points a, b, x, y are of the same period, x, y have distinct orbits, and a, b have distinct orbits.*
(4) *There exists $i > 0$ such that $A = \{\ell_a, \ell_x, \sigma_d^i(\ell_x)\}$ and either $x < a < y \leqslant \sigma_d^i(x) < b < \sigma_d^i(y) \leqslant x$ or $x \leqslant \sigma_d^i(y) < a < \sigma_d^i(x) \leqslant y < b$, as shown in Figure 7.*

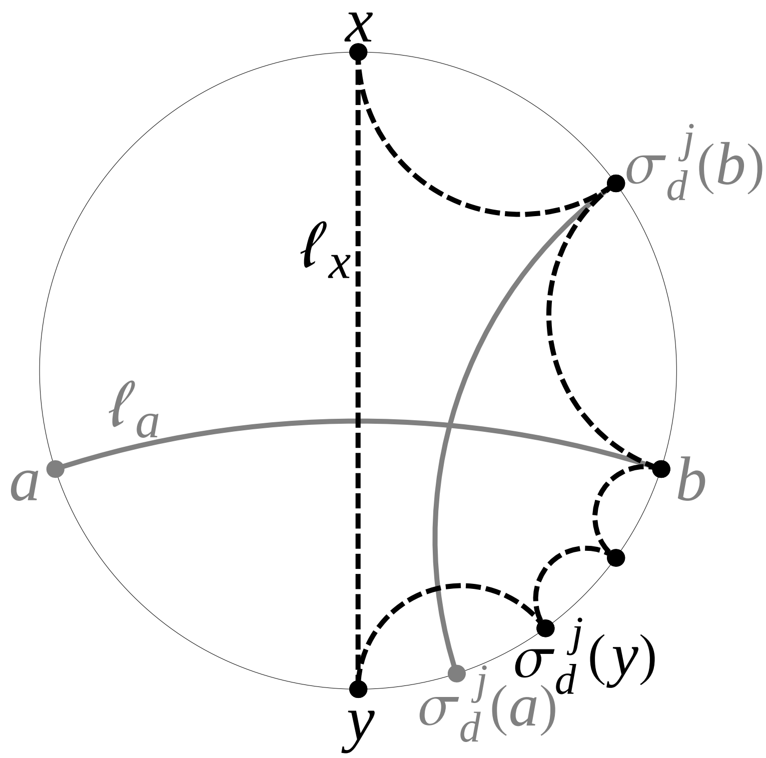

FIGURE 6. This figure illustrates the proof of Corollary 3.38.

PROOF. Three **distinct** images of ℓ_x cannot cross ℓ_a as if they do, then it is impossible for the separation required in Proposition 3.36 to occur for all of the pairs of images of ℓ_x. Hence at most two images of ℓ_x cross ℓ_a.

If two distinct leaves from the orbit of ℓ_x cross ℓ_a, then, by Proposition 3.36 and the order preservation, case (4) holds. Thus we can assume that $A = \{\ell_a, \ell_x\}$. If no forward image of ℓ_x is linked with ℓ_a, then we have case (1).

In all remaining cases we have $\sigma_d^k(\ell_x) = \ell_x$ for some $k > 0$. By Lemma 2.25, points x and y are of the same period. Suppose that x, y belong to the same periodic orbit. Choose the least j such that $\sigma_d^j(x) = y$.

Let us show that then $\sigma_d^j(y) = x$. Indeed, assume that $\sigma_d^j(y) \neq x$. Since by the assumption the only leaf from the forward orbit of ℓ_x, linked with ℓ_a, is ℓ_x, we may assume (for the sake of definiteness) that $y < \sigma_d^j(y) \leqslant b$. Then a finite concatenation of further σ_d^j-images of ℓ_x will connect y with x. Again, since $A = \{\ell_a, \ell_x\}$, one of their endpoints will coincide with b. Thus, $y < \sigma_d^j(y) \leqslant b < \sigma_d^j(b) \leqslant x$, see Figure 6. Let us now apply σ_d^j to A; by the order preservation $y < \sigma_d^j(a) < \sigma_d^j(y) \leqslant b < \sigma_d^j(b) \leqslant x < a$. Hence, $\sigma_d^j(\ell_a)$ is linked with ℓ_a, a contradiction.

Thus, $\sigma_d^j(y) = x$ (that is, σ_d^j flips ℓ_x onto itself), $k = j$, the points x and y are of period $2j$ and, by Lemma 3.37, the points a and b are also of period $2j$. If $\sigma_d^j(a) = b$, then $\sigma_d^j(b) = a$, and if $\sigma_d^j(b) = a$, then $\sigma_d^j(a) = b$ (since both points have period $2j$). Now, if

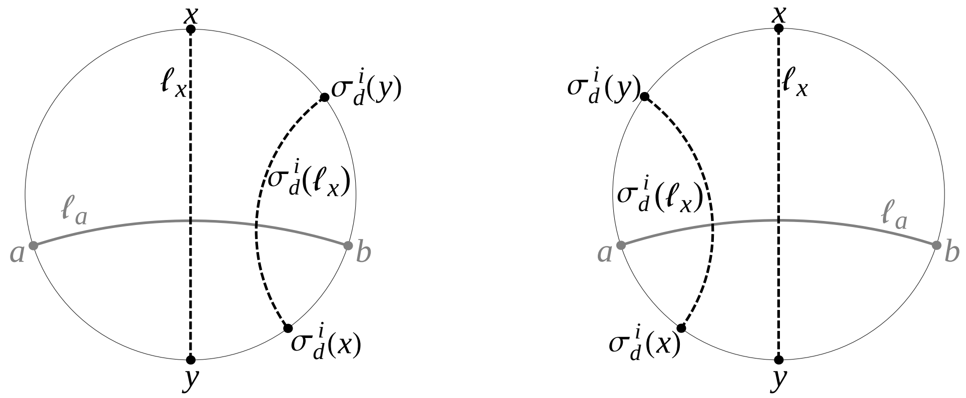

FIGURE 7. This figure shows two cases listed in Corollary 3.38, part (4).

$\sigma_d^j(a) \neq b$ and $\sigma_d^j(b) \neq a$, then, by the order preservation, ℓ_x separates the points a, $\sigma_d^j(b)$ from the points b, $\sigma_d^j(a)$. So, case (2) holds.

Assume that x and y belong to distinct periodic orbits of period k. By Lemma 3.37, the points a, b are of period k. Let points a and b have the same orbit. Then, if $k = 2i$ and σ_d^i flips ℓ_a onto itself, it would follow from the order preservation that $\sigma_d^i(\ell_x)$ is linked with ℓ_a. Since ℓ_x is the unique leaf from the orbit of ℓ_x linked with ℓ_a this would imply that σ_d^i flips ℓ_x onto itself, a contradiction with x, y having disjoint orbits. Hence we may assume that, for some j and $m > 2$, we have that $\sigma_d^j(a) = b$, $jm = k$, and a concatenation of leaves ℓ_a, $\sigma_d^j(\ell_a)$, $\dots$, $\sigma_d^{j(m-1)}(\ell_a)$ forms a polygon P.

If one of these leaves distinct from ℓ_a (say, $\sigma_d^{js}(\ell_a)$) is linked with ℓ_x, we can apply the map $\sigma_d^{j(m-s)}$ to $\sigma_d^{js}(\ell_a)$ and ℓ_x; by order preservation we will see then that ℓ_a and $\sigma_d^{j(m-s)}(\ell_x) \neq \ell_x$ are linked, a contradiction with the assumption that $A = \{\ell_a, \ell_x\}$. If none of the leaves $\sigma_d^j(\ell_a)$, $\dots$, $\sigma_d^{j(m-1)}(\ell_a)$ is linked with ℓ_x, then P has an endpoint of ℓ_x as one of its vertices. As in the argument given above, we can then apply σ_d^j to A and observe that, by the order preservation, the σ_d^j-image of ℓ_x is forced to be linked with ℓ_x, a contradiction. Hence a and b have disjoint orbits, and case (3) holds. □

3.3.3. Accordions are (pre-)periodic or wandering. Here we prove Theorem 3.42, which is the main result of Section 3.3.

Definition 3.39. A finite sequence of points $x_0, \dots, x_{k-1} \in \mathbb{S}$ is *positively ordered* if $x_0 < x_1 < \dots < x_{k-1} < x_0$. If the inequality is reversed, then we say that points x_0, $\dots$, $x_{k-1} \in \mathbb{S}$ are *negatively ordered*. A sequence y_0, y_1, $\dots$ is said to be *positively circularly ordered* if it is either positively ordered or there exists k such that $y_i = y_{i \bmod k}$ and $y_0 < y_1 < \dots < y_{k-1} < y_0$. Similarly we define sequences that are *negatively circularly ordered*.

A positively (negatively) **circularly** ordered sequence that is not positively (negatively) ordered is a sequence, whose points repeat themselves after the initial collection of points that are positively (negatively) ordered.

Definition 3.40. Suppose that the chords $\bar{t}_1$, $\dots$, $\bar{t}_n$ are edges of the closure Q of a single component of $\mathbb{D} \setminus \bigcup \bar{t}_i$. For each i, let m_i be the midpoint of the hole $H_Q(\bar{t}_i)$. We write

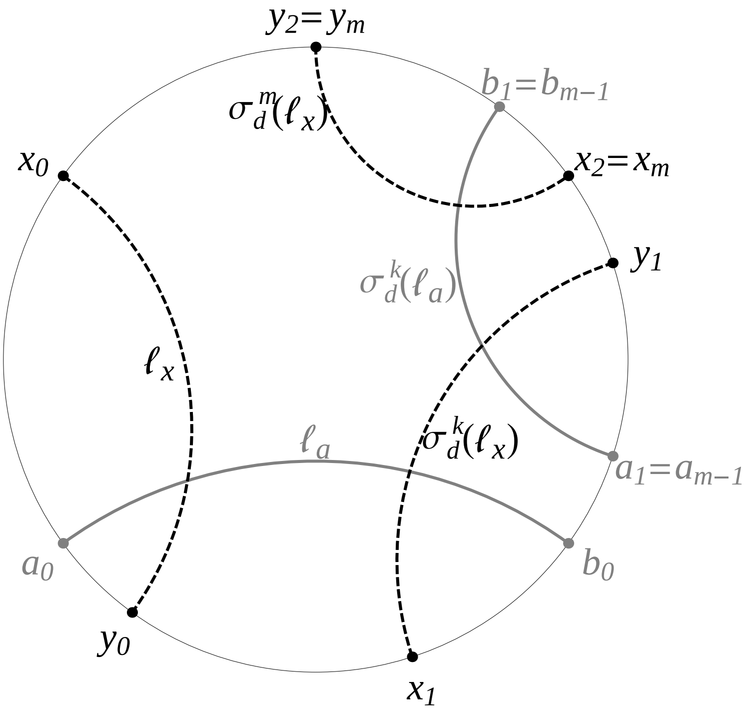

FIGURE 8. This figure illustrates Lemma 3.41. Images of ℓ_a cannot cross other images of ℓ_a, neither can they cross images of ℓ_x that are already linked with two images of ℓ_a (by Corollary 3.38). Similar claims hold for ℓ_x.

$\bar{t}_1 < \bar{t}_2 < \cdots < \bar{t}_n$ if the points m_i form a positively ordered set and call the chords $\bar{t}_1, \ldots, \bar{t}_n$ *positively ordered*. If the points m_i are positively circularly ordered, then we say that $\bar{t}_1, \ldots, \bar{t}_n$ are *positively circularly ordered*. *Negatively ordered* and *negatively circularly ordered* chords are defined similarly.

Lemma 3.41 is used in the main result of this section.

Lemma 3.41. *If ℓ_a and ℓ_x are linked, have mutually order preserving accordions, and $\sigma_d^k(\ell_x) \in A_{\ell_x}(\ell_a)$ for some $k > 0$, then, for every $j > 0$, the leaves $\sigma_d^{ki}(\ell_x)$, $i = 0, \ldots, j$, are circularly ordered, and ℓ_a, ℓ_x are periodic with endpoints of the same period.*

PROOF. By Lemma 3.37, we may assume that case (4) of Corollary 3.38 holds (and so $\sigma_d^k(\ell_x) \neq \ell_x$). Set $B = \{\ell_a, \ell_x\}$, $\ell_a = \overline{ab}, \ell_x = \overline{xy}$ and let a_i, b_i, x_i, y_i denote the σ_d^{ik}-images of a, b, x, y, respectively ($i \geqslant 0$). We may assume that the first possibility from case (4) holds and $x_0 < a_0 < y_0 \leqslant x_1 < b_0 < y_1 \leqslant x_0$ (see the left part of Figure 7 and Figure 8). By the assumption of mutually order preserving accordions applied to B, we have $x_i < a_i < y_i \leqslant x_{i+1} < b_i < y_{i+1} \leqslant x_i$ ($i \geqslant 0$), in particular $x_1 < a_1 < y_1$.

There are two cases depending on the location of a_1. Consider one of them as the other one can be considered similarly. Namely, assume that $b_0 < a_1 < y_1$ and proceed by induction for m steps observing that

$$x_0 < a_0 < y_0 \leqslant x_1 < b_0 \leqslant a_1 < \ldots \leqslant x_m < b_{m-1} < a_m < y_m \leqslant x_0.$$

Thus, the first m iterated σ_d^k-images of ℓ_x are circularly ordered and alternately linked with the first $m-1$ iterated images of ℓ_a under σ_d^k (see Figure 8). In the rest of the proof, we exploit the following fact.

Claim A. *Further images of ℓ_a or ℓ_x distinct from the already existing ones cannot cross the leaves $\ell_a, \sigma_d^k(\ell_x), \ldots, \sigma_d^{k(m-1)}(\ell_a), \sigma_d^{km}(\ell_x)$ because either it would mean that leaves from the same invariant geodesic lamination are linked, or it would contradict Corollary 3.38.*

By Claim A, we have $b_m \in (y_m, a_0]$. Consider possible locations of b_m.

(1) If $x_0 < b_m \leqslant a_0$, then $\overline{a_m b_m}$ is linked with $\overline{x_m y_m}$, $\overline{x_{m+1} y_{m+1}}$ and $\overline{x_0 y_0}$, which, by Corollary 3.38, implies that $\overline{x_{m+1} y_{m+1}} = \overline{x_0 y_0}$, and we are done (observe that, in this case, by Lemma 3.37, points a_0, b_0 are periodic of the same period as x_0, y_0).

(2) The case $x_0 = b_m$ is impossible because if $x_0 = b_m$, then, by the order preservation and by Claim A, the leaf $\overline{x_{m+1} y_{m+1}} = \sigma_d^{k(m+1)}(\ell_x)$ is forced to be linked with ℓ_a, a contradiction.

(3) Otherwise we have $y_m < b_m < x_0$ and hence, by the order preservation, $y_m \leqslant x_{m+1} < b_m$. Then, by Claim A and because images of ℓ_x do not cross, $b_m < y_{m+1} \leqslant x_0$. Suppose that $y_{m+1} = x_0$ while $y_0 \neq x_1$. Applying σ_d^k to leaves $\overline{x_{m+1} x_0}$ and $\overline{x_0 y_0}$ and using Claim A we see that $y_0 \leqslant x_{m+2} < x_1$. However, the order preservation then implies that $\overline{a_{m+1} b_{m+1}}$ crosses both $\overline{x_{m+1} x_0}$ and $\overline{x_{m+2} x_1}$ and therefore crosses ℓ_a itself, a contradiction. Hence the situation when y_{m+1} coincides with x_0 can only happen if $y_0 = x_1$. It follows that then $\sigma_d^k(\overline{x_{m+1} y_{m+1}}) = \overline{x_0 y_0}$, and we are done (as before, we need to rely on Lemma 3.37 here).

Otherwise $b_m < y_{m+1} < x_0$ and the arguments can be repeated as leaves $\sigma_d^{ki}(\ell_x), i = 0, \ldots, m+1$ are circularly ordered. Thus, either ℓ_x is periodic, $\overline{x_n y_n} = \overline{x_0 y_0}$ for some n, and all leaves in the σ_d^k-orbit of ℓ_x are circularly ordered, or the leaves $\overline{x_i y_i}$ converge monotonically to a point of $\mathbb{S}$. The latter is impossible since σ_d^k is expanding. By Lemma 3.37, the leaf ℓ_a is periodic and its endpoints have the same period as the endpoints of ℓ_x. $\square$

Theorem 3.42 is the main result of this section.

Theorem 3.42. *Consider linked chords $\ell_a = \overline{ab}$, $\ell_x = \overline{xy}$ with mutually order preserving accordions, and set $B = \mathrm{CH}(\ell_a, \ell_x)$. Suppose that not all forward images of B have pairwise disjoint interiors. Then there exists a finite periodic stand alone gap Q such that all vertices of Q are in the forward orbit of $\sigma_d^r(B)$ for some minimal r, they belong to two, three, or four distinct periodic orbits of the same period, and the remap of $Q \cap \mathbb{S}$ is not the identity unless $Q = \sigma_d^r(B)$ is a quadrilateral.*

PROOF. We may assume that there are two forward images of B with non-disjoint interiors. Choose the least r such that the interior of $\sigma_d^r(B)$ intersects some forward images of B. We may assume that $r = 0$ and, for some (minimal) $k > 0$, the interior of the set $\sigma_d^k(B)$ intersects the interior of B so that $\sigma_d^k(\ell_x) \in A_{\ell_x}(\ell_a)$. We write x_i, y_i for the endpoints of $\sigma_d^{ik}(\ell_x)$, and a_i, b_i for the endpoints of $\sigma_d^{ik}(\ell_a)$.

By Lemma 3.41 applied to both leaves, by the assumption of mutually order preserving accordions, and because leaves in the forward orbits of ℓ_a, ℓ_x are pairwise unlinked, we may assume without loss of generality that, for some $m \geqslant 1$,

$$x_0 < a_0 < y_0 \leqslant x_1 < b_0 \leqslant a_1 < \ldots \leqslant x_m < b_{m-1} \leqslant a_m < y_m < b_m$$

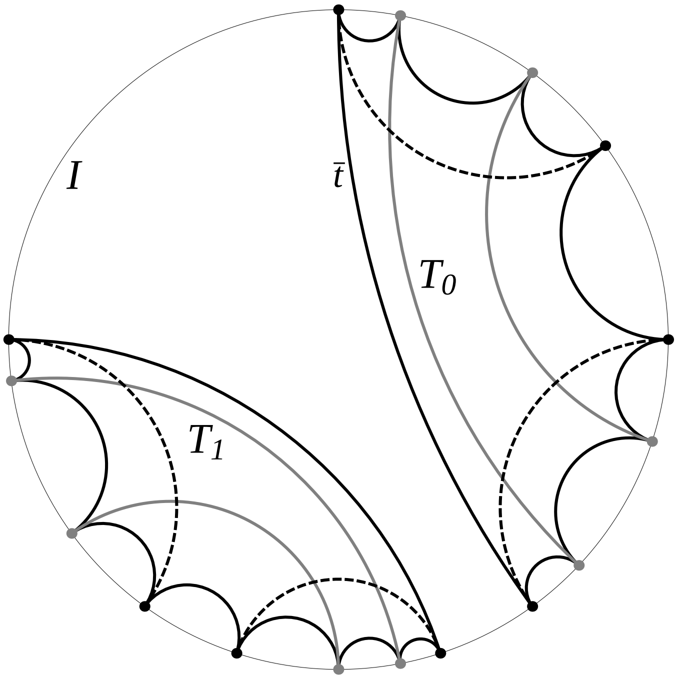

FIGURE 9. This figure illustrates the proof of Theorem 3.42 in the case $m > 1$.

and $x_m = x_0$, $y_m = y_0$, $a_m = a_0$, $b_m = b_0$, that is, we have the situation shown in Figure 8. Thus, for every $i = 0, \dots, k-1$, there is a loop L_i of alternately linked σ_d^k-images of $\sigma_d^i(\ell_a)$ and $\sigma_d^i(\ell_x)$. If the σ_d^k-images of $\sigma_d^i(\ell_a)$ are concatenated to each other, then their endpoints belong to the same periodic orbit, otherwise they belong to two distinct periodic orbits.

A similar claim holds for σ_d^k-images of $\sigma_d^i(\ell_x)$. Thus, the endpoints of B belong to two, three or four distinct periodic orbits of the same period (the latter follows by Corollary 3.38 and Lemma 3.41). Set $\mathrm{CH}(L_i) = T_i$ and consider some cases.

(1) Let $m > 1$ (this includes the "flipping" case from part (2) of Corollary 3.38). Let us show that the sets T_i either coincide or are disjoint. Every image $\hat{\ell}$ of ℓ_a in L_i crosses two images of ℓ_x in L_i (if $m = 2$ and ℓ_x is "flipped" by σ_d^k, we still consider ℓ_x and $\sigma_d^k(\ell_x)$ as distinct leaves). By Corollary 3.38, no other image of ℓ_x crosses $\hat{\ell}$.

Suppose that interiors of T_i and T_j intersect. Let $\bar{t}$ be an edge of T_i and $I = H_{T_i}(\bar{t})$ be the corresponding hole of T_i. Then the union of two or three images of ℓ_a or ℓ_x from L_i **separates** I from $\mathbb{S} \setminus I$ in $\overline{\mathbb{D}}$ (meaning that any curve connecting I with $\mathbb{S} \setminus I$ must intersect the union of these two or three images of ℓ_a or ℓ_x, see Figure 9). Hence if there are vertices of T_j in I **and** in $\mathbb{S} \setminus I$ then there is a leaf of L_j crossing leaves of L_i, a contradiction with the above and Corollary 3.38.

Thus, the only way $T_i \neq T_j$ can intersect is if they share a vertex or an edge. We claim that this is impossible. Indeed, $T_i \neq T_j$ cannot share a vertex as otherwise this vertex must

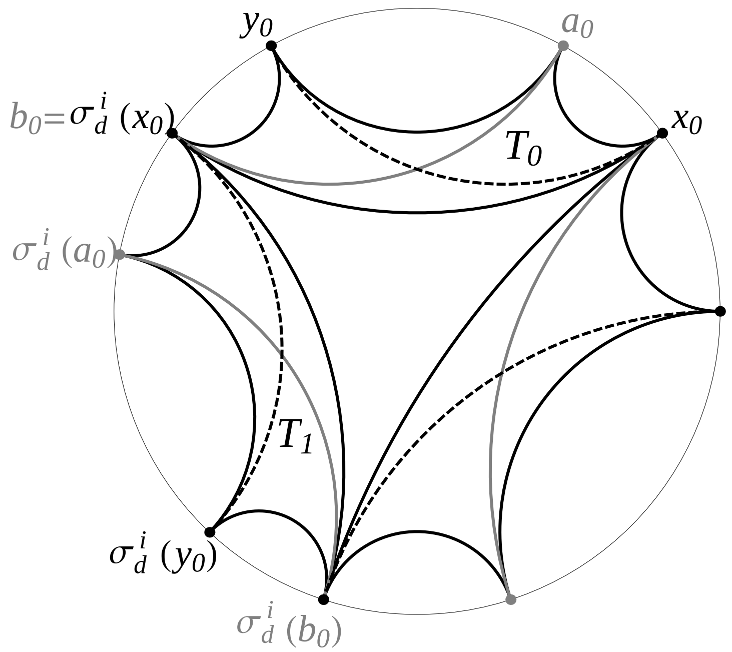

FIGURE 10. This figure illustrates the proof of Theorem 3.42 in the case $m = 1$.

be σ_d^k-invariant while all vertices of any T_r map to other vertices (sets T_r "rotate" under σ_d^k). Finally, if T_i and T_j share an edge ℓ then the same argument shows that σ_d^k cannot fix the endpoints of ℓ, hence it "flips" under σ_d^k. However this is impossible as each set T_r has at least four vertices and its edges "rotate" under σ_d^k.

So, the component Q_i of $X = \bigcup_{i=0}^{k-1} T_i$ containing $\sigma_d^i(\ell_a)$ is T_i. By Lemma 3.41, the map $\sigma_d|_{T_i \cap \mathbb{S}}$ is order preserving or reversing. As σ_d preserves order on any single accordion, $\sigma_d|_{T_i \cap \mathbb{S}}$ is order preserving. The result now follows; note that the first return map on Q is not the identity map.

(2) Let $m = 1$. This corresponds to part (3) of Corollary 3.38: both ℓ_a and ℓ_x have endpoints of minimal period k, and the orbit of ℓ_a (ℓ_x) consists of k pairwise disjoint leaves. Note that T_0 is a quadrilateral, and the first return map on T_0 is the identity map. Consider the case when not all sets T_i are pairwise disjoint. Note that, by the above, T_0 is a periodic stand alone gap satisfying the assumptions of Proposition 2.30. It follows that every component of the union of T_i is a concatenation of gaps sharing edges with the same polygon. See Figure 10, in which the polygon is a triangle.

□

For a leaf $\ell_1 \in \mathcal{L}_1$, let $\mathcal{B}_{\mathcal{L}_2}(\ell_1)$ be the collection of all leaves $\ell_2 \in \mathcal{L}_2$ that are linked with ℓ_1 and have mutually order preserving accordions with ℓ_1. Observe that if ℓ_1 is (pre)critical, then $\mathcal{B}_{\mathcal{L}_2}(\ell_1) = \varnothing$ by Definition 3.34. Similarly, no leaf from $\mathcal{B}_{\mathcal{L}_2}(\ell_1)$ is (pre)critical.

Corollary 3.43. *The collection $\mathcal{B}_{\mathcal{L}_2}(\ell_1)$ is finite.*

PROOF. Suppose first that ℓ_1 is not (pre)periodic. Let us show that the convex hull B of ℓ_1 and leaves $\overline{n}_1, \dots, \overline{n}_s$ from $\mathcal{B}_{\mathcal{L}_2}(\ell_1)$ is wandering. By Theorem 3.42, for each

i, the set $B_i = \mathrm{CH}(\ell_1, \overline{n}_i)$ is wandering (because ℓ_1 is not (pre)periodic). This implies that if $i \neq j$ then $\sigma_d^i(\ell_1)$ and $\sigma_d^j(\overline{n}_t)$ are disjoint (otherwise $\sigma_d^i(B_t)$ and $\sigma_d^j(B_t)$ are non-disjoint). Moreover, $\sigma_d^i(\ell_1)$ and $\sigma_d^j(\ell_1)$ are disjoint as otherwise, by Lemma 2.27, the leaf ℓ_1 is (pre)periodic. Therefore $\sigma_d^j(\ell_1)$ is disjoint from $\sigma_d^i(B)$.

Suppose that $\sigma_d^i(B)$ and $\sigma_d^j(B)$ are non-disjoint. By the just proven then, say, $\sigma_d^j(\overline{n}_1)$ is non-disjoint from $\sigma_d^i(B)$. Again by the just proven $\sigma_d^j(\overline{n}_1)$ is disjoint from $\sigma_d^i(\ell_1)$. Hence the only possible intersection is between $\sigma_d^j(\overline{n}_1)$ and, say, $\sigma_d^i(\overline{n}_2)$. Moreover, since $\sigma_d^j(\ell_1)$ is disjoint from $\sigma_d^i(B)$, then $\sigma_d^j(\overline{n}_1) \neq \sigma_d^i(\overline{n}_2)$ and, moreover, as distinct leaves of the same invariant geodesic lamination, the leaves $\sigma_d^j(\overline{n}_1), \sigma_d^i(\overline{n}_2)$ cannot cross. Hence the only way $\sigma_d^j(\overline{n}_1)$ and $\sigma_d^i(\overline{n}_2)$ are non-disjoint is that $\sigma_d^j(\overline{n}_1)$ and $\sigma_d^i(\overline{n}_2)$ are concatenated.

Assume that $\sigma_d^t(\overline{n}_2)$ is concatenated with $\overline{n}_1$ at an endpoint x of $\overline{n}_1$. Clearly, x is a common vertex of B and of $\sigma_d^t(B)$. Hence $\sigma_d^t(x)$ is a common vertex of $\sigma_d^t(B)$ and $\sigma_d^{2t}(B)$, etc. Connect points x, $\sigma_d^t(x)$, $\sigma_d^{2t}(x)$, $\dots$ with consecutive chords $\overline{m}_0$, $\overline{m}_1$, $\dots$. These chords are pairwise unlinked because, as it follows from the above, the sets $\sigma_d^r(B)$, $r = 0, 1, \dots$ have pairwise disjoint interiors. Hence, by Lemma 2.26, the point x is (pre)periodic, a contradiction with the fact that all sets $B_i = \mathrm{CH}(\ell_1, \overline{n}_i)$ are wandering. Thus, B is wandering. Hence, by [**Kiw02**], the collection $\mathcal{B}_{\mathcal{L}_2}(\ell_1)$ is finite. In fact, [**Kiw02**] implies a nice upper bound on the number of vertices of B. Indeed, it is proven in [**Kiw02**] that a wandering non-(pre)critical gap of a lamination has at most d vertices; in particular, B has at most d vertices (notice that by the assumptions any power of σ_d in B is one-to-one).

Suppose now that ℓ_1 is periodic. Then by Theorem 3.42 any leaf of $\mathcal{B}_{\mathcal{L}_2}(\ell_1)$ is periodic with the same periods of endpoints. This implies that in this case the collection $\mathcal{B}_{\mathcal{L}_2}(\ell_1)$ is finite. Finally, if $k > 0$ is the minimal number such that $\sigma_d^k(\ell_1)$ is periodic and $\ell_2 \in \mathcal{B}_{\mathcal{L}_2}(\ell_1)$ then $\sigma_d^k(\ell_2)$ is linked with $\sigma_d^k(\ell_1)$, which implies that ℓ_2 is a σ_d^k-preimage of one of finitely many leaves from $\mathcal{B}_{\mathcal{L}_2}(\sigma_d^k(\ell_1))$. Thus, in this case $\mathcal{B}_{\mathcal{L}_2}(\ell_1)$ is finite too. □

3.4. Smart criticality

Throughout this section, we assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ are linked or essentially equal geodesic invariant laminations with quadratically critical portraits, see Definition 3.10. Our aim in Section 3.4 is to introduce *smart criticality*, a principle that allows one to use a flexible choice of critical chords of $\mathcal{L}_1$ and $\mathcal{L}_2$ in order to treat certain sets of linked leaves of $\mathcal{L}_1$ and $\mathcal{L}_2$ as if they were sets of one invariant geodesic lamination. However, first we need simple claims dealing with critical clusters and special critical leaves; these claims follow from the definitions almost immediately. Critical clusters are defined in 3.9 and special critical leaves and special critical clusters in 3.10.

Lemma 3.44. *Suppose that ℓ_1 is a special critical leaf of $\mathcal{L}_1$. Then the only leaves of $\mathcal{L}_2$ it can be linked with are special critical leaves of $\mathcal{L}_2$. Moreover these leaves have the same image as ℓ_1. Otherwise ℓ_1 may have a common endpoint with some leaves of $\mathcal{L}_2$, in which case its forward images are endpoints of the corresponding images of these leaves.*

PROOF. By definition, if ℓ_1 is a special critical leaf then $\ell_1 \subset C$ where C is a critical cluster common for both $\mathcal{L}_1$ and $\mathcal{L}_2$. Since edges of C are leaves of $\mathcal{L}_2$, it follows that the only leaves of $\mathcal{L}_2$ that are linked with ℓ_1 are chords of C connecting vertices of C. This implies the first claim of the lemma. The second claim is left to the reader. □

In the next several lemmas we study the dynamics of a leaf ℓ_1 of $\mathcal{L}_1$ assuming that ℓ_1 is not a special critical leaf of $\mathcal{L}_1$.

Lemma 3.45. *If $\ell_1 \in \mathcal{L}_1$ is not a special critical leaf, then each critical set C of QCP_2 has a spike $\overline{c}$ unlinked with ℓ_1; these spikes form a full collection $\mathcal{E}$ of spikes of $\mathcal{L}_2$ unlinked with ℓ_1. If an endpoint x of ℓ_1 is neither a vertex of a special critical cluster nor a common vertex of associated critical quadrilaterals of the invariant geodesic laminations $\mathcal{L}_1$ and $\mathcal{L}_2$, then $\mathcal{E}$ can be chosen so that x is not an endpoint of a spike from $\mathcal{E}$.*

PROOF. Since ℓ_1 is not a special critical leaf, spikes of $\mathcal{L}_2$ from special critical clusters are unlinked with ℓ_1. Otherwise take a pair of associated critical quadrilaterals $A \in \mathcal{L}_1, B \in \mathcal{L}_2$ with vertices alternating non-strictly on $\mathbb{S}$

$$a_0 \leqslant b_0 \leqslant a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant a_3 \leqslant b_3 \leqslant a_0$$

and observe, that ℓ_1 is contained, say, in $[a_0, a_1]$ and hence is unlinked with the spike $\overline{b_1 b_3}$ of B.

The second claim follows because by the assumptions, as we choose a spike from a critical quadrilateral of QCP_2, we can always choose it to avoid x. This completes the proof. □

We apply Lemma 3.45 to studying accordions. Denote by $\mathcal{E}_{\mathcal{L}_2}(\ell_1)$ a full collection of spikes from Lemma 3.45.

Corollary 3.46. *If $\ell_1 = \overline{ab} \in \mathcal{L}_1$ is not a special critical leaf, then $A = A_{\mathcal{L}_2}(\ell_1)$ is contained in the closure of a component of $\mathbb{D} \setminus \mathcal{E}_{\mathcal{L}_2}(\ell_1)^+$, and $\sigma_d|_{A\cap\mathbb{S}}$ is (non-strictly) monotone. Let $\ell_2 = \overline{xy} \in \mathcal{L}_2$ and $\ell_1 \cap \ell_2 \neq \varnothing$. Then:*

1. *if ℓ_1 and ℓ_2 are concatenated at a point x that is neither a vertex of a special critical cluster nor a common vertex of associated critical quadrilaterals of our invariant geodesic laminations, then σ_d is (non-strictly) monotone on $\ell_1 \cup \ell_2$;*
2. *if ℓ_2 crosses ℓ_1, then, for each i, we have $\sigma_d^i(\ell_1) \cap \sigma_d^i(\ell_2) \neq \varnothing$, and one the following holds:*
   - (a) *$\sigma_d^i(\ell_1) = \sigma_d^i(\ell_2)$ is a point or a leaf shared by $\mathcal{L}_1, \mathcal{L}_2$;*
   - (b) *$\sigma_d^i(\ell_1), \sigma_d^i(\ell_2)$ share an endpoint;*
   - (c) *$\sigma_d^i(\ell_1), \sigma_d^i(\ell_2)$ are linked and have the same order of endpoints as ℓ_1, ℓ_2;*
3. *points a, b, x, y are either all (pre)periodic of the same eventual period, or are all not (pre)periodic.*

PROOF. Set $\mathcal{E} = \mathcal{E}_{\mathcal{L}_2}(\ell_1)$. If ℓ_1 coincides with one of spikes from $\mathcal{E}$, then the claim follows (observe that then by definition $A = \ell_1$ as spikes of sets of $\mathcal{L}_2$ do not cross leaves of $\mathcal{L}_2$). Otherwise there exists a unique complementary component Y of $\mathcal{E}^+$ with $\ell_1 \subset Y$ (except perhaps for the endpoints). The fact that each leaf of $\mathcal{L}_2$ is unlinked with spikes from $\mathcal{E}$ implies that $A_{\mathcal{L}_2}(\ell_1) \subset \overline{Y}$. This proves the main claim of the lemma.

(1) By Lemma 3.45, the collection $\mathcal{E}$ can be chosen so that x is not an endpoint of a chord from $\mathcal{E}$. The construction of Y then implies that σ_d is monotone on $\ell_1 \cup \ell_2$.

(2) We use induction. By Definition 3.10, if a critical leaf $\overline{n}_1 \in \mathcal{L}_1$ crosses a leaf $\overline{m}_2 \in \mathcal{L}_2$ and comes from a special critical cluster (see Definition 3.9), then both $\overline{n}_1$ and $\overline{m}_2$ come from a special critical cluster and have the same image. Thus we may assume that neither $\sigma_d^i(\ell_1)$ nor $\sigma_d^i(\ell_2)$ are from a special critical cluster. We may also assume that $\sigma_d^i(\ell_1)$ and $\sigma_d^i(\ell_2)$ do not share an endpoint as otherwise the claim is obvious. Hence it remains to consider the case when $\sigma_d^i(\ell_1)$ and $\sigma_d^i(\ell_2)$ are linked and are not special critical leaves. Then by the main claim either their images are linked or at least they share an endpoint.

(3) By Lemma 2.27, if an endpoint of a leaf of an invariant geodesic lamination is (pre)periodic, then so is the other endpoint of the leaf. Consider two cases. Suppose first that an image of ℓ_1 and an image of ℓ_2 "collide" (that is, have a common endpoint z). By the above, if z is (pre)periodic, then all endpoints of our leaves are, and if z is not (pre)periodic, then all endpoints of our leaves are not (pre)periodic. Suppose now that no two images of ℓ_1, ℓ_2 collide. Then it follows that ℓ_1 and ℓ_2 have mutually order preserving accordions, and the claim follows from Theorem 3.42. □

Lemma 3.45 and Corollary 3.46 implement smart criticality. Indeed, given an invariant geodesic lamination $\mathcal{L}$, a finite gap or leaf G of it is such that the set $G \cap \mathbb{S}$ (loosely) consists of points whose orbits avoid critical sets of $\mathcal{L}$. It follows that any power of the map is order preserving on $G \cap \mathbb{S}$. It turns out that we can treat sets X formed by linked leaves of two linked or essentially equal invariant geodesic laminations similarly by varying our choice of the full collection of spikes at each step so that the orbit of X avoids **that particular** full collection of spikes at **that particular** step (thus **smart** criticality). Therefore, similarly to the case of one invariant geodesic lamination, any power of the map is order preserving on X. This allows one to treat such sets X almost as sets of one invariant geodesic lamination.

Combining Corollary 3.46 and Corollary 3.44 we obtain Corollary 3.47.

Corollary 3.47. *Suppose that $\ell_1 \in \mathcal{L}_1, \ell_2 \in \mathcal{L}_2$; moreover, let ℓ_1 and ℓ_2 be non-disjoint. Then $\sigma_d^n(\ell_1)$ and $\sigma_d^n(\ell_2)$ are non-disjoint for any $n \geqslant 0$.*

The proof of Corollary 3.47 is left to the reader.

The purpose of our investigation is to see how much two linked (or essentially equal) geodesic laminations can differ. In other words, we study the rigidity of geodesic laminations with respect to their quadratically critical portraits (we consider quadratically critical portraits as critical data associated with the corresponding geodesic lamination). In fact, we can already discuss the extent to which geodesic laminations $\mathcal{L}_1$ and $\mathcal{L}_2$ differ in the particular case of periodic Siegel gaps. Recall the notions of the skeleton, decorations and the extension of an infinite gap G introduced in Definitions 2.32 and 2.34. The skeleton of G is the convex hull of the maximal Cantor subset of $G \cap \mathbb{S}$. Decorations of G are the convex hulls of maximal connected unions of leaves attached to edges of G. The extension of G is the union of G and all its decorations. Recall also that, by Lemma 2.33, for every edge ℓ of a decoration of G, there is a gap in the grand orbit of G that has ℓ on its boundary.

Lemma 3.48. *Gaps from the grand orbits of periodic Siegel gaps of $\mathcal{L}_1$ and $\mathcal{L}_2$ can be paired up so that gaps in the same pair have the same skeletons and the same decorations.*

PROOF. Let G be a periodic Siegel gap of $\mathcal{L}_1$ of period n. Let H be the skeleton of G. Below when talking about fibers we mean fibers (point-preimages) of the semiconjugacy between σ_d^n restricted onto the extension of G and the corresponding irrational rotation. Suppose that ℓ_2 is a leaf of $\mathcal{L}_2$ that intersects the extension of G. If ℓ_2 intersects two distinct fibers of the extension of G, then, by Corollary 3.47, the σ_d^n-images of ℓ_2 will keep intersecting the σ_d^n-images of these fibers. The fact that σ_d^n restricted onto the extension of G is semiconjugate to an irrational rotation implies then that some images of ℓ_2 are linked with each other, a contradiction.

Thus, ℓ_2 and all its images intersect exactly one fiber. For geometric reasons this is equivalent to the fact that no leaf of $\mathcal{L}_2$ intersects the interior of H. Therefore, there exists a gap G_2 of $\mathcal{L}_2$ that contains H. Since $\mathcal{L}_2$ is a quadratically critical geodesic lamination, it cannot have infinite gaps of degree two. It follows that G_2 is also a periodic Siegel gap of period n with the same skeleton H as G.

Consider a gap T of $\mathcal{L}_1$ such that for some minimal $m \geqslant 0$ we have that $\sigma_d^m(T) = G$. The properties of geodesic laminations, the fact that $\mathcal{L}_1$ has no infinite critical gaps, and the fact that $H \cap \mathbb{S}$ is a Cantor set imply that T maps onto G with degree one and the maximal Cantor subset S of $T \cap \mathbb{S}$ maps onto $H \cap \mathbb{S}$ one-to-one except, perhaps, for the endpoints a, b of σ_d^m-critical chords such that (a, b) is a complementary arc of S (clearly, there are at most finitely many such pairs of points a, b). Each hole (a, b) of S corresponds to a finite concatenation of leaves of T connecting a and b with endpoints in $[a, b]$. Properties of geodesic laminations imply that there are maximal connected finite concatenations of leaves growing from a and b, and their unions map onto the corresponding decorations of G.

Now, if there exists a leaf ℓ_2 of $\mathcal{L}_2$ that intersects the convex hull of S, then by Corollary 3.47 the leaf $\sigma_d^m(\ell_2)$ connects two distinct fibers on the boundary of G, a contradiction. Therefore there exists a gap T_2 of $\mathcal{L}_2$ that contains S in $T_2 \cap \mathbb{S}$ as its maximal Cantor subset (observe that the arguments can be repeated in the opposite direction, which shows that S is a common maximal Cantor set in $T \cap \mathbb{S}$ and in $T_2 \cap \mathbb{S}$). It follows that σ_d^m maps T_2 onto G_2 with degree one. Since this argument does not depend on the choice of T and the corresponding choice of m, we see that the grand orbit of the gap G and the grand orbit of the gap G_2 consist of pairs of infinite gaps that share the same skeleton (because they share maximal Cantor subsets of their intersections with the unit circle).

The description of the dynamics of extensions of periodic Siegel gaps implies that given a decoration A of a periodic Siegel gap Q, we see a finite collection of eventual preimages of Q attached to this decoration so that the following holds: the convex hull of A has finitely many edges at each of which a skeleton of the corresponding preimage of Q is attached. Since by the above the family of skeletons of gaps from the grand orbits of periodic Siegel disks is the same for both $\mathcal{L}_1$ and $\mathcal{L}_2$, we conclude that the family of convex hulls of decorations of these gaps is also the same. □

Let us now continue studying orbits of pairs of non-disjoint leaves of geodesic laminations $\mathcal{L}_1$ ad $\mathcal{L}_2$. Lemma 3.49 describes how σ_d can be **non-strictly** monotone on $A \cap \mathbb{S}$ taken from Corollary 3.46. A concatenation $\mathcal{R}$ of spikes of an invariant geodesic lamination $\mathcal{L}$ such that the endpoints of its chords are monotonically ordered on the circle will be called a *chain of spikes* (*of* $\mathcal{L}$). Recall that for a collection of chords of $\mathbb{D}$ such as $\mathcal{R}$ we use $\mathcal{R}^+$ to denote $\bigcup \mathcal{R}$.

Lemma 3.49. *Suppose that $\ell_a = \overline{ab} \in \mathcal{L}_1$ and $\ell_x = \overline{xy} \in \mathcal{L}_2$, where $a < x < b \leqslant y < a$ (see Figure 11) and, if $b = y$, then b is neither a vertex of a special critical cluster nor a common vertex of associated critical quadrilaterals of our invariant geodesic laminations. Suppose also that $\sigma_d(a) = \sigma_d(x)$. Then either both ℓ_a, ℓ_x are contained inside the same special critical cluster, or there are chains of spikes $\mathcal{R}_1$ of $\mathcal{L}_1$ and $\mathcal{R}_2$ of $\mathcal{L}_2$ connecting a with x. If one of the leaves ℓ_a, ℓ_x is not critical, then we may assume that $\mathcal{R}_1^+ \cap \mathbb{S} \subset [a, x]$ and that $\mathcal{R}_2^+ \cap \mathbb{S} \subset [a, x]$. In any case, the points a and x belong to the critical sets of both laminations.*

Recall that, according to our terminology, a chord is contained *inside* S if it is a subset of S intersecting the interior of S.

PROOF. First assume that one of the leaves ℓ_a, ℓ_x (say, ℓ_a) is a special critical leaf. Then both a and b are vertices of a special critical cluster. By the assumptions, this implies that $b \neq y$ and hence ℓ_a and ℓ_x are linked and are inside a special critical cluster. Assume from now on that neither ℓ_a nor ℓ_x is a special critical leaf.

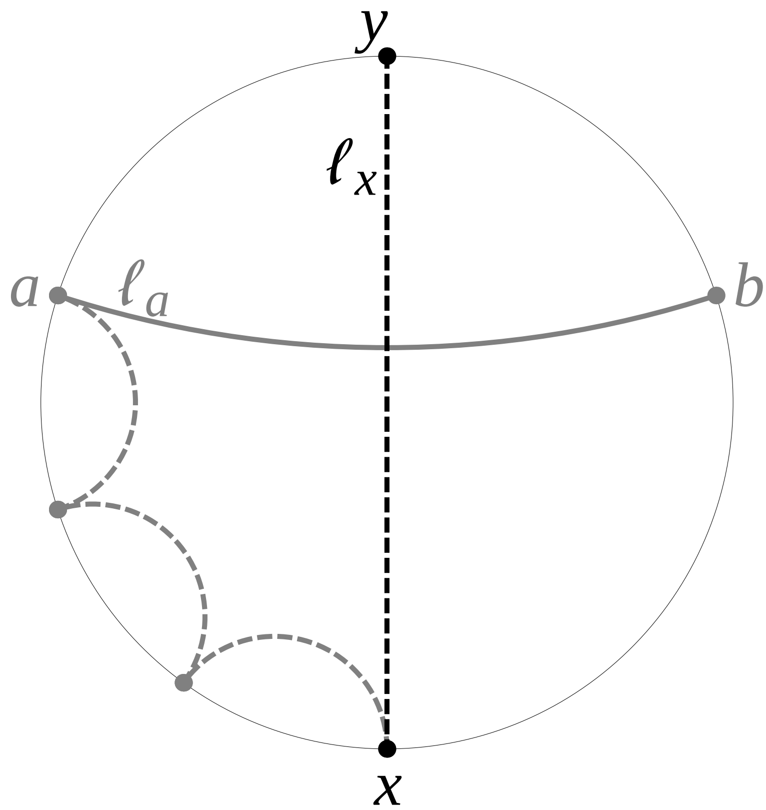

FIGURE 11. This figure illustrates Lemma 3.49. Here the leaves ℓ_a, ℓ_x collapse around a chain of spikes shown as dashed grey geodesics.

By Lemma 3.45, choose a full collection $\mathcal{A}_2$ of spikes of $\mathcal{L}_2$ unlinked with ℓ_a and a full collection $\mathcal{A}_1$ of spikes of $\mathcal{L}_1$ unlinked with ℓ_x. By the assumptions and Lemma 3.45, we may choose these collections so that if $b = y$, then $b = y \notin \mathcal{A}_1^+ \cup \mathcal{A}_2^+$. Thus in any case the point $\ell_a \cap \ell_x = w \in \mathbb{D}$ does not belong to $\mathcal{A}_1^+ \cup \mathcal{A}_2^+$.

It follows that there is a well-defined component Y of $\overline{\mathbb{D}} \setminus [\mathcal{A}_1^+ \cup \mathcal{A}_2^+]$ containing $\ell_a \cup \ell_x$ except perhaps for the endpoints. Since $\sigma_d(a) = \sigma_d(x)$, there is a chain of spikes $\mathcal{R}_2 \subset \mathcal{A}_2$ of $\mathcal{L}_2$ and a chain of spikes $\mathcal{R}_1 \subset \mathcal{A}_1$ of $\mathcal{L}_1$ connecting a and x. In particular, $a \in \mathcal{A}_1$, $x \in \mathcal{A}_2$, and both a and x must belong to the critical sets of both laminations.

Suppose that, say, $\mathcal{R}_1^+ \cap \mathbb{S} \subset [x, a]$. Since all spikes are critical chords that cross neither ℓ_a nor ℓ_x, this implies that both ℓ_a and ℓ_x are critical. Therefore, if at least one of the leaves ℓ_a, ℓ_x is not critical, then we may assume that $\mathcal{R}_1^+ \cap \mathbb{S} \subset [a, x]$ and that $\mathcal{R}_2^+ \cap \mathbb{S} \subset [a, x]$. $\square$

The assumptions of Lemma 3.49 automatically hold if leaves ℓ_a, ℓ_x are linked and one of them (say, ℓ_a) is critical; in this case, by Corollary 3.46, the point $\sigma_d(\ell_a)$ is an endpoint of $\sigma_d(\ell_x)$, and, renaming the points, we may assume that $\sigma_d(a) = \sigma_d(x)$.

Definition 3.50. Non-disjoint leaves $\ell_1 \neq \ell_2$ are said to *collapse around chains of spikes* if there are two chains of spikes, one in each of the two invariant geodesic laminations, connecting two adjacent endpoints of ℓ_1, ℓ_2 as in Lemma 3.49.

Smart criticality allows one to treat accordions as gaps of one invariant geodesic lamination provided images of leaves do not collapse around chains of spikes.

Lemma 3.51. *Let ℓ_1, ℓ_2 be linked leaves from $\mathcal{L}_1$, $\mathcal{L}_2$ such that there is no t with $\sigma_d^t(\ell_1)$, $\sigma_d^t(\ell_2)$ collapsing around chains of spikes (in particular, this holds if the endpoints of $\sigma_d^t(\ell_1)$ are disjoint from the endpoints of $\sigma_d^t(\ell_2)$ for all t). Then there exists an N such that the σ_d^N-images of ℓ_1, ℓ_2 are linked and have mutually order preserving accordions. Conclusions of Theorem 3.42 hold for ℓ_1, ℓ_2, and $B = \mathrm{CH}(\ell_1, \ell_2)$ is either wandering or (pre)periodic so that ℓ_1, ℓ_2 are (pre)periodic of the same eventual period of endpoints.*

PROOF. By way of contradiction, suppose that there exists the minimal t such that $\sigma_d^{t+1}(\ell_1)$ is not linked with $\sigma_d^{t+1}(\ell_2)$. Then $\sigma_d^t(\ell_1)$ crosses $\sigma_d^t(\ell_2)$ while their images have a common endpoint. Hence Lemma 3.49, applied to $\sigma_d^t(\ell_1)$ and $\sigma_d^t(\ell_2)$, implies that $\sigma_d^t(\ell_1)$, $\sigma_d^t(\ell_2)$ collapse around a chain of spikes, a contradiction. Thus, $\sigma_d^t(\ell_1)$ and $\sigma_d^t(\ell_2)$ cross for any $t \geqslant 0$. In particular, no image of either ℓ_1 or ℓ_2 is ever critical.

By Lemma 2.27, choose N so that leaves $\sigma_d^N(\ell_1) = \overline{ab}$ and $\sigma_d^N(\ell_2) = \overline{xy}$ are periodic or have no (pre)periodic endpoints. If $\overline{ab}$ and $\overline{xy}$ are periodic, then no collapse around chains of critical leaves on **any** images of $\overline{ab}, \overline{xy}$ is possible (for set-theoretic reasons). Hence $\sigma_d^N(\ell_1), \sigma_d^N(\ell_2)$ are linked and have mutually order preserving accordions as desired.

Suppose now that our leaves have non-(pre)periodic endpoints. Evidently, the set E of all endpoints of all possible chains of spikes is finite. Thus, there exists an N such that if $n \geqslant N$, then $\sigma_d^n(a)$ is disjoint from E as otherwise by the pigeonhole principle a would have to be (pre)periodic. The same holds for b, x and y, so we may assume that, for $n \geqslant N$, no endpoint of $\sigma_d^n(\ell_1)$ or $\sigma_d^n(\ell_2)$ is in E. Hence, the σ_d^N-images of ℓ_1, ℓ_2 are linked and have mutually order preserving accordions. □

3.5. Linked quadratically critical invariant geodesic laminations

The main results of Section 3.5 are based on the principle of Smart Criticality and the results describing the dynamics of accordions. Basically, we are studying two linked or essentially equal invariant geodesic laminations with quadratically critical portraits and establish the extent to which they must resemble each other. Therefore our results can be viewed as rigidity results of certain subsets (or certain dynamical properties) of geodesic invariant laminations with respect to their linked perturbations. For instance, we show that two linked or essentially equal invariant geodesic laminations with quadratically critical portraits have the same *perfect parts* (see Definition 3.11). We also show that two linked or essentially equal invariant geodesic laminations with quadratically critical portraits have the same *Siegel parts* defined below (see also page 7).

Definition 3.52. The closure of the union of the grand orbits of all periodic Siegel gaps of an invariant geodesic lamination $\mathcal{L}$ is denoted by $\mathcal{L}^{Sie}$ and is called the *Siegel part* of $\mathcal{L}$.

However the relations between the remaining parts of two linked or essentially equal invariant geodesic laminations are less rigid. We study them in the next section concentrating upon the case when invariant geodesic laminations are generated by invariant laminational equivalence relations.

In this section, we will always assume that the invariant geodesic laminations with quadratically critical portraits $(\mathcal{L}_1, \mathrm{QCP}_1)$ and $(\mathcal{L}_2, \mathrm{QCP}_2)$ are linked or essentially equal.

The next lemma studies cardinalities of certain collections of leaves.

Lemma 3.53. *Suppose that invariant geodesic laminations $(\mathcal{L}_1, \mathrm{QCP}_1)$ and $(\mathcal{L}_2, \mathrm{QCP}_2)$ with quadratically critical portraits are linked or essentially equal. The set $\mathcal{T}$ of all leaves of $\mathcal{L}_2$ non-disjoint from a leaf ℓ_1 of $\mathcal{L}_1$ is at most countable. Thus, if ℓ is a leaf on which uncountably many leaves of one of the geodesic laminations $\mathcal{L}_1$ or $\mathcal{L}_2$ accumulate then ℓ is unlinked with any leaf of the other geodesic lamination.*

PROOF. If ℓ_1 has (pre)periodic endpoints, then, by Corollary 3.46, any leaf of $\mathcal{L}_2$ non-disjoint from ℓ_1 has (pre)periodic endpoints implying the first claim of the lemma in this case. Let ℓ_1 have no (pre)periodic endpoints. Then, by Corollary 3.46, leaves of $\mathcal{L}_2$ non-disjoint from ℓ_1 have no (pre)periodic endpoints. By Lemma 2.29, for every eventual image x of an endpoint of ℓ_1 there are finitely many leaves with endpoint x. Hence the set of all leaves of $\mathcal{L}_2$ with endpoint whose orbit collides with the orbit of an endpoint of ℓ_1 is countable. If we remove them from $\mathcal{T}$, then we obtain a new collection $\mathcal{T}'$ of leaves; by Lemma 3.51, they have mutually order preserving accordions with ℓ_1. By Corollary 3.43, the collection $\mathcal{T}'$ is finite. This completes the proof of the first claim of the lemma. The second claim follows immediately. □

Let QCP be a quadratically critical portrait of an invariant geodesic lamination $\mathcal{L}$. Since, by Corollary 3.16, distinct critical sets of the perfect part $\mathcal{L}^p$ are disjoint, each critical set of $\mathcal{L}$ is contained in a unique critical set of $\mathcal{L}^p$. Hence QCP generates the **critical pattern** $\mathcal{Z}(\mathrm{QCP})$ **of** QCP **in** $\mathcal{L}^p$, and so each invariant geodesic lamination with critical portrait $(\mathcal{L}, \mathrm{QCP})$ gives rise to the perfect invariant geodesic lamination with critical pattern $(\mathcal{L}^p, \mathcal{Z}(\mathrm{QCP}))$.

Definition 3.54 (Perfect-Siegel part)**.** The union $\mathcal{L}^{pS}$ of the perfect and the Siegel parts of an invariant geodesic lamination $\mathcal{L}$ is called the *perfect-Siegel part* of $\mathcal{L}$ (it is easy to see $\mathcal{L}^{pS}$ is a geodesic lamination).

In fact, $\mathcal{L}^{pS}$ is a proper geodesic lamination (see Definition 2.22) because critical leaves with periodic endpoints or critical wedges with periodic vertices are impossible in the perfect-Siegel part of $\mathcal{L}$ (hence, they are not present in $\mathcal{L}^{pS}$). Hence $\mathcal{L}^{pS}$ induces the corresponding laminational equivalence relation $\approx_{\mathcal{L}^{pS}}$, which in turn defines its geodesic lamination $\mathcal{L}_{\approx_{\mathcal{L}^{pS}}}$.

Periodic Fatou gaps of $\mathcal{L}^{pS}$ and $\mathcal{L}_{\approx_{\mathcal{L}^{pS}}}$ may differ. Indeed, let U be a periodic Fatou gap of $\mathcal{L}^{pS}$ of degree greater than one. There may exist a finite chain of edges of U. Since U is a gap of $\mathcal{L}^{pS}$, these edges must be non-isolated from the outside of U. It follows that they all are (pre)periodic. On the other hand, by definition the initial and the terminal points of this chain of edges are connected by a leaf of $\mathcal{L}_{\approx_{\mathcal{L}^{pS}}}$ that is **not** a leaf of $\mathcal{L}^{pS}$.

With respect to the Siegel parts, the geodesic lamination $\mathcal{L}_{\approx_{\mathcal{L}^{pS}}}$ contains convex hulls of the skeletons of the periodic Siegel gaps and their pullbacks and convex hulls of decorations attached to these gaps and their pullbacks while the geodesic lamination $\mathcal{L}^{pS}$ may contain finite chains of leaves inside the decorations and their pullbacks, but contains no leaves inside convex hulls of the periodic Siegel gaps and their pullbacks. These are the only two types of differences between $\mathcal{L}^{pS}$ and $\mathcal{L}_{\approx_{\mathcal{L}^{pS}}}$. Observe that if the original two geodesic laminations are generated by laminational equivalence relations, the latter phenomenon (concerning the Siegel parts) is impossible because by definition there is no erasing of leaves related to it.

Definition 3.55. A laminational equivalence relation $\sim$ is said to be *perfect-Siegel* if $\sim = \approx_{\mathcal{L}^{pS}_\sim}$.

Some conditions immediately imply that for an equivalence $\sim$ its perfect-Siegel part generates the equivalence relation $\approx_{\mathcal{L}^{pS}_\sim}$ that coincides with $\sim$.

Lemma 3.56. *If all critical sets of $\sim$ are finite, then $\sim$ is a perfect-Siegel equivalence relation.*

PROOF. If $\sim \neq \approx_{\mathcal{L}^{pS}_\sim}$, then there must exist a periodic Fatou gap U of $\approx_{\mathcal{L}^{pS}_\sim}$ of degree $k > 1$ such that all its edges are leaves of $\mathcal{L}_\sim$ and there is a countable non-empty set of leaves of $\mathcal{L}_\sim$ inside U; moreover, U contains no Siegel gaps of $\sim$. However then we can consider $\mathcal{L}_\sim$ restricted on U, and by the assumptions it would follow that there are no infinite gaps of $\mathcal{L}_\sim$ inside U at all. It is well-known that in this case $U/\sim$ is a dendrite and $\mathcal{L}_\sim$ must have uncountably many leaves inside U, a contradiction. $\square$

Theorem 3.57 studies the perfect-Siegel parts of invariant geodesic laminations with quadratically critical portraits. Recall that for brevity we call a gap G uncountable (countable, finite) if $G \cap S^1$ is uncountable (countable, finite).

Theorem 3.57. *If $(\mathcal{L}_1, \mathrm{QCP}_1)$ and $(\mathcal{L}_2, \mathrm{QCP}_2)$ are invariant geodesic laminations with quadratically critical portraits that are linked or essentially equal, then we have the following equality:*

$$(\mathcal{L}^p_1, \mathcal{Z}(\mathrm{QCP}_1)) = (\mathcal{L}^p_2, \mathcal{Z}(\mathrm{QCP}_2)).$$

Also, the Siegel parts $\mathcal{L}^{Sie}_1$ of $\mathcal{L}_1$ and $\mathcal{L}^{Sie}_2$ of $\mathcal{L}_2$ coincide, and so $\mathcal{L}^{pS}_1 = \mathcal{L}^{pS}_2$.

PROOF. By way of contradiction, assume that $\mathcal{L}^p_1 \not\subset \mathcal{L}^p_2$; then $\mathcal{L}^p_1 \not\subset \mathcal{L}_2$, and there exists a leaf $\ell^p_1 \in \mathcal{L}^p_1 \setminus \mathcal{L}_2$. Then, by Lemma 3.53, the leaf ℓ^p_1 is inside a gap G of $\mathcal{L}_2$. Since $\mathcal{L}^p_1$ is perfect, from at least one side all one-sided neighborhoods of ℓ^p_1 contain uncountably many leaves of $\mathcal{L}^p_1$. Hence G is uncountable (if G is finite or countable, then there must exist edges of G that cross leaves of $\mathcal{L}^p_1$, a contradiction as above). We claim that this is impossible. Indeed, by [**Kiw02**] G is (pre)periodic. Hence we may assume that G is periodic and still contains uncountably many leaves from $\mathcal{L}^p_1$. Since our geodesic laminations have quadratically critical portraits, it follows that G is a Siegel gap. This contradicts Corollary 2.35. Finally, the claim of the lemma dealing with Siegel parts of geodesic laminations $\mathcal{L}_1$ and $\mathcal{L}_2$ follows from Lemma 3.48. $\square$

Jan Kiwi showed in [**Kiw04**] that if all critical sets of an invariant geodesic lamination $\mathcal{L}$ are critical leaves with **aperiodic kneading**, then its perfect part $\mathcal{L}^p$ is completely determined by these critical leaves (he also showed that this defines the corresponding laminational equivalence relation $\sim$ such that $\mathcal{L}^p = \mathcal{L}_\sim$ and that $\sim$ is dendritic). Our results are related to Kiwi's. Indeed, by Theorem 3.57, if $\mathcal{L}$ is an invariant geodesic lamination with a quadratically critical portrait QCP, then $\mathcal{L}^p \subset \mathcal{L}$ is completely defined by QCP; in other words, if there is another invariant geodesic lamination $\widehat{\mathcal{L}}$ with the same quadratically critical portrait QCP, then still $\widehat{\mathcal{L}}^p = \mathcal{L}^p$.

Theorem 3.57 takes the issue of how critical data impacts the perfect part of an invariant geodesic lamination further as it considers the dependence of the perfect parts upon critical data while relaxing the conditions on critical sets and allowing for "linked perturbation" of the critical data. Therefore, Theorem 3.57 could be viewed as a rigidity result: "linked perturbation" of critical data does not change the perfect invariant geodesic lamination.

3.6. Invariant geodesic laminations generated by laminational equivalence relations

In the previous section, we investigated the relation between two given quadratically critical invariant geodesic laminations that are linked or essentially equal. However in general geodesic laminations are not quadratically critical. Thus we need to develop tools allowing us to adjust arbitrary geodesic laminations to produce quadratically critical ones. In this section we do so first without any restrictions upon the degree of the map and its Fatou gaps, and then adding some restrictions to obtain more precise results.

3.6.1. Linked invariant geodesic laminations of any degree. The above analysis justifies the next definition.

Definition 3.58. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be invariant geodesic laminations. Suppose that there are invariant geodesic laminations with quadratically critical portraits $(\mathcal{L}_1^m, \mathrm{QCP}_1)$, $(\mathcal{L}_2^m, \mathrm{QCP}_2)$ such that $\mathcal{L}_1 \subset \mathcal{L}_1^m$, $\mathcal{L}_2 \subset \mathcal{L}_2^m$. Then we say that the modifications $\mathcal{L}_1^m$ of $\mathcal{L}_1$ and $\mathcal{L}_2^m$ of $\mathcal{L}_2$ are *induced* by quadratically critical portraits QCP_1 and QCP_2, respectively. If this can be done so that $(\mathcal{L}_1^m, \mathrm{QCP}_1)$ and $(\mathcal{L}_2^m, \mathrm{QCP}_2)$ are linked or essentially equal, then we say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are *intrinsically linked* (*essentially equal*, respectively).

Two invariant geodesic laminations are intrinsically linked or essentially equal if and only if we can "tune" them into two quadratically critical geodesic laminations by inserting into their critical sets critical quadrilaterals in a dynamically consistent way so that the thus constructed quadratically critical portraits of the two geodesic laminations are linked/essentially equal.

However arbitrary quadratically critical modifications of invariant geodesic laminations may yield a significant increase of the corresponding perfect-Siegel parts of these invariant geodesic laminations. Thus, in order to implement our results we need to agree upon the way arbitrary invariant geodesic laminations should be modified (tuned) into geodesic laminations with quadratically critical portraits. This can only be done by inserting critical quadrilaterals into critical sets of given geodesic laminations. Moreover, ideally this quadratically critical tuning should not increase the size of the original geometric lamination too much.

Recall that laminational equivalence relations $\sim$ appear in complex dynamics in a very natural way (see Section 2.2). For many polynomials P with connected Julia sets, they give rise to semiconjugacies between $P|_{J(P)}$ and the topological polynomial $f_{\sim_P} : \mathbb{S}/\sim_P \to \mathbb{S}/\sim_P$ with a special choice of $\sim_P$.

Therefore, from the point of view of complex dynamics, laminational equivalence relations are of main interest. We want to use our tools to study them, in particular to study the mutual location of their critical sets. Thus, we need to develop methods of quadratically critical tuning for invariant geodesic laminations generated by laminational equivalences.

Definition 3.59 (Invariant non-capture geodesic lamination)**.** Consider an invariant geodesic lamination $\mathcal{L}_\sim$ generated by an invariant laminational equivalence relation $\sim$. Suppose that there are no **preperiodic** Fatou gaps that map onto their image k-to-1 with $k > 1$. Then both $\sim$ and $\mathcal{L}_\sim$ are said to be of *non-capture* type. Otherwise $\sim$ and $\mathcal{L}_\sim$ are said to be of *capture* type. Any periodic Fatou gap U with a preperiodic pullback that maps forward k-to-1, where $k > 1$, is itself said to be of *capture type*.

The reason for our interest in invariant non-capture geodesic laminations is the following. An invariant geodesic lamination $\mathcal{L}_\sim$ of capture type will have at least one preperiodic Fatou gap U that maps onto its image in a k-to-1 fashion. This allows for a variety of ways critical quadrilaterals can be inserted in U. It is therefore impossible to associate with $\mathcal{L}_\sim$ a unique (or finitely many) quadratically critical portrait(s).

Similarly, there exists an ambiguity related to the issue of how critical quadrilaterals can be inserted into **periodic** Fatou gaps of degree greater than one. We will tackle this issue later on, however first we want to simplify the picture and consider the case with no periodic Fatou gaps of degree $k > 1$. An easier version here is that of two dendritic geodesic laminations $\mathcal{L}_{\sim_1}$ and $\mathcal{L}_{\sim_2}$ generated by laminational equivalence relations $\sim_1$ and $\sim_2$. However, by Theorem 3.57, we can work with a wider class of geodesic laminations.

Lemma 3.60 (Laminations with finite critical sets)**.** *Let $\mathcal{L}_\sim$ be an invariant geodesic lamination. Then all critical sets of $\mathcal{L}_\sim$ are finite if and only if $\mathcal{L}_\sim$ has no Fatou gaps of degree greater than one. Moreover, then $\mathcal{L}_\sim = \mathcal{L}_\sim^{pS}$.*

PROOF. The first claim of the lemma is left to the reader. To prove the last claim of the lemma, recall that by [**BOPT14**] the perfect part $\mathcal{L}_\sim^p$ of $\mathcal{L}_\sim$ is itself an invariant geodesic lamination with some (possibly empty) collection of periodic Fatou gaps of degree greater than one (so-called *super-gaps* of $\mathcal{L}_\sim$) and their pairwise disjoint preimages. By the assumption, $\mathcal{L}_\sim$ can have neither periodic Fatou gaps of degree greater than one nor Siegel gaps of capture type. Therefore, if U is a periodic super-gap of $\mathcal{L}_\sim$, then U lies entirely in the Siegel part of $\mathcal{L}_\sim$. Clearly, pullbacks of the intersection of U and the Siegel part of $\mathcal{L}_\sim$ fill up pullbacks of periodic super-gaps of $\mathcal{L}_\sim$. We conclude that $\mathcal{L}_\sim = \mathcal{L}_\sim^{pS}$. $\square$

Observe that if $\mathcal{L}_\sim$ is an invariant geodesic lamination generated by a laminational equivalence relation $\sim$ such that all critical sets of $\mathcal{L}_\sim$ are finite, then $\mathcal{L}_\sim$ is regular because finite critical sets correspond to $\sim$-classes of equivalence and hence either coincide or are disjoint. Therefore, by Definitions 3.17 and 3.18 one can talk about critical patterns of quadratically critical portrait in $\mathcal{L}_\sim$ or of invariant geodesic lamination $\mathcal{L}_\sim$ with critical pattern.

Combining Lemma 3.60 with Theorem 3.57, we obtain Corollary 3.61 (the last claim of Corollary 3.61 is left to the reader).

Corollary 3.61. *If geodesic laminations $\mathcal{L}_{\sim_1}$ and $\mathcal{L}_{\sim_2}$ with no Fatou gaps of degree greater than one are intrinsically linked or essentially equal, then $\sim_1=\sim_2=\sim$ and $\mathcal{L}_{\sim_1} = \mathcal{L}_{\sim_2} = \mathcal{L}_\sim$ are equal. If QCP_1 and QCP_2 are two quadratically critical portraits of $\mathcal{L}_{\sim_1}$ and $\mathcal{L}_{\sim_2}$ that are linked or essentially equal, then the critical patterns of QCP_1 and QCP_2 in $\mathcal{L}_\sim$ coincide.*

Observe that this generalizes results by Kiwi [**Kiw04**]. In our terms his results state that if two dendritic geodesic laminations are essentially equal then they coincide. We weaken the assumptions here and allow for **linked** geodesic laminations generated by laminational equivalence relations from a **wider** class while the conclusion remains the same.

The general case is more complicated. Consider two perfect non-empty geodesic laminations $\mathcal{L}_{\sim_1}$ and $\mathcal{L}_{\sim_2}$ that are intrinsically linked or essentially equal. By definition, there are modifications $\mathcal{L}_{\sim_1}^m$ of $\mathcal{L}_{\sim_1}$ and $\mathcal{L}_{\sim_2}^m$ of $\mathcal{L}_{\sim_2}$ that are quadratically critical and linked. This means that critical quadrilaterals were inserted into critical sets of both geodesic laminations forming two linked quadratically critical portraits. However this may have resulted in a significant growth of the corresponding geodesic lamination as together with the inserted quadrilaterals we have to add their images, preimages and their limits. Therefore in general we cannot conclude that $\mathcal{L}_{\sim_1}$ and $\mathcal{L}_{\sim_2}$ are equal if they are intrinsically linked or essentially equal.

Thus, when studying two invariant geodesic laminations $\mathcal{L}_1$ and $\mathcal{L}_2$ with infinite gaps of degree greater than one, it may be useful to consider a restricted family of their modified invariant geodesic laminations $\mathcal{L}_1^m$ and $\mathcal{L}_2^m$, designed to increase the basic invariant geodesic laminations $\mathcal{L}_1$ and $\mathcal{L}_2$ as little as possible so that the fact that $\mathcal{L}_1^m$ and $\mathcal{L}_2^m$ are linked implies more information about $\mathcal{L}_1$ and $\mathcal{L}_2$ themselves. Accordingly, we would like to specify quadratically critical tuning of invariant geodesic laminations.

Definition 3.62 (Admissible quadratically critical modifications)**.** Suppose that $\mathcal{L}_\sim$ is an invariant geodesic lamination generated by an invariant laminational equivalence relation $\sim$ and $\mathcal{L}^m$ is its quadratically critical modification. We say that $\mathcal{L}^m$ is an *admissible quadratically critical modification* of $\mathcal{L}$ if the set $\mathcal{L}^m \setminus \mathcal{L}$ consists of a countable family of leaves.

Then the following theorem easily follows.

Theorem 3.63. *Suppose that invariant geodesic laminations $\mathcal{L}_{\sim_1}$ and $\mathcal{L}_{\sim_2}$ have admissible quadratically critical modifications $\mathcal{L}^m_{\sim_1}$ of $\mathcal{L}_{\sim_1}$ and $\mathcal{L}^m_{\sim_2}$ of $\mathcal{L}_{\sim_2}$ that are linked or essentially equal. Then $\mathcal{L}^p_{\sim_1} = \mathcal{L}^p_{\sim_2}$. If either lamination has no eventual preimages U of periodic Siegel gaps with $\sigma_d|_U$ being k-to-1 with $k > 1$, then their Siegel parts coincide too so that $\mathcal{L}^{pS}_{\sim_1} = \mathcal{L}^{pS}_{\sim_2}$*

PROOF. By Theorem 3.57 the perfect parts and the Siegel parts of the admissible modified invariant geodesic laminations $\mathcal{L}^m_{\sim_1}$ and $\mathcal{L}^m_{\sim_2}$ coincide. By definition, these perfect parts coincide with perfect parts of original invariant geodesic laminations $\mathcal{L}_{\sim_1}$ and $\mathcal{L}_{\sim_2}$. Clearly, this implies the first claim of the theorem. Suppose now that either lamination does not have eventual preimages U of periodic Siegel gaps such that $\sigma_d|_U$ is k-to-1 with $k > 1$. Then Siegel parts of modified geodesic laminations and of the original geodesic laminations coincide, which implies the last claim of the theorem. □

Corollary 3.64 easily follows from definitions and Theorem 3.63.

Corollary 3.64. *In the situation of Theorem 3.63, two linked or essentially equal quadratically critical portraits* QCP_1 *and* QCP_2 *of $\mathcal{L}_{\sim_1}$ and $\mathcal{L}_{\sim_2}$, respectively, generate the same critical pattern in the common perfect part of both geodesic laminations. If either lamination does not have eventual preimages U of periodic Siegel gaps such that $\sigma_d|_U$ is k-to-1 with $k > 1$, then* QCP_1 *and* QCP_2 *generate the same critical pattern in $\mathcal{L}^{pS}_{\sim_1} = \mathcal{L}^{pS}_{\sim_2}$.*

The proof is left to the reader.

Observe that some assumption concerning eventual preimages of periodic Siegel gaps is necessary for the conclusion of the theorem to hold. Indeed, consider an invariant geodesic lamination $\mathcal{L}_{\sim_1}$ generated by an invariant laminational equivalence relation $\sim_1$. Assume that there exists an eventual preimage U of a periodic Siegel gap that maps onto its image in the k-to-1 fashion with $k > 1$. Consider a different laminational equivalence relation $\sim_2$ that identifies k (possibly degenerate) edges of U with the same image, and, accordingly, identifies preimages of these edges, which themselves are edges of the same pullbacks of U. In terms of invariant geodesic laminations, this means that a critical set (gap or leaf) that coincides with the convex hull A of k newly identified edges of U is inserted in U, and then this set is pulled back according to Thurston's pullback construction.

Clearly, $\sim_1$ and $\sim_2$ can be supplied with two quadratically critical portraits that are admissible for both and, in fact, coincide themselves. Indeed, choose an appropriate collection of critical quadrilaterals in A and use it as a part of a quadratically critical portrait; then define the remaining critical quadrilaterals so that altogether we will get an admissible quadratically critical portrait Q for $\mathcal{L}_{\sim_1}$; it follows that Q serves as an admissible quadratically critical portrait for $\sim_2$ as well. Thus, we must make assumptions clarifying the way periodic Siegel gaps are pulled back.

On the other hand, no assumption concerning periodic Fatou gaps of perfect parts of $\mathcal{L}_\sim$ is necessary. Indeed, suppose that U is a periodic Fatou gap of $\mathcal{L}^p_\sim$. Then any eventual preimage-gaps of U cannot share edges or finite gaps separating them, unlike in the Siegel case, because if they do, then this will give rise to isolated leaves in $\mathcal{L}^p_\sim$, a contradiction. Hence the ambiguity described above for Siegel parts is impossible in the case of perfect parts of invariant geodesic laminations.

3.6.2. Counterexamples of type B. There are examples showing that the assumptions of Theorem 3.63 in its current form cannot be relaxed. That is, in general, we cannot make conclusions concerning the coincidence of the given invariant geodesic laminations as a whole if they have linked or essentially equal admissible modifications. Indeed, let us consider a class of invariant geodesic laminations studied in [**BOPT13**]. Every geodesic lamination of this class has a unique invariant finite gap G. By [**Kiw02**], there are at most two periodic orbits (of the same period) forming the set of vertices of G, and if vertices of G form two periodic orbits, points of these orbits alternate on $\mathbb{S}$. For each edge ℓ of G denote by $H_G(\ell)$ the circle arc with the same endpoints as ℓ, separated from G by ℓ, and call it the *hole of G behind ℓ*.

Moreover, assume that at each edge ℓ of G there exists a Fatou gap, say, U, attached to G (that is, sharing with G a common edge ℓ) and having the maximal possible degree depending on its location (i.e., if the map σ_3 is two-to-one, respectively, three-to-one on $H_G(\ell)$, then the map σ_3 is two-to-one, respectively, three-to-one on U). It is not difficult to explicitly construct such Fatou gaps. Indeed, let G have m edges ℓ_0, $\dots$, ℓ_{m-1}. For each i, let FG_i be the convex hull of all points $x \in \overline{H_G(\ell_i)}$ with $\sigma_3^j(x) \in \overline{H_G(\sigma_3^j(\ell_i))}$ for every $j \geqslant 0$. It is straightforward to see that FG_i are infinite gaps such that FG_i maps to FG_j if $\ell_j = \sigma_3(\ell_i)$. These gaps are called the *canonical Fatou gaps attached to G*.

It is shown in [**BOPT13**] that, given a gap G, the corresponding invariant geodesic lamination with the listed properties exists and is unique. It is then called the *canonical invariant geodesic lamination of G* and is denoted by $\mathcal{L}_G$. By [**BOPT13**], the invariant geodesic lamination $\mathcal{L}_G$ is generated by an invariant laminational equivalence relation, which we will denote $\sim_G$. Observe that G can also be an invariant leaf $\overline{0\frac{1}{2}}$ in which case the definitions are similar to the above.

Finite invariant gaps G are classified in [**BOPT13**] into several categories called gaps of type A, B and D. This classification mimics Milnor's classification of hyperbolic components in slices of cubic polynomials and quadratic rational functions [**Mil93, Mil09**]. In the present paper we are interested in gaps of type B (from "Bi-transitive") and their canonical invariant geodesic laminations.

Definition 3.65 (Gaps of type B)**.** Suppose that G is a σ_3-invariant gap. Assume that its vertices form one periodic orbit (so that the edges of G form one periodic orbit of edges too). Moreover, suppose that there is an edge of G, denoted by M_1, that separates G from 0, and another edge of G, denoted by M_2, that separates G from $\frac{1}{2}$. The edges M_1 and M_2 are said to be *major edges (leaves)* or simply *majors* of G. Then G is said to be an *invariant finite gap of type B*.

It is easy to see that major holes of an invariant gap G of type B are of length greater than $\frac{1}{3}$ but less than $\frac{2}{3}$. The next example illustrates the definitions just given.

Example 3.66. *Consider the finite gap G with vertices $\frac{7}{26}$, $\frac{11}{26}$ and $\frac{21}{26}$. This is a gap of type B. The first major leaf M_1 connects $\frac{21}{26}$ with $\frac{7}{26}$ and the second major leaf M_2 connects $\frac{11}{26}$ with $\frac{21}{26}$. The edges of G form one periodic orbit to which both M_1 and M_2 belong. The major hole $H_G(M_1)$ contains 0 and the major hole $H_G(M_2)$ contains $\frac{1}{2}$.*

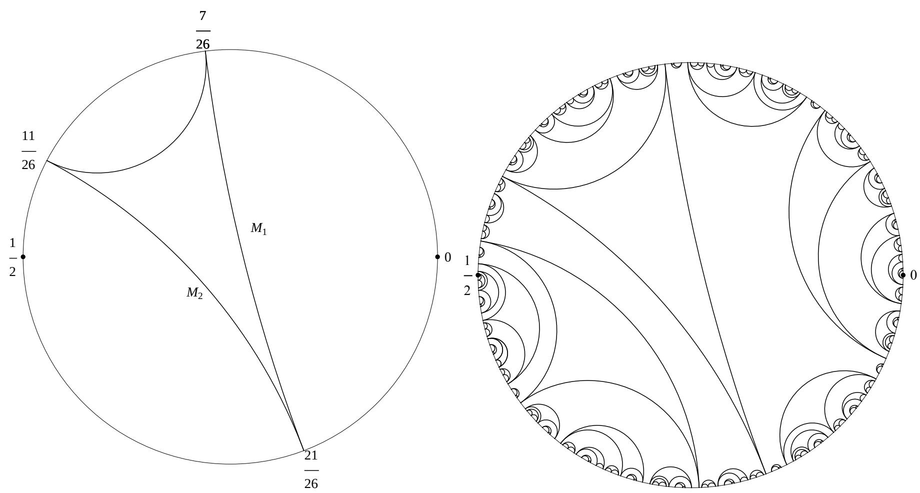

FIGURE 12. The rotational gap described in Example 3.66 and its canonical lamination.

Consider an invariant finite gap G of type B. Denote by T and N its majors and by U_T and U_N the corresponding canonical Fatou gaps of its canonical invariant geodesic lamination $\mathcal{L}_G$. Let G have m edges $\ell_0, \dots, \ell_{m-1}$ ordered in the positive direction around the circle. For each $0 \leqslant i \leqslant m-1$, let FG_i be the canonical Fatou gap attached to G at ℓ_i. Let $T = \ell_t$ and $N = \ell_n$ with $0 < t < n < m-1$ (we may always achieve this by renumbering the edges of G). Then there are $n-t-1$ edges of G in the positive direction from T and N and $m-n-1+t$ edges of G in the positive direction from N to T.

Let $T = \overline{ab}$ with $a < 0 < b$. Let a', b' be the points on the boundary of U_T such that $\sigma_3(a') = \sigma_3(a)$ and $\sigma_3(b') = \sigma_3(b)$. It is easy to see [**BOPT13**] that we have $a < b' < 0 < a' < b$. Choose two σ_3^m-fixed points on the boundary of U_T and denote them x and y so that $a < x < y < b$. Clearly, $x < 0 < y$, and, moreover, $a < x < b' < 0 < a' < y < b'$. Then the orbit of the point y has exactly one point in every hole of G. If we connect them in the positive order, then we obtain a new σ_3-invariant finite gap H.

It is easy to see that H is of type B. Indeed, let $\hat{y} \in H_G(\ell_{t-1})$ be a unique point from the orbit of y in $H_G(\ell_{t-1})$. Then $\overline{\hat{y}y}$ is an edge of H, which separates 0 from H because $\hat{y} < 0 < y$. A similar edge of H can be found on the opposite side of it cutting $\frac{1}{2}$ off H. By definition this implies that H is of type B. Clearly, the canonical invariant geodesic laminations $\mathcal{L}_G$ and $\mathcal{L}_H$ are distinct. However it is not difficult to show that they have admissible quadratically critical modifications that are linked. Moreover those modifications are very natural, if not the only natural, quadratically critical modifications of the respective invariant geodesic laminations.

More precisely, insert a critical quadrilateral $R_G = \mathrm{CH}(a, b', a', b)$ in U_T. Insert a similarly defined critical quadrilateral L_G in U_N. Using Thurston's pullback construction we can complete the non-invariant geodesic lamination formed by $\mathcal{L}_G$ together with R_G

and L_G to the invariant geodesic lamination $\mathcal{L}_G^m$. In the same fashion we can modify the canonical invariant geodesic lamination $\mathcal{L}_H$ into an invariant geodesic lamination $\mathcal{L}_H^m$. We claim that $\mathcal{L}_G^m$ and $\mathcal{L}_H^m$ are linked.

Indeed, to observe that we need to figure out where on $(\hat{y}, y)$ the other vertices of R_H are located. Now, since the point $\sigma_3(y)$ belongs to the arc $(\sigma_3(a), \sigma_3(b'))$, it follows that the vertex y' of R_H with the same σ_3-image as y must belong to (a, b'). Similarly the vertex $\hat{y}'$ of R_H with the same image as $\hat{y}$ must belong to (b', a'). Therefore the quadrilaterals R_G and R_H are strongly linked. In the same fashion one can show that the quadrilaterals L_G and L_H are strongly linked. Therefore, $\mathcal{L}_G^m$ and $\mathcal{L}_H^m$ are strongly linked whereas they are certainly not equal. This shows that apparently there are no general claims analogous to Theorem 3.63 applicable to all invariant geodesic laminations (with admissible linked or essentially equal modifications) rather than only to their perfect or Siegel parts.

3.6.3. Quadratically almost perfect-Siegel non-capture case. As we explained above, an important issue is that of assigning admissible quadratically critical portraits to geodesic laminations with periodic Fatou gaps of degree greater than one (as before we consider *non-capture* geodesic laminations). In what follows we define *legal quadrilaterals*; they give rise to quadratically critical portraits with desired properties. A quadratically critical pattern formed by legal quadrilaterals will be called a *legal quadratically critical pattern*. Let us emphasize that the choice of legal quadratically critical patterns should be laminational in the following sense: together with a legal quadratically critical pattern we should be able to choose an invariant geodesic lamination that has this quadratically critical pattern but also is in essence the same or very close to the original geodesic lamination. Two laminational equivalence relations will be called *linked* if they have linked legal quadratically critical patterns. We aim at defining the above notions so that two laminational equivalence relations that are linked will have to coincide. We achieve this goal in the current section in particular case of so-called *quadratically almost perfect-Siegel* laminational equivalence relations.

Now we need to define the concepts listed above. In doing so we are motivated by the case of σ_2, which we will now consider. To begin with, choose a critical leaf $\overline{\mathrm{Di}} = \overline{0\frac{1}{2}}$ with fixed endpoint 0. Then apply classic Thurston's step-by-step pullback construction agreeing that, in case of ambiguity, we will choose **all** possible consistent pullbacks of the existing leaves. Thus, at the first step we will add to $\overline{\mathrm{Di}}$ the leaves, $\overline{0\frac{1}{4}}$, $\overline{\frac{1}{4}\frac{1}{2}}$, $\overline{\frac{1}{2}\frac{3}{4}}$, $\overline{\frac{3}{4}0}$. This creates a quadrilateral (actually a square) with vertices $0, \frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$ and diagonal $\overline{\mathrm{Di}}$ so that the quadrilateral is represented as a union of two triangles.

If we continue in the same fashion, we will add to these two triangles four more triangles adjacent to the original two at their "outer" edges, i.e. at edges not equal to $\overline{\mathrm{Di}}$. Continuing in this fashion we will in the end **tile** the closed disk $\overline{\mathbb{D}}$ into triangles, each of which eventually maps to one of the original "big" triangles, then collapses to $\overline{\mathrm{Di}}$ and then collapses further to the singleton $\{0\}$. The leaves that we add form a null sequence and accumulate to points of the unit circle $\mathbb{S}$. The constructed invariant geodesic lamination has countably many leaves so that its perfect part coincides with the empty geodesic lamination. We will call this invariant geodesic lamination *basic quadratic geodesic lamination* and denote it by $\mathcal{L}_2^{bas}$. One can consider other versions of Thurston's pullback construction starting with $\overline{\mathrm{Di}}$, but they will all be subsets of $\mathcal{L}_2^{bas}$ and will all have the empty geodesic lamination as the perfect part.

On the other hand, it is well-known that with any other choice of a critical leaf ℓ there exists a non-trivial invariant laminational equivalence $\sim$ such that the corresponding invariant geodesic lamination $\mathcal{L}_\sim$ is compatible with ℓ in the following sense: leaves of $\mathcal{L}_\sim$ are not linked with ℓ. In other words, $\overline{\mathrm{Di}}$ is the unique critical leaf of σ_2 that is compatible only with the trivial invariant laminational equivalence relation.

A similar construction can be implemented inside any σ_2-periodic critical Fatou gap U of period, say, n, however there will be an important distinction. Indeed, it is well-known that U has a refixed edge $M_0 = M(0)$ and all other edges of U are appropriate pullbacks of $M(0)$. To clarify the picture, we will denote those pullbacks of $M(0)$ using the semiconjugacy between $\sigma_2^n|_{\overline{U}}$ and σ_2 that collapses all edges of U to points of the unit circle so that $M(\frac{1}{2})$ is the pullback of $M(0)$ since $\frac{1}{2}$ is the σ_2 pullback of 0. Insert in U a critical quadrilateral Q coinciding with the convex hull of $M(0)$ and its sibling edge $M(\frac{1}{2})$ of U (clearly, $M(\frac{1}{2})$ is the unique edge of U distinct from $M(0)$ and such that $\sigma_2^n(M(\frac{1}{2})) = \sigma_2^n(M(0))$.

Then there are two edges of U, namely, $M(\frac{1}{4})$ and $M(\frac{3}{4})$, that map to $M(\frac{1}{2})$ under σ_2^n. We connect $M(\frac{1}{4})$ with the closest edge of Q to it (evidently, this edge connects endpoints of $M(0)$ and $M(\frac{1}{2})$) and thus construct a quadrilateral that maps onto Q under σ_2^n. In the same fashion we treat $M(\frac{3}{4})$ and construct one more quadrilateral that maps onto Q under σ_2^n. Then we continue to implement Thurston's pullback construction and in the end construct a countable invariant geodesic sublamination inside U. In fact, this sublamination is the preimage of the invariant geodesic lamination constructed in the previous paragraph under the semiconjugacy collapsing edges of U to points of the unit circle.

Suppose now that a σ_d-invariant geodesic lamination $\mathcal{L}_\sim$ generated by a laminational equivalence relation $\sim$ is given. Recall that a periodic Fatou gap U of period n of a laminational equivalence relation $\sim$ is said to be *quadratic* if $\sigma_d^n|_{\mathrm{Bd}(U)\cap\mathbb{S}}$ is two-to-one (except perhaps for points of $\mathrm{Bd}(U)\cap\mathbb{S}$ that are images of σ_d^n-critical edges of $\mathrm{Bd}(U)$; the latter may have more than two σ_d^n-preimages). We need a construction similar to the above in the case of critical quadratic periodic Fatou gaps U of period n. The construction is inspired by the fact that $\sigma_d^n : \mathrm{Bd}(U) \to \mathrm{Bd}(U)$ is monotonically semiconjugate to σ_2 and in a lot of cases the semiconjugacy can be extended onto finite gaps attached to U. First though we study the dynamics of periodic Fatou gaps of degree greater than one.

In the following lemma, we talk about topological polynomials and their bounded Fatou domains instead of talking about laminations and their Fatou gaps. However, a translation from one language into the other is straightforward.

Lemma 3.67 (Dynamics of periodic Fatou domains). *Let Ω be a periodic bounded Fatou domain of a topological polynomial f of degree greater than one. Suppose that the period of Ω is n. Then one of the following holds.*

(1) *All the sets $f^i(\overline{\Omega})$, where $i = 0, \dots, n-1$, are pairwise disjoint.*
(2) *There exists $m < n$ such that $n = mk$ for some integer $k > 1$, the set $Y = \bigcup_{l=0}^{k-1} f^{lm}(\overline{\Omega})$ is connected, and the sets Y, $f(Y)$, $\dots$, $f^{m-1}(Y)$ are pairwise disjoint. Moreover, the intersection $\bigcap_{l=0}^{k-1} f^{ml}(\overline{\Omega})$ is a singleton.*

PROOF. Consider the orbit of $\overline{\Omega}$, that is the union $X = \bigcup_{i=0}^{n-1} f^i(\overline{\Omega})$. Clearly, $X = \bigcup_{j=0}^{m-1} f^j(Y)$ where Y is the component of X containing Ω, and $n = mk$ for some integer $k \geqslant 1$; it follows, that $f^m(Y) = Y$. If $k = 1$, then $Y = \overline{\Omega}$, which simply means that the sets $\overline{\Omega}$, $f(\overline{\Omega})$, $\dots$, $f^{n-1}(\overline{\Omega})$ are pairwise disjoint. This corresponds to case (1) of the lemma.

Suppose that $k > 1$. Then $Y = \bigcup_{l=0}^{k-1} f^{lm}(\overline{\Omega})$. Now, it is well-known that $f^m|_Y$ must have a fixed point, say, a; since by definition the sets $Y, \dots, f^{m-1}(Y)$ are pairwise disjoint, the point a is of period m. Clearly, $a \in \bigcap_{l=0}^{k-1} f^{lm}(\overline{\Omega})$. Geometrically this means that $\overline{\Omega}$ "rotates" around a under iterations of f^m so that after k steps it maps back onto itself. Observe that sets $f^{im}(\overline{\Omega})$ and $f^{jm}(\overline{\Omega})$ with $0 \leqslant i < j < k$ cannot intersect other than at a because otherwise this will create points of $\mathrm{Bd}(\Omega)$ "shielded' from infinity, which is impossible. $\square$

Based upon Lemma 3.67, we can give the following definition.

Definition 3.68. In case (1) of Lemma 3.67 we say that the Fatou domain U (and the corresponding Fatou gap of the corresponding invariant geodesic lamination/laminational equivalence relation) is of *non-rotational type*. In case (2) of Lemma 3.67 we say that the Fatou domain U (and the corresponding Fatou gap of the corresponding invariant geodesic lamination/laminational equivalence relation) is of *rotational type*.

Now let us discuss well-known facts concerning finite periodic gaps of invariant geodesic laminations and stated here without proof. Let U be a Fatou gap of an invariant geodesic lamination. Say that *a finite gap G is attached to U at a leaf M* if U and G share an edge M. Recall also that a gap (leaf) G of $\mathcal{L}_\sim$ is said to be *periodic (of period s)* if s is the least number such that $\sigma_d^s(G) = G$. The nature of $\sigma_d^s|_G$ may be different though.

Definition 3.69 (Types of finite periodic gaps and leaves)**.** Let G be a finite periodic gap of period s. Then $\sigma_d^s : G \to G$ may be the identity map. In this case we say that G is a *fixed return* gap . Otherwise $\sigma_d^s : G \to G$ is "cyclic", and we say that G is a *cyclic return gap*. Similar analysis and terminology apply to periodic leaves. If a periodic leaf ℓ of period s is such that $\sigma_d^s : \ell \to \ell$ is the identity map, then we say that ℓ is a *fixed return* leaf. If $\sigma_d^s : \ell \to \ell$ flips ℓ, then we say that ℓ is a *flip return leaf*. If $\sigma_d^i(\ell) \cap \ell \neq \varnothing$ for some $0 < i < s$, then ℓ is an edge of a finite cyclic return gap G and we say that ℓ is a *fixed return leaf of cyclic type*.

The analysis above shows that if a finite periodic gap G is attached to a periodic Fatou gap U then the following cases are possible.

(1) The gap G is fixed return. In this case the edge of U, at which G is attached to U, has period that is a multiple of the period of U.
(2) The gap G is cyclic return. Then such gap G is unique for U and U is a Fatou gap of rotational type. Moreover, the edge of U at which G is attached to U, must be a fixed return leaf of cyclic type of the same period as U.

We are ready to define legal critical quadrilaterals associated with a critical quadratic n-periodic Fatou gap U of a σ_d-invariant laminational equivalence relation $\sim$. Recall that we also want to define the geodesic lamination containing the critical quadrilaterals in question in such a way that it is not very different from the original geodesic lamination containing U. Let M be the refixed edge of U, and let M' be edge of M such that $\sigma_d(M) = \sigma_d(M')$ (M' is the *sibling* edge of M). Critical quadrilaterals Q that we associate with U will be either the convex hull of M and M' (the construction of Q is then analogous to the case of σ_2, and Q is called *trivial*) or convex hulls of certain edges of a finite gap G attached to U at M and their sibling edges coming from the sibling gap of G attached to U at M' (then Q is called *non-trivial*). In the latter case the construction of Q and the corresponding new geodesic lamination containing Q must involve erasing the entire grand orbit of M.

Let us now proceed with the construction. First assume that there is no finite gap attached to U at M. In this case, we associate with U only the critical quadrilateral obtained as the convex hull of M and its sibling edge M'. This is similar to the case of σ_2 considered above. Otherwise suppose that G is a finite periodic gap attached to U at M. Then there is a sibling-gap G' of G that is attached to U at M'. Clearly, σ_d^n maps G' to G. Consider two cases.

(1) If G is a fixed return gap, we first erase the grand orbit of M from $\mathcal{L}$. Then any remaining edge of G can be connected to its sibling edge of G' to create a critical quadrilateral Q. Finally, Q must be pulled back to create the corresponding geodesic lamination. Then such critical quadrilateral Q is said to be *legal* and the corresponding invariant geodesic lamination is called a *legal modification* of $\mathcal{L}_\sim$. Observe that if we simply erase the grand orbit of M we get a new geodesic lamination $\widehat{\mathcal{L}}$ that has a periodic Fatou gap containing U and of the same period and the same degree as U; basically, in the new gap finite concatenations of leaves replace appropriate leaves from the grand orbit of M. Clearly, $\widehat{\mathcal{L}}$ is proper and generates $\sim$ (and $\mathcal{L}_\sim$) in the usual way: two points are $\sim$-equivalent if and only if they can be connected with finite chain of leaves of $\widehat{\mathcal{L}}$. And in the sense of $\widehat{\mathcal{L}}$ we do literally the same as was done in the σ_2-case: construct a critical quadrilateral based upon a refixed edge of a gap and its sibling.

(2) Assume that the period of G is $m < n$ and $n = mk$ for some $k > 1$. By Lemma 3.67, the gap G is attached to U at the refixed edge M of U, the gap G is of cyclic type, σ_d^m acts on G as "rotation", and only after σ_d^m is k times applied to G will we have the identity map on G. Thus, each edge of G "rotates" under σ_d^m, and there are k edges in its orbit under this "rotation". Hence the number kl of edges of G is a multiple of k. Consider now two separate cases: $l = 1$ and $l > 1$.

(a) Suppose that $l = 1$. Then, as usual, we insert a critical quadrilateral Q based upon M and M', and pull Q back to construct the corresponding geodesic lamination. In particular, in this case there is a unique critical quadrilateral Q associated with M.

(b) Suppose that $l > 1$. We can think of σ_d^m and its action on G as follows. Choose l consecutive edges of G; they will all be in different orbits, and under the action of σ_d^m this segment of $\mathrm{Bd}(G)$ maps so that its images are pairwise disjoint (except for the endpoints) segments of $\mathrm{Bd}(G)$ until under $\sigma_d^n = (\sigma_d^m)^k$ is maps back to itself as the identity map.

One can insert an $l + 1$-gon into G as follows. Choose a segment I of $\mathrm{Bd}(G)$ concatenating l consecutive edges of G so that one of these edges is M. Take the convex hull $\mathrm{CH}(I)$ of I. It follows that $\mathrm{CH}(I)$ is an $(l + 1)$-gon, which "rotates" inside G under the action of σ_d^m until it comes back to itself under σ_d^n, the first return map being the identity. Observe that the choice of I is by no means unique. The orbit of $\mathrm{CH}(I)$ under σ_d^m consists of k gaps with pairwise disjoint interior; these gaps are "concatenated" at their appropriate common vertices. The complement in G to the union of σ_d^m-images of $\mathrm{CH}(I)$ is another finite gap T with k edges, which form one cycle under σ_d^m.

This construction is not unique as one can choose a segment I in several ways. In fact, it is easy to see that there are l distinct choices of a gap T inside G, and, accordingly, l distinct cycles of sets like $\mathrm{CH}(I)$ "rotating" around T. In each case there is exactly one segment I of the boundary of G that contains M. If we now erase M and its entire grand orbit, then we obtain an invariant geodesic lamination similar to the one described above in case (1). The gap U will again be enlarged, and all leaves from the grand orbit of M will be replaced by finite concatenations of leaves (for example, M itself will be replaced by the remaining edges of $\mathrm{CH}(I)$, etc). As before, the period of U and the period of the newly constructed gap containing U are equal, and the same can be said about their degrees.

Thus, we can erase the grand orbit of M but on the other hand add the gap T as above and its grand orbit. This yields a new geodesic lamination $\widehat{\mathcal{L}}$ similar to the geodesic lamination $\widehat{\mathcal{L}}$ from case (1). Observe that in $\widehat{\mathcal{L}}$ U is enlarged at the expense of G, and G is replaced by a smaller gap T. As before, $\widehat{\mathcal{L}}$ is a proper geodesic lamination, which generates $\sim$ in the usual way; we simply insert a critical quadrilateral Q based upon an edge of $\mathrm{CH}(I)$ and its sibling edge, and pull Q back to construct the corresponding geodesic lamination. This can be done in $2l-1$ ways.

Since in case (2)(b) we are replacing the gap G by a smaller gap T, then in all pullbacks of G the corresponding pullbacks of T will have to appear. The original gap U is enlarged by adding to it $\mathrm{CH}(I)$ and all its pullbacks. It follows that the new geodesic lamination (which is evidently proper as any geodesic lamination remains proper after one erases some of its leaves) generates the same laminational equivalence relation.

However, in case (1), the picture is more sensitive. In this case, there is a situation in which erasing M and its grand orbit leads to a significant change in the geodesic lamination in question and contradicts our desire to not change it too much. Indeed, suppose that G is a finite gap attached to U at M, and there exists a critical gap H that eventually maps to G. Then there must exist several Fatou gaps attached to H at its edges that are pullbacks of M. Erasing M and its pullbacks will result into these Fatou gaps merging into one Fatou gap of higher degree. Thus, the structure of the new geodesic lamination with the critical quadrilateral Q described above will be very different from the original. Hence the construction above is not applicable if there exist critical gaps that are preimages of G. Therefore when defining legal modifications of periodic quadratic Fatou gaps we always assume that *no critical gap is mapped to a fixed return gap attached to U at its refixed edge M or to M itself.*

We are ready to define legal quadrilaterals and legal modification of geodesic laminations. However first we need a useful general definition.

Definition 3.70 (Geolaminational collections)**.** If $\mathcal{Y}=(Y_1,\dots,Y_k)$ is a collection of gaps or leaves and there exists a σ_d-invariant geodesic lamination $\mathcal{L}$ such that Y_1, …, Y_k are gaps or leaves of $\mathcal{L}$, then we will call $\mathcal{Y}$ *(σ_d-)geolaminational*.

Recall that by a *full* collection of critical quadrilaterals we mean a collection such that on the boundaries of components of its complement the map σ_d is one-to-one except perhaps for boundary critical chords (clearly, such a collection must consist of $d-1$ critical quadrilaterals).

Definition 3.71 (Legal objects)**.** Critical quadrilaterals constructed in (1) and (2) for $\sim$ and $\mathcal{L}_\sim$ are said to be *legal*. A critical quadrilateral is also called *legal* if it is contained in a finite critical set of a geodesic lamination. An full ordered geolaminational collection of legal critical quadrilaterals of $\sim$ (and $\mathcal{L}_\sim$) is called a *legal quadratically critical portrait* of $\sim$ (and $\mathcal{L}_\sim$). The corresponding geodesic pullback laminations are called *legal modification* of $\mathcal{L}_\sim$.

Recall that legally modifying a lamination is a two step process. At the first step, we replaced all periodic quadratic Fatou gaps by possibly larger gaps. A quadratic gap U gets larger if a refixed edge of it and all edges in its grand orbit are erased. In this case the larger gap $\widetilde{U}$ is said to be a *legal modification* of U.

Let us again emphasize that legal quadratically critical portraits of geodesic laminations do not always exist. However they can definitely be constructed if *no critical gap is mapped to a fixed return gap attached to U at its refixed edge M or to M itself.*

If they do exist then, by definition, the corresponding geodesic lamination is a quadratically critical geodesic lamination. It is easy to see that legal modifications differ from the original geodesic lamination $\mathcal{L}_\sim$ in that they tune critical gaps of $\mathcal{L}_\sim$ and gaps from their grand orbit, but otherwise $\mathcal{L}_\sim$ remains the same. Even though legal quadrilaterals and legal modifications of geodesic laminations are not always well-defined, we can define the class of laminational equivalence relations and their geodesic laminations for which legal quadrilaterals and legal modifications are not only well-defined but also have the following crucial property: for them the fact that legal modifications are linked implies that the geodesic laminations coincide.

Definition 3.72. Let $\sim$ be an invariant non-capture laminational equivalence relation such that all periodic Fatou gaps of $\mathcal{L}^{pS}_\sim$ of degree greater than one are quadratic. Then we say that $\mathcal{L}_\sim$ and $\sim$ are *quadratically almost perfect-Siegel non-capture* invariant geodesic lamination and *quadratically almost perfect-Siegel non-capture* laminational equivalence relation.

It is easy to see that $\sim$ is quadratically almost perfect-Siegel non-capture if and only if all critical sets of $\mathcal{L}^{pS}_\sim$ are either finite sets or periodic quadratic Fatou gaps. The terminology is a little awkward but explicit; indeed, in the above definition $\mathcal{L}_\sim$ is not itself perfect-Siegel as $\mathcal{L}^{pS}_\sim$ does not have to coincide with $\mathcal{L}_\sim$ but the difference between the two is of "quadratic periodic nature": $\mathcal{L}^{pS}$ has no capture Fatou domains and all periodic Fatou domains U of $\mathcal{L}^{pS}$ are quadratic. Clearly, U may contain leaves of the original geodesic lamination $\mathcal{L}_\sim$, but by definition there are no Siegel gaps of $\mathcal{L}_\sim$ in U, and no more than countably many leaves of $\mathcal{L}_\sim$ in U.

Observe that if an invariant laminational equivalence relation $\sim$ is quadratically almost perfect-Siegel non-capture then all its periodic Fatou gaps are either Siegel or quadratic, and there are no critical preperiodic Fatou gaps. However the opposite statement is not true. Indeed, consider a cubic invariant geodesic lamination obtained as follows. Let $\overline{0\frac{1}{2}} = \overline{\mathrm{Di}}$ be the unique chord in $\mathbb{D}$ with σ_3-invariant endpoints. Let FG_a be the convex hull of all points with orbits **a**bove $\overline{\mathrm{Di}}$ and FG_b be the convex hull of all points with orbits **b**elow $\overline{\mathrm{Di}}$. Then $\overline{\mathrm{Di}}$ is a common edge of both gaps. Using Thurston's pullback scheme one can construct a unique cubic invariant geodesic lamination $\mathcal{L}_{ab}$ that has both gaps. Clearly, $\mathcal{L}_{ab}$ has two invariant critical quadratic Fatou gaps and no other critical sets. In particular, all its periodic Fatou gaps are either Siegel or quadratic, and there are no critical preperiodic Fatou gaps. However, the perfect part of $\mathcal{L}_{ab}$ is the empty geodesic lamination, which, evidently, has an invariant cubic gap (the entire closed disk). Hence $\mathcal{L}_{ab}$ is **not** quadratically almost perfect-Siegel non-capture.

Let us fix a quadratically almost perfect-Siegel non-capture geodesic lamination $\mathcal{L}_\sim$. Say that a gap G is *almost attached to U* if either G is attached to U, or G is attached to a gap H and H is attached to U.

Lemma 3.73. *Suppose that U is a quadratic periodic Fatou gap of $\mathcal{L}_\sim$. Then there are no critical sets of $\mathcal{L}_\sim$ mapped to edges of U or to gaps attached to U; in particular, there are no critical gaps attached to U and U has no critical edges. Moreover, if W is a Fatou gap of $\approx_{\mathcal{L}^{pS}_\sim}$ then no Fatou gap of $\mathcal{L}_\sim$ is almost attached to W. In particular, only finite gaps G of fixed return type can be attached to a Fatou gap W of $\approx_{\mathcal{L}^{pS}_\sim}$ at edges of W, and all their other edges are non-isolated in $\mathcal{L}_\sim$ from outside of G.*

Observe that W above must be a quadratic gap as follows from the definitions and assumptions.

PROOF. Let a critical set (gap or leaf) H map to a leaf or to a gap G attached to U. We may assume that U is critical. If H is attached to U, then there must exist a Fatou gap V attached to H and such that $\sigma_d(V) = \sigma_d(U)$. Thus, the Fatou gap W of $\mathcal{L}^{pS}_\sim$ containing

$U \cup H \cup V$ maps forward under σ_d in at least three-to-one fashion, a contradiction. Hence H is not attached to U. Then there are at least two Fatou gaps attached to H and mapped to U under the appropriate power of σ_d. Similar to the above, consider the Fatou gap W of $\mathcal{L}^{pS}_{\sim}$ containing H and these two gaps. Clearly, W is critical. If it contains U, then the degree of $\sigma_d|_{\mathrm{Bd}(W)}$ is at least three, a contradiction. If W does not contain U, then either W is not periodic, or W is a critical periodic Fatou gap of $\mathcal{L}^{pS}_{\sim}$ such that in its orbit there is another critical gap, namely the one that contains U. In all these cases we arrive at a contradiction.

Now, suppose that W is a Fatou gap of $\approx_{\mathcal{L}^{pS}_{\sim}}$. Thus there are at most countably many leaves of $\mathcal{L}_\sim$ inside W. If there exists a Fatou gap V of $\mathcal{L}_\sim$ attached to W at an edge G, or a Fatou gap V attached to a gap G that is attached to W, then we can unite W, G and V to create a "non-dynamic" gap containing W, G and V and therefore containing at most countably many leaves of $\mathcal{L}_\sim$. This would show that W cannot be a gap of $\approx_{\mathcal{L}^{pS}_{\sim}}$ a contradiction. The last claim of the lemma now easily follows. $\square$

Let U be a critical Fatou gap of $\approx_{\mathcal{L}^{pS}_{\sim}}$ of degree greater than one. By the assumptions, U is periodic, say, of period n. By Lemma 3.73 all periodic gaps attached to U are finite fixed return gaps. Such periodic gaps G give rise to the difference between the geodesic lamination generated by $\approx_{\mathcal{L}^{pS}_{\sim}}$ and the geodesic lamination $\mathcal{L}^{pS}$. Indeed, a gap G attached to U has an edge ℓ separating U and G that is not a part of $\mathcal{L}^{pS}$; the same holds for similar leaves and their pullbacks. Otherwise the geodesic lamination generated by $\approx_{\mathcal{L}^{pS}_{\sim}}$ and $\mathcal{L}^{pS}$ coincide.

Corollary 3.74. *Legal modifications of $\mathcal{L}_\sim$ are well-defined.*

This justifies the next definition.

Definition 3.75. Suppose that $\sim_1$ and $\sim_2$ are quadratically almost perfect-Siegel non-capture laminational equivalence relations such that some legal modifications $\mathcal{L}^{leg}_{\sim_1}$ and $\mathcal{L}^{leg}_{\sim_2}$ are linked or essentially equal. Then we say that $\sim_1$ and $\sim_2$ are *linked* or *essentially equal*.

Now we can state the main theorem of this section.

Theorem 3.76. *Suppose that $\sim_1$ and $\sim_2$ are quadratically almost perfect-Siegel non-capture laminational equivalence relations that are linked or essentially equal. Then $\sim_1=\sim_2$. Moreover, if QCP_1 and QCP_2 are linked or essentially equal legal quadratically critical portraits of $\mathcal{L}_{\sim_1}$ and $\mathcal{L}_{\sim_2}$ then critical patterns of QCP_1 and QCP_2 in $\mathcal{L}_{\sim_1} = \mathcal{L}_{\sim_2}$ coincide.*

PROOF. By Theorem 3.63, we have $\mathcal{L}^{pS}_{\sim_1} = \mathcal{L}^{pS}_{\sim_2}$. Therefore, $\approx_{\mathcal{L}^{pS}_{\sim_1}}=\approx_{\mathcal{L}^{pS}_{\sim_2}}=\approx$. Consider a periodic gap U of period n of $\approx$. Collapse all edges of U by a map ψ that semiconjugates $\sigma_d^n|_{\mathrm{Bd}(U)}$ and σ_2. Consider the two induced under the action of ψ by the restrictions of $\sim_1$ and $\sim_2$ onto U **quadratic** laminational equivalence relations $\approx_1$ and $\approx_2$ and the corresponding **quadratic** invariant geodesic laminations $\mathcal{L}_{\approx_1}$ and $\mathcal{L}_{\approx_2}$. Since $\mathcal{L}^{leg}_1$ and $\mathcal{L}^{leg}_2$ are linked or essentially equal, it follows that $\mathcal{L}_{\approx_1}$ and $\mathcal{L}_{\approx_2}$ are linked or essentially equal. By [**Thu85**], we have $\mathcal{L}_{\approx_1} = \mathcal{L}_{\approx_2}$ and hence the restrictions of $\sim_1$ and $\sim_2$ on U coincide. Applying this argument to all periodic gaps of $\mathcal{L}^{pS}_{\sim_1} = \mathcal{L}^{pS}_{\sim_2}$, we conclude that $\sim_1=\sim_2$ as desired. The last claim of the lemma follows from definitions, so we leave it to the reader. $\square$

We can also establish a simple but useful version of Theorem 3.76. To this end we need Definition 3.77. Recall that invariant geodesic laminations are called *regular* if their critical sets are pairwise disjoint except for the case when a critical leaf is a boundary edge of an all-critical set.

Definition 3.77. Suppose that $(\mathcal{L}_{\sim_1}, \mathcal{Z}_1)$ and $(\mathcal{L}_{\sim_2}, \mathcal{Z}_2)$ are regular invariant geodesic laminations with critical patterns. Then we say that $(\mathcal{L}_{\sim_1}, \mathcal{Z}_1)$ and $(\mathcal{L}_{\sim_2}, \mathcal{Z}_2)$ are *linked* or *essentially equal* if there are two linked or essentially equal quadratically critical portraits QCP_1 and QCP_2 inserted, respectively, in sets from $\mathcal{Z}_1$ and $\mathcal{Z}_2$ (by definition then it follows that $\mathcal{L}_{\sim_1}$ and $\mathcal{L}_{\sim_2}$ are intrinsically linked or essentially equal).

Corollary 3.78 now easily follows.

Corollary 3.78. *If $(\mathcal{L}_{\sim_1}, \mathcal{Z}_1)$ and $(\mathcal{L}_{\sim_2}, \mathcal{Z}_2)$ are quadratically almost perfect-Siegel non-capture invariant geodesic laminations with critical patterns that are linked or essentially equal, then $(\mathcal{L}_{\sim_1}, \mathcal{Z}_1) = (\mathcal{L}_{\sim_2}, \mathcal{Z}_2)$.*

CHAPTER 4

Applications: spaces of topological polynomials

In this section we apply the tools developed above.

4.1. The local structure of the space of all simple dendritic polynomials

The results obtained in this subsection were described in the Introduction in Subsection 1.5. Recall that a (*critically*) *marked polynomial* P is a polynomial of degree d equipped with an ordered $(d-1)$-tuple $C(P)$ of critical points of P such that the number of entries of every critical point in $C(P)$ reflects its multiplicity, i.e., equals the multiplicity minus one. In what follows when talking about a marked polynomial we use the notation $(P, C(P))$ with $C(P) = (c_1, \dots, c_{d-1})$ being critical points of P. The space of all marked polynomials $(P, C(P))$ is endowed with the natural product topology.

If P is dendritic, then by Theorem 2.18 (due to Kiwi [**Kiw04**]) there exists an invariant laminational equivalence relation $\sim_P$ such that the filled Julia set $J(P)$ of the polynomial $P|_{J(P)}$ is monotonically semiconjugate by a map ψ_P to the associated topological polynomial $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ induced by σ_d on the topological Julia set $J_{\sim_P} = \mathbb{S}/\sim_P$; let $\varphi_P : \mathbb{S} \to \mathbb{S}/\sim_P$ be the corresponding quotient map . For every point $z \in J(P)$ we set $G_z = \varphi_P^{-1}(\psi_P(z))$; the set G_z is a laminational counterpart of the point z.

A *simple dendritic* polynomial is defined as a dendritic polynomial P with only simple critical points and the following property: every pair of distinct critical points of P can be separated by a pair of (pre)periodic external rays together with their common landing point. In more combinatorial terms, a dendritic polynomial P is simple dendritic if there are $d-1$ distinct (and hence disjoint) critical sets $G_{c_1}, \dots, G_{c_{d-1}}$ of $\mathcal{L}_P$. The equivalence of the two definitions follows from Lemma 3.25 and Theorem 2.18. It follows that if $(P, C(P))$ is a simple marked dendritic polynomial, then all points $c_1, \dots, c_{d-1}$ in $C(P)$ must be distinct. Denote by $\mathcal{CMD}_d^{sim}$ the family of all simple (critically) marked dendritic polynomials of degree d.

Lemma 4.1. *The family $\mathcal{CMD}_d^{sim}$ of all simple marked dendritic polynomials is an open subset of the space of all marked dendritic polynomials.*

PROOF. Suppose that $(P, C(P))$ is a simple marked dendritic polynomial. We need to show that all marked dendritic polynomials in a sufficiently small neighborhood $\mathcal{U}$ of $(P, C(P))$ are simple. Consider a dendritic topological Julia set $J_{\sim_P}$. By definition all $d-1$ critical points of the topological polynomial $f_{\sim_P}$ are distinct. By definition, there is a collection of (pre)periodic external rays of P such that the union Γ_P of these rays and their landing points divides the plane into finitely many pieces, and each piece contains no more than one critical point of P. We may assume that the landing points are all (pre)periodic but not (pre)critical.

By Lemma 3.22, if an open neighborhood $\mathcal{U}$ of $(P, C(P))$ is sufficiently small, then, for any marked polynomial $(Q, C(Q)) \in \mathcal{U}$, there is a union Γ_Q of (pre)periodic external rays and their landing points that is close to Γ_P and has the following property: the external rays in Γ_Q have the same arguments as the external rays in Γ_P, and pairs of rays in Γ_Q land together if and only if the corresponding pairs of rays in Γ_P land together. Since $C(Q)$ is also close to $C(P)$, it follows that any two elements of $C(Q)$ are separated by Γ_Q, hence Q is simple dendritic. $\square$

Definition 4.2 (Local parameterization of dendritic polynomials)**.** Define the following map Ψ_d from $\mathcal{CMD}_d^{sim}$ to the space $2^{\overline{\mathbb{D}}^{d-1}}$ of compact subsets of $\overline{\mathbb{D}}^{d-1}$:

$$\Psi_d(P) = G_{P(c_1)} \times G_{P(c_2)} \times \cdots \times G_{P(c_{d-1})}$$

and call the sets $\Psi_d(P)$ *postcritical tags of critically marked dendritic polynomials*.

We show that if we fix a simple critically marked dendritic polynomial P and a sufficiently small neighborhood $\mathcal{U}$ of P in $\mathcal{CMD}_d^{sim}$, then the space of all corresponding tags has nice properties. However first we need to introduce a few new notions and quote a useful topological result.

Definition 4.3. A collection $\mathcal{D} = \{D_\alpha\}$ of compact and disjoint subsets of a metric space X is *upper semicontinuous* (*USC*) if, for every D_α and every open set $U \supset D_\alpha$, there exists an open set V containing D_α so that for each $D_\beta \in \mathcal{D}$, if $D_\beta \cap V \neq \varnothing$, then $D_\beta \subset U$. A decomposition of a metric space is said to be *upper semicontinuous* (*USC*) if the corresponding collection of sets is upper semicontinuous.

Upper semicontinuous decompositions of separable metric spaces are studied in [**Dav86**, p. 13].

Theorem 4.4 ([**Dav86**])**.** *If $\mathcal{D}$ is an upper semicontinuous decomposition of a separable metric space X, then the quotient space $X/\mathcal{D}$ is also a separable metric space.*

In the above situation we call $X/\mathcal{D}$ the space *generated by* $\mathcal{D}$ and denote by $\pi_{\mathcal{D}} : X \to X/\mathcal{D}$ the corresponding quotient map. In what follows we will use a well-known fact given below without a proof. Recall that by Definition 3.29 a map $F : A \to 2^B$ from a topological space A to the space of all compact subsets of a compactum B is *upper semicontinuous* if for every $x \in A$ and every neighborhood $\mathcal{U}$ of $F(x)$ there exists open neighborhood V of x such that $y \in V$ implies $F(y) \subset \mathcal{U}$.

Lemma 4.5. *Let $F : A \to 2^B$ be an upper semicontinuous map from a topological space A to the space 2^B of all compact subsets of a compactum B. Suppose that for any two points $x, y \in A$ either $F(x) = F(y)$, or $F(x) \cap F(y) = \varnothing$. Then the partition $\mathcal{D}$ of the set $\bigcup_{x \in A} F(x)$ into sets $F(x)$ is upper semicontinuous and the map $\pi_{\mathcal{D}} \circ F : A \to X/\mathcal{D}$ is continuous.*

The next theorem is the main theorem of this subsection. It combines the Theorems on Local Charts for Dendritic Polynomials and Local Pinched Polydisk Model for Dendritic Polynomials.

Theorem 4.6. *For a simple critically marked dendritic polynomial $(P, C(P)) \in \mathcal{CMD}_d^{sim}$ of degree d there exists a neighborhood $\mathcal{U}$ of $(P, C(P))$ such that for any two critically marked dendritic polynomials $(Q, C(Q)), (R, C(R)) \in \mathcal{U}$ either $\Psi_d(Q)$ and $\Psi_d(R)$ are disjoint or $\Psi_d(Q) = \Psi_d(R)$.*

Moreover, the map Ψ_d from $\mathcal{U}$ to the space $2^{\overline{\mathbb{D}}^{d-1}}$ of all compact and connected subsets of the polydisk $\overline{\mathbb{D}}^{d-1}$ is upper semicontinuous, the partition $\mathcal{D}$ of the set $\bigcup_{T \in \mathcal{U}} \Psi_d(T)$ into subsets $\Psi_d(T), T \in \mathcal{U}$, is upper semicontinuous, and the map $\pi_{\mathcal{D}} \circ \Psi_d : \mathcal{U} \to \bigcup_{T \in \mathcal{U}} \Psi_d(T)/\mathcal{D}$ is continuous.

PROOF. Recall that the map $\widehat{\Psi}_d$ was introduced in Subsection 3.2.2; this map associates to $(P, C(P))$ the dendritic invariant geodesic lamination with critical pattern $(\mathcal{L}_{\sim_P}, \mathcal{Z}(P, C(P)))$ where $\mathcal{Z}(P, C(P)) = (G_{c_1}, G_{c_2}, \dots, G_{c_{d-1}})$. By Corollary 3.30 and by definition the map $\widehat{\Psi}_d$ is upper semicontinuous. Since by definition

$$\Psi_d(P, C(P)) = \sigma_d(G_{c_1}) \times \dots \sigma_d(G_{c_{d-1}})$$

it follows that Ψ_d is upper semicontinuous too. By Lemma 4.5 to prove the theorem it remains to prove the first claim of the theorem, that is to prove that there exists a neighborhood $\mathcal{U}$ of $(P, C(P))$ such that for any two critically marked dendritic polynomials $(Q, C(Q)), (R, C(R)) \in \mathcal{U}$ either $\Psi_d(Q)$ and $\Psi_d(R)$ are disjoint or $\Psi_d(Q) = \Psi_d(R)$. By Lemma 4.1 we may assume that $\mathcal{U}$ is chosen sufficiently small so that any marked dendritic polynomial in $\mathcal{U}$ is simple. Moreover, since Ψ is upper semi-continuous, we may assume that for each critical set $C_i \in C(P)$ and each $Q \in \mathcal{U}$ there exists an open set W_i so that $\sigma_d|_{W_i}$ is two-to-one and the corresponding critical set $D_i \in C(Q)$ is contained in W_i.

By way of contradiction assume that two simple critically marked dendritic polynomials $(Q, C(Q)), (R, C(R)) \in \mathcal{U}$ have non-disjoint sets $\Psi_d(Q, C(Q))$ and $\Psi_d(R, C(R))$. Let us show that if Q and R are sufficiently close to P then this actually implies that $\Psi_d(Q) = \Psi_d(R)$. Choose a critical set C of $\sim_Q$. Then σ_d maps C forward in a two-to-one fashion. The vertices of C can be divided between two segments, each of which contains, say, m vertices of C. We can order vertices of C on the circle so that their collection is the set $x_1 < \dots < x_m < y_1 < \dots < y_m$ where the segments mentioned above are $[x_1, x_m]$ and $[y_1, y_m]$. Thus, for any given j with $1 \leqslant j \leqslant m$, we have that $\sigma_d(x_j) = \sigma_d(y_j) = z_j$. Then the σ_d-image $\sigma_d(C)$ of C coincides with the convex hull of the points $z_1 < z_2 < \dots < z_m$.

Now, by the assumption $\Psi_d(Q, C(Q))$ and $\Psi_d(R, C(R))$ are non-disjoint. Choose a critical set D of $\mathcal{L}_{\sim_Q}$ and a critical set E of $\mathcal{L}_{\sim_R}$ that have the same position in $\mathcal{Z}(Q, C(Q))$, respectively, in $\mathcal{Z}(R, C(R))$, as C does in $\mathcal{Z}(P, C(P))$. Since $\Psi_d(Q, C(Q))$ and $\Psi_d(R, C(R))$ are non-disjoint, we have $\sigma_d(D) \cap \sigma_d(E) \neq \varnothing$. By the assumptions on $\mathcal{U}$ there exists a tight neighborhood W of C such that $D \cup E \subset W$ and W maps onto its image exactly two-to-one.

The fact that $\sigma_d(D) \cap \sigma_d(E) \neq \varnothing$ implies that there is an edge ℓ_D of $\sigma_d(D)$ and an edge ℓ_E of $\sigma_d(E)$ such that $\ell_D \cap \ell_E \neq \varnothing$. Thus, either ℓ_D and ℓ_E share an endpoint, or they are linked. Since $D \cup E \subset W$, it follows that there exists a unique critical quadrilateral $Q_D \subset W$ that maps two-to-one onto ℓ_D. Clearly, $Q_D \subset D$. Similarly, there exists a unique critical quadrilateral $Q_E \subset W$ that maps two-to-one onto ℓ_E. Moreover, $Q_E \subset E$. If ℓ_D and ℓ_E share an endpoint, then the full preimage of this endpoint inside W is a critical diagonal shared by Q_D and Q_E. If ℓ_D and ℓ_E are linked, then it easily follows that Q_D and Q_E are strongly linked.

This argument can be repeated for all critical sets of $\mathcal{L}_{\sim_P}$. Therefore, we see that if $\Psi_d(Q, C(Q))$ and $\Psi_d(R, C(R))$ are non-disjoint, then the critical sets from $\mathcal{Z}(Q, C(Q))$ and $\mathcal{Z}(R, C(R))$ that occupy the same position in the respective critical patterns contain critical quadrilaterals that are strongly linked or share a critical chord. By definition this implies that $(\mathcal{L}_{\sim_Q}, \mathcal{Z}(Q, C(Q)))$ and $(\mathcal{L}_{\sim_R}, \mathcal{Z}(R, C(R)))$ are linked or essentially equal. By Corollary 3.78 $(\mathcal{L}_{\sim_Q}, \mathcal{Z}(Q, C(Q))) = (\mathcal{L}_{\sim_R}, \mathcal{Z}(R, C(R)))$ □

4.2. Two-dimensional spaces of σ_d-invariant geodesic laminations

The second application of our tools extends the results of [**BOPT15a**] where we studied the space $\mathbb{LP}_3^{np}(\overline{ab})$ of all cubic invariant geodesic laminations generated by cubic invariant laminational equivalence relations $\sim$ compatible with fixed critical leaf $D = \overline{ab}$ with non-periodic endpoints; in other words, we consider all laminational equivalence relations $\sim$ with $a \sim b$. The main result of [**BOPT15a**] is that this family of cubic invariant geodesic laminations is itself a lamination. This result resembles a laminational description of the combinatorial Mandelbrot set. First we study Thurston's invariant geodesic pullback laminations.

4.2.1. Invariant geodesic pullback laminations. We use [**BMOV13**] where *proper* invariant geodesic laminations were introduced (see the necessary definitions and claims in Subsection 2.2.2 of the present paper).

Lemma 4.7. *Suppose that $\mathcal{L}$ is an invariant geodesic lamination. Then the following claims hold.*

(1) *There is no critical leaf $\ell = \overline{ab} \in \mathcal{L}$ with a periodic endpoint a that is approximated by leaves of $\mathcal{L}$ disjoint from a.*
(2) *There is no critical wedge W of $\mathcal{L}$ with a periodic vertex v such that both sides of W are approximated by leaves of $\mathcal{L}$ disjoint from v.*

PROOF. (1) If a critical leaf $\ell = \overline{ab} \in \mathcal{L}$ with a n-periodic is approximated by leaves of $\mathcal{L}$ disjoint from a, then the fact that a repels close by points under σ_d^n implies that leaves approximating ℓ and disjoint from a will have σ_d^n-images crossing these leaves, a contradiction.

(2) Let a critical wedge W consist of leaves $\overline{vu}$ and $\overline{vt}$ where v is n-periodic. We may assume that $v < u < t < v$ and that $\sigma_d^n(v) = v < t \leqslant \sigma_d^n(u) = \sigma_d^n(t) \leqslant v$. Then leaves approximating $\overline{vt}$ and disjoint from v will have σ_d^n-images crossing these leaves, a contradiction. □

We will need the following definition and notation.

Definition 4.8 (Admissible critical collection)**.** Let $\mathcal{C} = \{\overline{c}_1 = \overline{a_1b_1}, \dots, \overline{c}_{d-1} = \overline{a_{d-1}b_{d-1}}\}$ be a full collection of $d-1$ critical chords. If each chord $\overline{c}_1, \dots, \overline{c}_r$ has a periodic endpoint while $\overline{c}_{r+1}, \dots, \overline{c}_{d-1}$ have non-periodic endpoints then we call $\mathcal{C}$ an *(r-)admissible critical collection*. Also, the closure of a component of $\overline{\mathbb{D}} \setminus \bigcup_{i=1}^{d-1} \overline{c}_i$ is called a *$\mathcal{C}$-domain*.

Let us fix an r-admissible critical collection $\mathcal{C}$ and use the notation from Definition 4.8. We want to associate to $\mathcal{C}$ a laminational equivalence relation. To this end we mimic Thurston's pullback construction [1] (Proposition II.4.5 [**Thu85**]) and define pullback (geodesic) laminations generated by $\mathcal{C}$.

By [**BMOV13**] geodesic laminations can be easily associated to laminational equivalence relations if they are proper (see Subsection 2.2.2). Therefore, taking into account the definition and properties of proper laminations, it is natural to pullback only leaves $\overline{c}_{r+1}, \dots, \overline{c}_{d-1}$ that have non-periodic endpoints. It is natural to expect that this will result in a proper lamination and thus will lead to a laminational equivalence relation.

However this construction may involve ambiguities. Consider one such possibility. Assume that on the n-th step a finite forward invariant (under σ_d) lamination $\mathcal{L}_n$ is obtained. Let a critical chain of leaves $\ell_1 = \overline{a_1a_2}, \ell_2 = \overline{a_2a_3}, \dots, \ell_k = \overline{a_ka_{k+1}}$ be a part of the boundary of a $\mathcal{C}$-component U, and that $\sigma_d(a_1) = x$ is an endpoint of several leaves $\overline{xy_1}, \dots, \overline{xy_s}$ of $\mathcal{L}_n$. For simplicity assume that points $y_1, \dots, y_s$ have unique σ_d-preimages $z_1, \dots, z_s \in \partial U$. Then when on the next step we pull back the leaves

[1] We are indebted to Gao Yan for drawing our attention to an inaccuracy in the original construction of a pullback lamination.

$\overline{xy_1}, \dots, \overline{xy_s}$ into U the point x can be pulled to any of the points $a_1, \dots, a_{k+1}$ which creates an ambiguous situation (despite the fact that points $y_1, \dots, y_s$ can be pulled in U uniquely only to points $z_1, \dots, z_s \in \partial U$). In other words, a number of points (namely, $k+1$ points $a_1, \dots, a_{k+1}$) can be connected with $z_1, \dots, z_s$, and this creates ambiguity. This ambiguity surfaces even if $k = 1$ (the boundary critical chain of U consists of one leaf) as even in that case the two endpoints of this critical leaf can be connected to $z_1, \dots, z_s$ in various ways. We will have to take care of this ambiguity as we define pullback laminations generated by $\mathcal{C}$.

Let us now give precise definitions. Suppose that on some step n a collection $\mathcal{L}_n$ of pullback leaves is constructed. On step $n+1$ each leaf $\ell = \overline{xy} \in \mathcal{L}_n$ can be pulled back into $\mathcal{C}$-domains B. The boundary of B maps forward covering $\mathbb{S}$ in the one-to-one fashion except for critical edges of B, which map to one point each. Then the pullback of ℓ in B is well-defined and unique in all cases except for the following.

Let us call maximal concatenations of (critical) edges of B *critical chains*. Endpoints of critical leaves in a critical chain Z are called *vertices* of Z. If $\ell = \overline{xy}$ has exactly one endpoint (say, x) that is the immediate image of a critical chain T while y pulls back to just one point $w \in B$, then the leaf ℓ can pull back to various leaves connecting w to vertices of T. On the other hand, if both endpoints x and y pull back to critical chains of B, then $\overline{xy}$ can pull back to various leaves connecting vertices of the first chain to vertices of the second chain. In what follows we call the convex hull of the union of all pullbacks of a leaf $\ell = \overline{xy}$ into a $\mathcal{C}$-domain U a *$\mathcal{C}$-maximal* pullback of ℓ; clearly, this is the convex hull of the full pullbacks of x and y to the boundary of U.

Thus, the ambiguity stems from the fact that critical chains can map to the endpoints of leaves of $\mathcal{L}_n$. We could resolve it by postulating our choices. However we prefer a different approach. Namely, we show that there *exists* a well-defined way of constructing the pullback $\mathcal{L}_{n+1}$ of $\mathcal{L}_n$ so that $\mathcal{L}_{n+1}$ is sibling forward invariant. In other words, we prove the *existence* of pullback laminations. Then we consider *any* sequence $\mathcal{L}_0 \subset \mathcal{L}_1 \subset \dots$ of finite invariant pullback laminations with $\mathcal{L}_0 = \{\overline{c}_{r+1}, \dots, \overline{c}_{d-1}\}$ and show that their limits generate the same equivalence relation.

Recall that by a *forward invariant lamination* we mean a collection of leaves satisfying Definition 2.5 (see remark right after this definition), in particular it has to satisfy our *sibling* condition: for each non-critical leaf ℓ of the collection there must exist $d-1$ other leaves of this collection so that all d leaves in question are *pairwise disjoint* and have the same image as ℓ.

Definition 4.9. Suppose that there exists a sequence of finite forward invariant laminations $\mathcal{L}_0 = \{\overline{c}_{r+1}, \dots, \overline{c}_{d-1}\}, \mathcal{L}_1, \dots$ such that $\sigma_d(\mathcal{L}_{n+1}) = \mathcal{L}_n$ for every $n > 0$. Then we say that $\mathcal{L}_0, \mathcal{L}_1, \dots$ is a sequence of *finite pullback laminations generated by* $\mathcal{C}$.

Lemma 4.10 follows from definitions.

Lemma 4.10. *Suppose that $\ell_1, \dots, \ell_d$ are arbitrary pairwise disjoint leaves that do not cross leaves from $\mathcal{C}$ and have the same image. Then each $\mathcal{C}$-domain contains exactly one leaf ℓ_i.*

Observe that the leaves $\ell_1, \dots, \ell_d$ are not assumed to be members of a forward invariant pullback lamination generated by $\mathcal{C}$.

Proof. Set $\mathcal{T} = \{\ell_i, i = 1, \dots, d\}$. Let $\sigma_d(\ell_1) = \overline{xy}$; for each i let $\ell_i = \overline{x_i y_i}$ with $\sigma_d(x_i) = x, \sigma_d(y_i) = y$. Call x_i's *x-points* and y_i's *y-points*. By the assumptions all preimages of x and all preimages of y form the set of all endpoints of leaves from $\mathcal{T}$.

Let U be a $\mathcal{C}$-domain and show that it contains a leaf from $\mathcal{T}$. Since all preimages of x and y are endpoints of leaves from $\mathcal{T}$, the claim is immediate if an x-point or a y-point belongs to ∂U but is not an endpoint of a critical leaf from $\mathcal{C}$. Hence we may assume that there are two critical chains in ∂U mapping to x and y respectively. Let $X = \hat{\ell}_1 \cup \dots \hat{\ell}_r$ be the critical chain in ∂U that maps to x. Let $\hat{\ell}_j = \overline{a_j a_{j+1}}, j = 1, \dots, r$. Assume that $a_1 < a_2 < \dots < a_{r+1}$ and (a_1, a_{r+1}) is the circle arc which intersects ∂U in exactly $\{a_2, \dots, a_r\}$.

Clearly, a_j's are x-points. Since the number of x-points in the closed circle arc $[a_1, a_2]$ is by one greater than the number of y-points in that arc, then at least one leaf from $\mathcal{T}$ has an x-endpoint inside $[a_1, a_2]$ and the other endpoint outside $[a_1, a_2]$. If this leaf is inside U, we are done. Otherwise the only possibility is that its endpoints are a_2 and some y-point inside $[a_2, a_3]$. In that case we can repeat the argument and continue it until we find the desired leaf. $\square$

Next we show that Definition 4.9 is not vacuous.

Lemma 4.11. *Sequences of finite pullback laminations generated by $\mathcal{C}$ exist.*

PROOF. As in Thurston's pullback construction, we define finite forward invariant laminations $\mathcal{L}_n, n = 0, 1, \dots$ step-by-step. We always include all points of the unit circle in them, so below we will only describe their non-degenerate leaves. Set $\mathcal{L}_0 = \{\overline{c}_{r+1}, \dots, \overline{c}_{d-1}\}$. Clearly, $\mathcal{L}_0$ is a finite forward invariant lamination. Assume now that $\mathcal{L}_n$ is constructed and describe how $\mathcal{L}_{n+1}$ is constructed. Observe that there are d $\mathcal{C}$-domains U, and on the boundary ∂U of each such U the map σ_d is one-to-one except for critical chains in ∂U that collapse to points.

Following Thurston, we pullback into U every leaf $\ell = \overline{xy}$ of $\mathcal{L}_n$. If neither endpoint of ℓ is the image of a critical chain from ∂U then such a pullback is unique. However one or both endpoints of ℓ may be images of critical chains in ∂U. Thus, to have a well-defined pullback of ℓ to U we need a more elaborate algorithm. Here is how we want to do it.

Choose the positive (counterclockwise) direction on ∂U; given an arc $T \subset \partial U$, we call this direction on T (on the entire ∂U) the *U-direction*. This induces specific direction on each critical chain T from ∂U and on each critical leaf from ∂U. Observe that positive direction on a critical leaf depends on the choice of U. Indeed, each critical leaf $\overline{c} = \overline{ab} \in \mathcal{C}$ is an edge of two $\mathcal{C}$-components, say, U and V; then if viewed as an edge of U it will have, say, initial endpoint a and terminal endpoint b while if viewed as an edge of V it will then have initial endpoint b and terminal endpoint a.

Clearly, ∂U maps onto $\mathbb{S}$ in a monotone fashion, with critical chains in ∂U being exactly the non-degenerate fibers of $\sigma_d|_{\partial U}$. Given a point $x \in \mathbb{S}$, denote by $I_x(U)$ the arc-preimage of x in ∂U, denote by $i_x(U)$ the *initial* point of $I_x(U)$ and by $t_x(U)$ the *terminal* point of $I_x(U)$ understood in terms of the U-direction on ∂U. Observe that the arcs $I_x(U)$ are in fact either points or critical chains on the boundary of U.

Claim. *Let U and V be two distinct $\mathcal{C}$-domains. Then for every point $x \in \mathbb{S}$ we have $t_x(U) \neq t_x(V), i_x(U) \neq i_x(V)$.*

PROOF OF THE CLAIM. We may assume that $Z = \overline{U} \cap \overline{V} \neq \varnothing$. Then either Z is a critical leaf shared by the boundaries of U and V, or $Z = \{z\}$ is a point of the circle which is a common point of two critical leaves $\ell_U \subset \partial U$ and $\ell_V \subset \partial V$. In the former case the U-direction on Z is opposite to the V-direction on Z, hence $t_x(U) \neq t_x(V)$ as desired.

Now, suppose that $Z = \{z\}$ is just a point. Then z is a common vertex of a critical chain $I_x(U)$ and of a critical chain $I_x(V)$. Suppose that $z = t_x(U) = t_x(V)$. Then there are distinct critical leaves $\overline{az} \subset I_x(U), \overline{bz} \subset I_x(V)$, and we may assume, without losing generality, that $a < b < z$. However, since the V-direction on $\overline{bz}$ must be from b to z it follows from the definitions that V must be located between $\overline{az}$ and $\overline{bz}$ which, because of the existence of the leaf $\overline{az}$, means that the critical chain $I_x(V)$ cannot terminate at z and must include at least one more critical edge $\overline{zc}$ of V growing out of z with $a \leqslant c < b$. This shows that z is *not* the terminal point of $I_x(V)$, a contradiction. Similar arguments show that $i_x(U) \neq i_x(V)$. □

Now, for a leaf $\ell = \overline{xy} \in \overline{\mathbb{D}}$, we postulate that as the endpoints of the pullback leaf of ℓ we choose the points $t_x(U)$ and $t_y(U)$. We claim that this yields a sequence of finite forward invariant laminations $\mathcal{L}_n$. Moreover, we will show inductively (it follows immediately from the construction) that for each point $x \in \mathbb{S}$ and each leaf ℓ of one of our laminations with endpoint x all leaves of our laminations contained in U and mapped to leaves with endpoint x have as the corresponding endpoint $t_x(U) \in \partial U \cap \mathbb{S}$.

Evidently, the base of induction holds. Suppose that $\mathcal{L}_n$ satisfies all the declared conditions and consider $\mathcal{L}_{n+1} \setminus \mathcal{L}_n$. Let us show that $\mathcal{L}_{n+1}$ is sibling invariant. To this end we need to verify that a non-critical leaf $\ell \in \mathcal{L}_{n+1} \setminus \mathcal{L}_n$ has $d-1$ sibling leaves, and all these d leaves are pairwise disjoint. We may assume that $\ell \in \mathcal{L}_{n+1} \setminus \mathcal{L}_n$. Hence ℓ was added on the last step in the construction. By construction this implies that each $\mathcal{C}$-domain contains exactly one preimage of $\sigma_d(\ell) = \overline{xy}$. Moreover, there are no other preimages of $\overline{xy}$ in $\mathcal{L}_{n+1}$ and by the Claim all these preimages of $\overline{xy}$ are pairwise disjoint. This implies the desired. □

The next definition is a step towards constructing a fully invariant (not just forward invariant) lamination generated by $\mathcal{C}$.

Definition 4.12. Consider a sequence $\mathcal{L}_0 = \{\overline{c}_{r+1}, \dots, \overline{c}_{d-1}\}, \mathcal{L}_1, \mathcal{L}_2, \dots$ of finite pullback laminations generated by $\mathcal{C}$. Call the union of all leaves from all laminations $\mathcal{L}_n, n = 0, 1, \dots$ a *pullback prelamination generated by* $\mathcal{C}$.

By Lemma 4.11, the family of such pullback prelaminations is non-empty. Since in Definition 4.12 we mean *any* sequence of pullback laminations, not just a particular sequence constructed in Lemma 4.11, there may exist several pullback prelaminations generated by $\mathcal{C}$.

Definition 4.13. The closure of a pullback prelamination generated by $\mathcal{C}$ is called a *pullback lamination generated by* $\mathcal{C}$

Observe that by Definition 4.13, the pullbacks of leaves from $\mathcal{C}$ are dense in any pullback lamination $\mathcal{L}$ generated by $\mathcal{C}$ while no leaf of $\mathcal{L}$ crosses a leaf from $\mathcal{C}$.

Theorem 4.14. *Any pullback lamination generated by* $\mathcal{C}$ *is invariant.*

PROOF. The claim follows from Corollary 3.20 [**BMOV13**]. □

By [**BMOV13**] a *proper* invariant geodesic lamination $\mathcal{L}$ defines a laminational equivalence relation $\approx_{\mathcal{L}}$ such that $a \approx_{\mathcal{L}} b$ if and only if there exists a finite chain of leaves of $\mathcal{L}$ connecting a and b. We will show that all pullback laminations are proper and that they all generate *the same laminational equivalence*.

Recall that in the pullback construction we only pullback the leaves $\overline{c}_{r+1}, \dots, \overline{c}_{d-1}$ (by definition these are exactly the leaves from our critical collection which do not have periodic endpoints).

Lemma 4.15. *Let $\mathcal{C} = (\overline{c}_1, \dots, \overline{c}_{d-1})$ be an r-admissible critical collection. Let $\mathcal{L}(\mathcal{C})$ be a geodesic lamination generated by $\mathcal{C}$. Then:*

(1) *$\mathcal{L}(\mathcal{C})$ is proper and generates a laminational equivalence relation $\approx_{\mathcal{L}(\mathcal{C})}$ (in particular, $a_j \approx_{\mathcal{L}(\mathcal{C})} b_j$ for each $j, r+1 \leqslant j \leqslant d-1$);*
(2) *if ℓ is a critical leaf of $\mathcal{L}(\mathcal{C})$, then its endpoints must be non-periodic vertices of one of the critical chains from $\mathcal{C}$ (in particular, if all critical chains are just leaves - e.g., if all leaves in $\mathcal{C}$ are pairwise disjoint - then ℓ must be $\overline{c}_j$ for some $j, r+1 \leqslant j \leqslant d-1$);*
(3) *if $\widehat{\mathcal{L}}$ is another pullback lamination generated by $\mathcal{C}$ then it is proper and $\approx_{\widehat{\mathcal{L}}} = \approx_{\mathcal{L}(\mathcal{C})}$ (and so the laminational equivalence relation $\approx_{\mathcal{L}(\mathcal{C})}$ depends only on $\mathcal{C}$ and is from now on denoted by $\approx_{\mathcal{C}}$);*
(4) *if $\widehat{\mathcal{L}}$ is an invariant geodesic lamination with leaves $\overline{c}_{r+1}, \dots, \overline{c}_{d-1}$ whose pullbacks do not cross leaves from $\mathcal{C}$, then $\widehat{\mathcal{L}}$ contains all limit leaves of $\mathcal{L}(\mathcal{C})$ and at least one pullback of each $\overline{c}_i, r+1 \leqslant i \leqslant d-1$ inside each $\mathcal{C}$-maximal pullback of $\overline{c}_i$;*
(5) *if $\sim$ is a laminational equivalence relation such that $a_j \sim b_j, r+1 \leqslant j \leqslant d-1$ and no leaf of $\mathcal{L}_\sim$ crosses $\overline{c}_i, 1 \leqslant i \leqslant r$, then for any two points a, b such that $a \approx_{\mathcal{C}} b$ we have $a \sim b$.*

PROOF. (1) The fact that $\mathcal{L}(\mathcal{C})$ is proper follows from the way we define it and Lemma 4.7.

(2) Since a critical leaf ℓ of $\mathcal{L}(\mathcal{C})$ must not cross leaves from $\mathcal{C}$, it must be contained in a $\mathcal{C}$-domain. This implies that its endpoints must be vertices of one of the critical chains from $\mathcal{C}$ because distinct critical chains from the boundary of the same $\mathcal{C}$-domain have distinct images. Using Lemma 4.7, the particular cases listed in the rest of the claim now easily follow.

(3) By definition, $\mathcal{L}(\mathcal{C})$ is the closure of an invariant prelamination that is the union of leaves from a sequence of finite pullback laminations $\mathcal{L}_0 \subset \mathcal{L}_1 \subset \dots$ generated by $\mathcal{C}$. For each n define the equivalence relation $\approx_n$ on $\mathbb{S}$ as follows: two points $x, y \in \mathbb{S}$ are $\approx_n$-equivalent if there exists a finite chain of concatenated leaves of $\mathcal{L}_n$ such that x and y are the endpoints of the first and the last leaf in the chain respectively. In cases like that we will simply say that the chain of leaves in question *connects* x and y.

Suppose that $\ell_1 \cup \ell_2 \cup \dots \ell_k$ is a chain of concatenated leaves of $\mathcal{L}_n$. Let x be the initial endpoint of ℓ_1; let y be the terminal point of ℓ_k. Choose a $\mathcal{C}$-domain U and points $x'', y'' \in \partial U$ such that $\sigma_d(x'') = x, \sigma_d(y'') = y$. Then, by Lemma 4.10, there exists a chain of leaves $\ell'_1 \cup \ell'_2 \cup \dots \cup \ell'_k$ of $\mathcal{L}_{n+1}$ such that $\ell'_i \subset U, \sigma_d(\ell'_i) = \ell_i$ and for each $i, 1 \leqslant i \leqslant k-1$ the terminal point of ℓ'_i and the initial endpoint of ℓ'_{i+1} are connected with a chain of critical edges of U; moreover, if x' is the initial endpoint of ℓ' and y' is the terminal point of ℓ'_k then there exists also a chain of critical edges of U connecting x'' and x' and a chain of critical ledges of U connecting y' and y''. This amounts to the following claim: *if x and y are connected with a chain of leaves of $\mathcal{L}_n$ and $x'', y'' \in \partial U$ are such that $\sigma_d(x'') = x, \sigma_d(y'') = y$ then there exists a chain of leaves of $\mathcal{L}_{n+1}$ contained in U and connecting x'' and y''.*

Now, let $\widehat{\mathcal{L}}$ be another pullback lamination generated by $\mathcal{C}$; denote by $\widehat{\mathcal{L}}_n$ the finite laminations whose sequence gives rise to $\widehat{\mathcal{L}}$. For each n the finite pullback lamination $\widehat{\mathcal{L}}_n$ generates the equivalence relation $\widehat{\approx}_n$ in the same fashion as above for $\approx_n$. We claim that $\approx_n = \widehat{\approx}_n$. Indeed, it is clear that $\approx_0 = \widehat{\approx}_0$ (after all, $\mathcal{L}_0 = \widehat{\mathcal{L}}_0 = \{\overline{c}_{r+1}, \dots, \overline{c}_{d-1}\}$). Assume that $\approx_n = \widehat{\approx}_n$ and prove that $\approx_{n+1} = \widehat{\approx}_{n+1}$. It suffices to show that if $x \approx_{n+1} y$, then $x \widehat{\approx}_{n+1} y$. By definition, to this end it is enough to show that if $\ell = \overline{xy}$ is a leaf of $\mathcal{L}_{n+1}$ then there exists a chain of leaves of $\widehat{\mathcal{L}}_{n+1}$ connecting x and y.

First, suppose that ℓ is critical. Then by definition $\ell \in \mathcal{L}_0 = \widehat{\mathcal{L}}_0$ and we are done. Second, suppose that ℓ is not critical. Then $\ell \subset U$ for some $\mathcal{C}$-domain U, and $\sigma_d(\ell)$ is a leaf of $\mathcal{L}_n$. By induction there exists a chain of leaves of $\widehat{\mathcal{L}}_n$ connecting $\sigma_d(x)$ and $\sigma_d(y)$. By the above claim there exists a chain of leaves of $\widehat{\mathcal{L}}_{n+1}$ connecting x and y (and contained in U). Hence $x \widehat{\approx}_{n+1} y$ as desired. By induction this implies that $\approx_n = \widehat{\approx}_n$ for every n. By definition of the laminational equivalence relation determined by a proper invariant lamination it then follows that $\approx_{\widehat{\mathcal{L}}} = \approx_{\mathcal{L}(\mathcal{C})}$ as desired.

(4) By definition of invariant lamination both $\widehat{\mathcal{L}}$ and $\mathcal{L}(\mathcal{C})$ have leaves contained in each $\approx_{\mathcal{C}}$-class. Therefore any limit leaf of $\mathcal{L}(\mathcal{C})$ is a limit leaf of $\widehat{\mathcal{L}}$ and vice versa; moreover, any such limit leaf is the limit of a sequence of convex hulls of $\approx_{\mathcal{C}}$-classes. The fact that $\widehat{\mathcal{L}}$ contains at least one pullback of each $\overline{c}_i, r+1 \leqslant i \leqslant d-1$ inside each $\mathcal{C}$-maximal pullback of $\overline{c}_i$ is immediate.

(5) The leaves $\overline{c}_j$ with $r+1 \leqslant j \leqslant d-1$ either are leaves of $\mathcal{L}_\sim$ or are contained in finite gaps of $\mathcal{L}_\sim$. Add them to $\mathcal{L}_\sim$ and then pull them back as leaves of $\mathcal{L}_\sim$ or inside pullbacks of gaps of $\mathcal{L}_\sim$; do it according to Definition 4.9. In this way, we can get an invariant pullback prelamination generated by $\mathcal{C}$. When we close it, by (4) the new leaves added all belong to $\mathcal{L}_\sim$. Thus, $\mathcal{L}(\mathcal{C})$ consists of either leaves of $\mathcal{L}_\sim$ or leaves contained inside finite gaps of $\mathcal{L}_\sim$. By definition of $\approx_{\mathcal{C}}$, this implies the desired. □

The following definition is central for this section.

Definition 4.16 (Pullback laminational equivalence relations)**.** The laminational equivalence relation $\approx_{\mathcal{C}}$ is called the *laminational equivalence relation generated by* $\mathcal{C}$.

In the next lemma, we study possible differences between certain invariant geodesic pullback laminations.

Lemma 4.17. *In the notation of Lemma 4.15 the following holds.*

(1) *There may exist leaves of $\mathcal{L}(\mathcal{C})$ which intersect the interiors of finite gaps of $\mathcal{L}_{\approx_{\mathcal{C}}}$; all such leaves are pullbacks of the leaves $\overline{c}_{r+1}, \dots, \overline{c}_{d-1}$. This is the only situation when a leaf of $\mathcal{L}(\mathcal{C})$ does not belong to $\mathcal{L}_{\approx_{\mathcal{C}}}$.*
(2) *Suppose that a leaf ℓ of $\mathcal{L}_{\approx_{\mathcal{C}}}$ is a common edge of an infinite Fatou gap and a finite gap. Then either ℓ is a pullback of a leaf $\overline{c}_j$ with $r+1 \leqslant j \leqslant d-1$, or ℓ does not belong to $\mathcal{L}(\mathcal{C})$. The latter is the only situation when a leaf from $\mathcal{L}_{\approx_{\mathcal{C}}}$ does not belong to $\mathcal{L}(\mathcal{C})$.*
(3) *A leaf ℓ of $\mathcal{L}(\mathcal{C})$ that is not a pullback of one of the leaves $\overline{c}_i$ with $r+1 \leqslant i \leqslant d-1$ (in particular, this holds if ℓ is periodic) is non-isolated in $\mathcal{L}(\mathcal{C})$, non-isolated in $\mathcal{L}_{\approx_{\mathcal{C}}}$, and belongs to $\mathcal{L}_{\approx_{\mathcal{C}}}$.*

PROOF. (1) If a leaf ℓ of $\mathcal{L}(\mathcal{C})$ is not a leaf of $\mathcal{L}_{\approx_{\mathcal{C}}}$, then there must exist a finite gap of $\mathcal{L}_{\approx_{\mathcal{C}}}$ containing ℓ. Moreover, since in that case ℓ is isolated in $\mathcal{L}(\mathcal{C})$, it itself must be a pullback of a leaf $\overline{c}_j$ with $r+1 \leqslant j \leqslant d-1$.

(2) Suppose that a leaf $\ell = \overline{ab}$ of $\mathcal{L}_{\approx_{\mathcal{C}}}$ is not a leaf of $\mathcal{L}(\mathcal{C})$. By definition there exists a finite chain of leaves of $\mathcal{L}(\mathcal{C})$ connecting a and b. Hence ℓ is an edge of a finite gap G of $\mathcal{L}_{\approx_{\mathcal{C}}}$. If ℓ is not isolated in $\mathcal{L}(\mathcal{C})$ from the outside of G, then it itself must belong to $\mathcal{L}(\mathcal{C})$, a contradiction. If there is a finite gap of $\mathcal{L}(\mathcal{C})$ outside of G that shares ℓ with G, then this gap and G must be united into one bigger gap of $\mathcal{L}_{\approx_{\mathcal{C}}}$, a contradiction to the definition of $\approx_{\mathcal{C}}$. Hence there is a Fatou gap U of $\approx_{\mathcal{C}}$ such that ℓ is an edge of U. Since ℓ is not a leaf

of $\mathcal{L}(\mathcal{C})$, it cannot be a pullback of a leaf $\overline{c}_j, r+1 \leqslant d-1$. On the other hand, if a leaf ℓ of $\mathcal{L}_{\approx_\mathcal{C}}$ is a common edge of an infinite Fatou gap and a finite gap, then, in case this leaf is not a pullback of a leaf $\overline{c}_j, r+1 \leqslant d-1$, it follows from definitions that it cannot be a leaf of $\mathcal{L}(\mathcal{C})$ as desired.

(3) If a leaf $\hat{\ell}$ does not belong to the union of the grand orbits of leaves $\overline{c}_i$ with $r+1 \leqslant i \leqslant d-1$, then by definition it is approximated by pullbacks of these leaves disjoint from $\hat{\ell}$ (as the endpoints of leaves $\overline{c}_i$ with $r+1 \leqslant i \leqslant d-1$ are non-periodic, only finitely many their pullbacks share an endpoint). By Lemma 4.15, each leaf $\overline{c}_i, r+1 \leqslant i \leqslant d-1$ is contained in the convex hull G of a $\approx_\mathcal{C}$-class where G is a leaf or a finite gap. Hence $\hat{\ell}$ is a limit leaf of leaves of $\mathcal{L}_{\approx_\mathcal{C}}$. This implies that $\hat{\ell}$ is a leaf of $\mathcal{L}_{\approx_\mathcal{C}}$. □

Let us study periodic Fatou gaps of $\mathcal{L}_{\approx_\mathcal{C}}$ and $\mathcal{L}(\mathcal{C})$.

Lemma 4.18. *Let U be a periodic Fatou gap of $\approx_\mathcal{C}$ of degree $s > 1$. Then all finite periodic gaps G attached to U at edges ℓ of U are fixed return so that U is of non-rotational type; moreover, all other edges of gaps G which are attached to U are non-isolated in $\mathcal{L}(\mathcal{C})$. Each critical chord $\overline{c}_i$ with $i \leqslant r$ is contained in a periodic critical Fatou gap W of $\mathcal{L}(\mathcal{C})$ of degree greater than one. Finally, there are no preperiodic critical Fatou gaps of $\approx_\mathcal{C}$ (or of $\mathcal{L}(\mathcal{C})$).*

PROOF. Let G be a finite periodic gap attached to U at an edge M. Then M is isolated in $\mathcal{L}_{\approx_\mathcal{C}}$ and periodic. This implies that M is not a leaf of $\mathcal{L}(\mathcal{C})$. All other edges of G must be non-isolated in $\mathcal{L}_{\approx_\mathcal{C}}$ because otherwise these edges are not leaves of $\mathcal{L}(\mathcal{C})$. This shows that vertices of G are not connected with finite chains of leaves of $\mathcal{L}(\mathcal{C})$, and G is not a gap of $\mathcal{L}_{\approx_\mathcal{C}}$, a contradiction. Since all other edges $\ell \neq M$ of G are non-isolated in $\mathcal{L}_{\approx_\mathcal{C}}$, then, clearly, G must be fixed return. Similarly, U cannot have periodic edges of flip type. This implies that U is of non-rotational type.

By Lemma 4.15, the chord $\overline{c}_i$ (with $i \leqslant r$) is not a leaf of $\mathcal{L}(\mathcal{C})$. Hence $\overline{c}_i$ is contained in a critical gap of $\mathcal{L}_{\approx_\mathcal{C}}$. Since critical gaps of laminational equivalence relations can only have periodic vertices if they are infinite, it follows that $\overline{c}_i$ is contained is a critical periodic Fatou gap W. Suppose that W is not periodic. Let $\overline{c}_i$ have a periodic endpoint a of period m. Since W is not periodic then a cannot be non-isolated on $\mathrm{Bd}(W)$ from either side. Hence a is an edge of a periodic leaf which implies that W is actually periodic. Thus, $\overline{c}_i$ is contained in a periodic critical Fatou gap W of $\mathcal{L}(\mathcal{C})$ of degree greater than one. The last claim is left to the reader. □

The following is a corollary of our results.

Corollary 4.19. *Let $\mathcal{C}$ be a 0-admissible critical collection of chords. Then $\mathcal{L}_{\approx_\mathcal{C}} \subset \mathcal{L}(\mathcal{C})$, all critical sets of $\mathcal{L}_{\approx_\mathcal{C}}$ are finite, all its infinite gaps (if any) are periodic Siegel gaps and their degree one preimages. If $\ell = \overline{ab}$ is a chord such that all its forward images do not cross leaves from $\mathcal{C}$ and one another then $a \approx_\mathcal{C} b$.*

PROOF. The proof of all the claims but the last one is left to the reader. Now, let $\ell = \overline{ab}$ be a chord such that all its forward images do not cross leaves from $\mathcal{C}$. Let us show that then $a \approx_\mathcal{C} b$. Observe that if ℓ is non-disjoint from a leaf $\hat{\ell}$ of $\mathcal{L}(\mathcal{C})$ then the corresponding images $\sigma_d^m(\ell)$ and $\sigma_d^m(\hat{\ell})$ are also non-disjoint (this is because both leaves do not have images crossing critical leaves from $\mathcal{C}$). Now, suppose that ℓ crosses infinitely many pullbacks of critical leaves from $\mathcal{C}$. Since by the assumption images of ℓ do not cross leaves from $\mathcal{C}$ this would imply that for each σ_d^N-pullback of a leaf of $\mathcal{C}$ crossing ℓ the leaf $\sigma_d^N(\ell)$ has a common endpoint with a leaf from $\mathcal{C}$. Since we assume the existence of infinitely many pullbacks of critical leaves from $\mathcal{C}$ crossing ℓ, then there are some leaves from $\mathcal{C}$ that have periodic endpoints, a contradiction.

Thus, there are only finitely many pullbacks of critical leaves from $\mathcal{C}$ that cross ℓ. This implies that in fact there is a finite string of gaps of $\approx_{\mathcal{C}}$, say, $G_1, G_2, \dots, G_k$ such that a is a vertex of G_1, b is a vertex of G_k, and each two gaps G_i and G_{i+1} share a common edge that crosses ℓ. This implies that at least one of these gaps is infinite and hence is a preimage of a periodic Siegel gap of $\approx_{\mathcal{C}}$ in which ℓ connects two edges. Hence a forward image ℓ' of ℓ connects edges of a periodic Siegel gap U. Since the canonical map that collapses edges of gaps semiconjugates the first return map and an irrational rotation, it follows that the image of ℓ' under this semiconjugacy will cross one of its eventual images, a contradiction. Hence the situation described above is impossible and $a \approx_{\mathcal{C}} b$ as desired. □

Definition 4.20 (Tuning of laminations)**.** Suppose that $\mathcal{L}$ is an invariant geodesic lamination. Suppose that there exists a periodic Fatou gap U of $\mathcal{L}$ of degree greater than one. Finally, suppose that there exists an invariant geodesic lamination $\widehat{\mathcal{L}} \supset \mathcal{L}$ such that $\widehat{\mathcal{L}} \setminus \mathcal{L}$ consists of leaves contained in the grand orbit of U. Then we say that $\widehat{\mathcal{L}}$ *tunes* $\mathcal{L}$ *(in the grand orbit of* U*)*. If $\mathcal{L}$ generates a laminational equivalence relation $\approx_{\mathcal{L}}$ while $\widehat{\mathcal{L}}$ generates a laminational equivalence relation $\approx_{\widehat{\mathcal{L}}}$, then we say that $\approx_{\widehat{\mathcal{L}}}$ *tunes* $\approx_{\mathcal{L}}$.

Corollary 4.21 deals with more specific pullback geodesic laminations.

Corollary 4.21. *Let $\mathcal{C}$ be an admissible critical collection with $r = 1$; let $\overline{c}_1 = \overline{a_1 b_1}$ where a_1 is of period n. Then the following holds.*

(1) *The chord $\overline{c}_1$ is a subset of a periodic quadratic critical Fatou gap V of $\mathcal{L}(\mathcal{C})$ of period m such that $n = mk$ is a multiple of m; the gap V contains a periodic quadratic critical Fatou gap U of $\approx_{\mathcal{C}}$ of period m.*
(2) *Except for U, all critical sets of $\approx_{\mathcal{C}}$ are finite. All infinite gaps of $\approx_{\mathcal{C}}$ are either in the grand orbit of U or in the grand orbits of (possibly existing) periodic Siegel gaps.*

Moreover, there exists a laminational equivalence relation $\sim$ that tunes $\approx_{\mathcal{C}}$ inside the grand orbit of U, has a critical quadratic periodic Fatou gap $T \subset U$, and is such that a_1 is a refixed vertex of T or of a finite periodic gap G attached to T at the refixed edge of T.

Observe that the gap V may contain finite concatenations of edges. The gap $U \subset V$ is obtained from V by replacing every such maximal concatenation of edges with a single leaf (an edge of the convex hull of the concatenation).

PROOF. (1) This claim follows immediately from Lemma 4.18.

(2) This claim follows from Lemma 4.15.

To prove the last claim, consider $\sigma_d^m|_V$. Since the point a_1 is an n-periodic vertex of V, then a_1 is of period k under the action of σ_d^m. Apply the standard monotone semiconjugacy ψ between $\sigma_d^m|_V$ and σ_2 (the map ψ simply collapses edges of V to points of $\mathbb{S}$). It follows that $\psi(a_1) = x$ is a k-periodic point of σ_2. Now, it is well-known that there exists a σ_2-invariant laminational equivalence relation $\simeq$ such that there exists a critical periodic Fatou gap W_2 of $\simeq$ and x is an endpoint of the major edge of W_2 (in particular, x is a refixed vertex of W_2).

Using ψ, we can lift the equivalence relation $\simeq$ to the entire V. This gives rise to a σ_d^m-equivalence relation $\sim^V$ on V and to the corresponding σ_d^m-invariant geodesic sublamination $\mathcal{L}_\sim^V$ of V. Note that we write $\mathcal{L}_\sim^V$ instead of $\mathcal{L}_{\sim^V}$. We can then pull this back under the action of σ_d to obtain a laminational equivalence relation $\sim$ on the entire circle.

Points that are $\approx_{\mathcal{C}}$-equivalent will be declared $\sim$-equivalent while otherwise two points are declared $\sim$-equivalent if the chord connecting them lies in a pullback of a finite gap of $\mathcal{L}^V_\sim$. This defines a σ_d-invariant laminational equivalence relation $\sim$ on the entire circle. Basically, we tune V and then pull back this tuning. The equivalence relation $\sim$ is closed because pullbacks of leaves of $\mathcal{L}^V_\sim$ can only accumulate on leaves of $\mathcal{L}^V_\sim$ or in the boundary of the grand orbit of V.

Let us show that $\sim$ satisfies all the necessary conditions. Clearly, all of them are automatically satisfied except for the claims concerning a_1. Now, the construction implies that the ψ-preimage of the gap W_2 is a stand-alone Fatou gap $H \subset V$ of period n. Observe that H is not necessarily a gap of $\sim$ because some vertices of W_2 may have non-degenerate ψ-preimages. However H contains a gap T of $\sim$, and one can obtain T by completing convex hulls of finite chains of edges on the boundary of H with extra edges. The relation of H and $T \subset H$ is like the relation of V and $U \subset V$. Now, if the major M_2 of W_2 has endpoints with degenerate ψ-preimages then $\psi^{-1}(M_2) = M$ is just a leaf of $\sim$ and a_1 is an endpoint of M as desired. If, however, one or both endpoints of M_2 have non-degenerate ψ-preimages, then the ψ-preimage of M_2 is the desired finite gap G. □

Theorem 4.22. *Let* $\mathrm{QCP} = \{C_1, \overline{c}_2 = \overline{a_2b_2}, \dots, \overline{c}_{d-1} = \overline{a_{d-1}b_{d-1}}\}$ *be a geolaminational quadratically critical portrait such that* C_1 *is a critical quadrilateral and* $\overline{c}_2, \dots, \overline{c}_{d-1}$ *are critical leaves with non-periodic endpoints. Then there exists a non-empty laminational equivalence relation* $\sim$ *such that* $a_2 \sim b_2, \dots, a_{d-1} \sim b_{d-1}$ *while* C_1 *is such that either* (1) *all its vertices are* $\sim$*-equivalent and non-periodic, or* (2) C_1 *has a periodic edge, and if* ℓ', ℓ'' *are diagonals of* C_1 *and* $\mathcal{T}' = (\ell', \overline{c}_2, \dots, \overline{c}_{d-1})$, $\mathcal{T}'' = (\ell'', \overline{c}_2, \dots, \overline{c}_{d-1})$, *then* $\mathcal{L}(\mathcal{T}') = \mathcal{L}(\mathcal{T}'') = \mathcal{L}$, $\approx_{\mathcal{T}'} = \approx_{\mathcal{T}''}$, *and* C_1 *is contained in a critical quadratic periodic Fatou gap of* $\mathcal{L}$.

PROOF. If C_1 has a diagonal ℓ with non-periodic endpoints, then we add ℓ to $\overline{c}_2, \dots, \overline{c}_{d-1}$ to form a collection $\mathcal{C}$. By Corollary 4.19, there exists a laminational equivalence relation $\approx_{\mathcal{C}}$ such that $\ell, \overline{c}_2, \dots, \overline{c}_{d-1}$ connect pairs of $\approx_{\mathcal{C}}$-equivalent points. Moreover, by the assumptions and by Corollary 4.19, all vertices of C_1 are $\approx_{\mathcal{C}}$-equivalent and non-periodic, as desired. Otherwise we may assume that there is an edge $\overline{ab}$ of C_1 such that a and b are periodic. Choose a diagonal ℓ' of C_1, say, the one that contains a. Suppose that a is of period n'. Set $\mathcal{T}' = \{\ell', \overline{c}_2, \dots, \overline{c}_{d-1}\}$. By Lemma 4.15 there exist an invariant geodesic pullback lamination $\mathcal{L}(\mathcal{T}')$ and the associated laminational equivalence relation $\approx_{\mathcal{L}_{\mathcal{T}'}}$. Moreover, by Corollary 4.21, the leaf ℓ' is contained in a periodic critical Fatou gap V' of $\mathcal{L}(\mathcal{T}')$. On the other hand, V' contains a Fatou gap U' of $\approx_{\mathcal{T}'}$. We may assume that the period of U' and V' is m' while $n' = m'k'$.

Let us discuss the location of b with respect to this picture; we may assume that $b \neq a$ and that C_1 is non-degenerate. We claim that b is a vertex of V'. Indeed, suppose otherwise. Then either $\overline{ab}$ intersects V' at only one point a, or $\overline{ab}$ crosses an edge of V'. Since all edges of V' are either pullbacks of leaves $\overline{c}_i$ with $2 \leqslant i \leqslant d-1$, or limits of such pullbacks, it follows that in either case there exists a pullback N of one of the leaves $\overline{c}_2 = \overline{a_2b_2}, \dots, \overline{c}_{d-1} = \overline{a_{d-1}b_{d-1}}$ that crosses $\overline{ab}$. Let us show that this leads to a contradiction.

Observe that Thurston's pullback construction implies the existence of a pullback geodesic lamination $\mathcal{L}'$ that contains **all** leaves from $\mathcal{T}'$. This geodesic lamination strictly contains $\mathcal{L}(\mathcal{T}')$ because (1) it must contain all pullbacks of leaves $\overline{c}_2, \dots, \overline{c}_{d-1}$ that generate $\mathcal{L}(\mathcal{T}')$, and (2) by Lemma 4.7, the leaf ℓ' is not a limit leaf of $\mathcal{L}'$. By definition, $\mathcal{L}^{\mathrm{QCP}}$ with quadratically critical portrait QCP and $\mathcal{L}'$ with quadratically critical portrait $\mathcal{T}'$ are essentially equal. Since the leaves $\overline{ab}$ and N are linked, they will either (1) stay linked under any iteration of σ_d, or (2) there will exist the minimal $i+1$ such that $\sigma_d^{i+1}(\overline{ab})$ and $\sigma_d^{i+1}(N)$ are not linked while $\sigma_d^i(\overline{ab})$ and $\sigma_d^i(N)$ are linked.

Now, in case (1), we will eventually obtain that the image of N that coincides with $\overline{c}_j$ for some $2 \leqslant j \leqslant d-1$ is linked with an image of $\overline{ab}$. This contradicts the fact that QCP is geolaminational. Consider case (2). Then $\sigma_d^i(N)$ cannot be critical again because QCP is geolaminational. Hence $\sigma_d^i(N)$ is precritical. This implies that $\sigma_d^{i+1}(N)$ is non-degenerate, (pre)critical, and has a periodic endpoint. Applying a suitable iteration of σ_d, we will observe that a certain image of N is a leaf $\overline{c}_j$ with $2 \leqslant j \leqslant d-1$ with a periodic endpoint, a contradiction. Thus, b is a vertex of V', which implies that $C_1 \subset V'$.

Clearly, the same construction can be implemented for $\mathcal{T}''$ based upon the other diagonal of C_1 passing through b. We may assume that the period of b is n''. It leads to a invariant geodesic pullback lamination $\mathcal{L}(\mathcal{T}'')$ and the associated laminational equivalence relation $\approx_{\mathcal{L}_{\mathcal{T}''}}$. Moreover, ℓ'' is contained in a periodic critical Fatou gap U'' of $\approx_{\mathcal{T}''}$, which is contained in the corresponding Fatou gap V'' of $\mathcal{L}(\mathcal{T}'')$. We may assume that the period of U'' and V'' is m'' while $n'' = m''k''$. As before for $\mathcal{T}'$ we will also have that $C_1 \subset V''$.

We need to show that $\approx_{\mathcal{T}'} = \approx_{\mathcal{T}''}$. To this end, observe that, since $C_1 \subset V'$, all pullbacks of leaves $\overline{c}_2, \dots, \overline{c}_{d-1}$ chosen for $\approx_{\mathcal{T}'}$ can be described as pullbacks outside of C_1, and the same can be said about pullbacks of $\overline{c}_2, \dots, \overline{c}_{d-1}$ chosen for $\approx_{\mathcal{T}''}$. Therefore, these pullbacks coincide. Since they are dense in $\mathcal{L}(\mathcal{T}')$ and in $\mathcal{L}(\mathcal{T}'')$, we have $\mathcal{L}(\mathcal{T}') = \mathcal{L}(\mathcal{T}'')$, which implies the other claims in the end of the lemma. □

4.2.2. The space of σ_d-invariant geodesic laminations compatible with a given generic collection of $d-2$ critical chords. We will now describe the results of [**BOPT15**] omitting technical details. Consider cubic geodesic laminations $\mathcal{L}$ with a critical leaf $D = \overline{ab}$ whose endpoints are non-periodic. Without loss of generality, we may assume that (a, b) is a positively oriented circle arc of length $\frac{1}{3}$. Then there are several possibilities concerning critical sets of $\mathcal{L}$. First, $\mathcal{L}$ can have a finite critical set $C \neq D$ contained in the convex hull of the circle arc $[b, a]$. By properties of invariant geodesic laminations, C is a gap or a leaf, on which σ_3 acts two-to-one (unless D is an edge of C and so the point $\sigma_3(D)$ has all three of its preimages in C). Thus, if C is finite, then there are two cases. First, C can be a $2n+1$-gon with D being one of its edges such that one can break down all its remaining edges into pairs of "sibling edges" (one can say that the "sibling edge" of D is the vertex of C not belonging to D and with the same image as D). Second, C can be a $2m$-gon such that D is not an edge of C; in this case $\sigma_3|_C$ is two-to-one.

Now, C could also be an infinite gap. Then it may be a periodic Fatou gap of period k and degree 2 (in this case D may well be an edge of C). Otherwise C may be preperiodic; then it cannot be eventually mapped onto a periodic gap of degree greater than one because we deal with the cubic case. Hence there must exist a periodic Siegel gap U with D being one of its edges and an infinite gap C such that $\sigma_3|_{\mathrm{Bd}(C)}$ is two-to-one and C eventually maps onto U. In other words, an invariant geodesic lamination with leaf D and of capture type must have a periodic Siegel gap U and a critical gap C that eventually maps onto U.

Now, let $\mathbb{LP}_3^{np}(D)$ be the family of all cubic geodesic laminations with a critical leaf D with non-periodic endpoints *except for geodesic laminations of capture type*. If $\mathcal{L} \in \mathbb{LP}_3^{np}(D)$, then a quadratically critical portrait QCP $= (Q, D)$ is said to be *privileged for* $\mathcal{L}$ if $Q \subset C$, where $C \neq D$ is defined above, and the additional requirement mentioned below is fulfilled. By the above analysis, either C is finite, or C is a periodic Fatou gap of degree two and period k. In the former case, the critical quadrilateral $Q \subset C$ can be arbitrary. In the latter case, we require that Q be a collapsing quadrilateral that is the convex hull of a (possibly degenerate) edge ℓ of C of period k and its sibling edge ℓ^* of C.

In [**BOPT15**] we show that for each $\mathcal{L} \in \mathbb{LP}_3^{np}(D)$ there are only finitely many privileged quadratically critical portraits. Let $\mathcal{S}_D$ denote the collection of all privileged for $\mathcal{L}$ quadratically critical portraits (Q, D) with D as the second element. To each such quadratically critical portrait (Q, D) we associate its *minor* (a chord or a point) $\sigma_3(Q) \subset \overline{\mathbb{D}}$. For each such chord we identify its endpoints, extend this identification by transitivity and define the corresponding equivalence relation $\simeq_D$ on $\mathbb{S}$. The main result of [**BOPT15**] is that $\simeq_D$ is itself a laminational equivalence (non-invariant!) whose quotient is a parameterization of $\mathbb{LP}_3^{np}(D)$.

The tools used in [**BOPT15**] are based upon accordions and smart criticality rather than upon Thurston's tools [**Thu85**]. Indeed, the main technical lemma used in [**Thu85**] is the Central Strip Lemma showing how "long" (based upon circle arcs longer than $\frac{1}{3}$) leaves of invariant geodesic laminations may enter the central strips between themselves and their siblings. The lemma has a multitude of consequences, including the fact that there are no wandering (i.e. non-preperiodic and non-precritical) triangles of quadratic invariant geodesic laminations, and the construction of QML. However, the Central Strip Lemma fails already in the cubic case (in particular, there are wandering triangles of cubic invariant geodesic laminations [**BO08**]). This shows the necessity of using new techniques in [**BOPT15**].

In order to generalize the results of [**BOPT15**] to the degree d case, we introduce appropriate spaces of laminations analogous to $\mathbb{LP}_3^{np}(D)$; these spaces depend not on one critical leaf but on a suitable *collection* of critical leaves.

Definition 4.23. Fix a collection $\mathcal{Y}$ of $d-2$ pairwise disjoint critical chords with non-(pre)periodic endpoints and pairwise disjoint forward orbits. Define a space $\mathbb{L}(\mathcal{Y})$ of *invariant geodesic laminations* as follows: $\mathcal{L} \in \mathbb{L}(\mathcal{Y})$ if $\mathcal{L}$ is generated by a *laminational equivalence relation* $\sim$ such that the endpoints of each critical chord from $\mathcal{Y}$ are $\sim$-equivalent, and $\mathcal{L}$ has no gaps of Siegel capture type.

Let $\mathcal{Y}^+$ be the union of all critical chords from $\mathcal{Y}$. It is easy to see that there is a unique component $A(\mathcal{Y}) = A$ of $\overline{\mathbb{D}} \setminus \mathcal{Y}^+$ such that $\sigma_d|_{\mathrm{Bd}(A)}$ is two-to-one except for its critical boundary edges (this map is one-to-one in the same sense on all other components of $\overline{\mathbb{D}} \setminus \mathcal{Y}^+$). Indeed, $d-2$ critical chords of $\mathcal{Y}$ split the disc into $d-2$ connected sets each of which has the boundary whose intersection with the circle maps onto the entire circle in almost one-to-one fashion (except for the endpoints of boundary edges that are critical chords). Hence the length of each such intersection is $\frac{a}{d}$ for some $a > 0$. Clearly, this implies that $d-3$ of them has the boundary whose intersection with the circle is of length $\frac{1}{d}$ while one of them has the boundary whose intersection with the circle is of length $\frac{2}{d}$. This is exactly the desired component A.

Denote by $\overline{y}_1, \ldots, \overline{y}_k$ all critical chords from $\mathcal{Y}$ contained in the boundary of A (clearly, $1 \leqslant k \leqslant d-2$). Consider $\overline{y}_1$; there exists exactly one point $a \in \mathrm{Bd}(A) \setminus \overline{y}_1$ with $\sigma_d(a) = \sigma_d(\overline{y}_1)$ (a chord $\overline{y}_t \subset \mathrm{Bd}(A)$ with $\sigma_d(\overline{y}_1) = \sigma_d(\overline{y}_t)$ would contradict the assumption of pairwise disjointness of forward orbits of critical chords from $\mathcal{Y}$) while other points of $\mathrm{Bd}(A)$ have images disjoint from $\sigma_d(\overline{y}_1)$. The same holds for other boundary critical chords of A. For any other component T of $\overline{\mathbb{D}} \setminus \mathcal{Y}^+$ it is easy to see that except for the collapsing of the boundary chords of T all other points of $\mathrm{Bd}(T)$ map forward in the one-to-one fashion.

Corollary 4.24. *The family $\mathbb{L}(\mathcal{Y})$ is non-empty.*

PROOF. Insert a critical chord $\overline{c}$ in A so that both endpoints of $\overline{c}$ are non-periodic, and $\overline{c}$ is disjoint from all chords from $\mathcal{Y}$. Then Corollary 4.19 implies the existence of the desired invariant laminational equivalence $\sim$ and the geodesic lamination generated by $\sim$. □

Let us study the critical sets of geodesic laminations from $\mathbb{L}(\mathcal{Y})$.

Lemma 4.25. *Let X be a critical set of $\mathcal{L}_\sim \in \mathbb{L}(\mathcal{Y})$. If X is infinite, then $X \subset \overline{A}$ is a periodic quadratic Fatou gap and all other critical sets of $\mathcal{L}$ are finite and non-preperiodic. If X is finite and preperiodic, then $X \subset A$ and all critical sets of $\mathcal{L}$ are finite.*

PROOF. By definition, the only possibly existing infinite critical set X of a geodesic lamination $\mathcal{L} \in \mathbb{L}(\mathcal{Y})$ is a critical Fatou gap contained in A. Clearly, X cannot be a preperiodic gap that maps onto a periodic Fatou gap of degree greater than one because then there will be at least two infinite critical sets of $\mathcal{L}$ (indeed, the orbit of a periodic Fatou gap must contain a critical Fatou gap [**BL02**]). On the other hand, by definition of the family of laminations $\mathbb{L}(\mathcal{Y})$, the set X cannot be a pullback of a periodic Siegel gap. Hence, the only possibility is that X is a periodic quadratic Fatou gap contained in A as desired. The rest of the lemma is immediate. □

Let us define tags for geodesic laminations from $\mathbb{L}(\mathcal{Y})$. Our approach is different from Thurston's: instead of considering *minor leaves*, or *minors*, of geodesic laminations we work with their *minor sets* basically defined as the images of critical sets.

Lemma 4.26. *If $\mathcal{L}_\sim \in \mathbb{L}(\mathcal{Y})$, then there is a unique critical set $C_\sim$ of $\mathcal{L}_\sim$ containing a critical chord $\overline{c}$, where $\overline{c} \subset A(\mathcal{Y})$ except for the endpoints. Any infinite gap non-disjoint from A is contained in $\overline{A}$. Finally, if $x \in \sigma_d(C_\sim) \cap \mathbb{S}$ then the entire set $\sigma_d^{-1}(x) \cap \overline{A}$ is contained in $C_\sim$.*

PROOF. Clearly, at least one critical set C of $\mathcal{L}_\sim$ contains a critical chord $\overline{c}$ contained in $A(\mathcal{Y})$ except for the endpoints. Let us show that this set C is unique. Indeed, it is easy to see that any two critical chords contained in $\overline{A}$ and non-disjoint from A are linked. Therefore, two *distinct critical* sets C_1 and C_2 of $\mathcal{L}_\sim$ with the properties from the lemma cannot exist. By Lemma 4.25, if $C_\sim$ is infinite then $C_\sim$ is a periodic quadratic Fatou gap.

Let U be an infinite gap non-disjoint from A. Since all boundary chords of A are contained in finite gaps of $\mathcal{L}_\sim$ or are themselves leaves of $\mathcal{L}_\sim$, it follows that $U \subset \overline{A}$. In particular, this holds for $C_\sim$ if it is an infinite gap.

Recall that $\overline{y}_1, \dots, \overline{y}_k$ are all critical chords from $\mathcal{Y}$ contained in the boundary of A. We claim that for each $\overline{y}_j, 1 \leqslant j \leqslant k$ either $C_\sim$ is disjoint from $\overline{y}_j$ or $\overline{y}_j \subset C_\sim$. This is clear if $C_\sim$ is a finite gap or leaf. Let $C_\sim$ be a periodic quadratic gap. If $\overline{y}_1 \cap C_\sim = \{z\}$ is a singleton, then the convex hull H of the $\sim$-class of z contains $\overline{y}_1$ and the edge ℓ of $\mathbb{C}_\sim$ with endpoint z. Since z is not (pre)periodic by the assumptions, then H cannot be (pre)periodic. By Lemma 2.28 this implies that ℓ must be (pre)critical. Thus, z is an endpoint of $\overline{y}_1$ that eventually maps to an endpoint of a critical leaf of $\mathcal{L}_\sim$, i.e. an endpoint of a leaf $\overline{y}_s$, a contradiction with our assumptions. Hence for each $\overline{y}_j$ either $C_\sim$ is disjoint from $\overline{y}_j$ or $\overline{y}_j \subset C_\sim$.

The fact that $C_\sim$ is critical implies that $\sigma_d|_{C_\sim \cap \overline{A}}$ is in fact the composition of the map that collapses all boundary chords of $C_\sim \cap \overline{A}$ to points and then an exactly two-to-one map. Let $x \in \sigma_d(C_\sim) \cap \mathbb{S}$. If x is not the image of one of the boundary chords of A, then it has exactly two preimages in $\overline{A}$, and both must belong to $C_\sim$. Otherwise, set $x = \sigma_d(\overline{y}_i)$, where $1 \leqslant i \leqslant k$. Then, by the above, there must still exist one more point in $C_\sim \cap A$ that maps to x. This proves that $\sigma_d^{-1}(x) \cap \overline{A}$ is contained in $C_\sim$, as desired. □

Observe that, while the set $C_\sim$ is typically contained in $\overline{A}$, some parts of it may "stick out" of $\overline{A}$. For example, it may happen that $\overline{y}_1$ is a diagonal of an all critical quadrilateral that has one vertex in A and the other one in a component of $\overline{\mathbb{D}} \setminus \mathcal{Y}^+$ adjacent to A at $\overline{y}_1$; clearly, the same can be said about $\overline{y}_2, \dots, \overline{y}_k$. In fact, any critical set $C_\sim$ not contained in $\overline{A}$ is finite and must contain $\overline{y}_i$ as a chord for some i. The set $C_\sim$ is important in defining *minor sets* of geodesic laminations from $\mathbb{L}(\mathcal{Y})$. Observe that any geodesic lamination from $\mathbb{L}(\mathcal{Y})$ admits legal modifications. Indeed, recall that legal modifications are well-defined if *no critical gap is mapped to a fixed return gap attached to a periodic critical Fatou U at its refixed edge M or to M itself.* However, by Lemma 4.25 in case U exists there are no (pre)periodic critical sets, and the desired follows.

Definition 4.27 (Minor sets of laminations from $\mathbb{L}(\mathcal{Y})$)**.** For $\mathcal{L}_\sim \in \mathbb{L}(\mathcal{Y})$ we define the *minor set* $m_\sim$ of $\mathcal{L}_\sim$ as follows.

(1) If $C_\sim$ is finite, set $m_\sim = \sigma_d(C_\sim)$.
(2) Suppose that $C_\sim$ is a quadratic periodic Fatou gap of period n. Then there is either one or several legal modifications U of $C_\sim$ associated with the corresponding legal modifications of $\mathcal{L}_\sim$ and corresponding legal quadrilaterals Q. In this case $m_\sim$ is defined as the convex hull of the union of σ_d-images of all these legal critical quadrilaterals.

Let us discuss the minor sets from Definition 4.27(2). Suppose that $C_\sim$ is a periodic critical quadratic Fatou gap. The easiest case is when there are no finite gaps attached to $C_\sim$ at the refixed edge M of $C_\sim$. In that case the corresponding legal critical quadrilateral is the convex hull of M and its sibling edge M' of $C_\sim$ so that $m_\sim = \sigma_d(M)$. Another simple case is when a fixed return gap G is attached to $C_\sim$ at its refixed edge M. In that case a unique legally modified gap is obtained by erasing M and its grand orbit from $C_\sim$ (thus, M and its pullbacks on the boundary of $C_\sim$ are replaced by concatenations of the remaining edges of G or appropriate preimages of G). The minor set in this case is the σ_d-image of G.

A more involved is the case when there exists a finite gap G of rotational type attached to $C_\sim$ at M. Then there are several images of M that are edges of G. Denote the two of them closest to M in $\mathrm{Bd}(G)$ by L and R. Then the minor set $m_\sim$ coincides with the convex hull of the image of the segment of the boundary of G containing M and stretching from L to R (not including either L or R). Observe that, unlike in the original paper by Thurston [**Thu85**] or in [**BOPT15a**], the minor set $m_\sim$ *is not* a gap or a leaf of the corresponding invariant geodesic lamination $\mathcal{L}_\sim$.

Our aim is to show that, as in the quadratic case with Thurston's quadratic minor lamination QML, the family of minor sets of invariant geodesic laminations from $\mathbb{L}(\mathcal{Y})$ can be viewed as the family of classes of equivalence of a laminational (non-invariant!) equivalence relation $\sim_\mathcal{Y}$ such that the quotient space $\mathbb{S}/\sim_\mathcal{Y}$ of $\mathbb{S}$ can be viewed as a parameter model of $\mathbb{L}(\mathcal{Y})$. Since we deal here with minor sets and the critical Fatou gaps involved are all quadratic, we will denote the geodesic (non-invariant!) lamination associated with $\sim_\mathcal{Y}$ by $\mathrm{QML}_\mathcal{Y}$. If we create the corresponding model in the plane, then we will have to "pinch" the unit disk, which would yield the associated quotient space of not only the unit circle but of the whole unit disk. This gives the "pinched disk" model, which will be denoted by $\mathcal{M}(\mathcal{Y})$. The boundary of $\mathcal{M}(\mathcal{Y})$ coincides with $\mathbb{S}/\sim_\mathcal{Y}$.

We will also interpret bounded interior components of $\mathcal{M}(\mathcal{Y})$ from the standpoint of dynamics. The description below is given without proofs.

A bounded connected component of the interior of $\mathcal{M}(\mathcal{Y})$ can be of two types: *quadratic* type and *Siegel capture* type. Components of quadratic type are similar to hyperbolic domains of the combinatorial Mandelbrot set $\mathcal{M}_2$. Let $\mathcal{U}$ be a component of quadratic type. It can be associated with an invariant geodesic lamination $\mathcal{L}_\sim$ with a periodic critical Fatou gap $C_\sim \subset A(\mathcal{Y})$ of period n such that $\sigma_d^n|_{C_\sim}$ is two-to-one. The association between $\mathcal{U}$ and $\mathcal{L}_\sim$ is uniquely defined by the properties described below. Consider the legal modification U of $C_\sim$. There is a continuous monotone map $\psi : \mathrm{Bd}(U) \to \mathbb{S}$ that collapses all edges of U and semi-conjugates the restriction of σ_d^n to $\mathrm{Bd}(U)$ with σ_2. Let Q be a critical quadrilateral in U such that $\psi(U)$ is a critical quadrilateral (possibly degenerate), whose σ_2-image lies in a minor set representing a boundary point of the combinatorial main cardioid. Then $\sigma_d(Q)$ lies in a minor set corresponding to a boundary point of $\mathcal{U}$. Conversely, any minor set corresponding to a boundary point of $\mathcal{U}$ includes $\sigma_d(Q)$ for some critical quadrilateral $Q \subset U$ such that $\sigma_2(\psi(U))$ lies in a minor set representing a boundary point of the combinatorial main cardioid. The lamination $\mathcal{L}_\sim$ itself can be viewed as representing a topological polynomial with an attracting periodic point inside the Fatou component corresponding to $C_\sim$. Alternatively, we can think of the corresponding topological polynomial as a topological polynomial with a parabolic periodic point.

Components of Siegel capture type are very different from those that appear in $\mathcal{M}_2$. Each such component is associated with an invariant geodesic lamination of Siegel capture type (such invariant geodesic laminations are excluded from $\mathbb{L}(\mathcal{Y})$). Let $\mathcal{U}$ be a component of Siegel capture type and $\mathcal{L}_\sim$ the associated lamination. The association between $\mathcal{U}$ and $\mathcal{L}_\sim$ is uniquely defined by the properties described below. There is an infinite critical set $C_\sim$ of $\mathcal{L}_\sim$ of Siegel capture type. Insert a finite critical set (a leaf or a quadrilateral) into $C_\sim$. In the new geodesic lamination instead of one gap $C_\sim$ we will have two adjacent gaps separated by a common finite critical set (e.g., a common edge). All such invariant geodesic laminations obtained by inserting various critical sets in $C_\sim$ give rise to finite minor sets, which form the boundary of $\sigma_d(C_\sim)$. These are precisely minor sets corresponding to points in the boundary of $\mathcal{U}$. Thus, the boundary of $\mathcal{U}$ can be naturally identified with $\sigma_d(C_\sim)$.

Recall that, given an invariant geodesic lamination $\mathcal{L}$ with finite critical sets $C_1, \ldots,$ C_r, we call a full quadratically critical portrait *legal* for $\mathcal{L}$ if it is geolaminational and its critical quadrilaterals are contained in the critical sets of $\mathcal{L}$. This holds automatically if the critical quadrilaterals have pairwise disjoint interiors and share opposite sibling edges with critical sets of $\mathcal{L}$.

Lemma 4.28. *Let $\mathcal{L}_{\sim_1}$ and $\mathcal{L}_{\sim_2}$ be two invariant geodesic laminations from $\mathbb{L}(\mathcal{Y})$. Suppose that their legal modifications $\mathcal{L}^{leg}_{\sim_1}$ and $\mathcal{L}^{leg}_{\sim_2}$ have legal quadratically critical portraits $\mathcal{T}_1 = (Q_1, \mathcal{Y})$ and $\mathcal{T}_2 = (Q_2, \mathcal{Y})$ respectively such that Q_1 and Q_2 are strongly linked. Then $\sim_1 = \sim_2$.*

Observe that if Q_1 and Q_2 share a diagonal then we can consider the common diagonal as a (degenerate) critical quadrilateral that is strongly linked with itself so that the conclusions of the lemma hold in this case too.

PROOF. Suppose first that at least one diagonal of a quadrilateral Q_1 or Q_2 (say, a diagonal of Q_1) has non-periodic endpoints. Then by construction $\sim_1$ has only finite critical sets. By Lemma 3.56 then $\approx^{pS}_{\sim_1} = \sim_1$. On the other hand, by Theorem 3.63 the perfect-Siegel parts of $\mathcal{L}_{\sim_1}$ and $\mathcal{L}_{\sim_2}$ are equal. This implies that $\mathcal{L}_{\sim_2}$ has only finite critical sets, and $\sim_1 = \sim_2$.

We may now assume that both Q_1 and Q_2 have periodic edges. By Theorem 4.22, quadratically critical portraits $\mathcal{T}_1$ and $\mathcal{T}_2$ give rise to invariant geodesic pullback laminations $\mathcal{L}(\mathcal{T}_1)$ and $\mathcal{L}(\mathcal{T}_2)$. Let us show that they coincide and $\approx_{\mathcal{T}_1}=\approx_{\mathcal{T}_2}$. Indeed, by definition, they are quadratically critical and linked (because Q_1 and Q_2 are linked). Consider a pullback $\overline{y}$ of a leaf $\overline{y}_i$ with $2 \leqslant i \leqslant d-1$ that belongs to $\mathcal{L}(\mathcal{T}_1)$ and prove that it is not linked with any edge of Q_2. Indeed, suppose otherwise. Then $\sigma_d(\overline{y})$ and $\sigma_d(Q_2)$ are linked as well; observe that $\sigma_d(Q_2)$ is a periodic leaf of $\mathcal{L}(\mathcal{T}_2)$. Recall that spikes of sets from $\mathcal{T}_2$ are sets $\overline{y}_i$ with $2 \leqslant i \leqslant d-1$ and two diagonals of Q_2. By the assumptions, there are no chains of spikes such that one endpoint of a chain is periodic and the other one is not. Hence no two images $\sigma_d^q(\overline{y})$ and $\sigma_d^q(Q_2)$ can have endpoint that coincide with distinct endpoints of a chain of spikes. By Lemma 3.45 then $\sigma_d^q(\overline{y})$ and $\sigma_d^q(Q_2)$ are linked for every q, a contradiction because for some q we have $\sigma_d^q(\overline{y}) = \overline{y}_i$, and $\sigma_d^q(Q_2)$ cannot be linked with $\overline{y}_i$.

Thus, pullbacks of the leaves $\overline{y}_i$ with $2 \leqslant i \leqslant d-1$ that belong to $\mathcal{L}(\mathcal{T}_1)$ are not linked with an edge of Q_2. By definition this implies that they actually belong to $\mathcal{L}(\mathcal{T}_2)$. Similarly, the pullbacks of leaves $\overline{y}_i$ with $2 \leqslant i \leqslant d-1$ that belong to $\mathcal{L}(\mathcal{T}_2)$ also belong to $\mathcal{L}(\mathcal{T}_1)$. Therefore, by definition, it follows that $\mathcal{L}(\mathcal{T}_2) = \mathcal{L}(\mathcal{T}_1)$. This implies that there exists a periodic critical quadratic Fatou gap U of period, say, m, that is common for both laminations and contains both Q_1 and Q_2. This allows us to use the standard semiconjugacy ψ of $\sigma_d^m : U \to U$ and $\sigma_2 : \mathbb{S} \to \mathbb{S}$. Then $\psi(Q_1)$ is either a diameter in $\overline{\mathbb{D}}$, or a σ_2-critical quadrilateral with one edge being a major of a periodic critical Fatou gap of σ_2. A similar observation applies to $\psi(Q_2)$. The fact that Q_1 and Q_2 are strongly linked implies then that the corresponding two σ_2-invariant geodesic laminations coincide, and therefore $\sim_1=\sim_2$ as desired. $\square$

We are ready to prove Theorem 4.29.

Theorem 4.29 (Parameter laminational equivalence $\sim_{\mathcal{Y}}$). *Minor sets of invariant geodesic laminations from $\mathbb{L}(\mathcal{Y})$ are classes of equivalence of a non-invariant laminational equivalence relation $\sim_{\mathcal{Y}}$. The corresponding "pinched" disk model $\mathcal{M}(\mathcal{Y})$ contains infinitely many pairwise disjoint copies of the combinatorial quadratic Mandelbrot set $\overline{\mathbb{D}}/\mathrm{QML}$. Components of the interior of $\mathcal{M}(\mathcal{Y})$ are either hyperbolic domains inside copies of $\overline{\mathbb{D}}/\mathrm{QML}$ or parameter components of Siegel capture type.*

PROOF. Suppose that two geodesic laminations $\mathcal{L}_\sim, \mathcal{L}_\approx$ belong to $\mathbb{L}(\mathcal{Y})$ and have non-disjoint minor sets $m_\sim$ and $m_\approx$. Consider cases. Recall that by $A(\mathcal{Y}) = A$ we denote the unique component of $\overline{\mathbb{D}} \setminus \mathcal{Y}^+$ such that $\sigma_d|_{\mathrm{Bd}(A)}$ is two-to-one except for its critical boundary edges.

First, it may happen that there exists a common vertex x of $m_\sim$ and $m_\approx$. Then by Lemma 4.26 the entire set $\sigma_d^{-1}(x) \cap \overline{A}$ is contained in $C_\sim \cap C_\approx$. We can choose one critical chord $\overline{c} \subset \sigma_d^{-1}(x) \cap \overline{A}$ that intersects A and add $\overline{c}$ to $\mathcal{Y}$ to form a new augmented quadratically critical portrait $\mathcal{Y}'$. It follows that the invariant geodesic laminations $\mathcal{L}_\sim$ and $\mathcal{L}_\approx$ are essentially equal. By Lemma 4.28 this implies that $\sim=\approx$ (the remark made right after the statement of Lemma 4.28 and before its proof shows that Lemma 4.28 applies in the case when $\mathcal{L}_\sim$ and $\mathcal{L}_\approx$ are essentially equal).

Second, it may happen that $m_\sim$ and $m_\approx$ do not have a common vertex. Then there must exist an edge $\ell_\sim$ of $m_\sim$ and an edge $\ell_\approx$ of $m_\approx$ that cross. Moreover, those edges can be chosen from legal modifications of $\mathcal{L}_\sim$ and $\mathcal{L}_\approx$ as images of their critical quadrilaterals. In other words, either edge pulls back to the corresponding collapsing quadrilateral ($Q_\sim \subset C_\sim$ and, respectively, $Q_\approx \subset C_\approx$) so that $Q_\sim$ and $Q_\approx$ are strongly linked. Then, by

Lemma 4.28, we have $\sim=\approx$. We conclude that in any case minor sets of distinct invariant geodesic laminations from $\mathbb{L}(\mathcal{Y})$ are pairwise disjoint.

Let us show that the family of minor sets of invariant geodesic laminations from $\mathbb{L}(\mathcal{Y})$ is upper semicontinuous. Consider a sequence of minor sets $m_1, m_2, \dots$ of invariant geodesic laminations $\mathcal{L}_1, \mathcal{L}_2, \dots$ generated by invariant laminational equivalence relations $\sim_1, \sim_2, \dots$. We may assume that the minor sets m_i converge in the Hausdorff sense, all these equivalence relations and invariant geodesic laminations are distinct, and, by the above, all the minor sets $m_1, m_2, \dots$ are pairwise disjoint.

Then the limit of the sets m_i is either a point or a leaf X. We need to show that in fact X is a subset of the minor set of an invariant geodesic lamination from $\mathbb{L}(\mathcal{Y})$ generated by the appropriate laminational equivalence relation, say, $\simeq$. To this end, we refine (if necessary) the sequence of geodesic laminations $\mathcal{L}_1, \mathcal{L}_2, \dots$ so that (by [**BMOV13**]) invariant geodesic laminations $\mathcal{L}_i$ will converge to some invariant geodesic lamination $\mathcal{L}$ in the Hausdorff sense. We have to find the desired laminational equivalence relation $\simeq$ using the existence of $\mathcal{L}$ and its properties as a tool.

Indeed, pull back X to the component $A(\mathcal{Y})$ of $\overline{\mathbb{D}} \setminus \mathcal{Y}^+$ on which the map σ_d is two-to-one. This will yield a (generalized) critical quadrilateral, say, Q, such that the quadratically critical portrait $\mathcal{T} = (Q, \mathcal{Y})$ is geolaminational (because $\mathcal{T}$ can be viewed as a quadratically critical portrait of $\mathcal{L}$). Hence, by Theorem 4.22, there exists an invariant geodesic lamination $\mathcal{L}_\simeq$ from $\mathbb{L}(\mathcal{Y})$ such that Q is a subset of the corresponding critical set of a legal modification of $\mathcal{L}_\simeq$. By definition, this implies that X is contained in the corresponding minor set $m_\simeq$ as desired.

The remaining claims concerning copies of the quadratic Mandelbrot set are rather standard and easily follow from the existence of invariant geodesic laminations $\mathcal{L}_\sim$ in $\mathbb{L}(\mathcal{Y})$ such that $\mathcal{L}_\sim$ has a periodic critical Fatou gap of degree greater than one (the latter in turn follows from Theorem 4.22). Finally, the existence of component of $\mathcal{M}(\mathcal{Y})$ associated with invariant geodesic laminations of Siegel capture follows from the analysis given right before Lemma 4.28. $\square$

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