

Locally connected models for Julia sets

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Abstract

Let P be a polynomial with a connected Julia set J . We use continuum theory to show that it admits a *finest monotone map* φ onto a *locally connected continuum* $J_{\sim P}$, i.e. a monotone map $\varphi : J \rightarrow J_{\sim P}$ such that for any other monotone map $\psi : J \rightarrow J'$ there exists a monotone map h with $\psi = h \circ \varphi$. Then we extend φ onto the complex plane \mathbb{C} (keeping the same notation) and show that φ monotonically semiconjugates $P|_{\mathbb{C}}$ to a *topological polynomial* $g : \mathbb{C} \rightarrow \mathbb{C}$. If P does not have Siegel or Cremer periodic points this gives an alternative proof of Kiwi's fundamental results on locally connected models of dynamics on the Julia sets, but the results hold for all polynomials with connected Julia sets. We also give a criterion and a useful sufficient condition for the map φ not to collapse J into a point.

Date: September 22, 2008.

2000 Mathematics Subject Classification. Primary 37F10; Secondary 37B45, 37C25, 37F20, 37F50, 54F15

Key words: Complex dynamics, Julia set, core decomposition

1 Introduction

A major idea in the theory of dynamical systems is that of modeling an arbitrary system by one which can be better understood and treated with the

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¹ This author was partially supported by NSF grant DMS-0456748.

² This author was partially supported by NSF grant DMS-0353825.

³ This author was partially supported by NSF grant DMS-0405774.

help of existing tools and methods. To an extent, the entire field of symbolic dynamics is so important for the rest of dynamical systems because symbolic dynamical systems serve as an almost universal model. A different example, coming from one-dimensional dynamics, is due to Milnor and Thurston who showed in [17] that any piecewise-monotone interval map f of positive entropy can be modeled by a piecewise-monotone interval map of constant slope h (i.e., f is *monotonically semiconjugate* to h). For us however the most interesting case is that of modeling complex polynomial dynamical systems on their connected Julia sets by so-called *topological polynomials* on their (topological) locally connected Julia sets. Let us now describe more precisely what we mean.

Consider a polynomial map $P : \mathbb{C} \rightarrow \mathbb{C}$; denote by J_P the *Julia set* of P , by K_P its *filled-in Julia set*, and by $U_\infty(P) = \mathbb{C} \setminus K_P$ its *basin of attraction of infinity*. In this paper we *always assume that J_P is connected*. A very-well known fact from complex dynamics (see, e.g., Theorem 9.5 from [16]) shows that there exists a conformal isomorphism Ψ from the complement of the closure of the open unit disk \mathbb{D} onto $U_\infty(P)$ which conjugates $z^d|_{\mathbb{C} \setminus \overline{\mathbb{D}}}$ and $P|_{U_\infty(P)}$. The Ψ -image R_α of the radial line of angle α in $\mathbb{C} \setminus \overline{\mathbb{D}}$ is called an (*external*) *ray*. By [9] external rays with rational arguments *land* at repelling (parabolic) periodic points or their preimages (i.e., the rays compactify onto such points). If J_P is locally connected, Ψ extends to a continuous function $\overline{\Psi}$ which semiconjugates $z^d|_{\mathbb{C} \setminus \mathbb{D}}$ and $P|_{\overline{U_\infty(P)}}$.

Set $\psi = \overline{\Psi}|_{\mathbb{S}^1}$ and define an equivalence relation \sim_P on \mathbb{S}^1 by $x \sim_P y$ if and only if $\psi(x) = \psi(y)$. The equivalence \sim_P is called the (*d-invariant*) *lamination* (*generated by P*). The quotient space $\mathbb{S}^1 / \sim_P = J_{\sim_P}$ is homeomorphic to J_P and the map $f_{\sim_P} : J_{\sim_P} \rightarrow J_{\sim_P}$ induced by $z^d|_{\mathbb{S}^1} \equiv \sigma$ is topologically conjugate to $P|_{J_P}$. The set J_{\sim_P} is a topological (combinatorial) model of J_P and is often called the *topological Julia set*. On the other hand, the induced map $f_{\sim_P} : J_{\sim_P} \rightarrow J_{\sim_P}$ serves as a model for $P|_{J_P}$ and is often called a *topological polynomial*. Moreover, one can extend the conjugacy between $P|_{J_P}$ and $f_{\sim_P} : J_{\sim_P} \rightarrow J_{\sim_P}$ (as the identity outside J_P) to the conjugacy on the entire plane. In fact, equivalences \sim similar to \sim_P can be defined abstractly, in the absence of any polynomial. Then they are called (*d-invariant*) *laminations* and still give rise to similarly constructed *topological Julia sets* J_\sim and *topological polynomials* f_\sim .

In his fundamental paper [14] Kiwi extended this to polynomials P with no irrational neutral periodic points (called *CS-points*). In the case of a polynomial P with connected Julia set he constructed a *d-invariant lamination* \sim on \mathbb{S}^1 such that $P|_{J_P}$ is semiconjugate to the induced map $f_\sim : J_\sim \rightarrow J_\sim$ by a monotone map $m : J_P \rightarrow J_\sim$ (*monotone* means a map with connected point preimages). Kiwi also proved that for all periodic points $p \in J_P$ the set J_P is locally connected at p and $m^{-1} \circ m(p) = \{p\}$.

However the results of [14] do not apply if a polynomial admits a CS-point. As an example consider the following. A *Cremer fixed point* is a neutral non-linearizable fixed point $p \in J$. A polynomial P is said to be *basic uniCremer* if it has a Cremer fixed point and no repelling/parabolic periodic point of P is bi-accessible (a point is called *bi-accessible* if at least two rays land it). In this case the only monotone map of J_P onto a locally connected continuum is a collapse of J_P to a point [4–6], strongly contrasting with [14].

The aim of this paper is to suggest a different (compared to [14]) approach to the problem of locally connected dynamical models for connected polynomial Julia sets J_P . Our approach works for *any* polynomial P , regardless of whether P has CS-points or not, and is based upon continuum theory. Accordingly, Section 3 does not deal with dynamics at all. To state its main result we need the following definitions. Let A be a continuum. Then an onto map $\varphi : A \rightarrow Y_{\varphi,A}$ is said to be a *finest (monotone) map (onto a locally connected continuum)* if for any other monotone map $\psi : A \rightarrow L$ onto a locally connected continuum L there exists a monotone map $h : Y_{\varphi,A} \rightarrow L$ such that $\psi = h \circ \varphi$. Observe, that in this situation the map h is automatically monotone because for $x \in L$ we have $h^{-1}(x) = \varphi(\psi^{-1}(x))$.

In general, it is not clear if a finest map exists. Yet if it does, it gives a finest locally connected model of A up to a homeomorphism. Suppose that $\varphi : A \rightarrow B$, $\varphi' : A \rightarrow B'$ are two finest maps. Then it follows from the definition that a map associating points $\varphi(x) \in B$ and $\varphi'(x) \in B'$ with x running over the entire A is a homeomorphism between B and B' . Hence all sets $Y_{\varphi,A}$ are homeomorphic and all finest maps φ are the same up to a homeomorphism. Thus from now on we may talk of **the finest model** $Y_A = Y$ of A and **the finest map** $\varphi_A = \varphi$ of A onto Y . In what follows we *always* use the just introduced notation for the finest map and the finest model. Call a planar continuum $Q \subset \mathbb{C}$ *unshielded* if it coincides with the boundary of the component of $\mathbb{C} \setminus Q$ containing infinity. The following is the main result of Subsection 3.1 of Section 3.

Theorem 1. *Let Q be an unshielded continuum. Then there exist the finest map φ and the finest model Y of Q . Moreover, φ can be extended to a map $\widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ which maps ∞ to ∞ , in $\widehat{\mathbb{C}} \setminus Q$ collapses only those complementary domains to Q whose boundaries are collapsed by φ , and is a homeomorphism elsewhere in $\widehat{\mathbb{C}} \setminus Q$.*

It may happen that the finest model is a point (e.g., this is so if the continuum is *indecomposable*, i.e. cannot be represented as the union of two non-trivial subcontinua). In Subsection 3.2 of Section 3 we establish a useful sufficient condition for this not to be the case. In Section 4 we apply Theorem 1 to a polynomial P with connected Julia set and prove the following theorem.

Theorem 2. *Let P be a complex polynomial with connected Julia set J_P . Then the finest map $\varphi_{J_P} = \varphi$ can be extended to a monotone map $\hat{\varphi} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ so that $\hat{\varphi}|_{\hat{\mathbb{C}} \setminus J_P}$ is one-to-one in $U_\infty(P)$ and in all Fatou domains whose boundaries are not collapsed to points by φ and $\hat{\varphi}$ semiconjugates P and a topological polynomial $g : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. There is a finest lamination \sim_P such that $g|_{\varphi(J_P)}$ is conjugate to $f_{\sim_P}|_{J_{\sim_P}}$.*

In particular, φ_{J_P} semiconjugates the dynamics on J_P , so we have the following diagram which commutes. (Here Φ is the quotient map corresponding to the lamination \sim_P .)

$$\begin{array}{ccccc}
 J_P & \xrightarrow{P|_{J_P}} & J_P & & \mathbb{S}^1 & \xrightarrow{\sigma_d} & \mathbb{S}^1 \\
 & \searrow \varphi & & \swarrow \varphi & & \swarrow \Phi & \\
 & & J_{\sim} & \xrightarrow{g|_{J_{\sim}}} & J_{\sim} & & \\
 & & & & & \swarrow \Phi & \\
 & & & & & & \mathbb{S}^1
 \end{array}$$

Finally, in Section 5 we suggest a criterion for the fact that the finest model is non-degenerate. Given a set of angles $A \subset \mathbb{S}^1$ denote by $\text{Imp}(A)$ the union of impressions of angles in A . Also, call a set *wandering* if all its images under a specified map are pairwise disjoint. Finally, call an attracting or parabolic Fatou domain of a polynomial *paratracting*. Essentially, the criterion is that the finest model is non-degenerate if and only if one of the following properties holds:

- (1) there are infinitely many bi-accessible P -periodic points;
- (2) J_P has a paratracting Fatou domain;
- (3) P admits a *Siegel configuration* defined later in Definition 41 — basically, it means that there are several collections of angles A_1, \dots, A_m such that for all i the eventual σ_d -image of A_i is a point and the sets $\text{Imp}(A_i)$ are wandering continua such that on the closures of their orbits the map is monotonically semiconjugate to an irrational rotation of the circle.

If P does not have Siegel or Cremer periodic points we deduce from our results an independent alternative proof of Kiwi's results [14]. We also obtain a few corollaries; to state them we need the following terminology. By a *(pre)periodic point* we mean a point with finite orbit and by a *preperiodic point* we mean a non-periodic point with finite orbit (similarly we define preperiodic and (pre)periodic sets as well as *(pre)critical* and *precritical* points). A set A is *(pre)critical* if there exists n such that $P^n|_A$ is not one-to-one and *non-(pre)critical* otherwise. Call K a *ray-continuum* if for some collection of angles, K is contained in the union of impressions of their external rays and contains the union of principal sets of their external rays; the cardinality of the set of rays whose principal sets are contained in K is said to be the *valence*

of K .

We show that if there is a wandering non-(pre)critical ray-continuum $K \subset J_P$ of valence greater than 1 then there are infinitely many repelling bi-accessible periodic points and the finest model is non-degenerate. In particular, these conclusions hold if there exists a non-(pre)periodic non-(pre)critical bi-accessible point of J_P . We also rely upon the finest model to study for what (pre)periodic points x we can guarantee that the Julia set J_P is locally connected at x ; to this end we apply a recent result [7] about the degeneracy of certain impressions.

2 Circle laminations

Consider an equivalence relation \sim on the unit circle \mathbb{S}^1 . Classes of equivalence of \sim will be called (\sim) -classes and will be denoted by boldface letters. A \sim -class consisting of two points is called a *leaf*; a class consisting of at least three points is called a *gap* (this is more restrictive than Thurston's definition in [26]; for the moment we follow [3] in our presentation). Fix an integer $d > 1$. Then \sim is said to be a (d) -invariant lamination if:

(E1) \sim is *closed*: the graph of \sim is a closed set in $\mathbb{S}^1 \times \mathbb{S}^1$;

(E2) \sim defines a *lamination*, i.e., it is *unlinked*: if \mathbf{g}_1 and \mathbf{g}_2 are distinct \sim -classes, then their convex hulls $\text{Ch}(\mathbf{g}_1), \text{Ch}(\mathbf{g}_2)$ in the unit disk \mathbb{D} are disjoint,

(D1) \sim is *forward invariant*: for a class \mathbf{g} , the set $\sigma_d(\mathbf{g})$ is a class too

which implies that

(D2) \sim is *backward invariant*: for a class \mathbf{g} , its preimage $\sigma_d^{-1}(\mathbf{g}) = \{x \in \mathbb{S}^1 : \sigma_d(x) \in \mathbf{g}\}$ is a union of classes;

(D3) for any gap \mathbf{g} , the map $\sigma_d|_{\mathbf{g}} : \mathbf{g} \rightarrow \sigma_d(\mathbf{g})$ is a *covering map with positive orientation*, i.e., for every connected component (s, t) of $\mathbb{S}^1 \setminus \mathbf{g}$ the arc $(\sigma_d(s), \sigma_d(t))$ is a connected component of $\mathbb{S}^1 \setminus \sigma_d(\mathbf{g})$.

The lamination in which all points of \mathbb{S}^1 are equivalent is said to *degenerate*. It is easy to see that if a forward invariant lamination \sim has a class with non-empty interior then \sim is degenerate. Hence equivalence classes of any non-degenerate forward invariant lamination are totally disconnected.

Call a class \mathbf{g} *critical* if $\sigma_d|_{\mathbf{g}} : \mathbf{g} \rightarrow \sigma_d(\mathbf{g})$ is not one-to-one, *(pre)critical* if $\sigma_d^j(\mathbf{g})$ is critical for some $j \geq 0$, and *(pre)periodic* if $\sigma_d^i(\mathbf{g}) = \sigma_d^j(\mathbf{g})$ for some $0 \leq i < j$. A gap \mathbf{g} is *wandering* if \mathbf{g} is neither (pre)periodic nor (pre)critical. Let $p : \mathbb{S}^1 \rightarrow J_\sim = \mathbb{S}^1 / \sim$ be the quotient map of \mathbb{S}^1 onto its quotient space

J_\sim , let $f_\sim : J_\sim \rightarrow J_\sim$ be the map induced by σ_d . We call J_\sim a *topological Julia set* and the induced map f_\sim a *topological polynomial*. The set J_\sim can be canonically embedded in \mathbb{C} and then the map p can be extended to the map $\hat{p} : \mathbb{C} \rightarrow \mathbb{C}$ [8]. Radial lines from \mathbb{S}^1 are then mapped by \hat{p} onto *topological external rays* of the topological Julia set J_\sim on which the map $z \mapsto z^d$ induces a well-defined extension of f_\sim onto the union of J_\sim and the component of $\mathbb{C} \setminus J_\sim$ containing infinity.

We need the following theorem [13]. Given a closed set $G' \subset \mathbb{S}^1$ let the ‘‘polygon’’ $G = \text{Ch}(G') \subset \overline{\mathbb{D}}$ be its convex hull. In this case we say that G' is the *basis* of G . In this situation let us call G (and G') *wandering* if the sets $G = \text{Ch}(G'), \text{Ch}(\sigma(G')), \text{Ch}(\sigma^2(G')), \dots$ are all pairwise disjoint (and so the sets $G', \sigma(G'), \dots$ are pairwise unlinked, see (E2) above). Also, call G (and G') *non-(pre)critical* if the cardinality $|\sigma^n(G')|$ of $\sigma^n(G')$ equals the cardinality $|G'|$ of G' for all n , and *(pre)critical* otherwise.

Theorem 3. *If G is a wandering polygon then $|G'| \leq 2^d$, and if G is not (pre)critical then $|G'| \leq d$.*

Consider a simple closed curve $S \subset J_\sim$. Call the bounded component $U(S) = U$ of $\mathbb{C} \setminus J_\sim$ enclosed by S a *Fatou domain*. By Theorem 3 S is (pre)periodic and for some minimal k the set $f_\sim^k(S) = Q$ is periodic of some minimal period m in the sense that pairwise intersections among sets $Q, \dots, f_\sim^{m-1}(Q)$ are at most finite. We cannot completely exclude such intersections; e.g., in the case of a parabolic fixed point a in a polynomial locally connected Julia set, there will be several Fatou domains ‘‘revolving’’ around a and containing a in their boundaries. However, it is easy to see that $U(Q), \dots, U(f_\sim^{m-1}(Q))$ are pairwise disjoint.

Lemma 4 ([3], Lemma 2.4). *There are only two types of dynamics of $f_\sim^m|_S$.*

- (1) *The map $f_\sim^m|_S$ can be conjugate to an appropriate irrational rotation.*
- (2) *The map $f_\sim^m|_S$ can be conjugate to $z^k|_{\mathbb{S}^1}$ with the appropriate $k > 1$.*

In the case (1) we call U a *(periodic) Siegel domain* and in the case (2) we call U a *(periodic) paratracting domain*.

The map f_\sim , which above was extended onto the unbounded component of $\mathbb{C} \setminus J_\sim$, can actually be extended onto the entire J_\sim -plane as a branched covering map. Indeed, it is enough to extend f_\sim appropriately onto any bounded component V of $\mathbb{C} \setminus J_\sim$. This can be done by noticing the degree k of $f_\sim|_{\text{Bd}(V)}$ and extending f_\sim onto V as a branched covering map of degree k so that the extension of f_\sim remains a branched covering map of degree d and behaves, from the standpoint of topological dynamics, just like a complex polynomial. In particular, if S is a Siegel domain of period m , we may assume that $U(S)$ is foliated by Jordan curves on which $f_\sim^m|_U$ acts as the rotation by the same

rotation number as that of f_{\sim}^m . On the other hand, if $k > 1$ then $f_{\sim}^m|_{U(S)}$ should have one attracting (in the topological sense) fixed point to which all points inside $U(S)$ are attracted under f_{\sim}^m . Any such extension of f_{\sim} onto \mathbb{C} will still be called a *topological polynomial* and denoted f_{\sim} . In Section 4 we relate P and the appropriate extension of f_{\sim} much more precisely, however here it suffices to guarantee the listed properties. Call a continuum $X \subset J_{\sim}$ *wandering* if all its iterates are disjoint.

Theorem 5. [3] *The map $f_{\sim}|_{J_{\sim}}$ has no wandering continua.*

The collection of chords in the boundaries of the convex hulls of all equivalence classes of \sim in \mathbb{D} is called a (*d-invariant*) *geometric lamination (of the unit disk)*. Denote the geometric lamination obtained from the lamination \sim by \mathcal{L}_{\sim} . In fact, geometric laminations - in what follows *geo-laminations* - can also be defined abstractly (as was originally done by Thurston [26]). A *geometric prelamination* \mathcal{L} is a collection of chords in the unit disk called (*geometric*) *leaves* and such that any two leaves meet in at most a common endpoint. If in addition the *union* $|\mathcal{L}|$ of all the elements of \mathcal{L} is closed, \mathcal{L} is said to be a *geometric lamination*. The closure of a component of $\mathbb{D} \setminus |\mathcal{L}|$ is called a (*geometric*) *gap*. If it is clear that we talk about a geo-lamination we will use *leaves* and *gaps*.

Denote a leaf $\ell = ab \in \mathcal{L}$ by its two endpoints. Given a geometric gap (leaf) G , set $G' = G \cap \mathbb{S}^1$ and call G' the *basis* of G . Clearly the boundary of each geometric gap is a simple closed curve S consisting of leaves of \mathcal{L} and points of \mathbb{S}^1 . As in [26] one can define the linear extension σ^* of σ over the leaves of \mathcal{L} which can then be extended over the entire unit disk (using, e.g., the baricenters) so that not only is $\sigma^*(ab) = \sigma(ab)$ the chord (or point) in \mathbb{D} with endpoints $\sigma(a)$ and $\sigma(b)$ but also for any geometric gap G we have that $\sigma^*(G)$ is the convex hull of the set $\sigma(G')$. Even though we denote this extension of σ by σ^* , sometimes (if it does not cause ambiguity) we use the notation σ for σ^* (e.g., when we apply σ^* to leaves).

A geometric prelamination \mathcal{L} is *d-invariant* if

- (1) (forward leaf invariance) for each $\ell = ab \in \mathcal{L}$, either $\sigma(\ell) \in \mathcal{L}$ or $\sigma(a) = \sigma(b)$,
- (2) (backward leaf invariance) for each leaf $\ell \in \mathcal{L}$ there exist d **disjoint** leaves $\ell_1, \dots, \ell_d \in \mathcal{L}$ such that for each i , $\sigma(\ell_i) = \ell$,
- (3) (gap invariance) for each gap G of \mathcal{L} , if $G' = G \cap \mathbb{S}^1$ is the basis of G and H is the convex hull of $\sigma(G')$ then either $H \in \mathbb{S}^1$ is a point, or $H \in \mathcal{L}$ is a leaf, or H is also a gap of \mathcal{L} . Moreover, in the last case $\sigma^*|_{\text{Bd}(G)} : \text{Bd}(G) \rightarrow \text{Bd}(H)$ is a positively oriented composition of a monotone map $m : \text{Bd}(G) \rightarrow S$, where S is a simple closed curve, and a covering map $g : S \rightarrow \text{Bd}(H)$.

Clearly, \mathcal{L}_\sim is a geometric lamination and \sim -gaps are bases of geometric gaps of \mathcal{L}_\sim . In general, the situation with leaves and geometric leaves is more complicated (e.g., the basis of a geometric leaf on the boundary of a finite gap of \mathcal{L}_\sim is not a \sim -leaf). Thus in what follows speaking of leaves we will make careful distinction between the two cases (that of a geometric leaf and that of a leaf as a class of a lamination). Call a gap G of a geometric prelamination *wandering* if all its iterates under σ^* are pairwise disjoint; then Theorem 3 applies to wandering (geometric) gaps of (geometric) laminations.

Slightly abusing the language, we sometimes use for gaps terminology applicable to their bases. Thus, speaking of a *finite/infinite* gap G we actually mean that G' is finite/infinite. Now we study infinite gaps (of geometric laminations) and establish some of their properties. We begin with a series of useful general lemmas in which we establish some properties of geometric laminations. Given two points $x, y \in \mathbb{S}^1$, set $\rho(x, y)$ to be the length of the smallest arc in \mathbb{S}^1 , containing x and y . There exists $\varepsilon_d > 0$ such that $\rho(\sigma_d(x), \sigma_d(y)) > \rho(x, y)$ whenever $0 < \rho(x, y) < \varepsilon_d$.

Lemma 6. *If $K \subset \mathbb{S}^1$ and $k > 0$ are such that $\lim_{i \rightarrow \infty} \text{diam}(\sigma_d^{ik}(K)) = 0$, then there exists i_0 such that $\text{diam}(\sigma_d^{i_0k}(K)) = 0$.*

Proof. If $\lim_{i \rightarrow \infty} \text{diam}(\sigma_d^{ik}(K)) = 0$, there exists i_0 such that $\text{diam}(\sigma_d^{ik}(K)) < \varepsilon_{kd}$ for all $i \geq i_0$. If $\text{diam}(\sigma_d^{i_0k}(K)) = 0$ then $(\text{diam}(\sigma_d^{ik}(K)))_{i=i_0}^\infty$ is an increasing sequence of positive numbers converging to 0, a contradiction. So $\text{diam}(\sigma_d^{i_0k}(K)) = 0$. \square

Let us study geometric leaves on the boundary of a periodic gap.

Lemma 7. *Suppose that G is a periodic gap of a geometric lamination. Then every leaf in $\text{Bd}(G)$ is either (pre)periodic from a finite collection of grand orbits of periodic leaves, or (pre)critical from a finite collection of grand orbits of critical leaves.*

Proof. We may assume that the gap G is fixed. Let ℓ be a leaf which is not (pre)periodic. Since $\text{Bd}(G)$ is a simple closed curve and $\sigma^i(\ell) \cap \sigma^j(\ell)$ may consist of at most a point, $\lim_{i \rightarrow \infty} \text{diam}(\sigma^i(\ell)) = 0$. Therefore, by Lemma 6, there exists i_0 such that $\text{diam}(\sigma^{i_0}(\ell)) = 0$, meaning that ℓ is (pre)critical. Now, there are only finitely many leaves $\alpha\beta$ in $\text{Bd}(G)$ such that $\rho(\alpha, \beta) \geq \varepsilon_d$, and there are only finitely many critical leaves in any geometric lamination. Since by the properties of ε_d any non-degenerate leaf in $\text{Bd}(G)$ maps to one of them, the proof of the lemma is complete. \square

In what follows for geometric leaves of a geometric lamination we will use self-explanatory terms *isolated (from one side)* and *limit (from one side)*. Let

us study critical leaves of geometric laminations. The following terminology is quite useful: a leaf is said to be *separate* if it is disjoint from all other leaves and gaps. Observe that if ℓ is a separate leaf then ℓ is a limit leaf from either side. Also, if a gap or a separate leaf is such that its image is a point we call it *all-critical*. Clearly, a gap is all-critical if and only if all its boundary leaves are critical. It may happen that two all-critical gaps are adjacent (have a common leaf). Moreover, there may exist several all-critical gaps whose union coincides with their convex hull. In other words, their union looks like a “big” all-critical gap inside which some leaves are added. Then we call this union an *all-critical union of gaps*. Clearly we can talk about boundary leaves of all-critical unions of gaps. Moreover, each all-critical gap is a part of an all-critical union of gaps, and there are only finitely many all-critical gaps.

Lemma 8. *Suppose that \mathcal{L} is a d -invariant geo-lamination and ℓ is its critical leaf. Then one of the following holds:*

- (1) ℓ is isolated in \mathcal{L} ;
- (2) ℓ is a separate leaf;
- (3) ℓ is a limit boundary leaf of an all-critical union of gaps G whose boundary consists of limit leaves.

In particular, if \mathcal{L} is the closure of a d -invariant prelamination \mathcal{L}' and ℓ lies on the boundary of a geometric gap G of \mathcal{L} then either $\ell \in \mathcal{L}'$, or $\sigma(G)$ is a point.

Proof. Suppose that neither (1) nor (2) holds. Then $\ell \in \mathcal{L}$ is a critical leaf lying on the boundary of a gap G which is the limit of a sequence of leaves ℓ_i approaching ℓ from outside of G . If $\sigma(G)$ is not a point, then $\sigma(\ell_i)$ must cross $\sigma(G)$, a contradiction. Hence $\sigma(G)$ is point and all leaves in the boundary of G are critical. Take the all-critical union of gaps H containing G . If all other boundary leaves of H are limit leaves we are done. Otherwise there must exist a boundary leaf ℓ of H and a gap T to whose boundary ℓ belongs. Then the leaves $\sigma(\ell_i)$ will cross the image $\sigma(H)$, a contradiction. This completes the proof. □

The next lemma gives a useful condition for an infinite gap to have nice properties. By two *concatenated* leaves we mean two leaves with a common endpoint, and by a *chain of concatenated leaves* we mean a (two-sided) sequence of leaves such that any consecutive leaves in the chain are concatenated (such chains might be both finite and infinite). For brevity we often speak of just *chains* instead of “chains of concatenated leaves”.

Lemma 9. *Let G be an infinite gap and on its boundary there are no leaves*

ℓ such that for some n, m we have that $\sigma^m(\ell)$ is a leaf while $\sigma^{m+n}(\ell)$ is an endpoint of $\sigma^m(\ell)$. Then the following claims hold.

- (1) There exists a number N such that any chain of concatenated leaves in $\text{Bd}(G)$ consists of no more than N leaves.
- (2) All non-isolated points of G' form a Cantor set G'_c , and so for any arc $[a, b] \subset \mathbb{S}^1$ such that $[a, b] \cap G'$ is not contained in one chain, the set $G' \cap [a, b]$ is uncountable (in particular, the basis G' of G is uncountable).
- (3) If G is σ^n -periodic then $\sigma^n|_{\text{Bd}(G)}$ is semiconjugate to $\sigma_k : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ with the appropriate $k > 0$ by the conjugacy which collapses to points all arcs in $\text{Bd}(G)$ complementary to G'_c . If $k = 1$ the map to which $\sigma^n|_{\text{Bd}(G)}$ is semiconjugate is an irrational rotation of the circle.

Proof. By Theorem 3, G is (pre)periodic. Since there are only finitely many gaps in the grand orbit of G on which the map σ is not one-to-one, we see that it is enough to prove the lemma with the assumption that G is fixed. Moreover, by Lemma 7 we may assume that all periodic leaves in $\text{Bd}(G)$ are fixed with fixed endpoints. Consider a chain of concatenated leaves from $\text{Bd}(G)$. By Lemma 7 under some power of σ this chain maps onto one of finitely many chains containing a critical or a fixed leaf. Thus, it remains to prove the lemma for chains containing a critical and/or a fixed leaf. By way of contradiction we may assume that L is a maximal infinite chain of concatenated leaves (it may be one-sided or two-sided).

First let $\ell \in L$ be a fixed leaf with fixed endpoints. By the assumptions of the lemma and by the properties of laminations each leaf concatenated to ℓ also has fixed endpoints. Repeating this argument we see that the chain consists of fixed leaves with fixed endpoints, hence L is a finite chain of fixed leaves with fixed endpoints. Second, consider the case when $\ell \in L$ is a critical leaf. Consider the points $a, b \in \mathbb{S}^1$ with $[a, b] \subset \mathbb{S}^1$ the smallest arc whose convex hull contains L . Then by Theorem 3 the convex hull $\text{Ch}(L)$ of L cannot be a wandering polygon. It follows that for some m we have that $\sigma^m(L) \subset \sigma^{m+n}(L)$. Since by the above there are no leaves with periodic endpoints in L and by the assumptions of the lemma no leaf of L can map into its endpoint, we see that all leaves of $\sigma^m(L)$ map under σ^n in the same direction, say, towards the point a so that every leaf has an infinite orbit converging to a . However then a is σ^n -fixed and must repel close points under σ^n , a contradiction. Since there exist only finitely many distinct chains containing a critical or periodic leaf, there exists a number N such that any chain of concatenated leaves in $\text{Bd}(G)$ consists of no more than N leaves. This immediately implies that any non-isolated point of G' is a limit point of other non-isolated points. Hence the set G'_c of all non-isolated points of G' is a Cantor set, and the claims (1) and (2) of the lemma are proven.

To prove (3) define $m : \text{Bd}(G) \rightarrow \mathbb{S}^1$ by collapsing to points all complementary

arcs to G'_c in $\text{Bd}(G)$. It follows that $(\sigma^*)^n|_{\text{Bd}(G)}$ is monotonically semiconjugate by the map m to a covering map f of the circle of a positive degree. It follows that for any non-degenerate arc $I \subset \mathbb{S}^1$ the set $m^{-1}(I) \cap \mathbb{S}^1$ is uncountable. Let us show that I is not wandering, i.e. the intervals $\{f^k(I) \mid k > 0\}$ are not pairwise disjoint. Indeed, if I wanders under f then so does $m^{-1}(I)$ under σ_d^* . Since $\text{Bd}(G)$ is homeomorphic to \mathbb{S}^1 , then $\lim_{k \rightarrow \infty} \text{diam}((\sigma_d^*)^k(m^{-1}(I))) = 0$, contradicting Lemma 6.

Also, let us show that I is not periodic. Suppose that $f^q(I) \subset I$. Then $f^m|_I$ is monotone preserving orientation and all points of I converge to an f^q -fixed point under $(\sigma^*)^q$. On the other hand, only countably many points of an uncountable set $m^{-1}(I) \cap \mathbb{S}^1$ map into a σ -periodic point. Thus, there exists a *non-(pre)periodic* point $y \in m^{-1}(I) \cap \mathbb{S}^1$ such that $m(y)$ converges under $(\sigma^*)^q$ to an f^q -fixed point z . Since m is monotone this implies that the orbit of y approaches the interval $m^{-1}(z)$ but does not map into it (because y is non-(pre)periodic). Thus, y must converge to an endpoint of $m^{-1}(z)$, which is impossible (e.g., it contradicts Lemma 6). A standard argument now implies that f is an irrational rotation or a map σ_k with appropriately chosen k , still we sketch it for the sake of completeness. Consider two cases.

Case 1: $\sigma^*|_{\text{Bd}(G)}$ is monotone.

Let us show that f has no periodic points. By way of contradiction, suppose $f^q(x) = x$, choose a point $y \neq x$ with $f^q(y) \neq y$, and let I be the component of $\mathbb{S}^1 \setminus \{x, y\}$ containing $f^q(y)$. Since $\sigma^*|_{\text{Bd}(G)}$ is monotone, it follows that I is a periodic interval a contradiction. Therefore, $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a positively oriented map with no periodic points and no wandering intervals, and is therefore conjugate to an irrational rotation by [15, Theorem 1.1]. By Lemma 7 all leaves in $\text{Bd}(G)$ are (pre)critical.

Case 2: $\sigma^*|_{\text{Bd}(G)}$ is not monotone.

Since f is a covering map of degree $k > 1$ without periodic and wandering intervals, f is conjugate to $z \mapsto z^k$ for some k . Indeed, that there is a monotone semiconjugacy between f and σ_k is well-known (see, e.g., [18] for the case $k = 2$). However if there are no wandering intervals and periodic intervals, then the semiconjugacy cannot collapse any intervals and is therefore a conjugacy. In what follows the semiconjugacy which we have just defined in both cases will be denoted ψ .

□

Given a geo-lamination \mathcal{L} , a periodic geometric gap G satisfying conditions of Lemma 9 is called a *Fatou gap (domain) of \mathcal{L}* . If G is a Fatou domain,

then by Theorem 3 G' is (pre)periodic. A Fatou domain G is called *periodic* (*preperiodic*, (*pre*)*periodic*) if so is G' . A periodic Fatou domain G of period m is called *parattracting* if $(\sigma^*)^m|_{\text{Bd}(G)}$ is not monotone (in the topological sense introduced earlier in the paper) and *Siegel* otherwise. Equivalently, G is parattracting (resp. Siegel) if $(\sigma^*)^m|_{\text{Bd}(G)}$ can be represented as the composition of a covering map of degree greater than 1 (resp. equal to 1) and a monotone map. The *degree of $\sigma^m|_G$* is then defined as the degree of the model map f defined in Lemma 9. Thus, the terms “parattracting Fatou domain” and “Siegel domain” are used both for the geometric laminations and for the topological polynomials. Since it will always be clear from the context which situation is considered, this will not cause any ambiguity in what follows.

There are several cases in which Lemma 9 applies. The first one is considered in Lemma 10. Recall, that given a lamination \sim we denote by p the corresponding quotient map $p : \mathbb{S}^1 \rightarrow J_\sim$.

Lemma 10. *Suppose that \mathbf{g} is an infinite gap of a non-degenerate lamination \sim . Then $B = \text{Bd}(\text{Ch}(\mathbf{g}))$ contains no geometric (pre)critical leaves and therefore is a Fatou gap. In addition to that, any chain of concatenated geometric leaves in B eventually homeomorphically maps to a periodic chain, and if \mathbf{g} is periodic of period n then the degree of $(\sigma^*)^n|_B$ is greater than 1.*

Proof. By Theorem 3 \mathbf{g} is (pre)periodic. Suppose that $\ell = \alpha\beta \subset B$ is a critical geometric leaf and that $\mathbf{g} \subset [\alpha, \beta]$. By Lemma 8 ℓ cannot be a limit leaf of \mathcal{L}_\sim . Hence there is a geometric gap H of \mathcal{L}_\sim on the side of ℓ opposite to B (so that $H' \subset [\beta, \alpha]$). The points α, β are limit points of H' for otherwise there must exist a geometric leaf $\beta\gamma$ or $\theta\alpha$ and hence γ must be added to \mathbf{g} , a contradiction. By the gap invariance then $\sigma(H) = \sigma(\text{Ch}(\mathbf{g}))$. Now, since H is a gap of \mathcal{L}_\sim , either H' is a class itself, or there are uncountably many distinct \sim -classes among points of H' . However the latter is impossible because all these classes map into one \sim -class \mathbf{g} . Thus, H' is one \sim -class which implies that it had to be united with \mathbf{g} in the first place, a contradiction. Hence Lemma 9 applies to \mathbf{g} . Clearly, it follows also that any chain of concatenated geometric leaves in B eventually homeomorphically maps to a periodic chain.

Let us now prove the last claim of the lemma. Since there are no critical leaves in B , $(\sigma^*)^n|_B$ is a covering map. If the degree of $(\sigma^*)^n|_B$ is 1, then $\sigma^n|_{\mathbf{g}}$ is one-to-one. By a well-known result from the topological dynamics (see, e.g., Lemma 18.8 from [16]) \mathbf{g} must be finite, a contradiction. \square

Lemma 10 shows that if \sim is a lamination, then there are two types of Fatou domains of \mathcal{L}_\sim : 1) Fatou domains whose basis (the intersection of the boundary with \mathbb{S}^1) is one \sim_E -class (one \sim_E -gap), 2) Fatou domains for which this is not true (and which correspond to a Fatou domain in the J_\sim -plane). However this distinction cannot always be made if we just look at the geometric lamination.

For a lamination \sim the induced geo-lamination \mathcal{L}_\sim has the property that every geometric leaf is either disjoint from all other geometric leaves and gaps, or contained in the boundary of a unique geometric gap G . For an arbitrary geometric lamination, this is no longer the case. Hence, in general distinct geometric gaps may intersect. If, given a geo-lamination \mathcal{L} , \sim is a lamination such that $a \sim b$ whenever $ab = \ell \in \mathcal{L}$, we say that the lamination \sim *respects the geo-lamination* \mathcal{L} . Given a d -invariant geo-lamination \mathcal{L} , let $\approx = \approx_{\mathcal{L}}$ be the *finest lamination which respects* \mathcal{L} . It is not difficult to see that \approx is unique and d -invariant. Let $\pi : \mathbb{S}^1 \rightarrow J_\approx$ be the corresponding quotient map. It may well be the case that \mathbb{S}^1 / \approx is a single point (see [2] for a characterization of quadratic geometric laminations \mathcal{L} with non-degenerate $J_{\approx_{\mathcal{L}}}$).

Let us discuss the properties of \approx . It is shown in [2] that \approx can be defined as follows: $a \approx b$ if and only if there exists a continuum $K \subset \mathbb{S}^1 \cup |\mathcal{L}|$ containing a and b such that $K \cap \mathbb{S}^1$ is countable. By Lemma 9 if G is a Fatou domain of \mathcal{L} , then G / \approx is a simple closed curve. In particular, whenever a d -invariant geo-lamination \mathcal{L} contains a Fatou gap, then $J / \approx_{\mathcal{L}}$ is non-degenerate. Moreover, if F is an invariant Fatou domain, then the restricted map $f_\approx : \pi(\text{Bd}(F)) \rightarrow \pi(\text{Bd}(F))$ coincides with the map f from Lemma 9 and is conjugate to a either an irrational rotation of a circle (if F is Siegel) or to the map σ_m for m equal to the degree of $\sigma|_F$ (in the paratracting case). The case of a periodic Fatou domain is similar.

Suppose that \mathcal{A} is a forward invariant family of pairwise disjoint periodic or wandering gaps/leaves with a given family of their preimages so that together they form a d -invariant geometric prelamination $\mathcal{L}_{\mathcal{A}}$. This prelamination and its closure $\overline{\mathcal{L}_{\mathcal{A}}}$ (which is a geo-lamination [26]) are said to be *generated* by \mathcal{A} (then \mathcal{A} is called a *generating family*). The following important natural case of this situation was studied by Kiwi in [14].

Given a point $y \in J_P$, denote by $A(y)$ the set of all angles whose rays land at y . If J_P is locally connected then $A(y) \neq \emptyset$ for any $y \in J_P$, however otherwise this is not necessarily so. A point $y \in J_P$ is called *bi-accessible* if $|A(y)| > 1$ (i.e., there are at least two rays landing at y). By Douady and Hubbard [9] if x is a repelling or parabolic periodic point (or a preimage of such point) then $A(x)$ is always non-empty, finite, and consists of rational angles. Denote by R the set of all its periodic repelling (parabolic) *bi-accessible* points and their preimages. Let $x \in R$; also, given a set A denote by $\text{Ch}(A)$ its convex hull. Then let $G_x = \text{Ch}(A(x))$ and let $|\mathcal{L}_{rat}|$ be the union of all the sets $G_x, x \in R$. Let \mathcal{L}_{rat} be the collection of all chords contained in the boundaries of all the sets G_x . Then \mathcal{L}_{rat} , called the *rational geometric prelamination*, is a d -invariant geometric prelamination. By [26] the closure $\overline{\mathcal{L}_{rat}}$ of \mathcal{L}_{rat} in the unit disk is a closed d -invariant geo-lamination called *rational geometric lamination*.

The situation described above may be considered in a more general way. Sup-

pose that we are given a geometric prelamination generated by (pre)periodic or wandering non-(pre)critical pairwise disjoint gaps and leaves. Then any result concerning its closure will serve as a tool for studying $\overline{\mathcal{L}_{rat}}$. The following theorem can be such a tool. If we require that all gaps or leaves in such prelamination map onto their images in a covering fashion, we can conclude that there are no critical leaves in the prelamination. Indeed, such leaves can only belong to gaps/leaves disjoint from other leaves and collapsing to a point (*all-critical*). However we assume that the generating family consists of leaves and gaps which are non-(pre)critical. Hence an all-critical element of the prelamination cannot come from the forward orbits of the elements of the generating family. On the other hand, it cannot come from their backward orbits since the generating family consists of gaps and leaves (and the image of an all-critical gap/leaf is a point).

In Lemma 11 we deal with geometric laminations. For simplicity, in its proof speaking of leaves and gaps we actually mean geometric leaves and gaps. By a *separate leaf* we mean a leaf disjoint from all other leaves or gaps.

Lemma 11. *Suppose that \mathcal{L}^- is a non-empty geometric d -invariant prelamination generated by a generating family \mathcal{A} such that no element of \mathcal{L}^- contains a critical leaf on its boundary. Let \mathcal{L} be the closure of the prelamination \mathcal{L}^- . Then the following holds.*

- (1) *If three leaves of \mathcal{L} meet at a common endpoint, then the leaf in the middle is either a leaf from \mathcal{L}^- or a boundary leaf of a gap from \mathcal{L}^- .*
- (2) *At most four leaves of \mathcal{L} meet at a common end point, and if they do then the two in the middle are on the boundary of a gap of \mathcal{L}^- .*
- (3) *Suppose G is a gap of \mathcal{L} and xa is a leaf of \mathcal{L} such that $xa \cap G = \{x\}$. Let $xb \subset G$ be the leaf such that xb separates $G \setminus xb$ from $xa \setminus \{x\}$. Then either xb is a leaf from \mathcal{L}^- , or $G \in \mathcal{L}^-$ or there exists a gap H of \mathcal{L}^- such that $xa \cup xb \subset H$. In particular, if two gaps G, H of \mathcal{L} meet only in a point, then there exists a gap $K \in \mathcal{L}^-$ such that both $G \cap K$ and $H \cap K$ are leaves from \mathcal{L} .*
- (4) *Suppose that ℓ is a critical leaf of \mathcal{L} . Then either $\ell = H$ is a separate leaf or ℓ is a boundary leaf of an all-critical gap H of \mathcal{L} , and all boundary leaves of H are limit leaves (in particular, ℓ is a limit leaf from at least one side). Moreover, $\sigma^n(H)$ is disjoint from H for all $n > 0$.*
- (5) *If G is a gap or leaf of \mathcal{L} and $\sigma^n(G) \subset G$ then $\sigma^n(G) = G$.*
- (6) *Any gap or leaf G of \mathcal{L} either wanders or is such that for some $m < n$ we have $\sigma^m(G) = \sigma^n(G)$.*
- (7) *If G is a gap of \mathcal{L} such that G' is infinite, then G is a Fatou gap.*

Proof. (1) Suppose that ax, bx, cx are three leaves of \mathcal{L} with $x < a < b < c$ in the counterclockwise order $<$ on S^1 . Then bx is isolated. Hence bx is either a separate leaf from \mathcal{L}^- or a boundary leaf of a gap from \mathcal{L}^- .

(2) Suppose that \mathcal{L} contains the leaves $a_1x, a_2x, a_3x, \dots, a_nx$ with $n \geq 4$. We may assume that $a_1 < a_2 \cdots < a_n < x$. Since the leaves $a_2x, a_3x, \dots, a_{n-1}x$ are isolated and must come from \mathcal{L}^- . Since elements of \mathcal{L}^- are pairwise disjoint, $n = 4$ and the leaves a_2x, a_3x are on the boundary of a gap of \mathcal{L}^- .

(3) Suppose that a gap G and a leaf xa of \mathcal{L} meet only at x . Let $xb \subset G$ be the leaf which separates $G \setminus xb$ from $xa \setminus \{x\}$. Then xb is isolated and, hence, either a separate leaf in \mathcal{L}^- or a boundary leaf of a gap of \mathcal{L}^- . If the former holds, or $G \in \mathcal{L}^-$, we are done. Otherwise there exists a gap $H \in \mathcal{L}^-$ which contains xb . It now follows easily that $xa \subset H$ as desired.

(4) The first part immediately follows from Lemma 8 and the assumption that there are no critical leaves in the prelamination \mathcal{L}^- . This implies that the point $\sigma(H)$ is separated by leaves of \mathcal{L}^- from all other points of \mathbb{S}^1 . Hence by the properties of geo-laminations $\sigma^n(H) \subset H$ is impossible.

(5) By (4) we may assume that G is a gap which contains no σ^n -critical leaves in $\text{Bd}(G)$ and $\sigma^n(G)$ is not a point. Now, if G is a gap and $\sigma^n(G) = ab$ is a boundary leaf of G then $\sigma^{2n}(a) = a, \sigma^{2n}(b) = b$ and G is a finite gap. Denote by ca the other leaf in $\text{Bd}(G)$ containing a . Suppose first that $G \in \mathcal{L}^-$. Then $\sigma^n(G) = ab \in \mathcal{L}^-$, a contradiction. Similarly, if ca is an element of \mathcal{L}^- then $\sigma^n(ca) = ab$ (because there are no critical leaves in $\text{Bd}(G)$), a contradiction with the fact that elements of \mathcal{L}^- are pairwise disjoint. Suppose that ca is on the boundary of a gap $H \neq G$. Then it follows that $H \in \mathcal{L}^-$ because $G \notin \mathcal{L}^-$ and ca is not an element of \mathcal{L}^- . Then since $\sigma^n(ca) = ab$ we see that $\sigma^n(H) \neq H$ whereas $\sigma^n(H) \cap H \neq \emptyset$, a contradiction. We conclude that ca is a limit leaf from the outside of G . However then the σ^{2n} -images of leaves converging to ca will cross G , a contradiction.

(6) Suppose that G is a gap or leaf from \mathcal{L} for which the conclusions of the lemma do not hold. If G is infinite, then by Theorem 3 G is preperiodic. So we may assume that G is finite and that $|G'| = |\sigma^i(G)'|$ for all $i > 0$. By the assumption about G we may assume that for some $n > 0$, $G \cap \sigma^n(G) \neq \emptyset$ and $\sigma^n(G) \neq G$ (in particular, G is not a point because otherwise we would have $\sigma(G) = G$) and for no $i \neq j$ we have $\sigma^i(G) = \sigma^j(G)$. Since the elements of \mathcal{L}^- are pairwise disjoint, $G \notin \mathcal{L}^-$. Moreover, it is easy to see that no leaf in $\text{Bd}(G)$ is periodic. Indeed, otherwise under the map which fixed the endpoints of this leaf G will have to be mapped onto itself.

Suppose now that G is a leaf. If an endpoint of G is σ^n -fixed then we would have more than 4 leaves of \mathcal{L} coming out of this point, contradicting (2). Hence $\sigma^n(G)$ must be a leaf, “concatenated” to G , $\sigma^{2n}(G)$ is a leaf “concatenated” to $\sigma^n(G)$, and so on. Since these leaves do not intersect inside \mathbb{D} , it follows that they converge to a leaf or to a point $\lim_i \sigma^{ni}(G)$ which is σ^n -fixed. This contradicts the fact that σ^n is locally repelling.

Suppose next that G is a gap such that G and $\sigma^n(G)$ meet along the (isolated) leaf ℓ . Since $G \notin \mathcal{L}^-$, $\ell \in \mathcal{L}^-$. Since there are no periodic leaves in $\text{Bd}(G)$, $\sigma^n(\ell) \cap \ell = \emptyset$. Repeating this argument we see that leaves $\sigma^{in}(\ell)$ are such that the gaps $\sigma^{in}(G)$ are “concatenated” (attached) to each other at these leaves. This again, as in the previous paragraph, implies that the limit $\lim \sigma^{ni}(\ell)$ exists and is either a σ^n -fixed leaf in \mathcal{L} or a σ^n -fixed point in \mathbb{S}^1 . which contradicts the fact that σ is locally repelling.

Hence it remains to consider the case when G and $\sigma^n(G)$ meet in a point $x \in \mathbb{S}^1$. By (2) there exist boundary leaves $xa \subset \text{Bd}(G)$ and $xb \subset \text{Bd}(\sigma^n(G))$ and there exists a gap $H \in \mathcal{L}^-$ which contains both of these leaves in its boundary. If H is periodic, then xa and xb are periodic too, a contradiction. Hence, H is not periodic. Since $H \in \mathcal{L}^-$, H must wander and $\sigma^n(H) \cap \sigma^m(H) = \emptyset$ when $n \neq m$. It follows that sets $\sigma^{in}(G)$ are all “concatenated” at points $x, \sigma^n(x), \dots$, the set $\bigcup_{i=0}^{\infty} \sigma^{ni}(G)$ is connected set, and $\lim \sigma^{ni}(G)$ exists and is either a leaf in \mathcal{L} or point in \mathbb{S}^1 which is fixed under σ^n . As before, this contradicts the fact that σ^n is locally repelling and completes the proof of (6).

(7) Follows immediately from (4) and Lemma 9.(2).

□

We are ready to construct a non-degenerate lamination compatible with \mathcal{L}^- (or, equivalently, with \mathcal{L}). Suppose that $\mathcal{A} = \{G_\alpha\}$ is a generating collection of finite gaps/leaves and \mathcal{L}^- is a non-empty geometric d -invariant pre-lamination generated by a generating family \mathcal{A} such that there are no critical leaves in \mathcal{L}^- . Set $\mathcal{L} = \overline{\mathcal{L}^-}$ and $\approx_{\mathcal{L}} = \approx_{\mathcal{A}}$.

Theorem 12. *We have that $\mathbb{S}^1 / \approx_{\mathcal{A}}$ is non-degenerate and any equivalence class of $\approx_{\mathcal{A}}$ is finite. Moreover, $\approx_{\mathcal{A}}$ has no Siegel domains. In particular, if $R \neq \emptyset$, then the finest lamination \approx_{rat} which respects \mathcal{L}_{rat} , is not degenerate and in the geometric lamination $\mathcal{L}_{\approx_{rat}}$ every leaf not contained in the boundary of a Fatou domain is a limit of leaves from \mathcal{L}_{rat} .*

Proof. Let \approx be the equivalence relation in \mathbb{S}^1 defined as follows: $x \approx y$ if and only if there exists a continuum $K \subset S^1 \cup \mathcal{L}^*$ such that $x, y \in K$ and $K \cap S^1$ is countable (such continua are called ω -continua). Then \approx is the finest closed equivalence relation which respects \mathcal{L} ; moreover, \approx is an invariant lamination ([2]).

Now, suppose first that \mathcal{L} has no gaps. Then the leaves from \mathcal{L} fill the entire disk. If there are two leaves coming out of one point, then there must be infinitely many leaves coming out of the same point which is impossible by Lemma 11. Hence all leaves of \mathcal{L} are pairwise disjoint and equivalence classes of \approx are endpoints of (possibly degenerate) leaves. From now in the proof we

assume that \mathcal{L} has gaps. It now follows easily that gaps of \mathcal{L} are dense in \mathbb{D} and so if an ω -continuum K meets a leaf $\ell \in \mathcal{L}$ and $\mathbb{D} \setminus \ell$, then K must contain one of the endpoints of ℓ .

In the proof below we construct so-called *super gaps* and associate them to some leaves and gaps of \mathcal{L} . If G is a leaf of \mathcal{L} disjoint from all gaps of \mathcal{L} we call it a *separate leaf* (of \mathcal{L}). In this case put $G^+ = G$ and call it a *super gap* associated with G . Clearly, G^+ is a two sided limit of leaves from \mathcal{L}^- . Let $\mathfrak{G} = \bigcup \{G \mid G \text{ is a finite gap of } \mathcal{L}\}$. For any gap G of \mathcal{L} , let G^+ be the closure of the convex hull of the component of \mathfrak{G} which contains G . Again, call G^+ a *super gap* associated with G . By Lemma 11.(6), a gap/leaf G of \mathcal{L} either wanders or is such that $\sigma^m(G) = \sigma^n(G)$ for some $m < n$.

Claim 1. *Suppose that G is a wandering gap/leaf of \mathcal{L} . Then G^+ is either a separate leaf or a finite union of finite gaps whose convex hull is a wandering polygon and every leaf in its boundary is a limit of leaves from \mathcal{L}^- .*

Proof of Claim 1. The case when G is a separate leaf immediately follows from the definition of a super gap; in this case $G^+ = G$ is a separate leaf. Suppose next that G is a leaf which meets a gap H of \mathcal{L} or G is a wandering gap. By Lemma 11.(2), there exist gaps G_0, \dots, G_n such that $G \subset G_0$, $G_i \cap G_{i+1}$ is a leaf and $G_n = H$ (if $G + H$ is a wandering gap, then we set $G_0 = H = G_n$). By Lemma 11.(6) G_0 is either wandering or (pre)periodic, and since G is wandering, so is G_0 .

Assume, by way of induction, that G' is a finite union of finite gaps which is a wandering polygon and H is a gap of \mathcal{L} which meets G' along the leaf ab . Then ab is wandering because it comes from G' . Again since by Lemma 11 H is either wandering or (pre)periodic, we see that H also wanders. In particular, by Theorem 3 H is finite. We claim that $H \cup G'$ is a wandering polygon. For suppose this is not the case. Then we may assume that $\sigma(G') \cap H \neq \emptyset$. Moreover, the common leaf ab of G' and H is isolated and hence comes from \mathcal{L}^- . Therefore it is not critical and its image $\sigma(ab)$ is a leaf again. Clearly, $\sigma(ab)$ is the leaf shared by $\sigma(G')$ and $\sigma(H)$. Repeating this argument, we get a sequence of gaps of \mathcal{L} “concatenated” at images of the leaf ab . Similarly to the arguments in the proof of Lemma 11.(6) it implies that the orbit of ab converges to a point or a leaf but never maps into it which is impossible because of repelling properties of σ .

It follows that G^+ is a wandering polygon and, by Theorem 3, $|G^+ \cap S^1| \leq 2^d$. Hence G^+ is finite union of finite gaps. Note that every leaf on the boundary of G^+ is a limit of leaves from \mathcal{L}^- as desired. This completes the proof of Claim 1. \square

Observe that by Lemma 11.(7) for every infinite gap G of \mathcal{L} the set G^+ consists

of a Cantor set $G'_c \subset \mathbb{S}^1$ and a countable collection of finite sets G'_1, G'_2, \dots of cardinality at most k (k depends on G) such that for every i the set G'_i is the intersection of G' and a complementary to G'_c subarc U_i of \mathbb{S}^1 . If $|G'_i| > 1$ we connect the endpoints of U_i with a leaf ℓ and add ℓ to the lamination \mathcal{L} . It is easy to see that the resulting extension of the geo-lamination \mathcal{L} is a geo-lamination itself. From now on we will use the notation \mathcal{L} for the new extended geo-lamination.

Suppose next that G is a finite (pre)periodic gap or a (pre)periodic leaf of \mathcal{L} . If some forward image of G contains a critical leaf on its boundary, then we may assume that $\sigma(G)$ is a point by Lemma 11.(4). Hence each leaf in the boundary of G is a limit of leaves from \mathcal{L}^- and $G^+ = G$. If no forward image of G contains a critical leaf on its boundary, then from some time on $|\sigma^k(G')| > 1$ stabilizes and by Lemma 11.(6) we may assume that $\sigma^m(G) = G$ for some $m > 0$ and $|G'| \geq 2$. Choose $n \geq 0$ such that $\sigma^n(G) = G$ and each leaf in the boundary of G is fixed.

Claim 2. *Suppose G is a (pre)periodic finite gap or (pre)periodic leaf of \mathcal{L} . Then G^+ is a finite polygon and any leaf in the boundary of G^+ is either a limit of leaves from \mathcal{L}^- or is contained in an uncountable gap of \mathcal{L} . Moreover, if G is an n -periodic gap/leaf then $G^+ \supset G$ is the convex hull of a subset of the component of the set of leaves from \mathcal{L} with σ^n -fixed endpoints.*

Proof of Claim 2. Suppose that $\sigma^n(G) = G$ and that all points of G' are fixed. If G is a separate leaf then $G^+ = G$ and we are done. If G is a non-separate leaf then it is a boundary leaf of a gap Q . Since the endpoints of G are σ^n -fixed, either $\sigma^n(Q) = G$ or, because the map $\sigma^n|_{\text{Bd}(Q)}$ is positively oriented, $\sigma^n(Q) = Q$. The former is impossible by Lemma 11.(5). Hence we find a gap $Q \supset G$ whose all vertices are σ^n -fixed. Finally, if G is a gap then we can set $Q = G$. Thus, if G is not a separate leaf, we can always find a gap $Q \supset G$ whose all vertices are σ^n -fixed.

Suppose, by induction, that G is a finite polygon which is a finite union of gaps from \mathcal{L} . Moreover, suppose that the boundaries of the gaps consist of leaves with σ^n -fixed endpoints. Let H be any gap of \mathcal{L} which meets G along the leaf ab . By Lemma 11.(5) $\sigma^n(H)$ cannot be equal to ab , and since $\sigma^n|_{\text{Bd}(H)}$ is a positively oriented covering map we see that $\sigma^n(H) = H$. If H is finite, all leaves in the boundary of H must also be fixed. Otherwise H is an infinite, and hence uncountable, gap. It follows that $G^+ \supset G$ is a finite union of σ^n -fixed gaps and that every leaf in the boundary of G^+ is either a limit of leaves from \mathcal{L}^- or is contained in an uncountable gap from \mathcal{L} .

Now let G be any (pre)periodic finite gap of \mathcal{L} . Then there exists n such that $\sigma^n(G) = H$ is periodic. If H is a point then by Lemma 11.(4) all leaves in the boundary of G are limit leaves and hence $G^+ = G$. Otherwise if H is a

separate leaf it follows that all boundary leaves of G are limit leaves and we are done. Thus by the previous paragraph we may assume that H^+ is a finite union $H = H_1, \dots, H_n$ of gaps of \mathcal{L} . Let G^+ be the component of $\bigcup_i \sigma^{-n}(H_i)$ which contains G . Then G^+ is a finite union of finite gaps and every leaf in the boundary of H^+ is either a limit of leaves from \mathcal{L}^- or on the boundary of an uncountable gap from \mathcal{L} . \square

We now pass on to the proof of the fact that $\approx_{\mathcal{L}} = \approx_{\mathcal{A}}$ is non-degenerate. Indeed, consider all the super gaps constructed in Claim 1 and Claim 2 (i.e., all the sets $G^+ \cap \mathbb{S}^1$ for different gaps and leaves G of the geo-lamination \mathcal{L}). Also, if a point $x \in \mathbb{S}^1$ does not belong to any gap or leaf of \mathcal{L} we call it a *separate point* and add it to the family of sets which we construct. Clearly, all sets in the just constructed family \mathcal{F} of super gaps and separate points are closed. Moreover, by the definition two sets in the family \mathcal{F} are disjoint. Indeed, two super gaps cannot meet over a leaf by the definition. If they meet at a vertex then by Lemma 11.(1), Lemma 11.(2) and by the construction of the extended lamination \mathcal{L} they again must be in one super gap. Hence all elements of \mathcal{F} are pairwise disjoint.

Considering elements of \mathcal{F} as equivalence classes we get a closed equivalence \approx on \mathbb{S}^1 which respects \mathcal{L}^- and \mathcal{L} (it is easy to see that \approx is indeed closed). By the construction and Claims 1 and 2, all \approx -classes are finite. Because of the definition of a super gap, if an equivalence respects \mathcal{L}^- (and hence \mathcal{L}), it cannot split a \approx -class (i.e., a set $G^+ \cap \mathbb{S}^1$ for some gap/leaf G of \mathcal{L}) into two or more classes of equivalence. Therefore \approx is the finest equivalence which respects \mathcal{L}^- . As was explained in the beginning of the proof of Theorem 12, by [2] there always exists the finest equivalence which respects a geometric lamination, and from what we have just proven it follows that this finest equivalence $\approx_{\mathcal{L}}$ coincides with \approx . By Claims 1 and 2 super gaps are finite, thus all \approx -classes are finite and hence \approx is non-degenerate.

Finally assume that U is a Siegel domain of \approx . Then $\text{Bd}(U)$ must contain a critical leaf because otherwise by a well-known result from the topological dynamics (see, e.g., Lemma 18.8 from [16]) $\text{Bd}(U) \cap \mathbb{S}^1$ must be finite, a contradiction. However by Lemma 11.(4) this is impossible. The rest of Theorem 12 which deals with the rational lamination follows immediately from the construction. This completes the proof of the theorem. \square

3 The existence of a locally connected model for unshielded planar continua

As outlined in Section 1, in this section we prove Theorem 1 and show the existence of the finest model and the finest map for any unshielded planar

continuum Q . We do this in Subsection 3.1. In Subsection 3.2 we suggest a topological condition sufficient for an unshielded continuum Q to have a non-degenerate finest model. This will be used later when in Theorem 2 we establish the criterion for the connected Julia set of a polynomial to have a non-degenerate finest model.

3.1 The existence of the finest map φ and the finest locally connected model

In what follows Q will always denote an unshielded continuum in the plane and U_∞ will always denote the corresponding simply connected neighborhood of infinity in the sphere, called the *basin of infinity* (so that $Q = \text{Bd}(U_\infty)$).

We begin by constructing the finest monotone map φ of Q onto a locally connected continuum. The map will be constructed in terms of impressions of the continuum Q . Since $Q = \text{Bd}(U_\infty)$, there is a unique conformal isomorphism $\Psi : U_\infty \rightarrow \mathbb{D}$ which has positive real derivative at ∞ . Define the *principal set of the external angle* $\alpha \in \mathbb{S}^1$ as

$$\text{Pr}(\alpha) = Q \cap \overline{\Psi^{-1}(\{re^{2\pi i\alpha} \mid r \in [0, 1)\})}.$$

Define the *impression of the external angle* $\alpha \in \mathbb{S}^1$ as

$$\text{Imp}(\alpha) = \left\{ \lim_{i \rightarrow \infty} \Psi^{-1}(\alpha_i) \mid \{\alpha_i \mid i > 0\} \subset \mathbb{D} \text{ and } \lim_{i \rightarrow \infty} \alpha_i = \alpha \right\}.$$

The *positive wing (of an impression)* is defined as follows:

$$\text{Imp}^+(\alpha) = \left\{ \lim_{i \rightarrow \infty} \Psi^{-1}(\alpha_i) \mid \{\alpha_i \mid i > 0\} \subset \mathbb{D} \text{ and } \lim_{i \rightarrow \infty} \alpha_i = \alpha \text{ with } \arg(\alpha_i) \geq \arg(\alpha) \right\}.$$

Similarly, the *negative wing (of an impression)* is defined as follows:

$$\text{Imp}^-(\alpha) = \left\{ \lim_{i \rightarrow \infty} \Psi^{-1}(\alpha_i) \mid \{\alpha_i \mid i > 0\} \subset \mathbb{D} \text{ and } \lim_{i \rightarrow \infty} \alpha_i = \alpha \text{ with } \arg(\alpha_i) \leq \arg(\alpha) \right\}.$$

The differences between these sets are illustrated in Figure 1. In a lot of applications it is crucial that in the above construction the map Ψ is conformal. However the construction can be carried out if instead of Ψ certain homeomorphisms $\Psi' : U_\infty \rightarrow \mathbb{D}$ are used. The definitions of the principle set, impression

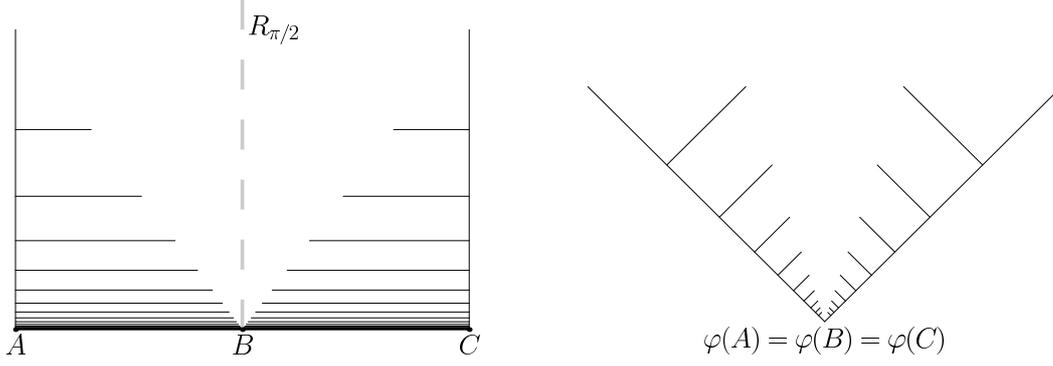


Fig. 1. On the left is depicted a continuum with an external ray for which the impression, positive wing, negative wing, and principle sets are distinct. The positive wing is the line segment joining A and B , while the negative wing is the line segment joining B and C . On the right is depicted the quotient by \mathcal{D} defined in Lemma 13, which is locally connected.

and wings of impression can be given in this case as well. Since some continua we construct have topological nature, we use this idea in what follows defining for them the map Ψ' in a topological way and then defining principle sets, impressions and wings of impressions accordingly.

Any angle's principle set, impression, wings of its impression are each subcontinua of Q . It is known that $\text{Pr}(\alpha) = \text{Imp}^+(\alpha) \cap \text{Imp}^-(\alpha) \subset \text{Imp}^+(\alpha) \cup \text{Imp}^-(\alpha) = \text{Imp}(\alpha)$. If Q is locally connected, the impression of every external angle is a point, and therefore impressions intersect only when they coincide. Non-locally connected continua may have impressions of different external angles which intersect and do not coincide.

Lemma 13. *There exists a partition $\mathcal{D}_Q = \mathcal{D}$ of Q which is finest among all upper semi-continuous partitions whose elements are unions of impressions of Q . Further, elements of \mathcal{D} are subcontinua of Q .*

Proof. Let Ξ be the collection of closed equivalence relations on Q such that, for any equivalence relation \approx from Ξ , $\text{Imp}(\alpha)$ is contained in one class of equivalence for any external angle α . Then the equivalence relation $\bigcap \Xi$ is also an element of Ξ (classes of equivalence of $\bigcap \Xi$ are intersections of classes of equivalence of all equivalence relations from Ξ). Let \mathcal{D} be the collection of equivalence classes of $\bigcap \Xi$.

To see that the elements of \mathcal{D} are connected, we can define a finer partition \mathcal{D}' whose elements are connected components of elements of \mathcal{D} . Then \mathcal{D}' is an upper semi-continuous monotone decomposition of Q [20, Lemma 13.2]. Since impressions are connected subsets of Q , that each impression belongs to an element of \mathcal{D} implies that it belongs to an element of \mathcal{D}' . Therefore, $\mathcal{D}' \in \Xi$ and \mathcal{D}' is a refinement of \mathcal{D} , so $\mathcal{D} = \mathcal{D}'$, and the elements of \mathcal{D} are connected. \square

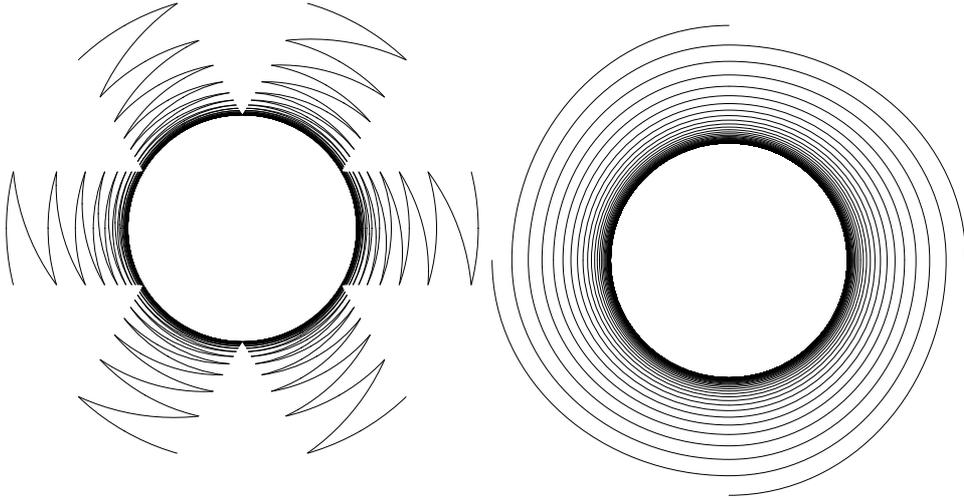


Fig. 2. Here are two examples of continua for which a fiber of φ is a simple closed curve. Notice that points of the simple closed curve in the figure on the left are accessible from both the bounded and unbounded complementary domains.

We will show that Q/\mathcal{D} is locally connected, and \mathcal{D} is the finest upper semi-continuous partition of X into connected sets with that property. Thus, the finest monotone map respecting impressions turns out to be the finest monotone map producing a locally connected model. To implement our plan we study properties of monotone maps of unshielded continua. First we suggest the canonic extension of any monotone map $m : Q \rightarrow Y$ onto the entire plane which respects the planar structure. Given any monotone map ψ , let call sets $\psi^{-1}(y)$ ψ -fibers, or just *fibers*.

Definition 14. Let $U \subset \widehat{\mathbb{C}}$ be a simply connected open set containing ∞ . If A is a continuum disjoint from U , the *topological hull* $\text{TH}(A)$ of A is the union of A with the bounded components of $\widehat{\mathbb{C}} \setminus A$. Equivalently, $\text{TH}(A)$ is the complement of the unique component of $\widehat{\mathbb{C}} \setminus A$ containing U . Note that $\text{TH}(A) \subset \mathbb{C}$ is a continuum which does not separate the plane.

Suppose that a monotone map $m : Q \rightarrow Y$ is given. Then m -fibers may be separating, as indicated in Figure 2, or non-separating. Denote by $T_m(Q)$ the union of Q and the topological hulls of all separating fibers. To extend our map m onto the plane as a monotone map, we must collapse topological hulls of separating fibers because otherwise the extension will not be monotone. This idea is implemented in the next lemma.

Lemma 15. *If Q is an unshielded continuum and $m : Q \rightarrow Y$ is a surjective monotone map, then there exists a monotone map $M : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ and an embedding $h : Y \rightarrow \mathbb{C}$ such that:*

- (1) $M|_{\widehat{\mathbb{C}} \setminus T_m(Q)}$ is a homeomorphism onto its image;

- (2) $M(U_\infty)$ is a simply connected open set whose boundary is $M(Q)$, with $M(\infty) = \infty$; and
(3) $M|_Q = h \circ m$.

Proof. We extend the map m by filling in its fibers. Define the collection

$$\widehat{\mathcal{D}} = \{\text{TH}(m^{-1}(y)) : y \in Y\} \cup \{\{p\} : p \notin T_m(Q)\}.$$

It is immediate that $\widehat{\mathcal{D}}$ is an upper semi-continuous partition of $\widehat{\mathbb{C}}$ whose elements are non-separating continua. Therefore, by [19], $\widehat{\mathbb{C}}/\widehat{\mathcal{D}}$ is homeomorphic to $\widehat{\mathbb{C}}$, and there exists a monotone map $M : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ whose fibers are sets from $\widehat{\mathcal{D}}$. Observe that by the construction $M^{-1}(Y) = T_m(Q)$.

Further, since points of $\widehat{\mathbb{C}} \setminus M^{-1}(Y)$ are elements of $\widehat{\mathcal{D}}$, invariance of domain gives that $M|_{\widehat{\mathbb{C}} \setminus M^{-1}(Y)}$ is a homeomorphism onto its image and that $M(U_\infty)$ is an open subset of $\widehat{\mathbb{C}}$ with $M(\infty) = \infty$. Also, $M(U_\infty) \cap M(Q) = \emptyset$, so $\text{Bd}(M(U_\infty)) = M(Q)$. Finally, notice that the fibers of $M|_Q$ are the same as the m -fibers so there exists a natural homeomorphism $h : Y \rightarrow M(Q)$. This is a homeomorphism of Y onto $M(Q)$ and an embedding of Y into \mathbb{C} since Y is compact. \square

Next we show that *any* monotone map of an unshielded continuum onto a locally connected continuum must collapse impressions to points. A *crosscut* of Q is a homeomorphic image $C \subset U_\infty$ of an open interval $(0, 1)$ under a homeomorphism $\psi : [0, 1] \rightarrow \mathbb{C}$ such that $\psi(0) \in Q \neq \psi(1) \in Q$. Define $\text{Sh}(C)$ (the *shadow of C*) as the closure of the bounded component of $U_\infty \setminus C$. Observe that in our definition of a crosscut and its shadow we always assume that the continuum is unshielded and that crosscuts are contained in the basin of infinity.

Lemma 16. *Suppose that $m : Q \rightarrow Y$ is a monotone map onto a locally connected continuum. Then $m(\text{Imp}(\alpha))$ is a point for every $\alpha \in \mathbb{S}^1$.*

Proof. Let M be as guaranteed in Lemma 15. Since $M|_{U_\infty}$ is one-to-one, it is then easy to see that a crosscut of Q maps by M either to a crosscut of $M(Q)$ or to an open arc in $M(U_\infty)$ whose closure is a simple closed curve meeting $M(Q)$ in a single point. Because $M(\infty) = \infty$, we see that $M(\text{Sh}(C)) = \text{Sh}(M(C))$ for any crosscut C whose image is a crosscut while if $\overline{M(C)}$ is a simple closed curve then $M(\text{Sh}(C))$ is the interior of the Jordan disk enclosed by $\overline{M(C)}$.

Choose any external angle α . There exists a null sequence of crosscuts $(C_i)_{i=1}^\infty$ such that $\bigcap_{i=1}^\infty \text{Sh}(C_i) = \text{Imp}(\alpha)$ [16, Lemma 17.9]. Since $\text{Sh}(C_i)$ are nested, we have

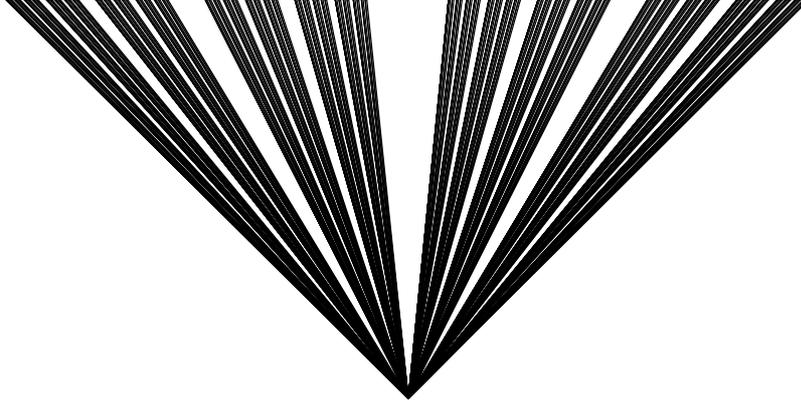


Fig. 3. In this continuum, constructed by joining every point of a Cantor set to a base point with a straight line segment, every pair of non-degenerate impressions intersect, and every point is contained in a non-degenerate impression. Therefore, Lemma 16 concludes that the finest locally connected model is a point.

$$\begin{aligned}
 M(\text{Imp}(\alpha)) &= M\left(\bigcap_{i=1}^{\infty} \text{Sh}(C_i)\right) \\
 &= \bigcap_{i=1}^{\infty} M(\text{Sh}(C_i)) \\
 &= \bigcap_{i=1}^{\infty} \text{Sh}(M(C_i)).
 \end{aligned}$$

By uniform continuity, $\lim_{i \rightarrow \infty} \text{diam}(M(C_i)) = 0$. Since $M(Q)$ is locally connected, $\bigcap_{i=1}^{\infty} \text{Sh}(M(C_i))$ is indeed a point, and so is $M(\text{Imp}(\alpha))$. \square

The next lemma is essentially a converse of Lemma 16.

Lemma 17. *Suppose that $m : Q \rightarrow Y$ is a monotone surjective map such that $m(\text{Imp}(\alpha))$ is a point for all $\alpha \in \mathbb{S}^1$. Then Y is locally connected. Moreover, the map $\Phi_m : \mathbb{S}^1 \rightarrow Y$ defined by $\Phi_m = m \circ \text{Imp}$ is a continuous single-valued onto function.*

Proof. Φ_m is a single-valued function, since by assumption m maps impressions to points of Y . Also, it is surjective, since m is surjective and every point is contained in the impression of some angle. To see sequential continuity, observe that

$$\begin{aligned}
\alpha_i \rightarrow \alpha &\implies \limsup_{i \rightarrow \infty} \text{Imp}(\alpha_i) \subset \text{Imp}(\alpha) \\
&\implies \limsup_{i \rightarrow \infty} m(\text{Imp}(\alpha_i)) \subset m(\text{Imp}(\alpha)) \\
&\implies \Phi(\alpha_i) \rightarrow \Phi(\alpha).
\end{aligned}$$

The continuous image of a locally connected continuum is locally connected, so Y is locally connected as the Φ_m -image of \mathbb{S}^1 . \square

The picture which follows from the above lemmas is as follows. Imagine that we have a monotone map m of an unshielded continuum $Q \subset \mathbb{C}$ onto a locally connected continuum Y . By Lemma 15 we can think of m as the restriction of a monotone map $M : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ which in fact is a homeomorphism on U_∞ as well as on the components of $\mathbb{C} \setminus Q$ whose boundaries are not collapsed by m . To avoid confusion, we call the plane containing Q the Q -plane, and the plane containing Y the Y -plane. Likewise, if there is no ambiguity we will call various objects in the Q -plane Q -rays etc while calling corresponding objects in the Y -plane Y rays etc.

Now, take external conformal Q -rays. Then the map M carries them over to the Y -plane as just continuous rays (obviously, our construction is purely topological and does not preserve the conformal structure in any way). The construction however forces all these Y -rays to land; moreover, the family of Y -rays can be used to define impressions in the sense of that family (see our explanation following the definition of the impression). By Lemma 16, these impressions must be degenerate.

We are ready to prove the existence of the finest locally connected model and the finest map for unshielded continua. Recall that $\mathcal{D}_Q = \mathcal{D}$ denotes the finest among all upper semi-continuous partitions of Q whose elements are unions of impressions of Q (it is provided by Lemma 13).

Theorem 18. *There exists a monotone map $\varphi : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ such that $\varphi|_Q$ is the finest monotone map of Q onto a locally connected continuum, $\varphi(Q)$ is the finest locally connected model of Q , and φ is a homeomorphism on $\widehat{\mathbb{C}} \setminus \varphi^{-1}(\varphi(Q))$. Moreover, the map $\varphi|_Q$ can be defined as the quotient map $Q \rightarrow Q/\mathcal{D}$.*

Proof. Let us show that the quotient map $m : Q \rightarrow Q/\mathcal{D}$ is the finest map of Q onto a locally connected continuum. Indeed, suppose that $\psi : Q \rightarrow A$ is a monotone map onto a locally connected continuum A . Then ψ generates an upper semi-continuous partition of Q whose elements, by Lemma 16, are unions of impressions of Q . By the choice of \mathcal{D} there exists a continuous map

$h : Q/\mathcal{D} \rightarrow A$ which associates to an element B of \mathcal{D} the point $x \in A$ such that $\psi^{-1}(x)$ contains B . To complete the proof we let $\varphi : \mathbb{C} \rightarrow \mathbb{C}$ be the extension of m guaranteed by Lemma 15. \square

Define $\Phi : \mathbb{S}^1 \rightarrow \varphi(Q)$ as $\Phi = \varphi \circ \text{Imp}$. From Lemma 17, Φ is a well-defined continuous function. According to the picture given after Lemma 17, Φ maps an angle α to the landing point of the corresponding $\varphi(Q)$ -ray (i.e., the φ -image of the external conformal ray to Q in the Q -plane). Then the *finest lamination* \sim_Q (corresponding to Q) is the equivalence relation \sim on \mathbb{S}^1 , defined by $\alpha_1 \sim \alpha_2$ if and only if $\Phi(\alpha_1) = \Phi(\alpha_2)$.

3.2 A constructive approach

Recall that the finest map of an unshielded continuum Q is always denoted by $\varphi = \varphi_Q$. Fibers under the finest map will be called *K-sets*. In the notation from Subsection 3.1 and Lemma 13, K-sets are exactly the elements of the partition $\mathcal{D}_Q = \mathcal{D}$, the finest among all upper semi-continuous partitions whose elements are unions of impressions of Q . Classes of equivalence in the lamination \sim_Q will be called *K-classes*. We are interested in the structure of K-sets, and will describe how to determine if two points lie in the same K-set. Given a set $A \subset \mathbb{S}^1$ let $\text{Imp}(A)$ be the union of impressions of all angles in A .

Lemma 19. *If $\{a\}$ is a degenerate K-set then Q is locally connected at a .*

Proof. Suppose that A is a K-class with a degenerate K-set $\text{Imp}(A) = \{a\}$ (by the definitions, this is equivalent to $\varphi^{-1}(\varphi(a)) = \{a\}$). Take the point $\varphi(a)$. Since $\varphi(Q)$ is locally connected, there is a nested sequence of open connected neighborhoods $U_1 \supset U_2 \supset \dots$ of $\varphi(a)$ such that $\bigcap_{i=1}^{\infty} U_i = \{\varphi(a)\}$. By the properties of φ , the sets $V_i = \varphi^{-1}(U_i)$ form a nested sequence of open connected neighborhoods of a with the intersection coinciding with $a = \varphi^{-1}(\varphi(a))$. So, Q is locally connected at a . \square

Now we introduce two important notions.

Definition 20. A *ray-compactum* (or *ray-continuum*) $X \subset Q$ is a compactum (respectively, a continuum or a point) for which there exists a closed set of angles $\Theta(X) \subset \mathbb{S}^1$ such that

$$\bigcup_{\theta \in \Theta(X)} \text{Pr}(\theta) \subset X \subset \bigcup_{\theta \in \Theta(X)} \text{Imp}(\theta).$$

Denote $X \cup \bigcup_{\theta \in \Theta(X)} R_\theta$ by \widetilde{X} .

One of the notions defined below is fairly standard. We give two equivalent definitions of second notion, one involving separation of sets and the other involving cutting the plane.

Definition 21. A set Y *separates* a space X between subsets A and B if $X \setminus Y = U \cup V$, where $A \subset U$, $B \subset V$, and $\overline{U} \cap V = U \cap \overline{V} = \emptyset$. We say that a ray-compactum C *ray-separates* subsets A and B of Q if \tilde{C} separates $\overline{U_\infty}$ between A and B . If $X \subset Q$ is a continuum and there are at least two points of X which are ray-separated by C , we say that C *ray-separates* X .

The definition of ray-separation can be equivalently given as follows: (a) a ray-compactum C *ray-separates* subsets A and B of Q if $C \cap (A \cup B) = \emptyset$ and there exists no component of $\overline{U_\infty} \setminus \tilde{C}$ containing points of A and B . All these notions are important ingredients of the central notion of *well-slicing*.

Definition 22. A continuum $X \subset Q$ is *well-sliced* if there exists a collection \mathcal{C} of pairwise disjoint ray-compacta in Q such that

- (1) each $C \in \mathcal{C}$ ray-separates X ,
- (2) for every different $C_1, C_2 \in \mathcal{C}$ there exists $C_3 \in \mathcal{C}$ which ray-separates C_1 and C_2 , and
- (3) \mathcal{C} has at least two elements.

The family \mathcal{C} is then a *well-slicing family* for X .

We will also use the following combinatorial (laminational) version of well-slicing.

Definition 23. Suppose that there is a collection \mathcal{C} of at least two pairwise disjoint geometric leaves or gaps in \mathbb{D} . Suppose that for every different $C_1, C_2 \in \mathcal{C}$ there exists $C_3 \in \mathcal{C}$ which separates \mathbb{D} between C_1 and C_2 . Then the family \mathcal{C} is then a *well-slicing family* for \mathbb{D} . Equivalently, consider the family \mathcal{C}' of closed pairwise unlinked subsets of \mathbb{S}^1 . Suppose that for every different $C'_1, C'_2 \in \mathcal{C}'$ there exists $C'_3 \in \mathcal{C}'$ which separates \mathbb{S}^1 between C'_1 and C'_2 . Then we say that \mathcal{C}' is a well-slicing family of \mathbb{S}^1 . Clearly, if \mathcal{C} is a well-slicing family of \mathbb{D} then the intersections of elements of \mathcal{C} with \mathbb{S}^1 (i.e., their bases) form a well-slicing family of \mathbb{S}^1 , and vice versa.

As an example of a well-slicing family, take $Q = \mathbb{S}^1$. We define the family of subsets

$$C_\alpha = \{e^{2\pi i\alpha}, e^{-2\pi i\alpha}\}$$

with α taking a rational value in $[0, 1/2)$. Each C_α is then a ray-compactum with the set of angles $\Theta(C_\alpha) = \{\alpha, -\alpha\}$. Then for $0 \leq \alpha < \beta < \frac{1}{2}$, we see that C_α and C_β are ray-separated by $C_{(\alpha+\beta)/2}$. Hence, \mathcal{C} is a well-slicing family for \mathbb{S}^1 . Set $\mathcal{C}_{\mathbb{S}^1} = \mathcal{C}$ and call this collection the *vertical collection*.

Suppose that a collection \mathcal{C}' of closed pairwise unlinked subsets of \mathbb{S}^1 is a well-slicing of \mathbb{S}^1 . Moreover, suppose that for each set $C' \in \mathcal{C}'$ the set $\text{Imp}(C')$ is a continuum in Q , and for distinct sets C'_1, C'_2 their impressions are disjoint. Then it follows from the definitions that the sets $\text{Imp}(C'), C' \in \mathcal{C}'$ form a well-slicing family of the entire Q . If $X \subset Q$ is such that all sets A from this collection cut X (i.e., $X \setminus A$ is disconnected) then it follows that this is a well-slicing family for X .

Lemma 24. *Suppose that C_1, C_2 are disjoint ray-compacta each of which ray-separates $A, B \subset Q$. If C_3 is a ray-compactum disjoint from $A \cup B$ which ray-separates C_1 and C_2 , then C_3 also ray-separates A and B .*

Proof. Suppose that C_3 does not ray-separate A and B . Then there exists a component V of $\mathbb{C} \setminus \tilde{C}_3$ containing points of both A and B . Since C_3 ray-separates C_1 and C_2 , one of these sets (say, C_1) is disjoint from V . Then V is contained in a component W of $\mathbb{C} \setminus \tilde{C}_1$. Hence W contains points of both A and B and so \tilde{C}_1 does not separate X between A and B , a contradiction. \square

The next lemma is close in spirit to Lemma 24.

Lemma 25. *Let $A, B \subset Q$. Suppose that K_1 is a ray-compactum which ray-separates A and B , and K_2 is a ray-compactum disjoint from B which ray-separates A and K_1 . Then K_2 ray-separates A and B .*

Proof. Suppose that K_2 does not ray-separate A and B . Then there exists a component V of $\overline{U_\infty} \setminus \tilde{K}_2$ containing points of both A and B . Since K_1 ray-separates A and B , there must be points of K_1 in V too. However this implies that K_2 does not ray-separate A and K_1 , a contradiction. \square

The next lemma shows that elements of a well-slicing family are separated by infinitely many elements of the same family.

Lemma 26. *If \mathcal{C} is a well-slicing family of a continuum $X \subset Q$ then, for any two elements C_1 and C_2 , infinitely many different elements of \mathcal{C} separate C_1 and C_2 .*

Proof. Choose $C_3 \in \mathcal{C}$ which ray-separates C_1 and C_2 . Then choose $C_4 \in \mathcal{C}$ which ray-separates C_3 and C_2 . It is easy to see that $C_4 \neq C_1$. By Lemma 25 C_3 ray-separates C_1 and C_2 . Inductively applying this argument, we will find a sequence of pairwise distinct elements of \mathcal{C} each of which ray-separates C_1 and C_2 as desired. \square

Now we prove the first theorem of this subsection which implies that in a few cases certain subcontinua of Q do not collapse under the finest map φ .

Theorem 27. *Suppose that \mathcal{C} is a well-slicing of a continuum $X \subset Q$. Then $\varphi(X)$ is not a point.*

Proof. Define $x \approx y$ whenever only finitely many elements of \mathcal{C} ray-separate x and y . Clearly, such a relation is symmetric and reflexive. To see that it is transitive, suppose $x \approx y$ and $x \not\approx z$. Then infinitely many elements of \mathcal{C} ray-separate x and z . However, only finitely many of these elements ray-separate x from y , and the rest then ray-separate y from z , so $y \not\approx z$.

Therefore, \approx is an equivalence relation. We will now show that \approx is a closed equivalence relation by showing that $\{(x, y) \in Q^2 \mid x \not\approx y\}$ is open. Suppose that $x \not\approx y$. In particular, there are two elements C_1 and C_2 which ray-separate x and y . Every subspace of \mathbb{C} is a normal space, so it is easy to see that sets \widetilde{C}_1 and \widetilde{C}_2 separate $\overline{U_\infty}$ between every point y in a neighborhood V of x and every point z in a neighborhood W of y . Then by Lemma 26 we can find infinitely many elements of \mathcal{C} which do not contain y or z and separate X between C_1 and C_2 . Each such element of \mathcal{C} separates X between y and z by Lemma 24. Hence no point in V is \approx -equivalent to any point in W , and \approx is closed. In particular, the partition of Q into \approx -classes is upper semi-continuous.

Now we show that, for any external angle α , the impression $\text{Imp}(\alpha)$ is contained in a \approx -class. To see this, suppose that $x, y \in \text{Imp}(\alpha)$ are ray-separated by two elements B, C of \mathcal{C} . Since $B \cap C = \emptyset$, we see that the set $\Theta(B)$ of angles associated with B is disjoint from $\Theta(C)$. Hence at most one of these sets of angles contains α , and we may assume that $\alpha \notin \Theta(C)$. Now, since C is a ray-compactum, then each component W of $\mathbb{C} \setminus \widetilde{C}$ corresponds to a well-defined open set of angles in \mathbb{S}^1 whose external rays are contained in W . Since $\alpha \notin \Theta(C)$, one such component V contains R_α together with rays of close to α angles. Hence $\text{Imp}(\alpha) \subset \overline{V}$ which means that $\text{Imp}(\alpha)$ is disjoint from all other components of $\overline{U_\infty} \setminus \widetilde{C}$ but V . However, by the assumption C ray-separates X between x and y , hence the points $x \in \text{Imp}(\alpha)$ and $y \in \text{Imp}(\alpha)$ must belong to distinct components of $\overline{U_\infty} \setminus \widetilde{C}$, a contradiction.

Finally, we show that $\varphi(X)$ is not a point. First we refine \approx to get an equivalence \approx' with connected classes. Indeed, as in the proof of Lemma 13 we can define a finer partition than that into \approx -classes whose elements are connected components of \approx -classes. Then the new partition is an upper semi-continuous monotone decomposition of Q [20, Lemma 13.2]. By the previous paragraph any impression is still contained in an \approx' -class. Thus the quotient map $m : Q \rightarrow Q/\approx'$ is a monotone surjective map collapsing impressions. By Lemma 17 Q/\approx' is locally connected. Now, let $C_1, C_2 \in \mathcal{C}$ be different. For all $x \in C_1 \cap X$ and $y \in C_2 \cap X$, we see that $x \not\approx y$ by Lemma 26 and hence $m(x) \neq m(y)$. Since φ is the finest monotone map, we see that $\varphi(x) \neq \varphi(y)$, and so $\varphi(X)$ is not a point. \square

Now we prove a related criterion: If an unshielded continuum $Q \subset \mathbb{C}$ has an uncountable family of disjoint ray-continua, each of which ray-separate Q , then there is a sub-family which is well-sliced, and therefore the finest model is non-degenerate.

Lemma 28. *Let \mathcal{C} be an uncountable collection of disjoint ray-continua of an unshielded continuum $Q \subset \mathbb{C}$, each of which ray-separates Q . Then there exist elements $C_0, C_1 \in \mathcal{C}$ such that uncountably many elements of \mathcal{C} ray-separate C_0 and C_1 .*

Proof. Assume by way of contradiction that this is not the case. For $A, B \in \mathcal{C}$, let Y_{AB} denote the set of points $x \in X \setminus (A \cup B)$ which are not ray-separated from A by B , nor vice-versa. We see that Y_{AB} is an open subset of X , each Y_{AB} contains every element of \mathcal{C} that it intersects, and by assumption each Y_{AB} may contain only countably many elements of \mathcal{C} . Then the open set $U = \bigcup_{A, B \in \mathcal{C}} Y_{AB}$ is an open subset of Q of which $\{Y_{AB}\}_{A, B \in \mathcal{C}}$ forms an open cover. Since Q is second countable, countably many Y_{AB} cover U . We therefore conclude that the set of elements of \mathcal{C} contained in (or intersecting) U is countable.

Consider now any $D \in \mathcal{C}$ contained in $Q \setminus U$. By the definition of U , D does not ray-separate any pair of elements in \mathcal{C} , so U must lie in a component of $\mathbb{C} \setminus \widetilde{D}$. Let V_D denote a different component of $\mathbb{C} \setminus \widetilde{D}$. Notice that, for any $D, E \in \mathcal{C}$ such that $D \cup E \subset Q \setminus U$, $V_D \cap V_E = \emptyset$, since any point in their intersection by definition belongs to $Y_{DE} \subset U$ while $V_D \cup V_E \subset \mathbb{C} \setminus U$. Therefore, $\{V_A \mid A \in \mathcal{C}, A \not\subset U\}$ is an uncountable collection of disjoint open subsets of X , contradicting that X is a metric continuum. \square

Theorem 29. *Suppose that \mathcal{C} is an uncountable collection of pairwise-disjoint ray-continua in an unshielded continuum $Q \subset \mathbb{C}$, each of which ray-separates Q . Then a subcollection of \mathcal{C} forms a well-slicing of Q , and the finest model of Q is non-degenerate.*

Proof. By Lemma 28, without loss of generality we may assume that there are elements $\alpha_0, \alpha_1 \in \mathcal{C}$ such that all other elements of \mathcal{C} ray-separate α_0 and α_1 . Clearly, a linear order \prec is induced on \mathcal{C} , where $\beta \prec \gamma$ whenever β ray-separates α_0 and γ (for if neither ray-separates the other from α_0 , one of them does not ray-separate α_0 and α_1).

To each element $\alpha \in \mathcal{C}$ we can associate a chord ℓ_α so that this collection of chords in the unit disk is uncountable and also linearly ordered. Hence there exists an element $\alpha_{1/2}$ such that both intervals $(\alpha_0, \alpha_{1/2})_\prec$ and $(\alpha_{1/2}, \alpha_1)_\prec$ in \mathcal{C} are uncountable. By induction we can define α_q for any dyadic rational $q, 0 < q < 1$. Then the collection $\{\alpha_q\}$ with q dyadic rational is a well-slicing family. By Theorem 27, the finest model of Q is non-degenerate. \square

4 The finest model for polynomial Julia sets is dynamical

Now we show that if $Q = J_P$ is a connected polynomial Julia set then the finest map φ (which we always canonically extend onto the entire plane as explained above) semiconjugates P to a branched covering map $g : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$, which we call the *topological polynomial*. Call $\varphi(J_P)$ the *topological Julia set*. In Section 1 by a topological polynomial we understood the map f_{\sim} induced by \sim on the quotient space of a lamination \sim ; since it will always be clear whether we deal with a topological polynomial considered on J_{\sim} or we deal with its canonic extension on the entire plane, our terminology will not cause ambiguity. Recall that $\mathcal{D}_Q = \mathcal{D}$ is the finest among all upper semi-continuous partitions whose elements are unions of impressions of Q , or, as we have shown above, the family of all fibers of the finest map φ (K-sets).

Theorem 30. *For any $D \in \mathcal{D}$, $P(D) \in \mathcal{D}$.*

Proof. Beginning from the relation

$$x \sim y \iff \exists \alpha : x, y \in \text{Imp}(\alpha),$$

we examine the equivalence which is its transitive closure \sim_T (defined by $x \sim_T y$ iff there is a connected finite union of impressions containing x and y), and then the topological closure \sim_P of \sim_T in $J_P \times J_P$.

Part 1: $x \sim_P y \implies P(x) \sim_P P(y)$ (hence \sim_P -classes map *into* classes).

Let d be the degree of P and define $\sigma_d = \sigma$ as the restriction of the map $z \mapsto z^d$ to the unit circle \mathbb{S}^1 . Knowing that $x \sim y$ immediately implies that $P(x) \sim P(y)$, because of the Böttcher relation $P(\text{Imp}(\alpha)) = \text{Imp}(\sigma(\alpha))$ for any external angle α . If $x \sim_T y$, let $\alpha_1, \dots, \alpha_n$ be impressions so that $x \in \text{Imp}(\alpha_1)$, $y \in \text{Imp}(\alpha_n)$, and $\text{Imp}(\alpha_i) \cap \text{Imp}(\alpha_{i+1}) \neq \emptyset$ for $1 \leq i < n$. Since $P(\text{Imp}(\alpha_i)) = \text{Imp}(\sigma(\alpha_i))$ then intersects $P(\text{Imp}(\alpha_{i+1})) = \text{Imp}(\sigma(\alpha_{i+1}))$, we have a connected finite chain of impressions (corresponding to $\sigma(\alpha_1), \dots, \sigma(\alpha_n)$) joining $P(x)$ to $P(y)$, so $P(x) \sim_T P(y)$. Now, suppose that $x \sim_P y$. Then there exist sequences $(x_i)_{i=1}^{\infty} \rightarrow x$, $(y_i)_{i=1}^{\infty} \rightarrow y$ such that $x_i \sim_T y_i$ for all i . Then $P(x_i) \sim_T P(y_i)$, so $P(x_i) \sim_P P(y_i)$ and $P(x) \sim_P P(y)$.

Part 2: \sim_P -classes map *onto* \sim_P -classes.

That impressions map onto impressions again follows immediately from the Böttcher relation. We prove that \sim_T -classes map onto \sim_T -classes by induction. Let $x \in J_P$ and suppose that $P(x) \sim_T y$. If $P(x)$ is contained in the same impression $\text{Imp}(\alpha)$ as y , let $\beta = \frac{\alpha+k}{d}$ be chosen so that $x \in \text{Imp}(\beta)$. Then $y \in P(\text{Imp}(\beta)) = \text{Imp}(\alpha)$, so y is in the image of the \sim_T -class of x .

Now, suppose that we have proven that for any connected union A of n impressions the fact that $P(x) \in A$ implies that the \sim_T -class B of x has the P -image containing A . Let $P(x) \sim_T y$, with a collection $\alpha_1, \dots, \alpha_{n+1}$ of angles such that $P(x) \in \text{Imp}(\alpha_1)$, $y \in \text{Imp}(\alpha_{n+1})$, and $\text{Imp}(\alpha_i) \cap \text{Imp}(\alpha_{i+1})$ non-empty for $1 \leq i < n + 1$. Choose $z \in \text{Imp}(\alpha_n) \cap \text{Imp}(\alpha_{n+1})$. Then by induction the $z \in P(B)$ and we can choose $z' \in B$ such that $P(z') = z$. On the other hand, by the same argument as in the basis of induction the fact that $z, y \in \text{Imp}(\alpha_{n+1})$ implies that the P -image of the \sim_T -class of z' must contain both z and y . Since $z' \in B$, the \sim_T -class of z' is actually B , hence $y \in P(B)$ as desired: y is in the image of the \sim_T -class of x .

Now, suppose that $x \in J_P$ with $P(x) \sim_P y$. Since \sim_P is the closure of \sim_T , there exist sequences $(z_i)_{i=1}^\infty \rightarrow P(x)$ and $(y_i)_{i=1}^\infty \rightarrow y$ with $z_i \sim_T y_i$ for each i . Since P is an open map, there exists a sequence $(x_i)_{i=1}^\infty \rightarrow x$ with $P(x_i) = z_i$ [20, Theorem 13.5]. Since by the proven above \sim_T -classes map onto each other, there exists a sequence $(w_i)_{i=1}^\infty$ such that $w_i \sim_T x_i$ and $P(w_i) = y_i$. If w is any limit point of $(w_i)_{i=1}^\infty$, we see then that $x \sim_P w$ and $P(w) = y$. Since y was arbitrary, the \sim_P -class of x maps onto the \sim_P -class of $P(x)$. It remains to observe that the \sim_P -classes are exactly the elements of \mathcal{D} . \square

The next theorem follows from Theorem 30.

Theorem 31. *The map φ semiconjugates P to a branched covering map $g : \mathbb{C} \rightarrow \mathbb{C}$.*

Proof. Let $m : \mathbb{C} \rightarrow \mathbb{C}$ be the quotient map corresponding to \mathcal{D} . The map $m \circ P : \mathbb{C} \rightarrow \mathbb{C}$ is continuous, and is constant on elements of \mathcal{D} by Theorem 30. Therefore, there is an induced function $g : \mathbb{C} \rightarrow \mathbb{C}$ such that $m \circ P = g \circ m$. Also, it is easy to see that g is continuous. Indeed, let $x_i \rightarrow x$. Then $\varphi^{-1}(x_i)$ converge into $\varphi^{-1}(x)$ and $P(\varphi^{-1}(x_i))$ converge into $P(\varphi^{-1}(x))$. Applying φ to this, we see that $g(x_i) = \varphi(P(\varphi^{-1}(x_i)))$ converge to $g(x) = \varphi(P(\varphi^{-1}(x)))$, and so g is continuous.

To see that g is open, let $U \subset \mathbb{C}$ be an open set. Then $m^{-1}(U)$ is a saturated open set, and $P(m^{-1}(U))$ is a saturated open set again by Theorem 30. Therefore, $m(P(m^{-1}(U))) = g(U)$ is open. Since U was arbitrary, g is an open map. By the Stoilow Theorem [25] g is branched covering. \square

In what follows we always denote by g the topological polynomial to which P is semiconjugate; the φ -image of J_P is denoted by J_{\sim_P} . Define $\Phi : \mathbb{S}^1 \rightarrow J_{\sim_P}$ as $\Phi = \varphi \circ \text{Imp}$. From Lemma 17, Φ is a well-defined continuous function.

Theorem 32. *The map Φ semiconjugates $z \mapsto z^d$ to $g|_{J_{\sim_P}}$.*

Proof. Define $\sigma_d = z \mapsto z^d$. Recall that g is defined so that (1) $g \circ \varphi = \varphi \circ P$ and also that the Böttcher uniformization gives that (2) $P \circ \text{Imp} = \text{Imp} \circ \sigma_d$. We then see that, as desired,

$$\begin{aligned}
g \circ \Phi &= g \circ \varphi \circ \text{Imp} \\
&= \varphi \circ P \circ \text{Imp} && \text{(by (1))} \\
&= \varphi \circ \text{Imp} \circ \sigma_d && \text{(by (2))} \\
&= \Phi \circ \sigma_d.
\end{aligned}$$

□

Then, as in the previous section, the *finest lamination* corresponding to J_{\sim_P} is the equivalence relation \sim_P on \mathbb{S}^1 , defined by $\alpha_1 \sim_P \alpha_2$ if and only if $\Phi(\alpha_1) = \Phi(\alpha_2)$.

5 A criterion for the polynomial Julia set to have a non-degenerate finest monotone model

Here we obtain the remaining main results of the paper. We give a necessary and sufficient condition for the existence of a non-degenerate locally connected model of the connected Julia set of a polynomial P . We give this criterion in terms of its rational lamination as well as the existence of specific wandering continua in the Julia set behaving in the fashion reminiscent of the irrational rotation on the unit circle.

5.1 Topological and laminational preliminaries

We need the following definition. A finite set A is said to be *all-critical* if $\sigma(A)$ is a singleton. A finite set B is said to be *eventually all-critical* if there exists a number n such that $\sigma^n(A)$ is a singleton. The following result is obtained in [7].

Theorem 33. *Suppose that J_P is the connected Julia set of a polynomial P such that its locally connected model J_{\sim} corresponding to the lamination $\sim = \sim_P$ is a dendrite. Then there are infinitely many periodic cutpoints of J_{\sim} and, respectively, \sim -classes, each of which consists of more than one point.*

We will also need another result from [7].

Theorem 34. *Suppose $f : \mathbb{C} \rightarrow \mathbb{C}$ is a holomorphic map, X is a non-separating continuum or a point such that $f(X) \subset X$, all fixed points in X are repelling or parabolic, and for each fixed point $x_i \in X$ there exists an external ray R_i of X , landing at x_i , such that $f(R_i) = R_i$. Then X is a single point.*

Theorem 33 applies in the proof of Theorem 35. Define an *all-critical* point as a cutpoint of J_\sim whose image is an endpoint of J_\sim .

Theorem 35. *Suppose \sim is a lamination. Then at least one of the following properties must be satisfied:*

- (1) J_\sim contains the boundary of a paratracting Fatou domain;
- (2) there are infinitely many periodic \sim -classes each of which consists of more than one angle;
- (3) there exists a finite collection of all-critical \sim -classes with pairwise disjoint grand orbits whose images under the quotient map form the set of all-critical points on the boundaries of Siegel domains from one cycle of Siegel domains so that all cutpoints of J_\sim on the boundaries of these Siegel domains belong to the grand orbits of these all-critical points.

Proof. Suppose that J_\sim is a dendrite. Then the result follows from Theorem 33. Suppose now that J_\sim is not a dendrite. Then J_\sim contains a simple closed curve S . By Lemma 4, there are two cases possible. First, we may assume that S is the boundary of a periodic paratracting Fatou domain. Then (1) holds.

Consider the case when S is of period 1 and $f_\sim|_S$ is conjugate to an irrational rotation (the case of higher period is similar). Consider a point $x \in S$ which is a cutpoint of J_\sim (x must exist since $S \neq J_\sim$). Then x is not (pre)periodic. Hence by Theorem 3 the \sim -class corresponding to x is finite. This implies that the number of components of $J_\sim \setminus \{x\}$ is finite. One such component contains $S \setminus \{x\}$ while all others have closures intersecting S exactly at x . Denote by B_x the union of x and all such components not containing points of S . Clearly the set B_x is closed and connected.

Let us show that x is eventually mapped into a point which is not a cutpoint of J_\sim . Indeed, otherwise all points $f_\sim^i(x)$ are cutpoints of J_\sim . Since there are finitely many critical points of f_\sim , we can then choose N such that no set $B_{f_\sim^m(x)}$ contains a critical point for $m \geq N$. On the other hand, $f_\sim^N(x)$ is a cutpoint of J_\sim by the above. Hence $B_{f_\sim^N(x)}$ is a wandering continuum in J_\sim , a contradiction with Theorem 5. Now the connection between \sim -classes and points of J_\sim implies that the \sim -class corresponding to x is eventually all-critical. Clearly, any all-critical point $y \in S$ corresponds to an all-critical \sim -class which meets the boundary of the corresponding Siegel domain U of \sim in a leaf (since \sim -classes are convex). Moreover, an all-critical point in S is a cutpoint of J_\sim whereas all forward images of an all-critical point are

endpoints of J_\sim . Hence the all-critical classes which are non-disjoint from \overline{U} have pairwise disjoint grand orbits. Clearly, this implies the properties listed in the case (3) of the theorem. Similar arguments go through if S is periodic rather than fixed. \square

By Theorem 2, if the finest model is not degenerate then it gives rise to a non-degenerate lamination \sim_P . Hence one of the three phenomena described in Theorem 35 will have to take place in J_{\sim_P} . Thanks to the existence of the finest map, this implies that corresponding phenomena will be present in the Julia set J_P . In other words, the presence of at least one of the phenomena is a *necessary* condition for the existence of a non-degenerate finest model (we will formalize this observation later on in Theorem 44). However now our main aim is to show that the presence of at least one of the phenomena is *sufficient* for the existence of a non-degenerate finest model. The main tool here is well-slicing studied in Subsection 3.2. We will describe three cases in which we establish sufficient conditions for the existence of well-slicing for the Julia set (and hence, by the results from Subsection 3.2, for the non-degeneracy of its finest model). The sufficient conditions are stated in a step by step fashion in a series of lemmas and propositions.

5.2 The case of infinitely many periodic cutpoints

Next we want to suggest a sufficient condition for the non-collapse of the entire J_P corresponding to the case (2) of Theorem 35. However this time we need a lot of preparatory work. First we study CS-points and CS-cycles (recall that a *CS-point* is a Siegel or Cremer periodic point and a *CS-cycle* is a cycle of CS-points). Call a set X *periodic (of period m)* if $X, \dots, P^{m-1}(X)$ are pairwise disjoint while $P^m(X) \subset X$. Then the union $\cup_{i=0}^{m-1} P^i(X)$ is said to be a *cycle of sets* (we can then talk about *cycles of continua* and the like).

Lemma 36. *If Y is a cycle of continua containing a CS-cycle and a periodic point not from this CS-cycle then it contains a critical point of P .*

Proof. We only consider the case when Y is an invariant continuum; the case of the cycle of continua can be dealt with similarly. Suppose that Y contains no critical points. Choose a neighborhood U of Y such that no critical points belong to \overline{U} , consider the set of all points never exiting \overline{U} , and then the component K of this set containing the given CS-point p ; clearly, $Y \subset K$. Such sets are called *hedgehogs* (see [21,22]) and have a lot of important properties. In particular, K cannot contain any other periodic points. On the other hand, $Y \subset K$, a contradiction with the assumption that there is a periodic point in Y , distinct from p . \square

Next we prove a few lemmas which discuss properties of J_P at (pre)periodic points. We need them for two reasons. First of all, they help us establish the next sufficient condition for the non-collapse of J_P under the finest map. Second, they give sufficient conditions on a (pre)periodic point to be a point of local connectivity of the Julia set. In that sense they generalize Kiwi's theorem [14] where he proves (using different methods) that in the absence of CS-points the Julia set is locally connected at its (pre)periodic points.

There are two competing laminations which both reflect the structure of J_P , the rational lamination \approx_{rat} and the finest lamination \sim_P . We use both of them to suggest sufficient conditions for J_P to be locally connected at a (pre)periodic point p . Recall that $A(y)$ is the set of all angles whose rays land at a point $y \in J_P$.

Lemma 37. *Suppose that $K = \text{Imp}(A)$ is the union of impressions of a finite set of periodic angles A which is periodic, connected and disjoint from impressions of all other angles. Then K is a repelling or parabolic periodic point. Thus, if p is a P -(pre)periodic point and $\Phi^{-1}(\varphi(p))$ is finite then $\varphi^{-1}(\varphi(p)) = \{p\}$ (i.e., $\{p\}$ is a K -set) and J_P is locally connected at p .*

Proof. If K contains a paratracting Fatou domain in its topological hull, then infinitely many periodic repelling points on its boundary (which exist by [23]) would give rise to infinitely many impressions non-disjoint from K , a contradiction. Let us show that the topological hull $\text{TH}(K)$ of K cannot contain a CS-point either. Indeed, otherwise by Lemma 36 it has to contain a critical point c . Moreover, since $\text{TH}(K)$ does not contain paratracting Fatou domains, $c \in J_P$. Then the symmetry around critical points implies that there are two angles in A which map into one (recall that the only angles whose impressions may contain c are the angles of A), a contradiction with the assumptions on A . Thus K is an invariant non-separating subcontinuum of J_P which contains no CS-points. On the other hand, by the assumptions, there are only finitely many periodic points in K . Hence all of these points are repelling or parabolic and together with the rays landing at them may be assumed to be fixed. By Theorem 34 this implies that K is a repelling or parabolic periodic point.

To establish the next implication of the lemma, assume that p is a P -periodic point and $\varphi(p) = x$. We may assume that x is g -fixed. Set $A = \Phi^{-1}(x)$. By the assumptions of the lemma $\Phi^{-1}(\varphi(p))$ is finite. Hence we may assume that all angles in A are fixed. Clearly, $B = \varphi^{-1}(x)$ is an invariant continuum. By the assumptions only angles of A can have impressions non-disjoint from B , and there are finitely many of them. Hence by the first part of the lemma B is a repelling or parabolic periodic point. The remaining claim that J_P is locally connected at p now follows from Lemma 19. \square

The next lemma relies upon Lemma 37. Recall, that by \approx_{rat} we denote the finest lamination which respects the geometric lamination \mathcal{L}_{rat} . Properties of \approx_{rat} are studied in Theorem 12 (in particular, it is shown there that \approx_{rat} is not degenerate).

Lemma 38. *Suppose that \mathbf{g} is a (pre)periodic finite gap or leaf of \approx_{rat} disjoint from boundaries of Fatou domains of \approx_{rat} . Then $\text{Ch}(\mathbf{g})$ is a gap or leaf of \mathcal{L}_{rat} , coinciding with the set $A(p)$ for a point $p \in R$, and J_P is locally connected at p .*

Proof. We may assume that \mathbf{g} is invariant. Also, since no leaf of $B = \text{Bd}(\text{Ch}(\mathbf{g}))$ can come from the boundary of a Fatou domain of \approx_{rat} , all leaves in B are limit leaves of \mathcal{L}_{rat} . The upper semi-continuity of impressions now implies that the union of impressions $\text{Imp}(\mathbf{g})$ of angles of \mathbf{g} is a continuum itself. Moreover, it is disjoint from impressions of all angles not in \mathbf{g} because for any such angle γ we can find a leaf of \mathcal{L}_{rat} which cuts $\text{Imp}(\gamma)$ off $\text{Imp}(\mathbf{g})$. Now the lemma follows from Lemma 37. \square

We need another preparatory result, dealing with laminations generated by collections of periodic gaps and leaves. If we fix a set A , then a set $B \subset A$ is said to be *cofinite (in A)* if $|A \setminus B|$ is finite. Given a generating family A of pairwise disjoint periodic gaps or leaves we then consider a geometric prelamination \mathcal{L}_A consisting of A and preimages of elements of A . By Theorem 12 we can construct the corresponding lamination \approx_A ; given a finite gap or leaf G from \mathcal{L}_A , there exists a finite \approx_A -class $\text{Cl}(G)$ containing G and called the \approx_A -class *generated* by G . Denote by p_A the quotient map from \mathbb{S}^1 to J_{\approx_A} .

Lemma 39. *Suppose that A is an infinite generating family of periodic gaps or leaves. Then there exists a cofinite subset $D' \subset A$ such that any cofinite set $E \subset D'$ has the following properties.*

- (1) *If $G \in \mathcal{L}_E$ then $\text{Cl}(G) \cap \text{Bd}(U) = \emptyset$ for any Fatou domain U in \mathcal{L}_{\approx_E} or \mathcal{L}_{\approx_A} .*
- (2) *The family of \approx_E -classes generated by the elements of \mathcal{L}_E is a well-slicing family of \mathbb{S}^1 .*

Proof. Clearly, A is countable. Denote the number of cycles of Fatou domains of the lamination \approx_A by f_A . Clearly, if $B \subset A$ then $\overline{\mathcal{L}_B} \subset \overline{\mathcal{L}_A}$. Hence by Theorem 12 each Fatou domain of \approx_A is contained in a Fatou domain of \approx_B . However there may exist cycles of Fatou domains of \approx_B disjoint from cycles of Fatou domains of \approx_A ; in other words, Fatou domains of \approx_B may appear in new places. Still, f_B is not necessarily greater than f_A because simultaneously with the above some Fatou domains of \approx_B may contain several Fatou domains of \approx_A . Now, the collection of Fatou domains of \approx_A and those Fatou domains

of \approx_B whose cycles are disjoint from the cycles of Fatou domains of \approx_A is a collection of several Fatou domains which are pairwise disjoint (recall, that Fatou domains are open). These Fatou domains are Fatou domains of two distinct laminations, \approx_A and \approx_B , but since they are disjoint, they can be considered together. This useful idea, which is an important ingredient of the arguments, motivates the following definition. Take a cofinite subset $B \subset A$ and consider all *intermediate* sets A' such that $A \supset A' \supset B$. Consider the family of orbits of all Fatou gaps of all laminations $\approx_{A'}$. There exists a maximal in terms of cardinality subcollection of this family of pairwise disjoint orbits of Fatou gaps; denote its cardinality by F_B .

Since $F_B < d$, we may choose a cofinite subset $B \subset A$ such that F_B is maximal possible. Clearly, any Fatou domain generated by an intermediate set A' is contained in a Fatou domain of B . Let us show that then, by the choice of B , for any cofinite subset $C \subset B$ any Fatou domain of \approx_C must contain a Fatou domain of B . Indeed, otherwise there exists a cycle of Fatou domains generated by C which contains no Fatou domains of any of the intermediate sets A' . Therefore, this cycle of Fatou domains can be added to the F_B cycles of Fatou domains, contradicting the maximality of F_B . In other words, Fatou gaps of C cannot pop up at new spots.

Denote the Fatou domains of \approx_B by U_1, \dots, U_k and choose n so that all U_i are σ^n -invariant. By the above, for any cofinite set $C \subset B$ a Fatou domain of \approx_C is σ^n -invariant and contains U_i for some i . Choose a cofinite set $C \subset B$ so that the number of its Fatou domains is minimal possible and denote them W_1, \dots, W_m . Then for any cofinite set $D \subset C$ there are m Fatou domains $V_1(D), \dots, V_m(D)$ of \approx_C each of which is σ^n -invariant. Now, choose a cofinite set $D \subset C$ so that the sum of degrees of $\sigma^n|_{V_i(D)}$ taken over $i = 1, \dots, m$ is maximal possible. Let us show that then for any cofinite set $E \subset D$ the Fatou domains of \approx_E coincide with Fatou domains of D . Indeed, otherwise we may assume that there exists a cofinite set $E \subset D$ which has a Fatou domain $Q \not\supseteq V_1(D)$. Then the degree of $\sigma^n|_Q$ is greater than that of $\sigma^n|_{V_1(D)}$ because otherwise *all* the preimages of points of $V_1(D)$ under $\sigma^n|_Q$ belong to $V_1(D)$ while being dense in Q , a contradiction with $Q \not\supseteq V_1(D)$. However then the sum of the degrees of $\sigma^n|_{V_i(E)}$ is greater than the sum of degrees of $\sigma^n|_{V_i(D)}$, a contradiction with the choice of D . We conclude that for any cofinite set $E \subset D$ the Fatou domains of \approx_E coincide with Fatou domains of \approx_D .

Each periodic Fatou domain of \approx_D has finitely many \approx_D -gaps/leaves non-disjoint from its boundary. Denote by $D' \subset D$ the family of all *other* \approx_D -gaps/leaves (observe that D' is cofinite and hence infinite). Suppose now that $E \subset D'$ is cofinite. Then by the previous paragraph the Fatou domains of \approx_E coincide with Fatou domains of \approx_D , and by the choice of E no element of \mathcal{L}_E intersects $\text{Bd}(U)$ where U is a Fatou domain of \mathcal{L}_{\approx_E} . Moreover, since $E \subset A$, $\overline{\mathcal{L}_E} \subset \overline{\mathcal{L}_A}$. Hence, Fatou domains of \mathcal{L}_{\approx_A} are contained in Fatou domains of

\mathcal{L}_{\approx_E} . This proves claim (1) of the lemma. To prove claim (2), let $G, H \in \mathcal{L}_E$ are such that $\text{Cl}(G)$ and $\text{Cl}(H)$ are distinct. Suppose that there are no \approx_E -classes, generated by elements of \mathcal{L}_E , separating \mathbb{S}^1 between $\text{Cl}(G)$ and $\text{Cl}(H)$. Since by the construction \approx_E -classes generated by elements of \mathcal{L}_E are dense in \mathcal{L}_{\approx_E} , there must be a Fatou gap of \mathcal{L}_{\approx_E} on whose boundary both $\text{Cl}(G)$ and $\text{Cl}(H)$ lie which is impossible by the above. This completes the proof of (2). \square

Proposition 40. *Suppose that $p \in J_{\sim_P} = \varphi(J_P)$ is a periodic point such that $\Phi^{-1}(p)$ is infinite. Then there are no more than finitely many periodic leaves of the rational geometric lamination \mathcal{L}_{rat} connecting angles of $\Phi^{-1}(p)$. In particular, (1) the set of all bi-accessible periodic repelling or parabolic points in $\varphi^{-1}(p)$ must be finite, and (2) if the set of **all** repelling bi-accessible periodic points of P is infinite then the finest model is non-degenerate.*

Proof. We may assume that p is a fixed point of g ; then $\Phi^{-1}(p)$ is an infinite gap of \sim_P . Set $G = \text{int}(\text{Ch}(\Phi^{-1}(p)))$; by Lemma 10 G is a Fatou gap of \mathcal{L}_{\sim_P} and hence by Lemma 9 there is a monotone semiconjugacy ψ of $\sigma^*|_{\text{Bd}(G)}$ and a map $\sigma_k : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ with the appropriately chosen $k > 1$. The map ψ collapses all chains of concatenated leaves in $\text{Bd}(G)$ to points; by Lemma 7 all leaves in the chains are (pre)periodic and by Lemma 9 and Lemma 10 each chain consists of at most N leaves (N depends on G). By way of contradiction suppose that there are infinitely many periodic leaves of the rational prelamination \mathcal{L}_{rat} connecting angles of $\Phi^{-1}(p)$. The idea is to use the map ψ in order to transport the restriction of \mathcal{L}_{rat} onto $\Phi^{-1}(p)$ to the entire circle \mathbb{S}^1 , then to use Lemma 39 to find a well-slicing family of \mathbb{S}^1 consisting of (pre)periodic geometric gaps and leaves of \mathbb{S}^1 corresponding to elements of \mathcal{L}_{rat} , and then to show that ray-continua corresponding those elements of \mathcal{L}_{rat} form a well-slicing family of $\varphi^{-1}(p)$. By Theorem 27 then $\varphi(\varphi^{-1}(p))$ is not a point, a contradiction.

The leaves of \mathcal{L}_{rat} which lie in the boundary of $\text{Ch}(\Phi^{-1}(p)) = G$ will produce just points under ψ . However, by Lemma 7 there are only finitely many periodic leaves in $\text{Bd}(G)$. Hence by the assumptions of the proposition there are infinitely many periodic geometric leaves or gaps of \mathcal{L}_{rat} contained in G and such that ψ does not identify points of their bases with other points at all. Denote their family by A ; also, denote the family of all their preimages under all powers of σ contained in G by $\widehat{\mathcal{L}}_A$ (recall, that the notation \mathcal{L}_A is reserved for the collection of *all* preimages of elements of A). Thus, $\widehat{\mathcal{L}}_A$ is the family of all (pre)periodic geometric leaves and gaps of \mathcal{L}_{rat} contained in G and not in $\text{Bd}(G)$. Define the geometric prelamination $\mathcal{L}' = \psi(\widehat{\mathcal{L}}_A)$ on the entire circle \mathbb{S}^1 as the family of convex hulls of ψ -images of bases of elements of $\widehat{\mathcal{L}}_A$ (recall that ψ is defined only on $\text{Bd}(G)$). It is easy to see that this indeed creates a geometric prelamination whose all leaves are (pre)periodic. By the choice of A in this way each gap/leaf of $\mathcal{L} = \widehat{\mathcal{L}}_A$ is transported by ψ to the corresponding

gap/leaf of \mathcal{L}' in a one-to-one fashion. Then $\psi(A)$ is the family of periodic geometric leaves and gaps of \mathcal{L}' . Clearly, $\psi(A)$ is infinite and the lamination \mathcal{L}' is the same as the lamination $\mathcal{L}_{\psi(A)}$ introduced right before Lemma 39 in which appropriate preimages of elements of $\psi(A)$ are used.

By Lemma 39 there exists a cofinite family $B \subset A$ satisfying both properties listed in Lemma 39. In particular, as in Lemma 39 for B the prelamination \mathcal{L}_B and the corresponding lamination \approx_B can be constructed. By the choice of A the map ψ then allows us to pull them back to G in a one-to-one fashion and without changing the order. Now, by claim (1) of Lemma 39 if $h \in \mathcal{L}_B$ then $\text{Cl}(h) \cap \text{Bd}(U) = \emptyset$ for any Fatou domain U in \mathcal{L}_{\approx_B} (here $\text{Cl}(h)$ is understood in the sense of the lamination \approx_B , i.e. $\text{Cl}(h)$ is the \approx_B -class containing h). Let us show that then in fact $h = \text{Cl}(h)$ and $\psi^{-1}(h) \in \mathcal{L}_{rat}$. Indeed, consider leaves on the boundary of $\text{Cl}(h)$. By Theorem 12 they all are limits of leaves of elements of \mathcal{L}_B . It follows that then leaves on the boundary of $\psi^{-1}(\text{Cl}(h))$ are limit leaves for leaves of ψ -preimages of elements of \mathcal{L}_B . Thus, leaves on the boundary of $\psi^{-1}(\text{Cl}(h))$ are limit leaves for leaves of \mathcal{L}_{rat} . This implies that the impression of any angle not from $\psi^{-1}(\text{Cl}(h))$ is cut off $\text{Imp}(\psi^{-1}(\text{Cl}(h)))$ by tails of the appropriate points of R and hence is disjoint from $\text{Imp}(\psi^{-1}(\text{Cl}(h)))$. By Lemma 37 then $h = \text{Cl}(h)$ and $\psi^{-1}(h) \in \mathcal{L}_{rat}$.

Now, by Lemma 39 \mathcal{L}_B is a well-slicing family of \mathbb{S}^1 . By the previous paragraph and by the properties of the map ψ it follows that the family of degenerate ray-continua $\text{Imp}(\psi^{-1}(h)), h \in \mathcal{L}_B$ is a well-slicing family of $\varphi^{-1}(p)$ and hence by Theorem 27 $\varphi(\varphi^{-1}(p))$ is not a point, a contradiction. This proves (1). Now, if the finest model is degenerate then the degenerate topological Julia set can play the role of the point p , the entire circle \mathbb{S}^1 plays the role of the \approx_p -class $\Phi^{-1}(p)$, and (1) implies that R is finite. Hence, (2) follows and the proof is completed. \square

5.3 The Siegel case

Now we establish the third sufficient condition for the non-degeneracy of the finest model, this time corresponding to the case (3) of Theorem 35. However first we need to introduce the appropriate terminology.

As was explained in Section 2, the closure of any invariant geometric prelamination is a geometric lamination. This idea was used when the geo-lamination $\overline{\mathcal{L}_{rat}}$ was constructed. However it can also be used in other situations. Suppose that there exists a finite collection \mathcal{K} of wandering ray-continua $K_i, i = 1, \dots, m$. We will call \mathcal{K} a *wandering collection* if distinct forward images of continua K_i are all pairwise disjoint. By the arguments similar those from Theorem 4.2 [6] one can associate to \mathcal{K} a geometric prelamination $\mathcal{L}_{\mathcal{K}}$ gen-

erated by \mathcal{K} , and then its closure - a geo-lamination $\overline{\mathcal{L}_{\mathcal{K}}}$ generated by \mathcal{K} . For completeness we will briefly explain the main ideas of this theorem.

First we need to construct the grand orbit of sets from \mathcal{K} . However it may happen that simply taking pullbacks of the forward images of these sets will lead to their growth. Indeed, suppose, for example, that K_1 contains a critical point c . Then already the first pullback of $P(K_1)$ may well be bigger than K_1 . If as we iterate the map K_1 hits several critical points, the same can take place several times. However since \mathcal{K} is a wandering collection we can choose a big N so that the continua $P^N(K_i), i = 1, \dots, m$ are non-precritical.

If we now take these continua, all their forward images, and then all pullbacks of these forward images we will get a “consistent” grand orbit of several sets meaning that for every set Q from the grand orbit in question the P^i -pullback of $P^i(Q)$ containing Q coincides with Q . As a result of the construction the initially given ray-continua may have grown, however they will have (eventually) the same images as the originally given continua. In particular, the continua K_i may have grown to new continua K'_i , and we can think of the just constructed grand orbit Γ as the grand orbit of the family of continua K'_1, \dots, K'_m . Observe that $\mathcal{K}' = \{K'_1, \dots, K'_m\}$ is still a wandering collection. Hence, since all Fatou domains are (pre)periodic, any set from Γ is a non-separating subcontinuum of J_P .

Since each K_i is a ray-continuum, by Definition 20 there is a set of angles associated to K_i in that the union of the principal sets of these angles is contained in K_i while the union of their impressions contains K_i . The new continuum K'_i is obtained as the union of K_i with some pullbacks of its images. Hence and because the collection of all principal sets and impressions is invariant we see that K'_i is also a ray-continuum. It follows that in fact any continuum $K' \in \mathcal{K}'$ is a ray-continuum, and if we define the set of angles $\Theta(K') = H_{K'}$ as the set of all angles whose principal sets are contained in K' then we will have

$$\bigcup_{\theta \in H_{K'}} \text{Pr}(\theta) \subset K' \subset \bigcup_{\theta \in H_{K'}} \text{Imp}(\theta)$$

which means that the set of angles $H_{K'}$ is associated with the continuum K' in the sense of the Definition 20. Observe that by Theorem 3 the sets of angles $H_{K'}, K' \in \mathcal{K}'$ cannot have more than 2^d angles (and therefore they are closed).

Now it is not hard to show (see Theorem 4.2 [6]) that the family of convex hulls of so defined sets of angles $H_{K'}, K' \in \Gamma$ form a geometric prelamination which we denote by $\mathcal{L}_{\mathcal{K}}$. By the definition each original ray-continuum K_i has the associated to it set of angles A_i , and it follows that $A_i \subset H_{K'_i} = H_i$. Therefore each A_i is contained in a leaf or gap of $\mathcal{L}_{\mathcal{K}}$. Then the closure $\overline{\mathcal{L}_{\mathcal{K}}}$ of $\mathcal{L}_{\mathcal{K}}$ is a geo-lamination. We are especially interested in collections of angles

which give rise, through the above construction, to specific geo-laminations reminiscent of the case (3) of Theorem 35.

Definition 41. Suppose that \mathcal{H} is a collection of finite sets of angles $H_i, i = 1, \dots, m$ such that the following holds.

- (1) Each set H_i is mapped into a one-angle set (i.e., is an *all-critical set*).
- (2) For each i the set $\text{Imp}(H_i)$ is a continuum disjoint from impressions of any angle not belonging to H_i .
- (3) The continua $\text{Imp}(H_i), i = 1, \dots, m$ form a wandering collection.
- (4) Consider the geo-lamination $\overline{\mathcal{L}_{\mathcal{H}}}$. Then there is a cycle of Siegel domains in $\overline{\mathcal{L}_{\mathcal{H}}}$ such that \mathcal{H} is the family of all-critical gaps/leaves on the boundaries of domains from the cycle. Moreover, each $\text{Ch}(H_i)$ meets the corresponding Siegel domain of $\overline{\mathcal{L}_{\mathcal{H}}}$ in a leaf and sets H_i have pairwise disjoint orbits.

In that case we say that the collection of sets of angles \mathcal{H} with their impressions and *all* their pullbacks form a *Siegel configuration*; the collection of sets of angles \mathcal{H} is said to *generate* the corresponding Siegel configuration. We will also say in this case that P *admits a Siegel configuration*.

The next proposition shows that such Siegel configuration cannot be admitted by the polynomial inside a periodic infinite K-class; in particular, if P admits a Siegel configuration, it implies that the finest model is non-degenerate.

Proposition 42. *Suppose that $p \in J_{\sim P} = \varphi(J_P)$ is a periodic point such that $\Phi^{-1}(p) = \mathbf{g}$ is infinite. Then no collection of subsets of \mathbf{g} generates a Siegel configuration. In particular, if P admits a Siegel configuration, then the finest model is non-degenerate.*

Proof. By way of contradiction let us assume that P admits a Siegel configuration, and the corresponding generating collection of sets of angles is $\mathcal{H} = \{H_1, \dots, H_m\}$. Set $\text{Imp}(H_i) = T_i$. First we simply analyze the corollaries of this assumption without assuming that sets from \mathcal{H} are contained in a periodic K-class.

We may assume that all sets H_i have common leaves with an *invariant* Siegel domain S . By Lemma 9 the map $\sigma^*|_{\text{Bd}(S)}$ is semiconjugate with an irrational rotation of the circle. Then there are no periodic leaves/points in $\text{Bd}(S)$ and by Lemma 7 every leaf $\ell \subset \text{Bd}(S)$ is (pre)critical. By Lemma 8 ℓ is not a limit leaf, hence ℓ belongs to an element Q of the grand orbit of \mathcal{H} . From part (4) of Definition 41 $Q \cap \text{Bd}(S) = \ell$. Since grand orbits of sets H_i are pairwise disjoint, all images of ℓ are two-sided limit points of $\text{Bd}(S) \cap \mathbb{S}^1$. Observe that there might exist chains of concatenated leaves in $\text{Bd}(S)$ (they may arise as a result of pulling back a set H_i through a critical gap on the boundary of

S). However by Lemma 9 any maximal chain of leaves in $\text{Bd}(S)$ consists of no more than N leaves with some uniform N . Points of $\text{Bd}(S)$ which are not contained in any leaf are angles whose impressions are also continua. Let us denote by \mathcal{A} the collection of elements of the grand orbit of \mathcal{H} non-disjoint from $\text{Bd}(S)$ as well as points in $\text{Bd}(S)$ which do not belong to leaves. Then all elements of \mathcal{A} have connected impressions.

Suppose now that $A, B \in \mathcal{A}$. Choose the arc $I \subset \text{Bd}(S)$ which contains $A \cap \text{Bd}(S), B \cap \text{Bd}(S)$ and runs in a counterclockwise direction from $A \cap \text{Bd}(S)$ to $B \cap \text{Bd}(S)$. Consider the union $T = T(A, B)$ of all elements of \mathcal{A} non-disjoint from I . Clearly, T is connected.

Claim A. *The set $\text{Imp}(T)$ is a continuum.*

It follows from the upper semi-continuity of impressions that $\text{Imp}(T)$ is closed. By way of contradiction suppose that $\text{Imp}(T) = X \cup Y$ where X, Y are disjoint non-empty closed sets. Since for every $Q \in \mathcal{A}$ such that $Q \subset T$ we have that the set $\text{Imp}(Q)$ is a continuum, every such Q has its impression either in X or in Y . Denote by X' the union of all elements of \mathcal{A} contained in T whose impressions are contained in X ; then X' is well-defined and disjoint from the union Y' of all elements of \mathcal{A} contained in T whose impressions are contained in Y . Now, by the upper semi-continuity of impressions the sets X', Y' are closed (every limit set of X' still comes from T and has its impression in X), and by the above they are disjoint and non-empty. However $X' \cup Y' = T$ is connected, a contradiction. This implies that $\text{Imp}(\mathcal{A})$ is a continuum.

Claim B. *Impressions of two distinct elements A, B of \mathcal{A} do not meet. The continuum $\text{Imp}(\mathcal{A})$ separates the plane.*

Indeed, suppose otherwise. Choose a set $H_1 \in \mathcal{H}$. Then $H_1 \cap \text{Bd}(S)$ is a leaf. By Lemma 9, there exists a sequence m_i such that $\sigma^{m_i}(A)$ will approach an endpoint of $H_1 \cap \text{Bd}(S)$ while $\sigma^{m_i}(B)$ will approach a point $y \in S'$. Now, $y \notin H_1$ because A is distinct from B and because the map σ on $\text{Bd}(S)$ acts like an irrational rotation. On the other hand, by the assumption $\text{Imp}(A) \cap \text{Imp}(B) \neq \emptyset$, hence by the upper semi-continuity of impressions $\text{Imp}(y) \cap \text{Imp}(H_1) \neq \emptyset$, a contradiction with the part (2) of Definition 41. Hence elements of \mathcal{A} have pairwise disjoint impressions. It implies that $\text{Imp}(\mathcal{A})$ separates the plane because otherwise by [1,10] $\text{Imp}(\mathcal{A})$ would contain a fixed point, and then an element of \mathcal{A} containing it and its image would have non-disjoint impressions, a contradiction.

Claim C. *The union of two impressions of distinct angles - images of elements of \mathcal{H} - ray-separates $\text{Imp}(\mathcal{A})$.*

Consider $\{\alpha\}, \{\beta\} \in \mathcal{A}, \alpha \neq \beta$, both α and β images of sets from \mathcal{H} which are non-isolated from either side in $\text{Bd}(S) \cap \mathbb{S}^1$ (we can do this by what we showed in the second paragraph of the proof). We need to show that if $Q = \text{Imp}(\alpha) \cup \text{Imp}(\beta)$ then $\text{Imp}(\mathcal{A})$ meets two distinct components of $\overline{U_\infty} \setminus \tilde{Q}$ (U_∞ is the basin of attraction of infinity, Q is a ray-compactum with the associated set of angles $\{\alpha, \beta\}$, and by \tilde{Q} we denote the union of Q and rays R_α, R_β , see Section 3 where this notation is introduced). Consider the union V of rays of all angles from $[\alpha, \beta]$ and the union W of rays of all angles from $[\beta, \alpha]$. Clearly, $V \cap W = R_\alpha \cup R_\beta$ and $V \cup W = U_\infty$. Also, it follows that $\overline{V} = V \cup \text{Imp}([\alpha, \beta])$ and $\overline{W} = W \cup \text{Imp}([\beta, \alpha])$.

Let us show that $\overline{V} \cap \overline{W} = \tilde{Q}$. It suffices to show that if $T' = \text{Imp}([\alpha, \beta]) \setminus Q$ and $T'' = \text{Imp}([\beta, \alpha]) \setminus Q$ then $T' \cap T'' = \emptyset$. Observe that by Claim B and by the choice of α, β we have that $\text{Imp}(\alpha)$ is disjoint from impressions of all angles not equal to α , and $\text{Imp}(\beta)$ is disjoint from impressions of all angles not equal to β . Hence it suffices to show that if $\gamma' \in (\alpha, \beta)$ and $\gamma'' \in (\beta, \alpha)$ then $\text{Imp}(\gamma') \cap \text{Imp}(\gamma'') = \emptyset$. By Claim B we may assume that at least one of the angles γ', γ'' (say, γ') does not belong to an element of \mathcal{A} . Then there exists a non-degenerate element L of \mathcal{A} such that $L \cap \mathbb{S}^1 \subset (\alpha, \beta)$ and γ' is contained in an arc $(\theta_1, \theta_2) \subset (\alpha, \beta)$ where $\theta_1, \theta_2 \in L$. This implies that $\text{Imp}(\gamma')$ is contained in the union of rays $R_{\theta_1}, R_{\theta_2}$ and the impression $\text{Imp}(L)$ of L . If γ'' belongs to $H \in \mathcal{A}$, put $M = H$ and $\theta_3 = \theta_4 = \gamma''$. Otherwise, there exists a set $M \in \mathcal{A}$ such that $M \cap \mathbb{S}^1 \subset (\beta, \alpha)$ and γ'' is contained in an arc $(\theta_3, \theta_4) \subset (\beta, \alpha)$ where $\theta_3, \theta_4 \in M$. Then $\text{Imp}(\gamma'')$ is contained in the union of rays $R_{\theta_3}, R_{\theta_4}$ and the impression $\text{Imp}(M)$ of M . Since by Claim B $\text{Imp}(L) \cap \text{Imp}(M) = \emptyset$, it follows that $\text{Imp}(\gamma') \cap \text{Imp}(\gamma'') = \emptyset$ as desired. Observe that $\overline{U_\infty} \setminus \tilde{Q} = (\overline{V} \setminus \tilde{Q}) \cup (\overline{W} \setminus \tilde{Q})$ where sets $\overline{V} \setminus \tilde{Q}$ and $\overline{W} \setminus \tilde{Q}$ are open in $\overline{U_\infty}$ and disjoint which proves the claim.

Let us now prove the theorem. Observe that by Claim A the set $\text{Imp}(A)$ is a continuum. Denote by \mathcal{Z} the family of impressions of singletons from \mathcal{A} which are angles-images of elements of \mathcal{H} . By Lemma 9 the map ψ semiconjugates $\sigma^*|_{\text{Bd}(S)}$ to an irrational rotation τ of \mathbb{S}^1 . This map allows us to associate to elements of \mathcal{Z} their ψ -images which are angles in \mathbb{S}^1 coming from a finite collection of orbits under τ . Choose pairs of angles from $\psi(\mathcal{Z})$ so that \mathbb{S}^1 with them is homeomorphic to \mathbb{S}^1 with the vertical collection of pairs $\mathcal{C}_{\mathbb{S}^1}$. This gives rise to the corresponding family of pairs of impression from \mathcal{Z} . By Claim C and by the construction these pairs of impressions form a well-slicing family of $\text{Imp}(A)$. Therefore by Theorem 27 $\varphi(\text{Imp}(A))$ is not a point. On the other hand, by the construction $\text{Imp}(A) \subset \varphi^{-1}(p)$, a contradiction. \square

5.4 The criterion

First we deal with paratracting Fatou domains. This sufficient condition for the non-collapse of a subset of J_P corresponds to case (1) of Theorem 35. Let us recall that by R we denote the set of all periodic repelling (parabolic) *bi-accessible* points and their preimages.

Proposition 43. *Suppose that U is paratracting Fatou domain of P . Then $\text{Bd}(U)$ is well-sliced in J_P and hence is not collapsed under the finest map φ . In particular, suppose that $p \in J_{\sim P}$ is a periodic point. Then $\varphi^{-1}(p)$ cannot contain the boundary of a paratracting Fatou domain of P .*

Proof. By [23], $R \cap \text{Bd}(U) = A$ is dense in $\text{Bd}(U) \subset X$ and each point of A is accessible from within and from without U . This implies that any pair of points of A ray-separates $\text{Bd}(U)$. Since A consists of points accessible from within U we can use the canonic Riemann map for U and parameterize points of A by the corresponding angles; denote the corresponding set of angles by \mathcal{A} . Since all points of A are accessible from outside U and A is dense in $\text{Bd}(U)$, it follows that \mathcal{A} is dense in \mathbb{S}^1 . Since R is countable, so is \mathcal{A} , and it is easy to see that we can choose pairwise disjoint pairs of angles from A so that \mathbb{S}^1 with this collection of pairs is homeomorphic to \mathbb{S}^1 with the vertical collection of pairs $\mathcal{C}_{\mathbb{S}^1}$ defined in the end of Section 3. Then the corresponding to these pairs of angles pairs of points from A form a well-slicing family of $\text{Bd}(U)$ and by Theorem 27 $\text{Bd}(U)$ is not collapsed under the finest φ as desired. \square

We are ready to state the main result of this section which gives a criterion of the finest model not be degenerate. It lists three conditions, and for the finest model to be non-degenerate it is necessary and sufficient that at least one of them must be satisfied. In a descriptive form it was given in Section 1.

Theorem 44. *The finest model of the Julia set of a polynomial P is not degenerate if and only if at least one of the following properties is satisfied.*

- (1) *The filled-in Julia set K_P contains a paratracting Fatou domain.*
- (2) *The set of all repelling bi-accessible periodic points is infinite.*
- (3) *The polynomial P admits a Siegel configuration.*

Proof. First we show that the fact that at least one of properties (1) - (3) holds is necessary for the non-degeneracy of $J_{\sim P} = \varphi(J_P)$. In other words, we assume that $J_{\sim P}$ is non-degenerate and deduce the appropriate properties of J_P using Theorem 35. Consider the cases (1) - (3) one by one.

- (1) Suppose that, according to Theorem 35.(1), $J_{\sim P}$ contains a simple closed curve S which is the boundary of a paratracting Fatou domain. Then $\varphi^{-1}(S)$

is a continuum which separates the plane and encloses an open set U complementary to J_P . Moreover, for a dense in S subset of g -periodic points their Φ -preimages are finite (there are no more than finitely many periodic points of $\varphi(J_P)$ whose Φ -preimages are infinite). By Lemma 37 this implies that full φ -preimages of these g -periodic points are P -periodic points at which J_P is locally connected. Thus, U is a Fatou domain of P whose boundary contains periodic points. This implies that U is a paratracting domain, and case (1) holds.

(2) Assume now that J_{\sim_P} does not contain simple closed curves, that is, that J_{\sim_P} is a dendrite. Consider the lamination \sim_P . Since J_{\sim_P} is a dendrite, \sim_P does not have Fatou domains. Hence by Theorem 35 there are infinitely many periodic \sim_P -classes each of which consists of more than one point. Moreover, we may assume that they are all finite (because there can only be finitely many infinite periodic classes of a lamination). Finally, by the construction the impression of each such class is disjoint from impressions of all angles not belonging to the class. Hence by Lemma 37 all their impressions are points. We conclude that there are infinitely many repelling bi-accessible periodic points as desired and case (2) holds.

(3) By Lemma 4 we may now assume that $\varphi(J_P)$ contains the boundary S of an invariant Siegel domain. By Theorem 35.(3), there exists a finite collection of all-critical \sim_P -classes $\mathcal{H} = \{H_1, \dots, H_m\}$ with pairwise disjoint grand orbits whose images x_1, \dots, x_m under the quotient map $\Phi : \mathbb{S}^1 \rightarrow \varphi(J_P) = J_{\sim_P}$ form the set of all-critical points in S so that *all* cutpoints of $\varphi(J_P)$ in S belong to the grand orbits of these all-critical points. Observe that by the construction for every i we have that $\text{Imp}(H_i) = \varphi^{-1}(x_i)$ is a continuum.

We want to show that this implies that P admits a Siegel configuration. As the collection of sets of angles needed to define a Siegel configuration we take exactly \mathcal{H} . Moreover, as in the definition of a Siegel configuration we take the grand orbit of \mathcal{H} then the corresponding sets of angles to form the geometric prelamination $\mathcal{L}_{\mathcal{H}}$. Observe that this will bring back *all* the leaves and gaps from the set $\Phi^{-1}(S)$ because all leaves and gaps in this set correspond to cutpoints of J_{\sim_P} in S and, by Theorem 35.(3), come from the grand orbits of all-critical points from S . Finally, by the construction the impressions $\text{Imp}(H_i)$ are disjoint from impressions of all angles not belonging to H_i . All this implies that P admits a Siegel configuration and completes the consideration of the case (3).

Now we consider the sufficiency of conditions (1) - (3). If (1) holds, then the finest map is not degenerate by Proposition 43. If (2) holds, then the finest map is not degenerate by Proposition 40. If (3) holds, then the finest map is not degenerate by Proposition 42. This completes the proof. \square

By Theorem 44 the finest model of a polynomial Julia set is degenerate if and only if there are no paratracting Fatou domains, the set of all repelling bi-accessible periodic points is finite, and there is no Siegel configuration. As an application let us first prove a sufficient condition for the finest model to be non-degenerate. Recall that the valence of a ray-continuum K is the cardinality of the set of all rays whose principal sets are contained in K .

Theorem 45. *Suppose that K' is a wandering ray-continuum such that the valence of $P^n(K')$ is greater than 1 for all $n \geq 0$. Then there are infinitely many repelling bi-accessible periodic points of J and the finest model is non-degenerate. In particular, these conclusions hold if there exists a non-(pre)periodic non-(pre)critical bi-accessible point of J .*

Proof. As explained in Subsection 5.3, the construction and the arguments similar to those from Theorem 4.2 [6] imply that there is a (possibly) bigger than K' but still wandering ray-continuum K (with the same eventual images as K') whose grand orbit Γ (i.e. the collection of pullbacks of its forward images) is well-defined. Moreover, to each element Q of Γ we can associate the set $\Theta(Q) = H_Q$ of all angles whose principal sets are contained in Q (by Theorem 3 the set H_Q is finite). Then all elements of Γ are non-separating and wandering ray-continua. Moreover, convex hulls of sets $H_Q, Q \in \Gamma$ form a prelamination which we denote \mathcal{L}_K . By the properties

By Theorem 12 we can consider its closure $\overline{\mathcal{L}_K}$ which is the geo-lamination generated by K and then the lamination \approx_K generated by K . By Theorem 12 $\approx_K = \approx$ has no Siegel domains. However it may have several paratracting Fatou domains.

Let us show that closures of Fatou domains of \approx are pairwise disjoint. Let U be a Fatou domain of $\overline{\mathcal{L}_K}$. By the construction from Theorem 12, U remains a Fatou domain of \approx . Let us study $\text{Bd}(U)$ in detail. By Theorem 12 in the geo-lamination $\overline{\mathcal{L}_K}$ and in the refined geo-lamination \mathcal{L}_\approx there are no critical leaves. Therefore by Lemma 7 all leaves in $\text{Bd}(U)$ are (pre)periodic. Thus, they do not come from \mathcal{L}_K and must be the limit leaves of \mathcal{L}_K . Choose a geometric leaf ℓ in $\text{Bd}(U)$. By Theorem 12 elements of \mathcal{L}_K cannot be contained in \overline{U} , hence they approach ℓ from outside of U . Moreover, we may assume that these elements of \mathcal{L}_K are contained in convex hulls of distinct \approx -classes. Therefore ℓ cannot lie on the boundary of any other gap of \mathcal{L}_\approx or on the boundary of another Fatou domain of $\overline{\mathcal{L}_K}$ (or, equivalently, of \approx), as desired.

Consider a new lamination $\approx'_K = \approx'$ obtained by identifying the boundary of each Fatou domain of \approx_K and show that $J_{\approx'}$ is a non-degenerate dendrite. It is easy to see that \approx' is a well-defined lamination. Then the corresponding topological Julia set $J_{\approx'}$ can be obtained from J_\approx by collapsing closures of all its Fatou domains into points. Clearly, there are no more than countably many

Fatou domains of \approx , their boundaries are continua, and these continua are pairwise disjoint by the previous paragraph. Then by the Sierpiński Theorem [24] the resulting (after this collapse) quotient space $J_{\approx'}$ is not degenerate. Hence the lamination \approx' is not degenerate. Moreover, since it no longer has Fatou domains, $J_{\approx'}$ is a dendrite.

By the results of [7] any dendritic topological Julia set has infinitely many periodic cutpoints. Hence there are infinitely many periodic cutpoints in $J_{\approx'}$. We now want to show that this implies that there are infinitely many periodic cutpoints of J . Let \mathbf{h} be a finite periodic class of \approx' which does not belong to the boundary of a Fatou domain of \approx . Then geometric leaves from $\text{Bd}(\text{Ch}(\mathbf{h}))$ cannot come from elements of \mathcal{L}_K (who are all wandering). Let us show that all geometric leaves on the boundary of \mathbf{h} are limit leaves of \mathcal{L}_K . Indeed, suppose that ℓ' is a boundary geometric leaf of $\text{Ch}(\mathbf{h})$ which is not such a limit leaf. Then there is a geometric gap \mathbf{g}' of \mathcal{L}_K on the side of ℓ' opposite to \mathbf{g} . By the choice of \mathbf{h} , the gap \mathbf{g}' cannot be a Fatou domain of $\overline{\mathcal{L}_K}$ which implies that it has a finite basis which should have been united with \mathbf{h} into one \approx -class, a contradiction. Thus, the set $\text{Imp}(\mathbf{h})$ is disjoint from impressions of all angles not in \mathbf{h} because these other impressions are cut off $\text{Imp}(\mathbf{h})$ by the ray-continua from the grand orbit of K corresponding to the appropriate elements of \mathcal{L}_K .

Consider now the set $\text{Imp}(\mathbf{h})$ and show that $\text{Imp}(\mathbf{h})$ is a continuum itself. If a geometric leaf ℓ'' belongs to the boundary of $\text{Ch}(\mathbf{h})$ then by the previous paragraph ℓ'' is the limit of a sequence of elements of \mathcal{L}_Θ . Taking the Hausdorff limit of a subsequence we see that the corresponding continua on the plane converge to a continuum. By the semi-continuity of impressions this continuum is contained in $\text{Imp}(\ell'')$. Hence $\text{Imp}(\ell'')$ is a continuum itself. Since the union of impressions of leaves ℓ'' from the boundary of $\text{Ch}(\mathbf{h})$ is in fact $\text{Imp}(\mathbf{h})$, the set $\text{Imp}(\mathbf{h})$ is a continuum. By Lemma 37 $\text{Imp}(\mathbf{h})$ is a repelling or parabolic periodic point, and since \mathbf{h} is a gap or leaf, it is a repelling or parabolic point of J at which at least two rays land, as desired. By Theorem 44 this implies that the finest model is non-degenerate. Clearly, the case when there exists a non-(pre)periodic bi-accessible point of J is a particular case of the above. This completes the proof. \square

Let us show how one can deduce Kiwi's results [14] from our results. Say that two angles α, β are *K-equivalent* if there exists a finite collection of angles $\alpha_0 = \alpha, \dots, \alpha_k = \beta$ such that $\text{Imp}(\alpha_i) \cap \text{Imp}(\alpha_{i+1}) \neq \emptyset$ for each $i = 0, \dots, k-1$. The notion (but not the terminology!) is due to Jan Kiwi [14] and is instrumental in his construction of locally connected models for connected Julia sets of polynomials without CS-points. Clearly, if two angles are K-equivalent, they must belong to the same K-class. Suppose that P does not have CS-points. Let us show first that the finest model is non-degenerate. Indeed, by the assumption P has no Siegel domains. If P has a paratracting domain then by

Theorem 44 the finest model is non-degenerate. It remains to consider the case when P has no Fatou domains (i.e., J_P is non-separating) and no CS-points. Then by [11,12] P has infinitely many repelling periodic bi-accessible points. Hence in this case the finest model is non-degenerate either.

Now, take any point p of P , consider the corresponding K-class $\Phi^{-1}(p)$ and show that it is finite. Indeed, suppose first that p is non-(pre)periodic. Then by Theorem 3 the corresponding K-class is finite. Now suppose that p is (pre)periodic; we may assume that it is periodic of period 1. Consider the set $Q = \varphi^{-1}(\varphi(p))$ and show that it is non-separating. Indeed, otherwise there is a paratracting domain U contained in the topological hull $\text{TH}(Q)$ (since P does not have CS-points it cannot be a Siegel domain). However by Lemma 43 the boundary $\text{Bd}(U)$ is not collapsed under φ , a contradiction. Hence Q is non-separating. Let us show that then it must contain infinitely many repelling periodic bi-accessible points. Indeed, suppose otherwise. Then replacing P by an appropriate power we may assume that all periodic points in Q and all the rays landing at them are invariant. By Theorem 34 this implies, that Q is a point, a contradiction to $\Phi^{-1}(p)$ being infinite by the assumption (at any repelling periodic point only finitely many rays land). So, if P has no CS-points then there are no infinite K-classes which implies that K-equivalence in fact coincides with the lamination \sim_P and thus produces the finest locally connected model of J_P .

Let us compare our approach and results with those of [14]. Kiwi uses direct arguments to construct the finest model for polynomials without CS-points. He also relies more upon combinatorial and related to symbolic dynamics arguments. Our approach, based upon continuum theory, is different. It allows us to show that Kiwi's locally connected model of a connected Julia set without CS-points is actually the *finest locally connected model of J_P* , the finest from the purely topological point of view. It also allows us to extend Kiwi's results [14] onto *all* polynomials with connected Julia sets. However we only tackle the case of connected Julia sets while in [14] disconnected Julia sets are also considered.

To conclude the paper we want to specify K-equivalence a little more. Namely, in the next theorem we obtain additional information about the way impressions of angles from finite K-classes can intersect. The theorem holds regardless of whether a polynomial has CS-points or not. However in the case when P has no CS-points it applies to *all* K-classes.

Theorem 46. *Suppose that $A = \{\alpha_1, \dots, \alpha_n\}$ is a finite K-class. Then impressions of adjacent on the circle angles in A meet. Moreover, any subset of A in which only adjacent angles have meeting impressions consists of no more than 3 elements.*

Proof. In the case when A is a (pre)periodic K -class (equivalently, \sim_P -class) it follows from Lemma 37 that $\text{Imp}(A)$ is a point which implies the conclusions of the lemma. Also, if A consists of two angles the conclusions of the lemma are obvious. Hence the remaining case is when $n \geq 3$ and A forms a so-called *wandering polygon*. Consider this case by way of contradiction. Assume that $\alpha_1, \dots, \alpha_n$ circularly ordered and $\text{Imp}(\alpha_1) \cap \text{Imp}(\alpha_2) = \emptyset$. Denote the open arc between α_1, α_2 which is complementary to A by I .

Let us show that there exists a Fatou domain U and a point of $x \in [\text{Imp}(A) \setminus (\text{Imp}(\alpha_1) \cup \text{Imp}(\alpha_n))] \cap \text{Bd}(U)$. Draw a curve L which starts at a point of a ray of an angle from I and ends at a point of a ray of an angle from $\mathbb{S}^1 \setminus \bar{I}$. Clearly, L separates $\text{Imp}(\alpha_1)$ from $\text{Imp}(\alpha_2)$. Since $\text{Imp}(A)$ is a continuum, L will have to intersect $\text{Imp}(A)$. Denote by x the first on L point of intersection between L and $\text{Imp}(A)$. Let us show that a sufficiently small open subarc T of L with one endpoint x and disjoint from $\text{Imp}(A)$ is in fact disjoint from J_P . Indeed, since α_1 and α_n are adjacent elements of A , the set $\cup_{\gamma \in I} \text{Imp}(\gamma)$ is disjoint from $\text{Imp}(A)$, and hence does not contain x . On the other hand, $x \notin \text{Imp}(\alpha_1) \cup \text{Imp}(\alpha_2)$ by the choice of L . Hence $x \notin \cup_{\gamma \in \bar{I}} \text{Imp}(\gamma) = Q$, and since Q is compact, we can find the desired arc T . On the other hand, the intersection $\text{Imp}(A) \cap Q = \text{Imp}(\alpha_1) \cup \text{Imp}(\alpha_n)$ is disconnected which implies that Q separates the plane. By the construction T must be contained in a bounded component U of $\mathbb{C} \setminus Q$. Since $Q \subset J_P$, it follows that U is a Fatou domain, and hence $x \in \text{Bd}(U)$.

Take a small ball B centered at x . By [23] there exists a (pre)periodic point $y \in B \cap \text{Bd}(U)$. Also, choose a (pre)periodic point $y' \in \text{Bd}(U)$ so that a ray of an angle belonging to I lands at Y' . Since $\text{Imp}(A)$ is wandering, $y, y' \notin \text{Imp}(A)$. As in the proof of Lemma 43, connect a point $z \in U$ with infinity by a curve E which intersects J_P only at y and y' . Then L' separates $\text{Imp}(\alpha_1)$ from $\text{Imp}(\alpha_2)$ on the plane and is disjoint from the continuum $\text{Imp}(A)$ which contains both $\text{Imp}(\alpha_1)$ and $\text{Imp}(\alpha_n)$, a contradiction. Thus, adjacent angles in A must have non-disjoint impressions.

To prove the rest, assume that there exist angles $\beta_1, \dots, \beta_r \in A, r \geq 4$ which are circularly ordered and such that all adjacent angles have non-disjoint impressions while otherwise the impressions of angles are disjoint. Consider two continua, $Y = \text{Imp}(\beta_1) \cup \text{Imp}(\beta_2)$ and $Z = \cup_{i=3}^r \text{Imp}(\beta_i)$. Then it follows that

$$Y \cap Z = [\text{Imp}(\beta_1) \cap \text{Imp}(\beta_r)] \cup [\text{Imp}(\beta_2) \cap \text{Imp}(\beta_3)]$$

which is disconnected because $\text{Imp}(\beta_1) \cap \text{Imp}(\beta_3) = \emptyset$ (recall that $r > 3$). Hence $\text{Imp}(A)$ separates the plane which is impossible. Indeed, if $\text{Imp}(A)$ separates the plane then its topological hull contains a Fatou domain and $\text{Imp}(A)$ is (pre)periodic. Assume that $\text{Imp}(A)$ (and A) are periodic of period 1. If $\text{Imp}(A)$

contains the boundary of an paratracting Fatou domain then by [23] $\text{Imp}(A)$ will have to intersect infinitely many impressions, a contradiction. If $\text{Imp}(A)$ contains the boundary of a Siegel domain then by Lemma 36 it contains a critical point $c \in J_P$ and A contains at least two angles with the same σ -image. However, as A is a finite invariant K-class, the map σ maps A onto itself in a one-to-one fashion, a contradiction. \square

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