PERFECT SUBSPACES OF QUADRATIC LAMINATIONS

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Dedicated to the memory of Tan Lei

ABSTRACT. The combinatorial Mandelbrot set is a continuum in the plane, whose boundary is defined as the quotient space of the unit circle by an explicit equivalence relation. This equivalence relation was described by Douady and, separately, by Thurston who used quadratic invariant geolaminations as a major tool. We showed earlier that the combinatorial Mandelbrot set can be interpreted as a quotient of the space of all limit quadratic invariant geolaminations with the Hausdorff distance topology. In this paper, we describe two similar quotients. In the first case, the identifications are the same but the space is smaller than that used for the Mandelbrot set. The resulting quotient space is obtained from the Mandelbrot set by "unpinching" the transitions between adjacent hyperbolic components. In the second case we identify renormalizable geolaminations that can be "unrenormalized" to the same hyperbolic geolamination while no two non-renormalizable geolaminations are identified.

INTRODUCTION

To study families of complex polynomials one may construct models for them. A famous case here is the *quadratic family* of polynomials $P_c(z) = z^2 + c$ where c belongs to the complex plane $\mathbb{C}$. The set $\mathcal{M}_2$ of all parameters c such that P_c has a connected Julia set is called the *filled Mandelbrot set*; we call its boundary the *Mandelbrot set* (notice that our terminology is not entirely standard). In his seminal preprint [Thu85], William Thurston constructed a combinatorial geometric model $\mathcal{M}_2^c$ of $\mathcal{M}_2$. There exists a monotone map from $\mathcal{M}_2$ onto $\mathcal{M}_2^c$. The *MLC conjecture* states that this map is a homeomorphism.

The set $\mathcal{M}_2^c$ contains a countable and dense family of homeomorphic copies of itself. Thus, $\mathcal{M}_2^c$ is an example of a so-called *fractal* set. According to Adrien Douady, the process of constructing $\mathcal{M}_2^c$ can be described

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as "pinching the closed unit disk $\overline{\mathbb{D}}$" which is why $\mathcal{M}_2^c$ is often called the "pinched disk model" of $\mathcal{M}_2$. "Pinching" refers to collapsing a chord of $\overline{\mathbb{D}}$ (or a polygon with vertices in $\mathbb{S}$); each additional act of pinching creates an increasingly complicated new quotient space of $\overline{\mathbb{D}}$. One can understand the "pinched disk model" by doing only some of the pinchings and ignoring other ones. The resulting partial quotient spaces of $\overline{\mathbb{D}}$ are steps towards understanding $\mathcal{M}_2^c$. This motivates our work. Also, producing similar models in the higher degree cases is a difficult problem that has not yet been solved. Partial quotients of $\overline{\mathbb{D}}$ constructed in this paper admit cubic analogs that may be viewed as simplified models of the cubic connectedness locus. This serves as our second motivation.

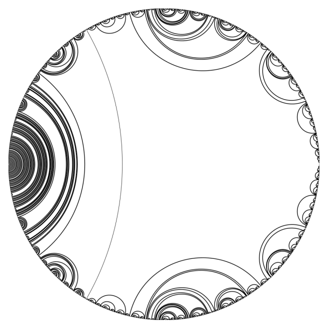

FIGURE 1. The geolamination QML

The main results of the paper use concepts related to laminational equivalence relations, geolaminations (geodesic laminations), etc. They require intimate knowledge of the structure of the combinatorial Mandelbrot set $\mathcal{M}_2^c$. All these notions and precise statement of our main results can be found in Section 1. Here we only describe our main results assuming the knowledge of the above mentioned concepts. Notice that when talking about σ_2-invariant objects (e.g. geolaminations) we often call them *quadratic*.

The combinatorial Mandelbrot set $\mathcal{M}_2^c$ is defined by Thurston [Thu85] as the quotient space of the unit circle $\mathbb{S}$ under the laminational equivalence relation $\sim_{\mathrm{QML}}$ generated by the *quadratic minor geolamination* QML. In [BOPT16a] we interpret this as follows. First we define the space $\mathbb{L}_2^q$ of all *quadratic laminational equivalence relations* $\sim$ on the unit circle $\mathbb{S}$ by defining, for each such equivalence relation $\sim$, the *geodesic lamination $\mathcal{L}_\sim$ generated by* $\sim$ which is the union of $\mathbb{S}$ and all the edges of convex hulls of all classes of $\sim$ (in what follows we often call geodesic laminations *geolaminations*); then we identify $\sim$ with $\mathcal{L}_\sim$. We define a metric on $\mathbb{L}_2^q$ by using the Hausdorff distance function on the set of geolaminations $\mathcal{L}_\sim$. Since the space in question in non-compact, we take its closure $\overline{\mathbb{L}_2^q}$. The space $\overline{\mathbb{L}_2^q}$

consists of Hausdorff limits of geolaminations $\mathcal{L}_\sim$ where $\sim$ belongs to $\mathbb{L}_2^q$. The main result of [BOPT16a] is that $\mathcal{M}_2^c$ is a quotient of the space $\overline{\mathbb{L}_2^q}$. More precisely, two geolaminations from $\overline{\mathbb{L}_2^q}$ are identified if their *minors* (see [Thu85]) are non-disjoint (we call it *minor equivalence*). We prove in [BOPT16a] that each class of equivalence in $\overline{\mathbb{L}_2^q}$ contains a unique geolamination $\mathcal{L}_\sim$. Hence the corresponding quotient of $\overline{\mathbb{L}_2^q}$ can be identified with $\mathbb{L}_2^q$ set-theoretically. Each laminational equivalence relation in $\mathbb{L}_2^q$ is identified with a point of $\mathcal{M}_2^c$, and we show in [BOPT16a] that the resulting one-to-one identification between classes of minor equivalence in $\overline{\mathbb{L}_2^q}$ and points of $\mathcal{M}_2^c$ is a homeomorphism.

In this paper we describe a similar quotient $\mathcal{M}_2^l$ of the space $\mathbb{L}_2^l \subset \overline{\mathbb{L}_2^q}$ consisting of all geolaminations which are non-isolated in $\overline{\mathbb{L}_2^q}$; the space $\mathcal{M}_2^l$ is obtained from $\mathcal{M}_2^c$ by "unpinching" all points of $\mathcal{M}_2^c$ at which two hyperbolic components of $\mathcal{M}_2^c$ meet. It is generated by the parametric geolamination QML^l obtained from QML by replacing all isolated leaves of QML by their endpoints. We also consider another modification $\mathcal{M}_2^{nr}$ of $\mathcal{M}_2^c$ obtained by replacing all maximal "baby-Mandelbrot" sets by the corresponding gaps of $\overline{\mathbb{D}}$ and thus defining yet another parametric geolamination QML^{nr}.

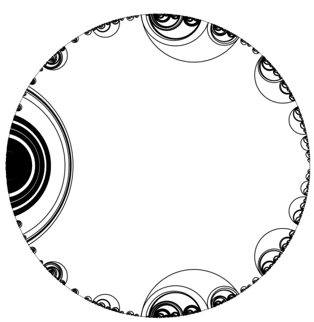

FIGURE 2. The geolamination QML^l

FIGURE 3. A zoom-in of QML^l

1. PRELIMINARIES

We write $\mathbb{D}$ for the open unit disk, and $\mathbb{S} = \mathrm{Bd}(\mathbb{D})$ for its boundary, the unit circle. Let $a, b \in \mathbb{S}$. By $[a, b]$, (a, b), etc., we mean the closed, open, etc., *positively oriented* circle arcs from a to b, and by $|I|$ the normalized length of an arc I in $\mathbb{S}$ (a normalization is made so that the length of $\mathbb{S}$ is 1).

1.1. **Laminational equivalence relations.** Denote by $\widehat{\mathbb{C}}$ the Riemann sphere. For a compactum $X \subset \mathbb{C}$, let $U^\infty(X)$ be the component of $\widehat{\mathbb{C}} \setminus X$ containing

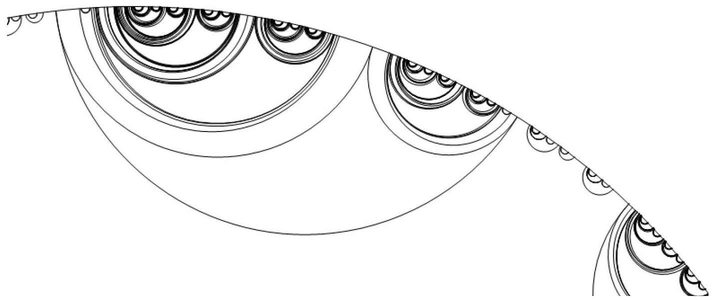

FIGURE 4. Another zoom-in of QML^l

infinity. If X is connected, there exists a Riemann mapping $\Psi_X : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to U^\infty(X)$; we always normalize it so that $\Psi_X(\infty) = \infty$, and $\Psi'_X(z)$ tends to a positive real limit as $z \to \infty$.

Consider a monic polynomial P of degree $d \geq 2$, i.e., a polynomial of the form $P(z) = z^d +$ lower order terms. Consider the Julia set J_P of P and the filled-in Julia set K_P of P. Extend the map $z \mapsto z^d$ to a map θ_d on $\widehat{\mathbb{C}}$. If J_P is connected, then $\Psi_{J_P} = \Psi : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to U^\infty(K_P)$ is such that $\Psi \circ \theta_d = P \circ \Psi$ on the complement of the closed unit disk [DH85, Mil00].

If J_P is locally connected, then Ψ extends to a continuous function

$$\overline{\Psi} : \widehat{\mathbb{C}} \setminus \mathbb{D} \to \overline{\widehat{\mathbb{C}} \setminus K_P},$$

and $\overline{\Psi} \circ \theta_d = P \circ \overline{\Psi}$ on the complement of the open unit disk. Thus, we obtain a continuous surjection $\overline{\Psi} \colon \mathrm{Bd}(\mathbb{D}) \to J_P$ (the *Carathéodory loop*). Identify $\mathbb{S} = \mathrm{Bd}(\mathbb{D})$ with $\mathbb{R}/\mathbb{Z}$. Set $\psi = \overline{\Psi}|_{\mathbb{S}}$. We will write σ_d for the restriction of θ_d to $\mathbb{S}$.

Define an equivalence relation $\sim_P$ on $\mathbb{S}$ by $x \sim_P y$ if and only if $\psi(x) = \psi(y)$, and call it the (σ_d-invariant) *laminational equivalence relation of* P; since Ψ defined above semiconjugates θ_d and P, the map ψ semiconjugates σ_d and $P|_{J(P)}$, which implies that $\sim_P$ is invariant. Equivalence classes of $\sim_P$ have pairwise disjoint convex hulls. The *topological Julia set* $\mathbb{S}/\sim_P = J_{\sim_P}$ is homeomorphic to J_P, and the *topological polynomial* $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$, induced by σ_d, is topologically conjugate to $P|_{J_P}$.

An equivalence relation $\sim$ on the unit circle, with similar properties to those of $\sim_P$ above, can be introduced with no references to polynomials.

Definition 1.1 (Laminational equivalence relations)**.** An equivalence relation $\sim$ on the unit circle $\mathbb{S}$ is said to be *laminational* if:
(E1) the graph of $\sim$ is a closed subset in $\mathbb{S} \times \mathbb{S}$;
(E2) convex hulls of distinct equivalence classes are disjoint;
(E3) each equivalence class of $\sim$ is finite.

For a closed set $A \subset \mathbb{S}$, let $\mathrm{CH}(A)$ be its convex hull. An *edge* of $\mathrm{CH}(A)$ is a closed straight segment I connecting two points of $\mathbb{S}$ such that $I \subset \mathrm{Bd}(\mathrm{CH}(A))$. By an *edge* of a $\sim$-class we mean an edge of its convex hull.

Definition 1.2 (Laminational equivalences and dynamics)**.** A laminational equivalence relation $\sim$ is (σ_d-)*invariant* if:
(D1) $\sim$ is *forward invariant*: for a class $\mathbf{g}$, the set $\sigma_d(\mathbf{g})$ is a class too;
(D2) for any $\sim$-class $\mathbf{g}$, the map $\tau = \sigma_d|_{\mathbf{g}}$ extends to $\mathbb{S}$ as an orientation preserving covering map $\hat{\tau}$ such that $\mathbf{g}$ is the full preimage of $\tau(\mathbf{g})$ under the covering map $\hat{\tau}$.

Definition 1.2 (D2) has an equivalent version. Given a closed set $Q \subset \mathbb{S}$, a (positively oriented) *hole* (a, b) of Q (or of $\mathrm{CH}(Q)$) is a component of $\mathbb{S} \setminus Q$. Then (D2) is equivalent to the fact that for a $\sim$-class $\mathbf{g}$ either $\sigma_d(\mathbf{g})$ is a point or for each positively oriented hole (a, b) of $\mathbf{g}$ the positively oriented arc $(\sigma_d(a), \sigma_d(b))$ is a hole of $\sigma_d(\mathbf{g})$. From now on, we assume that, unless stated otherwise, $\sim$ is a σ_d-invariant laminational equivalence relation.

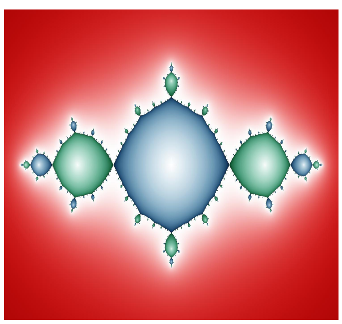

FIGURE 5. The Julia set of $f(z) = z^2 - 1$ (so-called "basilica")

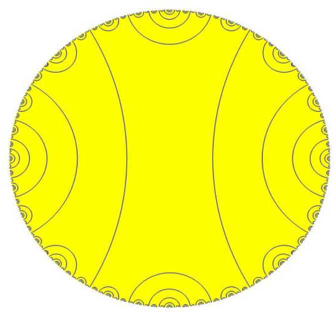

FIGURE 6. The geolamination for the Julia set of $z^2 - 1$

Given $\sim$, consider the *topological Julia set* $\mathbb{S}/\sim = J_\sim$ and the *topological polynomial* $f_\sim : J_\sim \to J_\sim$ induced by σ_d. Since $\mathbb{S} \subset \mathbb{C}$, we can use Moore's Theorem to embed $J_\sim$ into $\mathbb{C}$ and then to extend the quotient map $\psi_\sim : \mathbb{S} \to J_\sim$ to a map $\psi_\sim : \mathbb{C} \to \mathbb{C}$ with the only non-singleton fibers being the convex hulls of non-degenerate $\sim$-classes. A *Fatou domain* of $J_\sim$ (or of $f_\sim$) is a bounded component of $\mathbb{C} \setminus J_\sim$. If U is a periodic Fatou domain of $f_\sim$ of period n, then $f^n_\sim|_{\mathrm{Bd}(U)}$ is either conjugate to an irrational rotation of $\mathbb{S}$ or to σ_k for some $1 < k$, cf. [BL02]. In the case of irrational rotation, U is called a *Siegel domain*. The complement of the unbounded component of $\mathbb{C} \setminus J_\sim$ is called the *filled-in topological Julia set* and is denoted by $K_\sim$. Equivalently, $K_\sim$ is the union of $J_\sim$ and its bounded Fatou domains. If the laminational equivalence relation $\sim$ is fixed, we may omit

$\sim$ from the notation. By default, we consider $f_\sim$ as a self-mapping of $J_\sim$. For a collection $\mathcal{R}$ of sets, denote the union of all sets from $\mathcal{R}$ by $\mathcal{R}^+$.

Definition 1.3 (Leaves)**.** If A is a $\sim$-class, call an edge $\overline{ab}$ of $\mathrm{CH}(A)$ a *leaf of* $\sim$. All points of $\mathbb{S}$ are also called (*degenerate*) leaves of $\sim$.

The family of all leaves of $\sim$ is closed (the limit of a converging sequence of leaves of $\sim$ is a leaf of $\sim$); the union of all leaves of $\sim$ is a continuum. For any subset $X \subset \mathbb{D}$ with the property $X = \mathrm{CH}(X \cap \mathbb{S})$, we set $\sigma_d(X) = \mathrm{CH}(\sigma_d(X \cap \mathbb{S}))$. In particular, for any leaf ℓ of $\sim$, the set $\sigma_d(\ell)$ is a (possibly degenerate) leaf.

1.2. **Geolaminations.** Assume that $\sim$ is a σ_d-invariant laminational equivalence relation.

Definition 1.4. The set $\mathcal{L}_\sim$ of all leaves of $\sim$ is called the *geolamination generated by* $\sim$.

Geolaminations "visualize" laminational equivalence relations.

Definition 1.5 (Geolaminations, cf. [Thu85])**.** Distinct chords in $\overline{\mathbb{D}}$ are *unlinked* if they meet at most in a common endpoint; otherwise they are *linked*, or *cross* each other. A *geodesic pre-lamination* $\mathcal{L}$ is a set of (possibly degenerate) chords in $\overline{\mathbb{D}}$ such that any two distinct chords from $\mathcal{L}$ are unlinked. A geodesic pre-lamination $\mathcal{L}$ is a *geolamination* if all points of $\mathbb{S}$ are elements of $\mathcal{L}$, and $\mathcal{L}^+$ is closed. Elements of $\mathcal{L}$ are *leaves* of $\mathcal{L}$. A *degenerate* leaf (chord) is a singleton in $\mathbb{S}$. The continuum $\mathcal{L}^+ \subset \overline{\mathbb{D}}$ is the *solid* of $\mathcal{L}$. Let $\mathcal{L}$ be a geolamination. The closure in $\mathbb{C}$ of a non-empty component of $\mathbb{D} \setminus \mathcal{L}^+$ is a *gap* of $\mathcal{L}$. If a leaf (a gap) satisfies all the properties of leaves (gaps) of geolaminations but are not a part of any geolamination, we will call them *stand alone leaves/gaps*. If G is a gap or a leaf, call the set $G' = \mathbb{S} \cap G$ the *basis of* G. A gap is *finite* (*infinite, countable, uncountable*) if its basis is finite (infinite, countable, uncountable). Uncountable gaps are also called *Fatou gaps*. Points of G' are called *vertices* of G. Geolaminations of the form $\mathcal{L}_\sim$, where $\sim$ is a laminational equivalence relation, are called *q-laminations* ("q" from "e**q**uivalence"). A chord is (σ_d-)*critical* if its endpoints have the same image under σ_d (we often omit σ_d from notation).

The notion of *sibling invariant geolaminations* introduced below is slightly different from the original notion of *invariant geolaminations* in the sense of Thurston. However, sibling invariant geolaminations form a closed set and include all q-laminations. Thus, for all our purposes, it will suffice to consider sibling invariant geolaminations only. Some advantage of working with sibling σ_d-invariant geolaminations is that they are defined through properties of their leaves; gaps are not involved in the definition. It was

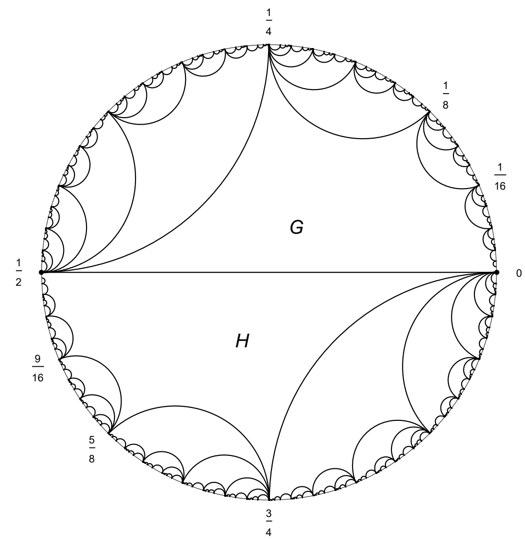

FIGURE 7. An example of a geolamination which is not a q-lamination

shown in [BMOV13] that all sibling invariant geolaminations are also invariant in the sense of Thurston [Thu85]. In particular for any gap G of a sibling invariant $\mathcal{L}$ the set $\sigma_d(G)$ is a point, or a leaf of $\mathcal{L}$, or a gap of $\mathcal{L}$. Moreover, if $\sigma_d(G) = H$ is a gap then $\sigma_d|_{\mathrm{Bd}(G)} : \mathrm{Bd}(G) \to \mathrm{Bd}(H)$ is a composition of a monotone map and a positively oriented covering map. In that case we call the degree of $\sigma_d|_{\mathrm{Bd}(G)}$ the *degree of* $\sigma_d|_G$.

Definition 1.6. A geolamination $\mathcal{L}$ is *sibling σ_d-invariant* provided that:

(1) for each $\ell \in \mathcal{L}$, we have $\sigma_d(\ell) \in \mathcal{L}$,
(2) for each $\ell \in \mathcal{L}$ there exists $\ell_1 \in \mathcal{L}$ so that $\sigma_d(\ell_1) = \ell$.
(3) for each $\ell \in \mathcal{L}$ so that $\sigma_d(\ell)$ is a non-degenerate leaf, there exist d **disjoint** leaves $\ell_1, \dots, \ell_d$ in $\mathcal{L}$ so that $\ell = \ell_1$ and $\sigma_d(\ell_i) = \sigma_d(\ell)$ for all $i = 1, \dots, d$.

Let us list a few properties of sibling σ_d-invariant geolaminations.

Theorem 1.7 ([BMOV13])**.** *The space of all sibling σ_d-invariant geolaminations is compact. All geolaminations generated by σ_d-invariant laminational equivalence relations are sibling σ_d-invariant.*

In what follows instead of "sibling σ_d-invariant geolaminations" we say "σ_d-invariant geolaminations". Also, we talk interchangeably about leaves (gaps) of $\sim$ or of $\mathcal{L}_\sim$. Let us now discuss gaps in the context of σ_d-invariant laminational equivalence relations and geolaminations.

Definition 1.8 (Critical gaps)**.** A gap G of a geolamination is called *(σ_d-)critical* if for each $y \in \sigma_d(G')$ the set $\sigma_d^{-1}(y) \cap G'$ consists of at least 2 points. If it does not cause ambiguity, we talk about *critical* gaps.

Definition 1.9 (Periodic and (pre)periodic gaps)**.** Let G be a gap of an invariant geolamination $\mathcal{L}$. A gap/leaf U of $\mathcal{L}_\sim$ is said to be *(pre)periodic* if $\sigma_d^{m+k}(U') = \sigma_d^m(U')$ for some $m \geq 0$, $k > 0$; if m, k are chosen to be minimal, then U is said to be *preperiodic* if $m > 0$ or *periodic (of period k)* if $m = 0$. If the period of G is 1, then G is said to be *invariant*. Define *precritical* and *(pre)critical* objects similarly to (pre)periodic and preperiodic objects defined above.

Consider infinite periodic gaps of σ_d-invariant geolaminations. Observe that, by [Kiw02], infinite gaps are eventually mapped onto periodic infinite gaps. First we state (without a proof) a well-known folklore lemma about the edges of preperiodic (in particular, infinite) gaps (see, e.g., Lemma 2.28 [BOPT17]).

Lemma 1.10. *Any edge of a (pre)periodic gap is either (pre)periodic or (pre)critical.*

Let us now classify infinite gaps. It is known that there are three types of such gaps: *caterpillar* gaps, *Siegel* gaps, and *Fatou gaps of degree greater than one*.

Definition 1.11. An infinite gap G is said to be a *caterpillar* gap if its basis G' is countable.

An example of a caterpillar gap is shown in Fig. 7. A general description of σ_3-invariant caterpillar gaps is given in [BOPT16b]. The fact that the basis G' of a caterpillar gap G is countable implies that there are lots of concatenated edges of G. Other properties of caterpillar gaps can be found in Lemma 1.12.

Lemma 1.12 (Lemma 1.15 [BOPT16a])**.** *Let G be a caterpillar gap of period k. Then the degree of $\sigma_d^k|_{\mathrm{Bd}(G)}$ is one, and G' contains some periodic points of period k.*

Definition 1.13. A periodic Fatou gap G of period n is said to be a periodic *Siegel* gap if the degree of $\sigma_d^n|_G$ is 1, and the basis G' of G is uncountable.

The next lemma is well known (see, e.g., [BOPT16a, Lemma 1.12]).

Lemma 1.14. *Let G be a Siegel gap of period n. Then $\sigma_d^n|_{\mathrm{Bd}(G)}$ is monotonically semiconjugate to an irrational circle rotation, contains no periodic points, and one of its iterated images has a critical edge.*

A period n Fatou gap is said to have *degree* $k > 1$ if the degree of $\sigma_d^n|_{\mathrm{Bd}(G)}$ is $k > 1$; if $k = 2$, then G is said to be *quadratic*. The next lemma is well known.

Lemma 1.15. *Let G be a Fatou gap of period n and of degree $k > 1$. Then the map $\sigma_d^n|_{\mathrm{Bd}(G)}$ is monotonically semiconjugate to σ_k.*

2. Limit geolaminations and their properties

Take the space E of all chords (including degenerate ones) in the unit disk with Hausdorff distance. Every geolamination $\mathcal{L}$ can be viewed as a closed subset of E (each leaf of $\mathcal{L}$ is a point of E). Define the Hausdorff distance between two geolaminations $\mathcal{L}_1$, $\mathcal{L}_2$ using the Hausdorff distance between the two closed subsets $\mathcal{L}_1$ and $\mathcal{L}_2$ of E. This defines a metric on the set of geolaminations. We speak of limits of geolaminations only in this sense.

Fix a degree d and consider limits of σ_d-invariant q-laminations. In lemmas below, we assume that a sequence of σ_d-invariant q-laminations $\mathcal{L}_i$ converges to a σ_d-invariant geolamination $\mathcal{L}_\infty$. By a *strip* we mean a (open) part of the unit disk contained between two disjoint chords. By a *strip around a chord* ℓ we mean a strip containing ℓ. In what follows, when talking about convergence of leaves/gaps, closeness of leaves/gaps, and closures of families of geolaminations, we always use the Hausdorff metric on E.

Definition 2.1. Let $\mathbb{L}_d^q$ be the family of all σ_d-invariant geodesic q-laminations. We will write $\overline{\mathbb{L}_d^q}$ for the closure of $\mathbb{L}_d^q$.

Even though we state below a few general results, we mostly concentrate on periodic objects of limit geolaminations.

Lemma 2.2 (Lemma 2.2 [BOPT16a]). *Let ℓ be a periodic leaf of $\mathcal{L} \in \overline{\mathbb{L}_d^q}$. If $\widehat{\mathcal{L}} \in \mathbb{L}_d^q$ is sufficiently close to $\mathcal{L}$, then any leaf of $\widehat{\mathcal{L}}$ sufficiently close to ℓ is either equal to ℓ or disjoint from ℓ.*

Definition 2.3 introduces the concept of rigidity.

Definition 2.3. A leaf/gap G of $\mathcal{L}$ is *rigid* if any q-lamination close to $\mathcal{L}$ has G as its leaf/gap.

Some lemmas proved in [BOPT16a] study rigidity of periodic leaves/gaps of geolaminations from $\overline{\mathbb{L}_d^q}$. These are combinatorial counterparts of the fact that repelling periodic points survive under small deformations of complex polynomials. By a (σ_d-)*collapsing polygon* we mean a polygon Q, whose edges map under σ_d to the same non-degenerate chord ℓ; if a point moves around Q, its σ_d-image moves back and forth along ℓ. If it does not cause ambiguity, we omit σ_d from notation. We say that Q is a *collapsing polygon* *of a geolamination* $\mathcal{L}$ if all edges of Q are leaves of $\mathcal{L}$; we also say that $\mathcal{L}$ *contains a collapsing polygon* Q. However, this does *not* imply that Q is a gap of $\mathcal{L}$ as Q might be further subdivided by leaves of $\mathcal{L}$ inside Q.

Lemma 2.4 (Lemma 2.5 - 2.10 [BOPT16a]). *Let $\mathcal{L} \in \overline{\mathbb{L}^q_d}$. If $\hat\ell \in \mathcal{L}$ is a non-degenerate rigid leaf, a leaf $\ell \in \mathcal{L}$ is such that $\sigma_d^k(\ell) = \hat\ell$ for some $k \ge 0$, and no leaf ℓ, $\sigma_d(\ell)$, $\dots$, $\sigma^{k-1}(\ell)$ is contained in a collapsing polygon of $\mathcal{L}$, then ℓ is rigid. Also, the following objects are rigid:*

(1) *periodic leaves that are not edges of collapsing polygons;*
(2) *finite periodic gaps;*
(3) (*pre*)*periodic leaves of a gap eventually mapped to a periodic gap;*
(4) *finite gaps that eventually map onto periodic gaps;*
(5) *periodic Fatou gaps whose images have no critical edges.*

Using these results and other tools, we characterize all σ_2-invariant limit geolaminations. Each such geolamination $\mathcal{L}$ can be described as a specific modification of an appropriate geolamination $\mathcal{L}^q$ from $\mathbb{L}^q_2$.

Definition 2.5. Geolaminations *coexist* if their union is a geolamination.

This notion was used in [BOPT16b]. If two geolaminations coexist, then a leaf of one geolamination is either also a leaf of the other geolamination or is located in a gap of the other geolamination.

For a σ_2-invariant geolamination $\mathcal{L}$, Thurston [Thu85] defines its *major* $M(\mathcal{L})$ as a longest leaf of $\mathcal{L}$; either $\mathcal{L}$ has a unique major (a diameter of $\overline{\mathbb{D}}$), or $\mathcal{L}$ has two distinct majors with equal σ_2-images. Thurston defines the *minor* of $\mathcal{L}$ as $m(\mathcal{L}) = \sigma_2(M(\mathcal{L}))$ and shows that *the family of the minors of all σ_2-invariant geolaminations is a geolamination itself*, called the *quadratic minor lamination* QML and generated by an equivalence relation $\sim_{\mathrm{QML}}$. Each class of $\sim_{\mathrm{QML}}$ is associated with a unique σ_2-invariant laminational equivalence relation and its topological polynomial. The quotient $\mathbb{S}/\sim_{\mathrm{QML}} = \mathcal{M}^c_2$ is called the *combinatorial Mandelbrot set.*

Definition 2.6. A σ_2-invariant geolamination is called *hyperbolic* if it has a periodic Fatou gap of degree two.

Clearly, if a σ_2-invariant geolamination $\mathcal{L}$ has a periodic Fatou gap U of period n and of degree greater than one, then the degree of $\sigma_2^n|_{\mathrm{Bd}(U)}$ is two. By [Thu85], there is a unique edge $M(\mathcal{L})$ of U with $\sigma_2^n(M(\mathcal{L})) = M(\mathcal{L})$. Either all leaves $M(\mathcal{L})$, $\dots$, $\sigma_2^{n-1}(M(\mathcal{L}))$ are pairwise disjoint, or their union can be broken down into several gaps permuted by σ_2, in each of which edges are "rotated" by the appropriate power of σ_2, or $n = 2k$ and σ_2^k flips $M(\mathcal{L})$ on top of itself while all leaves $M(\mathcal{L})$, $\dots$, $\sigma_2^{k-1}(M(\mathcal{L}))$ are pairwise disjoint. In fact, $M(\mathcal{L})$ and its sibling $M^*(\mathcal{L})$ are the two *majors* of $\mathcal{L}$ while $\sigma_2(M(\mathcal{L})) = \sigma_2(M^*(\mathcal{L})) = m(\mathcal{L})$ is the *minor* of $\mathcal{L}$

[Thu85]. Any σ_2-invariant hyperbolic geolamination $\mathcal{L}$ is actually a geolamination $\mathcal{L}_\sim$ generated by the appropriate *hyperbolic σ_2-invariant laminational equivalence relation $\sim$.*

Definition 2.7. A *critical set* $\mathrm{Cr}(\mathcal{L})$ of a σ_2-invariant geolamination $\mathcal{L}$ is either a critical leaf, or a collapsing quadrilateral which is a gap of $\mathcal{L}$, or a gap G with $\sigma_2|_G$ of degree two. A gap is said to be *critical* if it is a critical set.

A σ_2-invariant q-lamination has a finite critical set (a critical leaf, or a finite critical gap) or is hyperbolic. In both cases, the critical set is unique.

Definition 2.8. A *generalized critical quadrilateral* Q is either a collapsing quadrilateral or a critical leaf.

If $\mathrm{Cr}(\mathcal{L})$ is a generalized critical quadrilateral of a geolamination $\mathcal{L}$, then $\sigma_2(\mathrm{Cr}(\mathcal{L})) = m(\mathcal{L})$. Theorem 2.9 describes geolaminations from $\overline{\mathbb{L}_2^q}$. A periodic leaf $\overline{z}$ is called a *fixed return* periodic leaf if the period of its endpoints is k and all leaves $\overline{z}$, $\sigma_2(\overline{z}), \dots, \sigma_2^{k-1}(\overline{z})$ are pairwise disjoint.

Theorem 2.9 (Theorem 3.8 [BOPT16a]). *A geolamination $\mathcal{L}$ belongs to $\overline{\mathbb{L}_2^q}$ if and only if there exists a unique maximal q-lamination $\mathcal{L}^q$ coexisting with $\mathcal{L}$ and such that either $\mathcal{L} = \mathcal{L}^q$ or $\mathrm{Cr}(\mathcal{L}) \subset \mathrm{Cr}(\mathcal{L}^q)$ is a generalized critical quadrilateral, and exactly one of the following holds.*

(1) *The critical set $\mathrm{Cr}(\mathcal{L}^q)$ is finite, and $\mathrm{Cr}(\mathcal{L})$ is the convex hull of two edges or vertices of $\mathrm{Cr}(\mathcal{L}^q)$ with the same σ_2-image;*
(2) *the geolamination $\mathcal{L}^q$ is hyperbolic with a critical Fatou gap $\mathrm{Cr}(\mathcal{L})$ of period n, and exactly one of the following holds:*
    (a) *the set $\mathrm{Cr}(\mathcal{L}) = \overline{ab}$ is a critical leaf with a periodic endpoint of period n, and $\mathcal{L}$ contains exactly two σ_2^n-pullbacks of $\overline{ab}$ that touch $\overline{ab}$ at the endpoints (one at a and one at b).*
    (b) *the critical set $\mathrm{Cr}(\mathcal{L})$ is a collapsing quadrilateral, and $m(\mathcal{L})$ is a fixed return periodic leaf.*

Thus, any σ_2-invariant q-lamination corresponds to finitely many geolaminations from $\overline{\mathbb{L}_2^q}$, and the union of all of their minors is connected.

Given a geolamination $\mathcal{L} \in \overline{\mathbb{L}_2^q}$, let $\mathcal{L}^q$ be the σ_2-invariant q-lamination associated with $\mathcal{L}$ as in Theorem 2.9.

Definition 2.10 ([BOPT16a]). Geolaminations $\mathcal{L}_0$, $\mathcal{L}_k \in \overline{\mathbb{L}_2^q}$ are said to be *minor equivalent* if there exists a finite collection of geolaminations $\mathcal{L}_1$, $\mathcal{L}_2$, $\dots$, $\mathcal{L}_{k-1}$ from $\overline{\mathbb{L}_2^q}$ such that for each i with $0 \leqslant i \leqslant k-1$, the minors $m(\mathcal{L}_i)$ and $m(\mathcal{L}_{i+1})$ of the geolaminations $\mathcal{L}_i$ and $\mathcal{L}_{i+1}$ are non-disjoint.

Theorem 2.11 interprets the Mandelbrot set as a quotient of $\overline{\mathbb{L}_2^q}$. Let $\psi : \overline{\mathbb{L}_2^q} \to \mathbb{S}/\sim_{\mathrm{QML}}$ be the map which associates to each geolamination $\mathcal{L} \in \overline{\mathbb{L}_2^q}$ the $\sim_{\mathrm{QML}}$-class of the endpoints of the minor $m(\mathcal{L})$ of $\mathcal{L}$.

Theorem 2.11 (Theorem 3.10 [BOPT16a]). *The map $\psi : \overline{\mathbb{L}_2^q} \to \mathbb{S}/\sim_{\mathrm{QML}}$ induces a homeomorphism between the quotient space of $\overline{\mathbb{L}_2^q}$ with respect to the minor equivalence and $\mathbb{S}/\sim_{\mathrm{QML}}$.*

For every geolamination $\mathcal{L}$ let its *minor set* be the image of its critical set unless $\mathcal{L}$ is hyperbolic in which case we call $m(\mathcal{L})$ the *minor set* of $\mathcal{L}$. Then ψ associates to each class A of minor equivalence in $\overline{\mathbb{L}_2^q}$ the minor set of the geolamination $\mathcal{L}^q$, the only q-lamination in A. The minor set of $\mathcal{L}^q$ is the convex hull of the union of minors of all geolaminations in A.

We modify this by considering the subset of $\overline{\mathbb{L}_2^q}$ consisting of all non-isolated geolaminations. In other words, we consider geolaminations which are limits of sequences of pairwise distinct σ_2-invariant q-laminations.

Corollary 2.12. *A geolamination $\mathcal{L} \in \overline{\mathbb{L}_2^q}$ is non-isolated in $\overline{\mathbb{L}_2^q}$ if and only if case (1) or (2) of Theorem 2.9 holds.*

In order to prove Corollary 2.12, we need the following lemma.

Lemma 2.13. *Suppose that $\mathcal{L}$ is a σ_2-invariant q-lamination whose critical set is a generalized critical quadrilateral. Then $\mathcal{L}$ is the only σ_2-invariant geolamination with critical set $\mathrm{Cr}(\mathcal{L})$.*

Proof of Lemma 2.13. Indeed, properties of σ_2-invariant geolaminations imply that pullbacks of $\mathrm{Cr}(\mathcal{L})$ are well defined on each finite step; moreover, these pullbacks are all sets from $\mathcal{L}$. Furthermore, the closure $\widehat{\mathcal{L}}$ of their entire family is a σ_2-invariant geolamination itself, and since $\mathcal{L}$ is closed it follows that $\widehat{\mathcal{L}} \subset \mathcal{L}$. We claim that $\widehat{\mathcal{L}} = \mathcal{L}$. Indeed, suppose otherwise. Then $\widehat{\mathcal{L}}$ must contain a gap, say, U that itself is the union of $s > 1$ gaps of $\mathcal{L}$ and, therefore, *U contains leaves of $\mathcal{L}$* ***inside***. If U is finite, it follows that there are non-disjoint finite gaps of $\mathcal{L}$. The latter is impossible as $\mathcal{L}$ is a q-lamination. Thus, U is infinite. Mapping U forward several times, we may assume without loss of generality that U is periodic of period k (indeed, by [Kiw02], all infinite gaps of geolaminations are (pre)periodic).

Consider several cases. First suppose that U is a caterpillar gap. Then the critical leaf of U (or of a gap in the forward orbit of U) must coincide with the critical set of $\mathcal{L}$. Therefore, $\mathcal{L}$ has a critical leaf with a periodic endpoint, which is impossible for a q-lamination.

Now, suppose that U is a Siegel gap. It is well-known (e.g., it follows from Lemma 1.10) that all edges of U are (pre)critical and that, therefore, some image $\sigma_2^t(U)$ of U has a critical edge ℓ; it then follows that $\mathrm{Cr}(\mathcal{L}) = \ell$, that all edges of U are pullbacks of ℓ, and that under the map ψ collapsing edges of U to points any chord $\hat{\ell}$ connecting vertices of U projects to a non-trivial chord $\psi(\hat{\ell})$ of the unit circle. Since ψ semiconjugates $\sigma_2^k|_{\mathrm{Bd}(U)}$ to an irrational rotation $\rho : \mathbb{S} \to \mathbb{S}$, the chord $\psi(\hat{\ell})$ in the unit disk will intersect its eventual image under ρ, which implies a similar statement for the chord $\hat{\ell} \subset U$. We see that $\hat{\ell}$ cannot be a leaf of any geolamination, a contradiction with the above.

Finally, suppose that $\sigma_2^k|_{\mathrm{Bd}(U)}$ is of degree 2. Then some iterated image of U is an infinite gap V such that $\sigma_2|_{\mathrm{Bd}(V)}$ has degree two. On the other hand, $\mathrm{Cr}(\widehat{\mathcal{L}}) = \mathrm{Cr}(\mathcal{L})$ is a generalized critical quadrilateral, a contradiction with the existence of V. Hence this case is impossible either, and so $\mathcal{L} = \widehat{\mathcal{L}} = \mathcal{L}^q$ is the unique geolamination with critical set $\mathrm{Cr}(\mathcal{L})$. □

Proof of Corollary 2.12. By Theorem 2.9, if $\mathcal{L}$ satisfies the conditions of the corollary, then $\mathcal{L} \in \overline{\mathbb{L}_2^q}$. Since geolaminations in case (2) do not belong to $\mathbb{L}_2^q$, they must be limits of sequences of pairwise distinct σ_2-invariant q-laminations.

Consider case (1). Then $\mathrm{Cr}(\mathcal{L}^q)$ is finite, and $\mathrm{Cr}(\mathcal{L})$ is the convex hull of two edges or vertices of $\mathrm{Cr}(\mathcal{L}^q)$ with the same σ_2-image. Suppose that $\mathrm{Cr}(\mathcal{L}^q)$ is a polygon with more than four vertices. Then $\mathcal{L} \neq \mathcal{L}^q$ (in fact, $\mathcal{L} \supsetneq \mathcal{L}^q$). Hence $\mathcal{L} \notin \mathbb{L}_2^q$, and, as above, $\mathcal{L}$ is a limit point of $\mathbb{L}_2^q$.

Consider now the case when $\mathcal{L}^q$ has a generalized quadrilateral as its critical set $\mathrm{Cr}(\mathcal{L}^q)$. It may happen that $\mathcal{L}$ has a critical leaf that is a diagonal of a quadrilateral $\mathrm{Cr}(\mathcal{L}^q)$ so that $\mathcal{L} \neq \mathcal{L}^q$; as before, then $\mathcal{L}$ is the limit of a sequence of pairwise distinct σ_2-invariant geolaminations.

It remains to consider the case when $\mathcal{L} = \mathcal{L}^q$ is generated by an equivalence relation $\sim$ and has a critical set $\mathrm{Cr}(\mathcal{L})$ that is either a critical quadrilateral or a critical leaf. Let us show that then $\mathcal{L}$ is the limit of a non-constant sequence of q-laminations. By Lemma 2.13, the geolamination $\mathcal{L}$ is the unique σ_2-invariant geolamination with critical set $\mathrm{Cr}(\mathcal{L})$. Now, the fact that $\mathcal{L}$ is the limit of a sequence of pairwise distinct q-laminations follows from the uniqueness of $\mathcal{L}$ and the fact that, due to well-known properties of the combinatorial Mandelbrot set, there is a sequence of q-laminations $\mathcal{L}_i$ with critical sets $\mathrm{Cr}(\mathcal{L}_i) \to \mathrm{Cr}(\mathcal{L})$ (recall that we are considering the case when $\mathrm{Cr}(\mathcal{L})$ is a generalized quadrilateral). This completes the proof. □

Thus, isolated geolaminations in $\overline{\mathbb{L}_2^q}$ are (a) dendritic geolaminations with critical sets that have more than four vertices, and (b) hyperbolic geolaminations. Removing them from $\overline{\mathbb{L}_2^q}$, we obtain the closed space $\mathbb{L}_2^l \subset \overline{\mathbb{L}_2^q}$ of all σ_2-invariant geolaminations that are non-isolated in $\overline{\mathbb{L}_2^q}$. The minor equivalence on $\mathbb{L}_2^l$ is defined as before: two geolaminations are *minor equivalent* if their minors can be connected by a chain of non-disjoint minors. Since we only consider minors of geolaminations from $\mathbb{L}_2^l$, the minor equivalence on $\mathbb{L}_2^l$ is not a restriction of the minor equivalence on $\overline{\mathbb{L}_2^q}$, and some classes of minor equivalence on $\mathbb{L}_2^l$ are slightly different from the restrictions of the corresponding classes of minor equivalence on $\overline{\mathbb{L}_2^q}$. Let us list all the cases.

(1) Take a dendritic geolamination $\mathcal{L}$ generated by a laminational equivalence relation $\sim$ such that $\mathrm{Cr}(\mathcal{L})$ has more than four vertices. Several geolaminations in $\mathbb{L}_2^l$ with critical sets being generalized critical quadrilaterals in $\mathrm{Cr}(\mathcal{L})$ form one class A of the minor equivalence in $\mathbb{L}_2^l$. Unlike for $\overline{\mathbb{L}_2^q}$, the geolamination $\mathcal{L}$ does not belong to $\mathbb{L}_2^l$ and is not included into A. Still, the convex hull of the union of all minors of geolaminations in A is the same for $\mathbb{L}_2^l$ and for $\overline{\mathbb{L}_2^q}$.

(2) Let $\mathcal{L}$ be a dendritic geolamination such that $\mathrm{Cr}(\mathcal{L})$ is either a quadrilateral or a critical leaf. By Corollary 2.12, we have $\mathcal{L} \in \mathbb{L}_2^l$. The corresponding class of minor equivalence in $\mathbb{L}_2^l$ consists of $\mathcal{L}$ itself and two geolaminations obtained by inserting a critical diagonal in $\mathrm{Cr}(\mathcal{L})$ and pulling it back. This class coincides with the corresponding class in $\overline{\mathbb{L}_2^q}$. The convex hull of the union of minors remains the same as for $\overline{\mathbb{L}_2^q}$.

(3) A Siegel geolamination $\mathcal{L}$ belongs to both $\mathbb{L}_2^l$ and $\overline{\mathbb{L}_2^q}$. The corresponding class of the minor equivalence consists of $\mathcal{L}$ only.

(4) Let $\mathcal{L}$ be a hyperbolic geolamination with a critical gap U of period n whose unique edge M of period n is a fixed return leaf. Then $\mathcal{L}$ does not belong to $\mathbb{L}_2^l$, but three closely related geolaminations form a class of minor equivalence. Two of them have critical leaves with endpoints at endpoints of M. The third one has a collapsing quadrilateral based on M. This yields the same convex hull of the union of minors as before in case of $\overline{\mathbb{L}_2^q}$.

(5) Finally, let $\mathcal{L}$ be a hyperbolic geolamination with a critical gap U of period n whose unique edge $M = \overline{ab}$ of U of period n is *not* a fixed return leaf. Then neither $\mathcal{L}$ nor the geolamination with a collapsing quadrilateral based on M belong to $\mathbb{L}_2^l$. Thus, there are ***two non-equivalent*** *geolaminations with critical leaves ℓ_a and ℓ_b with endpoints a and b, respectively* that can be associated with $\mathcal{L}$, and so there are ***two classes of minor equivalence****, generated by ℓ_a and ℓ_b, respectively*, that can be associated with $\mathcal{L}$.

Let A be a class of minor equivalence in $\mathbb{L}_2^l$. Define $m(A)$ as the convex hull of the union of the corresponding minors. The association $A \mapsto m(A)$ is similar to that made in [BOPT16a] for $\overline{\mathbb{L}_2^q}$. Let A' be the minor equivalence class in $\overline{\mathbb{L}_2^q}$ containing A. The above analysis implies that, in cases (1) – (4), we have $m(A) = m(A')$. In cases (2) and (3), we have $A = A'$. In cases (1) and (4), the class A' consists of A and the geolamination $\mathcal{L}^q$ generated by the corresponding laminational equivalence.

In case (5) the situation is different. The two distinct classes of minor equivalence in $\mathbb{L}_2^l$ correspond to critical leaves ℓ_a and ℓ_b and give rise to singletons $\{\sigma_2(a)\}$ and $\{\sigma_2(b)\}$ replacing the minor $m(\mathcal{L}) = \overline{\sigma_2(a)\sigma_2(b)}$ that corresponds to $\mathcal{L}$ in QML. Thus, the leaf $m(\mathcal{L})$ is erased from QML and replaced by its two endpoints. This "unpinching" of the circle yields a new parametric geolamination QML^l, the laminational equivalence $\sim_{\mathrm{QML}^l}$, and the quotient space $\mathcal{M}_2^l$. Let $\psi^l : \mathbb{L}_2^l \to \mathbb{S}/\sim_{\mathrm{QML}^l}$ be the quotient map. Then Theorem 2.11 implies Theorem 2.14.

Theorem 2.14. *The map ψ^l induces a homeomorphism between the quotient space of $\mathbb{L}_2^l$ by the minor equivalence and the space $\mathbb{S}/\sim_{\mathrm{QML}^l}$.*

To visualize our results we describe the gap CA^l of $\mathcal{M}_2^l$ containing the Main Cardioid CA. First though we need to define the Main Cardioid. We do so by defining the *filled Main Cardioid* as the set of all parameters c such that the polynomial $P_c(z) = z^2 + c$ has an attracting fixed point. The *Main Cardioid* then is defined as the boundary of the filled Main Cardioid (equivalently, this is the set of all parameters c such that the polynomial $P_c(z) = z^2 + c$ has a neutral fixed point (i.e., a fixed point with multiplier of modulus one). Notice that our terminology is a little unusual, but intuitive and completely consistent with the classic notions of the Julia set and filled Julia set. It is well known that the Main Cardioid is homeomorphic to its laminational model, constructed in [Thu85] as a part of the construction of the combinatorial Mandelbrot set $\mathcal{M}_2^c$. Therefore in what follows we do not make a distinction between the Main Cardioid and its combinatorial counterpart, a subset of $\mathcal{M}_2^c$.

Now we define the *growing tree of $f_\sim$* [Lev98, BL02] (in [BL02] this is done for topological polynomials of *any degree*, yet for the sake of simplicity here we consider only the quadratic case). Given $\theta \in \mathbb{S}$ and laminational equivalence relation $\sim$, let $\psi_\sim(\theta)$ be the point of $J_\sim$ associated with the $\sim$-class containing θ. In the dendritic topological Julia set $J_\sim$, connect the points $\psi_\sim(0)$ and $\psi_\sim(1/2)$ by an arc $I_\sim$. Clearly, $I_\sim$ consists of $\sim$-classes that separate angles 0 and $1/2$, and if $c_\sim$ is the critical point of $f_\sim$ then $c_\sim \in I_\sim$ because $f_\sim(\psi(0)) = f_\sim(\psi(1/2)) = \psi(0)$. Denote the union of all images of $I_\sim$ under $f_\sim$ by $T_\sim^\infty$ and call it the *growing tree of $f_\sim$*; clearly, $T_\sim^\infty$ is an invariant connected set. In what follows we may omit $\sim$ from the notation if it does not cause ambiguity. Slightly abusing the language, in what follows by an interval we will mean any set homeomorphic to $[0, 1]$. If all images of a set B are pairwise disjoint, then the set is called *wandering*. Some useful for us results of [BL02] are collected in the next lemma.

Lemma 2.15 ([BL02]). *Suppose that $f_\sim$ is a topological polynomial of any degree. Then it has finitely many periodic Fatou domains. All other Fatou domains are their eventual preimages. Any continuum in $J_\sim$ is non-wandering. If $J_\sim$ is dendritic, and the images of all critical $\sim$-classes are non-degenerate, then there exists a finite invariant tree containing all critical points of $f_\sim$. In particular, if $f_\sim$ is quadratic, $J_\sim$ is dendritic, and the critical $\sim$-class consists of more than two points, then $T^\infty_\sim$ is a finite invariant tree.*

In what follows, for a dendrite D and points $x, y \in D$ we denote by $[x, y]_D$ the unique arc in D connecting x and y. If it clear, what D is, we will omit it from our notation.

Lemma 2.16. *If $J_\sim$ be a dendrite, the following claims are equivalent.*

(1) *The minor $m(\mathcal{L}_\sim)$ is vertical.*
(2) *The growing tree $T^\infty_\sim$ is an interval.*
(3) *The critical point of $f_\sim|_{J_\sim}$ belongs to an invariant interval.*

Moreover, if these claims hold then every branchpoint of J that belongs to $T^\infty_\sim$ must be (pre)critical.

Proof. To simplify notation, assume that $\sim$ is given and omit it from our notation (thus, we set $f = f_\sim, \mathcal{L} = \mathcal{L}_\sim$, etc). Observe that some of the notation was introduced above when we discussed growing trees.

To prove $(1) \implies (2)$, observe that the majors of $\mathcal{L}$ are vertical. Indeed, only a vertical or a horizonal leaf can map to a vertical leaf. Horizonal majors are impossible since they would cross their minors. Therefore, there is a finite critical gap G of $\mathcal{L}$ such that the two vertical majors of $\mathcal{L}$ are edges of G. It follows that I contains both the critical point c of f and its image $f(c)$ (the $\sim$-classes of points from I are exactly the $\sim$-classes whose convex hulls separate 0 from $1/2$). This in turn implies that I is invariant (indeed, $[\psi(0), c]_J$ is mapped to $[\psi(0), f(c)]_J \subset I$, and similarly for $[c, \psi(1/2)]_J$), and so the growing tree T^∞ is an interval.

Clearly, $(2) \implies (3)$.

Finally, assume that (3) holds. Let $I_0 \subset J$ be an invariant interval. First we will show that then the last claim of the lemma holds, i.e., that any branchpoint $b \in I_0$ of J must be (pre)critical. Indeed, otherwise an eventual image b' of b is a periodic branchpoint of J still belonging to I_0. Then the orbit of b' cannot contain c, and the power of f that fixes b', must rotate small one-sided interval neighborhoods of b' in J (which follows from [Kiw02]). Since at least one of these neighborhoods is contained in I_0 and I_0 is invariant, it follows that all of them are contained in I_0, a contradiction with the fact that I_0 is an interval.

Let us now prove that (3) $\implies$ (1). Clearly, $c \in I_0$. Observe that $c \in I_0 \cap I$, and hence $I \cap I_0 \neq \varnothing$. If $I_0 \subset I$, then all points of I_0 separate $\psi(0)$ from $\psi(1/2)$. Thus, all iterated images $\sigma_2^n(m_\sim)$ of $m_\sim$ cross Di. This property, in turn, implies that $m_\sim$ is vertical as desired. Now, suppose that $I_0 \not\subset I$ and set $Z = I \cup I_0$. It follows that Z is invariant. Indeed, if $z \in I_0$ then $f(z) \in I_0 \subset Z$. Suppose now that $z \in I$. Then $f(z) \in [\psi(0), f(c)] \subset Z$. Hence Z is invariant. Denote by $C_\sim = C$ the critical $\sim$-class.

The mutual location of some $\sim$-classes and the way they separate other $\sim$-classes is well-known. Indeed, if Q is the invariant $\sim$-class such that $0 \notin Q$ then Q separates $1/2$ from C, the class C separates Q from 0, and Q separates 0 from $\sigma_2(C)$. If we set $q = \psi(Q)$ then we see that $\psi(0) < c < q < \psi(1/2)$ where " $<$" is the natural order on I from $\psi(0)$ to $\psi(1/2)$. Clearly, $Z = I_0 \cup X \cup Y$ where X is the arc in J connecting $\psi(0)$ with I_0, and Y is the arc in J connecting $\psi(1/2)$ with I_0. We may assume that $X = [\psi(0), x]$ and $Y = [\psi(1/2), y]$. On the other hand, $q \in I_0$ (by the Brouwer fixed point theorem), $c \in I_0$, and hence $[q, c] \subset I_0 \cap I$. The mutual location of points $\psi(0) < c < q < \psi(1/2)$ now implies that $y \neq c$.

On the other hand, the fact that $I_0 \not\subset I$ implies that Z is not an interval, by construction Z has one or two branchpoints, and any branchpoint of Z is either x or y. Let $b \in Z$ be a branchpoint of Z. By the above, b is not periodic (in fact, no branchpoint of J in I_0 is periodic). Now, if b is not critical, then $f(b)$ is also a branchpoint of Z. Repeating it and relying upon the fact that no branchpoint of Z is periodic, we see that all branchpoints of Z are (pre)critical, and c is a branchpoint of Z. Since by the above $y \neq c$, it follows that $x = c$. Consider now three pairwise disjoint (except for the common point c) intervals: $K_0 = [c, \psi(0)]$ and K_1, $K_2 \subset I_0$ connecting c with two endpoints of I_0. Since $f|_{I_0}$ is not a homeomorphism, c is a critical point of $f|_{I_0}$. Hence $f(K_1) \cap f(K_2)$ contains a small interval starting at $f(c)$ and pointing towards q. On the other hand, the fact that Q separates 0 from $\sigma_2(C)$ implies that $f(K_0) \supset [c, q]$. Clearly, this is impossible as f is two-to-one. $\square$

If $\mathcal{L}_\sim$ is hyperbolic (equivalently, if $m_\sim$ is periodic) then it is well-known that $m_\sim$ coincides with a $\sim_{\mathrm{QML}}$-class. Otherwise $J_\sim$ is a dendrite and the critical $\sim$-class is finite. Suppose, in addition, that $m_\sim$ is vertical. Let us show that then $m_\sim$ again coincides with a $\sim_{\mathrm{QML}}$-class. For, if this is not the case, then $m_\sim$ is an edge of the convex hull G of a larger $\sim_{\mathrm{QML}}$-class and, moreover, G is a non-periodic $\sim$-class. Hence $g = \psi_\sim(G)$ is a non-periodic branchpoint of J belonging (by Lemma 2.16) to an invariant interval $I_0 \subset J_\sim$. By the last claim of Lemma 2.16, the point g must be (pre)critical which makes g periodic, a contradiction. We conclude that vertical minors are always full $\sim_{\mathrm{QML}}$-classes. If a minor $m_\sim$ is vertical, then the corresponding $\sim_{\text{QML}}$-class is also said to be *vertical*. The corresponding topological polynomials and Julia sets will be called *real* (they correspond to complex polynomials $z^2 + c$ with $c \in \mathbb{R}$). For *any* laminational equivalence relation $\sim$ denote by $x_\sim$ the point of $\mathcal{M}_2^c$ corresponding to $\sim$ ($x_\sim$ is the image of the minor class of $\sim$ under the quotient map). The set of all points $x_\sim$ corresponding to the images of vertical $\sim_{\text{QML}}$-classes under the quotient map is called a *real line*.

In the next several paragraphs we consider q-laminations of *arbitrary degree* d and study their *infinitely renormalizable* sets. This is justified as the results concerning infinitely renormalizable sets are obtained almost literally in the same way in the quadratic case and in the general case.

Definition 2.17 (Infinitely-renormalizable laminations)**.** A σ_d-invariant q-lamination $\mathcal{L}_\sim$ is said to be *infinitely renormalizable* if there is an infinite sequence of q-laminations $\mathcal{L}_1 \subset \mathcal{L}_2 \subset \dots$ with $\mathcal{L}_\sim = \bigcup_i \mathcal{L}_i$ and a nested sequence of critical Fatou gaps U_i of $\mathcal{L}_i$ of period m_i such that $m_1 < m_2 < \dots$. If $\mathcal{L}_\sim$ is infinitely renormalizable, then the corresponding topological polynomial $f_\sim$ is also said to be *infinitely renormalizable*. Let $\psi_\sim$ be the projection of $\mathbb{S}$ onto $J_\sim = \mathbb{S}/\sim$. Set $Z_i = \psi_\sim(\text{Bd}(U_i))$. The nested sequence $Z_1 \supset Z_2 \supset \dots$ is called a *generating sequence of continua*. Moreover, the set $Z = \bigcap_{i=1}^\infty \text{orb} Z_i$ is said to be a *infinitely renormalizable* set.

The notation introduced in Definition 2.17 will be used in what follows. The next lemma establishes a useful property of infinitely renormalizable topological polynomials.

Lemma 2.18. *Let $f_\sim$ be an infinitely renormalizable topological polynomial, and $Z_1 \supset Z_2 \supset \dots$ a generating sequence of continua. Then, for all sufficiently large i, Z_i are dendrites. Moreover, the infinitely renormalizable set Z contains no periodic points.*

Proof. Indeed, otherwise the fact that there are finitely many periodic Fatou domains, and all Fatou domains eventually map to periodic ones, implies that there must exist a periodic Fatou domain V of $f_\sim$ of period, say, k such that $\text{Bd}(V) \subset Z_i$ for any i. Since pairwise intersections of distinct Fatou domains are finite, this implies that $m_i \leqslant k$ for all i, a contradiction. Now, suppose that a periodic point y belongs to Z. Denote by Y the convex hull of the $\sim$-class associated to y. Consider several cases.

First assume that Y is a singleton (a degenerate $\sim$-class) of period N. Then Y is a degenerate $\approx_i$-class in every i (here $\approx_i$ is the laminational equivalence relation associated with q-lamination $\mathcal{L}_i$ from Definition 2.17). Hence, if $m_i > N$, then in the σ_d-orbit of U_i two distinct Fatou gaps have a common point that is a degenerate class of $\mathcal{L}_i$ which is clearly impossible.

Now assume that Y is a periodic leaf of period N. Then, if $m_i > 2N$, then there will be two distinct Fatou gaps in the σ_d-orbit of U_i that are located on the same side of Y, a contradiction. Finally, if Y is a periodic gap and its edges are of period N, then, if $m_i > N$, then there will be two distinct Fatou gaps in the σ_d-orbit of U_i that are "attached" to Y at the same edge of Y, a contradiction. □

In what follows, by a *continuum* we mean a connected compact set consisting of more than one point. By an (f-)*periodic continuum* we mean a continuum A such that for some $m > 0$ the pairwise intersections of A, $f(A)$, $\dots$, $f^{m-1}(A)$ are at most finite while $f^m(A) \subset A$. The integer m is called the *period* of A. Since a continuum is infinite, the period is well defined. Given a periodic continuum A of period m we set $\mathrm{orb} A = \bigcup_{j=0}^{m-1} f^j(A)$ and call $\mathrm{orb} A$ a *cycle of continua*. Evidently, continua Z_i from a generating sequence of continua of an infinitely renormalizable set are periodic (because closures of distinct Fatou domains in a cycle of Fatou domains intersect over sets that are at most finite and, in fact, consist of periodic points).

Lemma 2.19. *Let $f = f_\sim$ be an infinitely renormalizable topological polynomial, and $Z_1 \supset Z_2 \supset \dots$ a corresponding generating sequence of continua. Then $Z = \bigcap_i \mathrm{orb} Z_i$ is a Cantor set.*

Proof. Obviously, Z is compact. Let Y be a component of Z. We claim that Y is *wandering*. Indeed, suppose otherwise. We may assume that $f^n(Y) \cap Y \neq \varnothing$ for some n. Fix a number i and assume that $Y \subset f^k(Z_i)$. It follows that $f^k(Z_i) \cap f^{k+n}(Z_i)$ is non-empty. On the other hand, $f^k(Z_i) \cap f^{k+n}(Z_i)$ is finite and consists of periodic points (see the remark right before the lemma). Since $f^n(Y) \cap Y \subset f^k(Z_i) \cap f^{k+n}(Z_i)$, it follows that $Y \subset Z$ contains periodic points, a contradiction with Lemma 2.18. Thus any component of Z is wandering, and hence, any component of Z is a point (recall that by Lemma 2.15 there are no wandering continua in $J_\sim$). There are no isolated points in Z since every $f^j(Z_i)$ contains infinitely many points of Z. Therefore, Z is a Cantor set. □

It follows that the topological polynomial on an infinitely renormalizable set is conjugate to a so-called *adding machine* and is *minimal* (every point in it has a dense orbit in the set). In particular two distinct infinitely renormalizable sets are either disjoint or coincide, and infinitely renormalizable sets are Cantor sets that do not contain periodic points.

The next proposition relies on [BL02] (see Lemma 2.15). A gap is said to be *all-critical* if all its edges are critical.

Proposition 2.20. *Let $f = f_\sim$ be an infinitely renormalizable topological polynomial, and $Z_1 \supset Z_2 \supset \dots$ a corresponding generating sequence of* *continua. Suppose that, for any critical point c of f in $Z = \bigcap_i \mathrm{orb} Z_i$, the point $f(c)$ separates $J = J_\sim$. Then there exists a finite periodic tree $T \subset J$ of period m such that $Z \subset \mathrm{orb}T$. In particular, one may find a periodic interval I such that all sets in the cycle of I are intervals, and $Z \subset \mathrm{orb}I$.*

Proof. Let d be the degree of f. Consider a sequence of q-laminations $\mathcal{L}_1 \subset \mathcal{L}_2 \subset \dots$ and a nested sequence of critical Fatou gaps U_i of $\mathcal{L}_i$ with $\psi_\sim(\mathrm{Bd}(U_i)) = Z_i$. Choose i so large that the critical points of f that belong to $\mathrm{orb}Z_i$ are exactly the critical points of f that belong to Z. In particular, by the assumption on critical points of f belonging to Z it follows then that no Fatou gap $\sigma_d^k(U_i)$ intersects an all-critical gap. Let $c \in Z_i$ be a critical point while C is the convex hull of the corresponding $\sim$-class. Then in general C is either a leaf or a gap, yet in our case C cannot be a leaf since $\sigma_d(C)$ is non-degenerate. Thus, C is a gap.

Let us show that $\sigma_d^k(C)$ crosses the interior of $\sigma_d^k(U_i)$, for every k. By the previous paragraph $E = \sigma_d^k(C)$ is *not* an all-critical gap. It follows that $\sigma_d^k(C)$ is always a non-degenerate leaf or gap of $\mathcal{L}_\sim$. Assume that $\sigma_d^k(C)$ does not cross the interior of $\sigma_d^k(U_i)$. Then $\sigma_d^k(C)$ is an edge of $\sigma_d^k(U_i)$ or a finite gap "attached" to an edge of $\sigma_d^k(U_i)$. The edge $\sigma_d^k(C) \cap \sigma_d^k(U_i)$ is (pre)periodic or (pre)critical. In the latter case, $\sigma_d^k(C)$ is eventually mapped to an *all-critical* gap, again a contradiction. Since $c \in Z$ and by Lemma 2.18, the critical point c cannot be (pre)periodic. Hence no edge of C can be (pre)periodic. We again arrive at a contradiction, which shows that $\sigma_d^k(C)$ crosses the interior of $\sigma_d^k(U_i)$, for every k.

Consider the map $\phi_{U_i} : \mathrm{Bd}(U_i) \to \mathbb{S}$ collapsing all edges of U_i. The restriction of $\sim$ to U_i is mapped under ϕ_{U_i} to some σ_{d_i}-invariant laminational equivalence $\sim_i$. We will write g_i for the corresponding topological polynomial, and J_i for the corresponding topological Julia set. Recall that g_i is conjugate to the map induced by $\sigma_d^{m_i}$ on $\mathbb{S}/\sim_i$. By the above, for any critical point $c \in J_i$ and the corresponding gap C, the gap $\sigma_d^{m_i}(C)$ crosses the interior of U_i, hence $g_i(c)$ separates J_i. If we now apply Lemma 2.15 to g_i, we see that the image of Z under the homeomorphism between Z_i and J_i is contained in a g_i-invariant finite tree. The corresponding finite tree $T \subset Z_i$ must then contain Z; it is easy to see that T has all the required properties.

To complete the proof, choose a large N so that each set $f^j(Z_N)$ contains at most one critical point of f. This is possible by Lemma 2.19. Observe that any critical point $c \in \mathrm{orb}T$ has a small neighborhood W_c in $\mathrm{orb}T$ (here W_c is an interval if c is not a branchpoint of the corresponding component of $\mathrm{orb}T$ or a k-od for some k otherwise) such that if $Q \subset W_c$ is an interval then $f(Q)$ is an interval too. Call such neighborhoods W_c *interval preserving*. Of course if a subinterval of T contains no critical points then its image is again an interval. Now, since the periods of sets Z_i grow to infinity, the tree T has only finitely many vertices, and by definition of a periodic continuum, it follows that if N is sufficiently large then some sets $f^j(Z_N \cap T)$ are intervals and all sets of the form $f^i(Z_N \cap T)$ containing a critical point are contained in this critical point's interval preserving neighborhood. Hence, *all* sets $f^i(Z_N \cap T)$ are intervals (as in our setting at no moment can a non-interval be the image of an interval). This completes the proof. □

Let us now go back to the quadratic case. The above stated general facts can be restated in the quadratic case as follows. Suppose that a quadratic topological polynomial $f_\sim$ is *infinitely renormalizable*; then every such topological polynomial is dendritic, and there is a nested sequence of periodic continua $Z_0 \supset Z_1 \supset \dots$ of periods $m_0 < m_1 < \dots$ such that the critical point c of $f_\sim$ belongs to $Z = \bigcap_i \mathrm{orb} Z_i$. Also, for each i, there exists a topological conjugacy between $f^{m_i} : Z_i \to Z_i$ and the restriction of some quadratic topological polynomial $g_i = f_{\sim_i}$ to its Julia set J_i. Moreover, it is well-known that in this case $x_\sim$ (recall that this is the point in $\mathcal{M}_2^c$ associated with $f_\sim$ and $\sim$) belongs to baby Mandelbrot sets $\widetilde{\mathcal{M}_2^c}(i) \subset \mathcal{M}_2^c$, and its location in $\widetilde{\mathcal{M}_2^c}(i)$ corresponds to the location of $x_{\sim_i}$ in $\mathcal{M}_2^c$. To describe locations relative to $\widetilde{\mathcal{M}_2^c}(i)$ we speak of "baby Main Cardioid", "baby real line" etc. The topological polynomials g_i are called *renormalizations* of $f_\sim$. Proposition 2.20 and Lemma 2.16 imply the following corollary.

Corollary 2.21. *If a quadratic topological polynomial $f_\sim$ is infinitely renormalizable and $Z = \bigcap_i \mathrm{orb} Z_i$, where Z_i are as above, then $Z = \omega(c)$, and there are only two possibilities.*

1. *The critical class C of $\sim$ consists of two points.*
2. *The critical class C of $\sim$ is a quadrilateral, there exists N such that, for $i \geq N$, all sets $f^k(Z_i)$, $k = 0, 1, \dots$ can be assumed to be intervals, and the corresponding topological Julia sets J_i are real. In particular, $x_{\sim_i}$ belongs to a baby real line in the corresponding baby Mandelbrot set.*

We are ready to visualize the gap CA^l of $\mathcal{M}_2^l$ containing the Main Cardioid CA. A topological polynomial $f_\sim$ is said to be *Feigenbaum* if it is infinitely renormalizable and the above defined sequence of periods can be chosen to be $m_0 = 1 < m_1 = 2 < \dots < m_i = 2^i < \dots$. It is known that there is a unique topological Feigenbaum polynomial, so from now on we will talk about *the* Feigenbaum topological polynomial. The corresponding laminational equivalence relation will be denoted $\sim_F$. It is well-known that the minor set $m_{\sim_F}$ is a leaf of QML approximated from one side by uncountably many leaves (minors) of QML. If a topological polynomial has a renormalization which is the Feigenbaum topological polynomial, we say that it has a *Feigenbaum renormalization*; by the above, all minors associated to topological polynomials with Feigenbaum renormalizations are limits of uncountable families of minors from QML from one side. A baby Main Cardioid Y is *finitely attached* to CA if there are finitely many baby Main Cardioids between Y and CA.

Proposition 2.22. *The boundary of* CA^l *consists of vertices and leaves of* QML*. The vertices of* CA^l *are vertices of baby Main Cardioids finitely attached to* CA*, or endpoints of edges of baby Main Cardioids finitely attached to* CA*, or minors associated to some infinitely renormalizable quadratic topological polynomials that do* not *have a Feigenbaum renormalization. The edges of* CA^l *are all associated to infinitely renormalizable topological polynomials that have Feigenbaum renormalizations.*

Proof. The process of creation of CA^l can be viewed as follows. First we erase all non-degenerate edges of CA; then we erase non-degenerate edges in the copies of CA that used to be attached to the Main Cardioid, etc. On each step we obtain bigger and bigger gaps containing CA. Observe that by construction any q-lamination (or topological polynomial) associated with the minors of QML erased after finitely many steps in the process of creating CA^l must have only finitely many periodic leaves. In the end of this process we get CA^l. Hence the degenerate edges of CA^l obtained after finitely many steps are endpoints of edges erased after a finite number of steps or Siegel points on the boundary of a baby Main Cardioid finitely attached to CA. The remaining edges of CA^l are infinitely renormalizable limits of sequences of non-degenerate edges of deeper and deeper baby Main Cardioids. These edges may be degenerate or non-degenerate.

By Corollary 2.21 if ℓ is a non-degenerate edge of CA^l then it is associated with an infinitely renormalizable topological polynomial $f_\sim$, and an m-periodic copy J' of a *real* quadratic dendritic topological Julia set $J_\approx$ of a topological polynomial $f_\approx$ is contained in $J_\sim$ where $f_\sim^m|_{J'}$ is topologically conjugate to $f_\approx|_{J_\approx}$ ($f_\approx$ is generated by a laminational equivalence relation $\approx$). If $f_\approx$ is not the Feigenbaum topological polynomial then the Sharkovsky Theorem implies that for some N and all $i \ge 0$ the geolamination $\mathcal{L}_\approx$ has periodic leaves of periods $2^N(2i+1)$. If we now choose a minor $\ell' \in \mathrm{QML}$ which is very close to ℓ and was erased when we constructed CA^l then it would follow that periodic leaves of periods $2^N(2i+1)$ with $i \ge t$ are still leaves of the q-lamination associated with ℓ'. However this contradicts the fact that this q-lamination can only have finitely many periodic leaves. $\square$

When we construct QML^l we remove countable concatenations of copies of CA finitely attached to CA itself and replace their union by CA^l. We have to do similar actions inside each baby Mandelbrot set, Thus, the only infinite gaps of QML^l associated to the bounded complementary domains of $\mathcal{M}_2^l$ are copies of CA^l from various baby Mandelbrot sets.

Proposition 2.23. *The geolamination* QML^l *is perfect.*

Proof. We need to show that QML^l has no isolated leaves. Suppose that ℓ is an isolated leaf of QML^l. Since it is a leaf of QML^l, it is not isolated in QML. Hence we may assume that there exists a one-sided semi-neighborhood U of ℓ that contains no leaves of QML^l but contains leaves $\ell_i \in \mathrm{QML} \setminus \mathrm{QML}^l$ converging to ℓ. We may think of U as the Jordan disk with the boundary formed by ℓ itself, two circular arcs T and R whose endpoints are endpoints of ℓ, and the remaining chord connecting the other two endpoints of T and R and disjoint from ℓ. Then leaves ℓ_i connect T and R.

Fix a number i. Since ℓ_i is isolated in QML, ℓ_i is an edge of a baby Main Cardioid contained in a baby Mandelbrot set $M^c_{2,i}$. Hence ℓ_i is contained in a copy $A_i \subset M^c_{2,i}$ of CA^l. Therefore, either there exists an edge of A_i with endpoints in T and R, or ℓ itself is an edge of A_i. The former is impossible by the assumption on U. Thus, ℓ is an edge of A_i. However then it follows from the last claim of Proposition 2.22 and the remark right before this proposition that ℓ is approximated by leaves of QML^l from the side opposite to U. $\square$

The geolamination QML^l is the visual counterpart of a laminational equivalence relation $\sim_{\mathrm{QML}^l}$ that can be defined as follows: two angles α ad β are $\sim_{\mathrm{QML}^l}$-equivalent if there exists a finite chain of leaves of QML^l connecting them. By the above, $\sim_{\mathrm{QML}^l}$ is a well-defined laminational equivalence relation (so that all its classes are finite). Almost all $\sim_{\mathrm{QML}^l}$-classes are in fact $\sim_{\mathrm{QML}}$-classes and correspond to the appropriate non-hyperbolic quadratic topological polynomials. The minors $m = \overline{ab}$ of QML that used to be associated to quadratic hyperbolic topological polynomials are erased from QML and replaced by pairs of their endpoints a and b. Moreover, the geolamination associated, say, with a, is obtained from the corresponding to m hyperbolic q-lamination $\sim_m$ by inserting a critical leaf ℓ_a in the critical Fatou gap U of $\sim_m$ such that $\sigma_2(\ell_a) = a$ and then pulling it back inside various gaps of $\sim_m$ that are pullbacks of U.

Let us suggest an interpretation of interiors of various filled copies of CA^l. Recall that the *perfect part* of a geolamination $\mathcal{L}$ is obtained by taking the maximal perfect subset of $\mathcal{L}$. In particular, all isolated leaves of $\mathcal{L}$ must be erased as we extract the perfect part of $\mathcal{L}$. Now, say that two geolaminations are *countably equivalent* if they have the same perfect parts (equivalently, if the symmetric difference between them is countable). For instance, all q-laminations from the Main Cardioid are countably equivalent. Their common perfect part is the unit circle. Other q-laminations with countably many non-degenerate leaves also have $\mathbb{S}$ as their perfect part and, hence, are countably equivalent. We can associate the interior of the filled CA^l to the corresponding class of countable equivalence among q-laminations. In fact, interiors of all baby versions of CA^l can be associated to corresponding classes of countable equivalence among all q-laminations.

3. NON-RENORMALIZABLE GEOLAMINATIONS

In Section 3 we consider another way to modify $\mathcal{M}_2^c$. The aim, again, is to uncover the structure of $\mathcal{M}_2^c$ by replacing more complicated parts of $\mathcal{M}_2^c$ with their simplified "unpinched" versions in which some leaves of Thurston's quadratic minor lamination QML are deleted (i.e., replaced by pairs of their endpoints). In other words, some q-laminations are still considered, but some are not. We explain our selection below.

Suppose that there exist q-laminations $\widehat{\mathcal{L}} \subset \mathcal{L}$ and $\widehat{\mathcal{L}}$ is non-empty. By definition this means that some leaves of $\mathcal{L}$ are contained in gaps of $\widehat{\mathcal{L}}$. Since both are q-laminations, no leaves of $\mathcal{L}$ are in finite gaps of $\widehat{\mathcal{L}}$. Moreover, if a leaf ℓ is inserted in a periodic Siegel gap then the semiconjugacy with an irrational rotation that collapses all edges of this gap will transport this leaf into a chord inside a unit disk on whose boundary the corresponding irrational rotation acts; this shows that ℓ crosses its eventual image, a contradiction. Hence there must exist an n-periodic critical Fatou gap U of $\widehat{\mathcal{L}}$ and all the leaves of $\mathcal{L} \setminus \widehat{\mathcal{L}}$ are contained in gaps of $\widehat{\mathcal{L}}$ from the grand orbit of U; evidently, $\sigma_2|_{\mathrm{Bd}(U)}$ is of degree two. Restricting $\mathcal{L}$ onto U and collapsing all edges of U to points one semiconjugates $\sigma_2^n|_{\mathrm{Bd}(U)}$ and σ_2 (intuitively, this "magnifies" U to the unit circle) and transforms $\mathcal{L}|_U$ to a q-lamination $\mathcal{L}_1$. Then $\mathcal{L}$ is said to be a *tuning* of $\widehat{\mathcal{L}}$ (one can also say that $\mathcal{L}$ *tunes* $\widehat{\mathcal{L}}$), and $\mathcal{L}_1$ is called a *renormalization* of $\mathcal{L}$. In particular, $\mathcal{L}$ is renormalizable; it follows that if a q-lamination is non-renormalizable, then it cannot be a tuning of a non-empty q-lamination. Observe that $\widehat{\mathcal{L}}$ here is a *hyperbolic* q-lamination.

We work with tunings of q-laminations rather than with their renormalizations. If a q-lamination $\mathcal{L}_1$ is a tuning of a q-lamination $\mathcal{L}_2$, then $\mathcal{L}_2$ is said to be an *ancestor* of $\mathcal{L}_1$. We say that $\mathcal{L}_2 \subset \mathcal{L}_1$ is the *oldest ancestor* (of $\mathcal{L}_1$) if every q-lamination $\mathcal{L}_3 \subset \mathcal{L}_2$ is either empty (has no non-degenerate leaves) or coincides with $\mathcal{L}_2$. We want to parameterize the family of all oldest ancestors similarly to QML. By the previous paragraph, a non-renormalizable q-lamination is an oldest ancestor. Observe that all Siegel q-laminations from the Main Cardioid are non-renormalizable, hence they are oldest ancestors (of themselves). On the other hand, hyperbolic oldest ancestors are renormalizable but in a unique way, and their unique renormalizations are empty. Evidently, any oldest hyperbolic ancestor has a critical Fatou gap U. We may say that an oldest ancestor $\mathcal{L}$ replaces all q-laminations that are tunings of $\mathcal{L}$. The entire family of oldest ancestors is denoted by $\mathbb{L}^{nr}$. We will characterize ("tag") *all* q-laminations from $\mathbb{L}^{nr}$ with their postcritical (i.e., minor) sets. In particular, an oldest ancestor with a critical Fatou gap U is tagged with its post-critical Fatou gap $V = \sigma_2(U)$.

Thus, postcritical gaps $V = \sigma_2(U)$ of hyperbolic oldest ancestors, pinched under the equivalence relation $\sim_{\mathrm{QML}}$ in the process of creation of $\mathcal{M}_2^c$, are now "unpinched". It is well-known that pinched gaps V are in fact baby Mandelbrot sets maximal by inclusion among all non-trivial (i.e., not coinciding with $\mathcal{M}_2^c$) baby Mandelbrot sets. Thus, in QML^{nr} baby Mandelbrot sets are replaced by the corresponding infinite gaps.

As before, let us first concentrate upon gaps of QML^{nr} closely related to the Main Cardioid CA. Let $x \in \mathrm{CA}$ be a vertex of CA which is not an endpoint of an edge of CA. Then the q-lamination $\mathcal{L}_x$ corresponding to x has an invariant Siegel gap G and is the oldest ancestor of itself. Hence $\mathcal{L}_x \in \mathbb{L}^{nr}$. Now, let ℓ be an edge of CA. Then the q-lamination $\mathcal{L}_\ell$ associated to ℓ has an invariant finite gap G_ℓ with ℓ as its shortest edge, and the periodic forward orbit of a postcritical Fatou gap V attached to G_ℓ; the grand orbits of G_ℓ and V form the family of all gaps of $\mathcal{L}_\ell$. It follows that the empty q-lamination is the only ancestor of $\mathcal{L}_\ell$, and so $\mathcal{L}_\ell$ belongs to $\mathbb{L}^{nr}$. By construction its tag is the post-critical gap V. Thus, in the center of the geolamination QML^{nr} we have a "countable flower" with CA in the center and countably many postcritical gaps V growing out of CA at its edges. The edges of CA are thus isolated in QML^{nr}. A natural choice is to associate the interior of CA with the empty q-lamination. The gaps V described above are associated with hyperbolic q-laminations from the Main Cardioid; vertices of the Main Cardioid remain vertices of the "countable flower" and are, as before, associated with q-laminations with an invariant Siegel disk.

Let V be a postcritical gap of a hyperbolic oldest ancestor $\mathcal{L}_\sim$. Then V is periodic of some period n, and it is well known that V has a unique edge m of period n, and all other edges of V are pullbacks of ℓ that are not edges of other gaps of QML. We call m the *root edge* of V. It is also well known that such m is the root edge of only one postcritical gap V, and the unique q-lamination associated to m is $\mathcal{L}_\sim$.

Lemma 3.1. *The space* QML^{nr} *is compact. All leaves of* QML^{nr} *not on the boundary of* CA *are non-isolated.*

Thus QML^{nr} is "almost" perfect.

Proof. Let us use the notation and terminology introduced right before Lemma 3.1. Then it is easy to see that the leaves we remove are exactly leaves of QML that intersect the interior of V. Hence the set of all leaves we removed is an open set and its complement is closed. Thus, QML^{nr} is compact.

Suppose that ℓ is a leaf of QML^{nr} that is not on the boundary of CA. Let us prove that ℓ cannot be a common edge of two gaps of QML^{nr}. Indeed, suppose that ℓ is a common edge of a gap G and a gap H. By construction, finite gaps of QML^{nr} are finite gaps of QML. Thus the fact that QML is generated by a laminational equivalence relation $\sim_{\mathrm{QML}}$ implies that at least one of the gaps G, H is infinite. Suppose that G is infinite. Then by construction G is a postcritical n-periodic gap of some oldest ancestor $\mathcal{L}_\sim$. By the remark right before the statement of the lemma, ℓ is either the only edge of G of period n, or a pullback of the only edge m of G of period n. If H is also infinite, we can apply a high iteration of σ_2 to ℓ and obtain that the root edge of H and the root edge of G coincide, a contradiction. If H is finite, then the situation will contradict the remark right before the claim of the lemma. Thus, all possibilities lead to a contradiction. This proves that QML^{nr} is perfect. $\square$

By construction, we associate infinite post-critical gaps of QML^{nr} to renormalizable oldest ancestors; otherwise QML^{nr} consists of finite gaps and leaves that are not edges of any gaps of QML^{nr}. All above listed sets are pairwise disjoint. Moreover, properties of QML (in particular the fact that QML is generated by a laminational equivalence relation $\sim_{\mathrm{QML}}$) imply that this family of sets is upper-semicontinuous. Hence QML^{nr} is in fact generated by an equivalence relation $\sim^{nr}$ that has properties (E1) and (E2) of laminational equivalence relations. Although we may choose $\sim^{nr}$ to also satisfy property (E3) (stating that all classes are finite), it would be more natural to admit infinite classes of $\sim^{nr}$. Namely, we assume that infinite post-critical gaps V of QML^{nr} corresponding to hyperbolic oldest ancestors give rise to infinite classes $V \cap \mathbb{S}$. Call an equivalence relation $\sim$ on the circle a *laminational equivalence relation with possibly infinite classes* if it has properties (E1) and (E2) of Definition 1.1. Then it follows that QML^{nr} is generated by a laminational equivalence relation with possibly infinite classes $\sim^{nr}$. Moreover, by Lemma 3.1, all infinite gaps of QML^{nr} are convex hulls of $\sim^{nr}$-classes. It follows that the corresponding quotient space $\mathbb{S}/\sim^{nr} = \mathcal{M}_2^{nr}$ is a dendrite.

Finally, similarly to how we reinterpreted the q-lamination QML^l and the corresponding quotient space $\mathcal{M}_2^l$ using the notion of countable equivalence, we can reinterpret $\mathcal{M}_2^{nr}$ as follows. Call two q-laminations *common ancestor equivalent* if they have the same oldest ancestor. Then the space $\mathcal{M}_2^{nr}$ can be viewed as the quotient space of the combinatorial Mandelbrot set $\mathcal{M}_2^c$ under the common ancestor equivalence; the corresponding quotient map from $\mathcal{M}_2^c$ to $\mathcal{M}_2^{nr}$ simply collapses to points all maximal by inclusion baby Mandelbrot sets.

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