

PERFECT SUBSPACES OF QUADRATIC LAMINATIONS

ALEXANDER BLOKH, LEX OVERSTEEGEN, AND VLADLEN TIMORIN

Dedicated to the memory of Tan Lei

ABSTRACT. The combinatorial Mandelbrot set is a continuum in the plane, whose boundary can be defined, up to a homeomorphism, as the quotient space of the unit circle by an explicit equivalence relation. This equivalence relation was described by Douady and, in different terms, by Thurston. Thurston used quadratic invariant laminations as a major tool. As has been previously shown by the authors, the combinatorial Mandelbrot set can be interpreted as a quotient of the space of all limit quadratic invariant laminations. The topology in the space of laminations is defined by the Hausdorff distance. In this paper, we describe two similar quotients. In the first case, the identifications are the same but the space is smaller than that taken for the Mandelbrot set. The result (the quotient space) is obtained from the Mandelbrot set by “unpinching” the transitions between adjacent hyperbolic components. In the second case, we do not identify non-renormalizable laminations while identifying renormalizable laminations according to which hyperbolic lamination they can be “unrenormalised” to.

INTRODUCTION

Understanding families of one-dimensional complex polynomials is a central objective in complex dynamics. An important step towards this objective involves constructing combinatorial models for such families. The most famous case when such a model is known is the *quadratic family*, i.e., the family of quadratic polynomials $P_c(z) = z^2 + c$. The set of all parameters c such that P_c has connected Julia set (equivalently, the critical P_c -orbit $0, P_c(0) = c, \dots$, is bounded) is called the *Mandelbrot set* and is denoted by \mathcal{M}_2 . In his seminal preprint [Thu85], William Thurston introduced a variety of (mostly combinatorial and geometric by nature) new tools and constructed a combinatorial geometric model \mathcal{M}_2^c of \mathcal{M}_2 . There exists a

Date: July 17, 2017.

2010 Mathematics Subject Classification. Primary 37F20; Secondary 37F10, 37F50.

Key words and phrases. Complex dynamics; laminations; Mandelbrot set; Julia set.

The third named author was partially supported by the Russian Academic Excellence Project '5-100'.

monotone map from \mathcal{M}_2 onto \mathcal{M}_2^c . A major open conjecture in complex dynamics, referred to as MLC, states that this map is a homeomorphism.

The set \mathcal{M}_2^c has a very rich and fascinating structure. It contains a countable and dense family of homeomorphic copies of itself whose precise location can be deduced from the description of \mathcal{M}_2^c . Thus, \mathcal{M}_2^c is an example of a so-called *fractal* set. According to Adrien Douady, the process of constructing \mathcal{M}_2^c can be described as “pinching the closed unit disk $\overline{\mathbb{D}}$ ” which is why \mathcal{M}_2^c is often said to be the “pinched disk model” of \mathcal{M}_2 . “Pinching” refers to collapsing a certain chord of $\overline{\mathbb{D}}$; evidently, each act of pinching creates an increasingly complicated new quotient space of $\overline{\mathbb{D}}$. Therefore it is natural to attempt to understand the structure of the “pinched disk model” by doing only some of the pinchings and ignoring the other ones. Somewhat loosely, one can say that in order to understand \mathcal{M}_2^c , one may try to partially “unpinch” \mathcal{M}_2^c and thus obtain partial quotient spaces of $\overline{\mathbb{D}}$ as steps towards understanding \mathcal{M}_2^c .

This by itself provides a motivation for studying partially “unpinched” versions of \mathcal{M}_2^c . There is however another no less important thought that motivated the authors. Namely, while in the quadratic case the problem of constructing an adequate combinatorial model of \mathcal{M}_2 was solved by Thurston in [Thu85], producing a similar combinatorial model in the cubic case, let alone in the arbitrary degree d case, presents a difficult problem that has not yet been solved. Partial quotients of $\overline{\mathbb{D}}$ admit cubic analogs. The latter may serve as simplified models of the cubic connectedness locus.

We begin with an informal overview. We assume basic knowledge of complex dynamics. Later on, especially in the last two sections of the paper, we will also use well-known facts concerning the structure of the combinatorial Mandelbrot set \mathcal{M}_2^c .

Laminational equivalence relations. *Laminational equivalence relations* are closed equivalence relations \sim on the unit circle \mathbb{S} in the complex plane \mathbb{C} such that all classes are finite and the convex hulls of all classes are pairwise disjoint; denote the corresponding quotient map of \mathbb{S} by ψ_\sim . Notice that ψ_\sim can be canonically extended over \mathbb{C} by assuming that all classes of points of \mathbb{C} outside $\overline{\mathbb{D}}$ are these points themselves while inside $\overline{\mathbb{D}}$ any convex hull of a \sim -class is declared to be one class of equivalence while the points inside infinite gaps of \mathcal{L}_\sim are treated in the same way as points outside of $\overline{\mathbb{D}}$, i.e. are declared to be degenerate classes. This extends \sim and ψ_\sim onto the entire complex plane. We are interested in quotient spaces of $\overline{\mathbb{D}}$ and \mathbb{S} under the map ψ_\sim .

Each laminational equivalence relation \sim gives rise to an associated geometric object \mathcal{L}_\sim called a *geodesic lamination generated by \sim* . It is the set of all chords (called *leaves*) in the boundaries of the convex hulls (in the

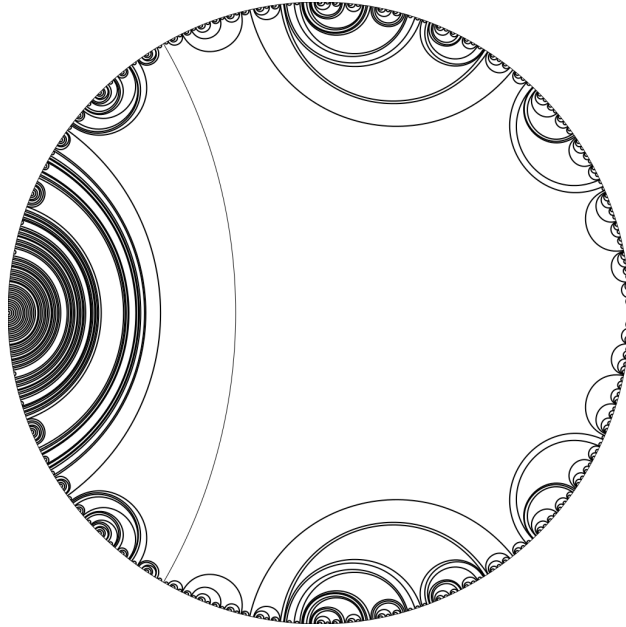


FIGURE 1. The geodesic lamination QML

closed unit disk) of all equivalence classes of \sim . The union of all leaves of \mathcal{L}_\sim and \mathbb{S} is a continuum \mathcal{L}_\sim^+ called the *solid* of \mathcal{L}_\sim (or of \sim). The closure in \mathbb{C} of a non-empty component of $\mathbb{D} \setminus \mathcal{L}_\sim^+$ is called a *gap* of \mathcal{L}_\sim . If G is a gap or a leaf, call the set $G' = \mathbb{S} \cap G$ the *basis* of G . A gap is said to be *finite* (*infinite*, *countable*, *uncountable*) if its basis is finite (infinite, countable, uncountable); countable gaps are only possible for generally defined geodesic laminations (see below) but not for the just defined geodesic lamination generated by \sim . For \mathcal{L}_\sim infinite gaps are associated with bounded complementary domains of \mathbb{S}/\sim by the corresponding quotient map.

Sometimes we relax the assumption of finiteness of classes of equivalence and talk of laminational equivalence relations *possibly with infinite classes* [BOPT17]. All related notions can be defined for either laminational equivalence relations or for laminational equivalence relations possibly with infinite classes, thus in what follows we only define them for laminational equivalence relations. In particular, this concerns the associated quotient maps and quotient spaces. In case when we deal with a laminational equivalence relation \sim , all the infinite gaps of \sim give rise to bounded complementary domains of the corresponding quotient space.

Assume now that \mathcal{L}_\sim is *perfect* (i.e., there are no isolated leaves in \mathcal{L}_\sim). Then, no two gaps of \mathcal{L}_\sim intersect. There are various laminational equivalence relations possibly with infinite classes that generate \mathcal{L}_\sim . This is because some infinite gaps of \mathcal{L}_\sim can be declared convex hulls of equivalence

classes (recall that \mathcal{L}_\sim is perfect!). Now, consider the laminational equivalence relation \sim^d in which *all* infinite gaps of \mathcal{L}_\sim are declared convex hulls of classes (equivalently, for each infinite gap G of \mathcal{L}_\sim its basis G' is declared one class). It is easy to see that $\mathbb{S}/\sim^d = \overline{\mathbb{D}}/\sim^d$ is a *dendrite* (a locally connected continuum without subsets homeomorphic to a circle). This dendrite can be obtained from $\overline{\mathbb{D}}/\sim$ by collapsing the closures of all bounded complementary domains of \mathbb{S}/\sim (recall that they are associated to infinite gaps of \mathcal{L}_\sim) to points. We call \sim^d and $\overline{\mathbb{D}}/\sim^d$ the *dendritic version* of \sim and $\overline{\mathbb{D}}/\sim$.

A laminational equivalence relation is (σ_d) -invariant if it is “respected” by the map $\sigma_d(z) = z^d : \mathbb{S} \rightarrow \mathbb{S}$; a formal definition will be given later. The map σ_d induces a *topological polynomial* $f_\sim : \mathbb{S}/\sim \rightarrow \mathbb{S}/\sim$ from the *topological Julia set* $J_\sim = \mathbb{S}/\sim$ to itself. If, for a polynomial P , its Julia set $J(P)$ is locally connected, then $J(P)$ can be identified with the topological Julia set \mathbb{S}/\sim of some laminational equivalence relation \sim such that ψ_\sim identifies with the extension to the unit circle of the normalized Riemann map of the basin of infinity of P . Thus, in this case, $P|_{J(P)}$ is topologically conjugate to (and even identifies with) f_\sim .

From now on, “invariant” means “invariant under the map σ_d ”. In particular, given a chord $\ell = \overline{ab} \in \mathcal{L}_\sim$, with endpoints $a, b \in \mathbb{S}$, the chord $\overline{\sigma_d(a)\sigma_d(b)}$ is also a (possibly degenerate) chord of \mathcal{L}_\sim , and we write $\sigma_d(\ell) = \overline{\sigma_d(a)\sigma_d(b)}$. Not only should the collection \mathcal{L}_\sim of chords be invariant but also boundaries of complementary to \mathcal{L}_\sim domains of $\overline{\mathbb{D}}$ should map forward in the orientation preserving fashion (except that some edges may collapse to points).

Invariant geodesic laminations. Although σ_d is only defined on the unit circle \mathbb{S} , it makes sense to talk about the images $\sigma_d(X)$ for subsets X of the closed unit disk obtained as convex hulls of $X \cap \mathbb{S}$; by definition, we set $\sigma_d(X)$ to be the convex hull of $\sigma_d(X \cap \mathbb{S})$. Thurston in his famous paper [Thu85] introduced and studied geometric objects similar to \mathcal{L}_\sim but more general, and called them *invariant geodesic laminations*. Thus, σ_d -invariant geodesic laminations \mathcal{L} are closed collections of chords in $\overline{\mathbb{D}}$ (*leaves*) with special properties relating the leaves and the map σ_d . It will be convenient to consider also unions of all leaves of \mathcal{L} together with all points of \mathbb{S} ; such unions are denoted \mathcal{L}^+ and are called *solids* of laminations. For any laminational equivalence relation \sim , the set \mathcal{L}_\sim is a σ_d -invariant geodesic lamination. Such geodesic laminations are called q -laminations (q from “equivalence”).

An important reason for defining geodesic laminations is to provide “visualization” of laminational equivalence relations and use the former to

“topologize” the latter. This is done by defining *Hausdorff metric* on geodesic laminations. To this end, consider the space of all chords (including degenerate ones) in the unit disk. Equivalently, we can consider the space of all unordered pairs of points of \mathbb{S} (so that a pair (α, β) is identified with the pair (β, α)). It follows that the space in question is a “half-torus” E . If a topological 2-torus is represented as a quotient space of the square $0 \leq x, y \leq 1$ by the equivalence relation identifying $(x, 0)$ with $(x, 1)$ and $(0, y)$ with $(1, y)$ for all $x, y \in [0, 1]$, then E is given by $x \leq y$ with the additional identification of $(0, x)$ with $(x, 1)$. Topologically, this is a Möbius strip. Every geodesic lamination \mathcal{L} can be viewed as a closed subset of E (each leaf of \mathcal{L} is a point of E). Define the Hausdorff distance between two geodesic laminations $\mathcal{L}_1, \mathcal{L}_2$ by viewing \mathcal{L}_1 and \mathcal{L}_2 as subsets of E . When speaking of geodesic laminations, we will always consider the topology on them defined by the Hausdorff metric. We will speak of limits of laminations only in this sense.

If $d = 2$, then the corresponding geodesic laminations, laminational equivalence relations, topological polynomials and Julia sets are said to be *quadratic*. However, for clarity of terminology, in this paper we address the corresponding geodesic laminations and laminational equivalence relations as σ_2 -invariant. Thurston [Thu85] introduced and studied properties of individual σ_d -invariant laminations while also obtaining striking results on the entire family of σ_2 -invariant geodesic laminations and parameterizing them with the help of his famous *quadratic minor lamination* QML.

For every σ_2 -invariant geodesic lamination \mathcal{L} , Thurston defines its *major* M as a longest leaf of \mathcal{L} . It is easy to see that either \mathcal{L} has a unique major, which is then a diameter of $\overline{\mathbb{D}}$, or \mathcal{L} has two distinct majors with equal σ_2 -images. Then Thurston defines $m = \sigma_2(M)$, calls it the *minor* of \mathcal{L} , and shows that *the family of the minors of all σ_2 -invariant geodesic laminations is a geodesic lamination itself!* This geodesic lamination is the *quadratic minor lamination* QML that can be generated by an equivalence relation \sim_{QML} . Each class of \sim_{QML} is associated with a unique σ_2 -invariant laminational equivalence relation and the corresponding topological polynomial. The quotient space $\mathbb{S}/\sim_{\text{QML}}$ parameterizes the space of all σ_2 -invariant laminational equivalence relations (equivalently, the space of all topological polynomials of degree two). This yields a set \mathcal{M}_2^ξ in the plane, defined up to a homeomorphism, whose boundary $\text{Bd}(\mathcal{M}_2^\xi)$ is homeomorphic to $\mathbb{S}/\sim_{\text{QML}}$.

Geodesic laminations were introduced not only to study dynamics of individual complex polynomials but also to construct combinatorial (laminational) models for spaces of complex polynomials. This is done with the help of the following approach. Firstly, with each complex polynomial from

the space under consideration, associate its laminational equivalence relation and the corresponding topological polynomial. Observe that, while this is straightforward in the locally connected case and in some other cases (see below), in case of more complicated topology of the Julia sets a special construction may be required. Secondly, describe (parameterize) the space of topological polynomials associated with all complex polynomials from the chosen space, and use the thus obtained space of topological polynomials as the model for the given space of complex polynomials.

In order to implement this plan, one has to overcome several difficulties. First, the association of a laminational equivalence relation with a complex polynomial may be hard. This is related to the fact that the Julia set $J(P)$ of a complex polynomial P may have complicated topology. Still, in many cases P gives rise to a lamination, even if $J(P)$ is not locally connected. For example, this happens in the case when $J(P)$ is connected and has no periodic non-repelling points. In this case, the association is due to Jan Kiwi [Kiw04]. The corresponding topological Julia sets are dendrites, and the corresponding complex polynomials are called *dendritic*. Notice that the actual Julia set of a dendritic polynomial does not have to be a dendrite.

Known facts from polynomial dynamics imply that dendritic polynomials are dense in the boundary of the Mandelbrot set. Thus, one can extend the correspondence between complex quadratic dendritic polynomials and their laminational equivalence relations onto $\text{Bd}(\mathcal{M}_2)$. We mean the extension by continuity, or the closure of the correspondence. This closure fails to be a map; one polynomial may correspond to several laminations.

Let us now discuss details of the above outlined plan focusing on the case of quadratic polynomials. A difficulty here is that geodesic laminations associated to laminational equivalence relations do not form a closed subspace. In [BOPT16a], we resolve this issue by taking the closure $\overline{\mathbb{L}}_2^q$ of the space \mathbb{L}_2^q of all q -laminations. Clearly, $\overline{\mathbb{L}}_2^q$ includes all q -laminations as well as limits of non-constant sequences of q -laminations.

Limit laminations. We proceed in [BOPT16a] by following Thurston [Thu85] and defining *minors* $m(\mathcal{L})$ of geodesic laminations from $\overline{\mathbb{L}}_2^q$. Two geodesic laminations $\mathcal{L}_1, \mathcal{L}_2 \in \overline{\mathbb{L}}_2^q$ are identified if $m(\mathcal{L}_1) \cap m(\mathcal{L}_2) \neq \emptyset$. This yields the associated quotient space of $\overline{\mathbb{L}}_2^q$ which is proven in [BOPT16a] to be homeomorphic to \mathcal{M}_2^c . However, we want to argue that there is another similar quotient space of a certain subspace of $\overline{\mathbb{L}}_2^q$ that is, from the point of view of geodesic laminations, in some sense even more natural than \mathcal{M}_2^c (viewed as a quotient space of $\overline{\mathbb{L}}_2^q$). Indeed, the space $\overline{\mathbb{L}}_2^q$ may contain isolated geodesic laminations. As we visualize the space $\overline{\mathbb{L}}_2^q$, isolated laminations can be placed anywhere in the picture. In other words, their location is

a matter of convenience and does not reflect the true structure of the space. Therefore, it is of interest to consider the space of all non-isolated geodesic laminations from $\overline{\mathbb{L}}_2^q$ with the same identification as above (two are identified if their minors are non-disjoint), and to describe the corresponding quotient space.

Limit laminations are defined as non-isolated points in $\overline{\mathbb{L}}_2^q$. It is proven in [BOPT16a] that a σ_2 -invariant limit lamination \mathcal{L} must contain a critical quadrilateral or a critical leaf. However, \mathcal{L} having a critical quadrilateral or leaf is only a necessary condition. In this paper, we give sufficient conditions and thus describe the entire space of limit laminations, which will be denoted by \mathbb{L}_2^l (“l” from “limit”). It turns out that the sufficient condition on \mathcal{L} is the existence of a q -lamination \mathcal{L}_\sim satisfying property (0) below and one of properties (1)–(3).

(0) The critical set of \mathcal{L} is included into the critical set of \mathcal{L}_\sim .

(1) The lamination \mathcal{L}_\sim is dendritic with a critical set X ; in this case, \mathcal{L} is obtained from \mathcal{L}_\sim by inserting a critical leaf/quadrilateral Q into X and inserting pullbacks of Q into pullbacks of X . Such geodesic lamination is a limit lamination. A *critical leaf* is by definition a diameter of the unit circle. A *critical quadrilateral* is by definition the convex hull of two critical leaves.

(2) The lamination \mathcal{L}_\sim has a periodic Siegel gap, and $\mathcal{L} = \mathcal{L}_\sim$.

(3) There is a periodic Fatou gap U of \mathcal{L}_\sim of period n that maps forward two-to-one under σ_2 . It is well known [Thu85] that then there exists a unique non-degenerate periodic edge M of U of exact σ_2 -period n . The leaf M is a *major* of \sim while $m = \sigma_2(M)$ is the *minor* of \sim (and of \mathcal{L}_\sim). There are two possible subcases.

(a) The leaves $M, \sigma_2(M), \dots, \sigma_2^{n-1}(M)$ are pairwise disjoint. The critical set of \mathcal{L} is either a critical leaf sharing an endpoint with M or the critical quadrilateral having M as an edge. All these laminations \mathcal{L} can be described explicitly, and all of them are identified.

(b) The leaves $M, \sigma_2(M), \dots, \sigma_2^{n-1}(M)$ are not pairwise disjoint. The lamination \mathcal{L} has a critical diameter sharing an endpoint with M .

Thus, the minors of laminations \mathcal{L} of type b) are endpoints of $m = \sigma_2(M)$. These laminations \mathcal{L} break down into two classes corresponding to the endpoints of m . These are distinct classes because no limit geodesic lamination has a critical quadrilateral with edge M .

Laminations of type b) are exactly laminations that correspond to points of \mathcal{M}_2^c at which the closures of two distinct hyperbolic domains in \mathcal{M}_2^c meet. According to the description in b), corresponding leaves in QML must be erased and replaced by their endpoints. Mimicking Thurston, we can call this process *cleaning of QML*. This is the only modification of

QML necessary to transform \sim_{QML} into a laminational equivalence relation parameterizing the above described quotient space of the space of all σ_2 -invariant limit laminations.

By the known properties of QML, this cleaning produces a new geodesic lamination QML^l with no isolated leaves (and hence perfect). The corresponding quotient space will be denoted by \mathcal{M}_2^l . Informally, one can say that while the “pinched disk model” of \mathcal{M}_2^c is obtained by pinching $\overline{\mathbb{D}}$ at various places prescribed by QML, we can then obtain \mathcal{M}_2^l by “unpinching” \mathcal{M}_2^c at places prescribed by $\text{QML} \setminus \text{QML}^l$. Alternatively, one can say that \mathcal{M}_2^l is obtained if certain parts of \mathcal{M}_2^c are replaced by topological disks. Two interior components of \mathcal{M}_2^c are said to be *associated* if they are connected by a chain of complementary components touching at boundary points. The closures of maximal unions of pairwise associated components are to be replaced with topological disks.

An alternative description of the space \mathbb{L}_2^l is that the latter consists of limits of sequences of *dendritic* geodesic laminations. It is easy to see that now we do not have to insist that the sequences be non-constant. Observe also that there are other, alternative ways to describe the laminations from \mathbb{L}_2^l by describing their properties intrinsically and without referring to the limits of geodesic laminations. Indeed, all laminations from \mathbb{L}_2^l have no isolated majors *unless* the critical set is just a critical leaf (as is the case of a Siegel gap). This can be turned into a characterization of \mathbb{L}_2^l .

After “unpinching”, the Main Cardioid CA turns into a bigger gap denoted here by CA^l . In fact, we can think of “unpinching” as follows. On the first step we erase all non-degenerate edges of the Main Cardioid itself. On the second step we erase non-degenerate edges in the copies of the Main Cardioid that used to be attached to the Main Cardioid. If we go on with this, we will in the end obtain CA^l . Observe that some degenerate edges of CA^l are obtained on finite steps in the process. These are endpoints of edge erased after a finite number of steps or Siegel points on the boundary of some copy of the Main Cardioid.

Evidently, the only remaining edges of CA^l that are not obtained after a finite number of steps are infinitely-renormalizable and are limits of sequences of non-degenerate edges of deeper and deeper copies of CA associated with CA. Some of these edges are non-degenerate, others are degenerate. Let us characterize those minors of infinitely renormalizable laminations that are non-degenerate (we provide here only a sketch of the proof). To this end we need a well-known description of canonical copies of \mathcal{M}_2^c contained in \mathcal{M}_2^c (sometimes these are informally called “baby Mandelbrot sets”).

Indeed, these copies are obtained by first choosing specific periodic (of period, say, k) infinite gaps V that are σ_2 -images of quadratic Fatou gaps. The standard monotone collapse of edges of V semiconjugates $\sigma_2^k|_{\text{Bd}(V)}$ with σ_2 (see details in Section 3). Using this correspondence, we lift QML to V . It turns out that the lift in fact coincides with the restriction of QML onto V . If we take the associated quotient space of V , then we will obtain a copy of \mathcal{M}_2^c inside \mathcal{M}_2^c . In other words, one chooses a special gap $V \subset \mathbb{D}$ and then considers a “pinched” version of V using QML transferred to V and inducing the corresponding lamination in V . This is what makes \mathcal{M}_2^c a *fractal*, i.e., a set that contains infinitely many homeomorphic copies of itself.

The part of \mathcal{M}_2^c important for us is the real line (i.e., one can draw the real line and consider its segment contained in \mathcal{M}_2^c). There are two equivalent ways of describing this part of \mathcal{M}_2^c . First, observe that all minors that intersect the real line are *vertical* and vice versa so that the fact that a minor is vertical is equivalent to it belonging to the real line in \mathcal{M}_2^c . To define the second way of describing the real line inside \mathcal{M}_2^c we need the notion of the *growing tree of f_\sim* [BL02]. It is defined as follows. In the topological Julia set J_\sim we connect the two points associated with angles 0 and $1/2$ (the unique fixed angle for σ_2 and its preimage). Denote this connecting arc I . Then consider the union of all images of I under f_\sim by T_∞ and call it the *growing tree of f_\sim* . It is easy to see ([BL02]) that the set T_∞ is an interval if and only if the minor of \mathcal{L}_\sim is vertical.

Now, if \mathcal{L}_\sim is infinitely renormalizable then J_\sim is a dendrite. Suppose that \mathcal{L}_\sim is associated with a real polynomial (equivalently, the two majors of \mathcal{L}_\sim are vertical). Clearly, these majors cannot coincide (and be critical leaves) as the only vertical diameter is $\frac{1}{4}\sqrt{\frac{3}{4}}$ which does not correspond to an infinitely renormalizable lamination. Thus, the critical set of \mathcal{L}_\sim is a critical quadrilateral with vertical edges and the minor of \mathcal{L}_\sim is non-degenerate. It follows, that if we choose a canonical copy of \mathcal{M}_2^c inside \mathcal{M}_2^c and then choose similar place inside this copy, the corresponding minor will be non-degenerate. For brevity let us call minors of this kind *renormalized real*. Thus, for every renormalized real minor of an infinitely renormalizable lamination \mathcal{L}_\sim is non-degenerate.

To prove the converse, consider J_\sim . If the critical set of \mathcal{L}_\sim is a non-degenerate critical quadrilateral Q , it follows that $f_\sim(c)$ is a cutpoint of J_\sim (here c is the critical point of f_\sim). Hence by [BL02] there exists a finite f_\sim -invariant tree T that contains c . If we consider sufficiently high renormalization of f_\sim it will have to be defined on the interval because there are only finitely many branchpoints in T . This implies that for this renormalization its growing tree is an interval. By the above this implies that the

corresponding minor is real. To summarize, the minor of an infinitely renormalizable lamination \mathcal{L}_\sim is non-degenerate if and only if it is renormalized real.

The geodesic lamination QML^l is a visual counterpart of a certain equivalence relation on the circle. As QML^l is perfect, there are two main alternative equivalence relations, \sim^l and \sim^{ld} , that generate the same geodesic lamination QML^l . Here \sim^l is a laminational equivalence relation (so that all its classes are finite), and since QML^l is perfect, \sim^l is well-defined. By construction, QML^l has a variety of pairwise disjoint infinite gaps. The interpretation of QML^l as the geodesic lamination generated by a laminational equivalence relation is based on the assumption that all classes are finite so that in the quotient, infinite gaps of QML^l give rise to bounded complementary domains of \mathcal{M}_2^l . As explained above, in an alternative dendritic version of QML^l we declare *every* gap of QML^l to be the convex hull of an equivalence class. This results into a well-defined laminational equivalence relation with possibly infinite classes \sim^{ld} . Hence, one can define yet another quotient space $\mathbb{S}/\sim^{ld} = \mathcal{M}_2^{ld}$, a dendritic version of \mathcal{M}_2^l , by collapsing the closures of all bounded complementary domains to \mathcal{M}_2^l .

This purely topological construction in fact defines a certain quotient space of \mathbb{L}_2^q (or $\overline{\mathbb{L}_2^q}$, or \mathbb{L}_2^l). Namely, recall that the *perfect part* of a geodesic lamination \mathcal{L} is obtained by taking the maximal perfect subset of \mathcal{L} . In particular, all isolated leaves of \mathcal{L} must be erased as we extract the perfect part of \mathcal{L} . Now, say that two geodesic laminations are *countably equivalent* if they have the same perfect parts (equivalently, if the symmetric difference between them is countable).

For instance, all laminations from the Main Cardioid are countably equivalent. Their common perfect part is the unit circle. Other countable laminations also have \mathbb{S} as their perfect part. All countable laminations are countably equivalent. However countable equivalence relation defined above is not closed. If we close it and consider the resulting class of equivalence of all countable laminations, we will get the class coinciding with the boundary of CA^l intersected with the unit circle (recall that when we talk about classes of equivalence we consider points of the unit circle). Slightly abusing the language, let us call the just defined closed equivalence relation *countable equivalence relation* and say that two geodesic laminations are *countably equivalent* if they belong to the same class in the sense of countable equivalence relation. Then CA^l is the convex hull of the class of countable equivalence relation corresponding to the Main Cardioid. The quotient space of the unit disk/circle under the countable equivalence relation is $\mathbb{S}/\sim^{ld} = \mathcal{M}_2^{ld}$ defined above.

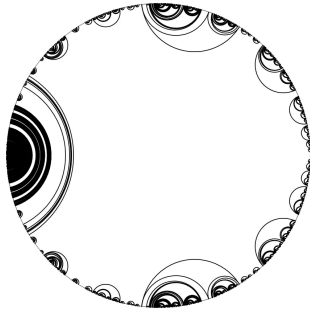


FIGURE 2. The lamination QML^l

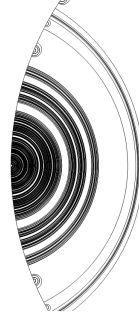


FIGURE 3. A zoom-in of QML^l

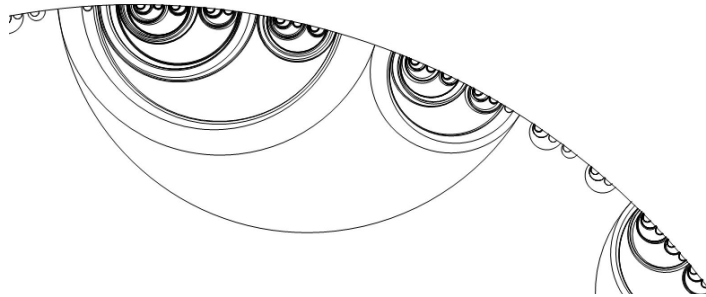


FIGURE 4. Another zoom-in of QML^l

Parameter space of non-renormalizable laminations. In the last section of the paper, Section 3, we consider another way to modify \mathcal{M}_2^c . The aim, as in the cases of \mathcal{M}_2^l and \mathcal{M}_2^{ld} above, is to uncover the structure of \mathcal{M}_2^c by replacing more complicated parts of \mathcal{M}_2^c by their simplified “unpinched” versions. Recall that “unpinching” works as follows. For certain leaves ℓ of \mathcal{M}_2^c , we allow laminations from $\overline{\mathbb{L}_2^q}$ with critical leaves sharing endpoints with ℓ and not allow laminations with the critical quadrilateral supported by ℓ . The name of the game is which critical quadrilaterals are allowed and which are not.

We now want to remove from QML all non-degenerate leaves corresponding to renormalizable laminations. Renormalizable q -laminations can be described as laminations \mathcal{L}_\sim such that J_\sim admits *non-trivial* periodic subcontinua (under the action of f_\sim). Equivalently, \mathcal{L}_\sim is renormalizable if \mathcal{L}_\sim^+ has a *non-trivial* σ_2 -periodic closed connected subset. Here “non-trivial” means non-degenerate and not equal to the entire Julia set (or, in the second case, not projected to a trivial continuum under the quotient map).

Now, suppose that \mathcal{L}_1 is a renormalization of \mathcal{L}_2 . It means the following. First, there exists a lamination $\widehat{\mathcal{L}}_2$ with an n -periodic critical Fatou gap U .

The lamination \mathcal{L}_2 contains $\widehat{\mathcal{L}}_2$ (i.e., $\widehat{\mathcal{L}}_2 \subset \mathcal{L}_2$) so that all the leaves of $\mathcal{L}_2 \setminus \widehat{\mathcal{L}}_2$ are contained in gaps of $\widehat{\mathcal{L}}_2$ that belong to the grand orbit of U . Now, restrict \mathcal{L}_2 onto U and, similar to the above, collapse all edges of U to points. This semiconjugates $\sigma_2^n|_{\text{Bd}(U)}$ and σ_2 (intuitively, one can say that this “magnifies U to the unit circle”) and transforms $\mathcal{L}_2|_U$ to \mathcal{L}_1 . In this setting \mathcal{L}_2 is said to be a *tuning* of $\widehat{\mathcal{L}}_2$.

In what follows it is easier for us to work with tunings of geodesic laminations rather than with their renormalizations (even though these two approaches are clearly equivalent). Thus, given a geodesic lamination \mathcal{L}_1 we may be able to find a lamination \mathcal{L}_2 with $\mathcal{L}_1 \supset \mathcal{L}_2$ such that \mathcal{L}_1 is a tuning of \mathcal{L}_2 . Now, \mathcal{L}_2 can itself be a tuning of a lamination $\mathcal{L}_3 \subset \mathcal{L}_2$, etc. If we go on with this, on the n -th step we will obtain a sequence of geodesic laminations $\mathcal{L}_1 \supset \mathcal{L}_2 \supset \mathcal{L}_3 \supset \dots \supset \mathcal{L}_{n+1}$ so that \mathcal{L}_i is a tuning of $\mathcal{L}_{i+1} \subset \mathcal{L}_i$ as long as $i \leq n$. In the obvious sense $\mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_{n+1}$ are *ancestors* of \mathcal{L}_1 while \mathcal{L}_1 is an *offspring* of $\mathcal{L}_2, \dots, \mathcal{L}_{n+1}$. Since we want to “unpinch” all minors associated with renormalizable laminations, we need to move along this “genealogical tree” as far back as possible in order to find the *oldest ancestor* of a given geodesic lamination $\mathcal{L} = \mathcal{L}_1$.

There is an important detail that should not be overlooked here. Namely, every geodesic lamination is a tuning of the *empty* lamination (the one without non-degenerate leaves). This trivializes the issue of looking for the oldest ancestor as the empty lamination is the oldest ancestor for all laminations. Thus, we need to decide in what cases the empty lamination can(not) be considered as the oldest ancestor of a lamination. To this end, consider a hyperbolic lamination \mathcal{L} with critical Fatou gap U such that \mathcal{L} is *not* a tuning of any non-empty lamination. (A hyperbolic lamination is a lamination corresponding to a hyperbolic polynomial). Making one step back from \mathcal{L} to the empty lamination should not be allowed if, as a result, a substantial piece of information is lost. Thus, if \mathcal{L} is uncountable, we consider \mathcal{L} as the *oldest ancestor lamination* and call \mathcal{L} *non-trivial*. However, if \mathcal{L} is countable (recall that \mathcal{L} is assumed to not be a tuning of any non-empty lamination so that its countability implies that \mathcal{L} belongs to the Main Cardioid) we consider the empty lamination as the corresponding *oldest ancestor lamination*. This defines the family of all oldest ancestor laminations. The empty lamination is called the *trivial* oldest ancestor lamination.

Consider now a new family of laminations. We keep the non-renormalizable laminations in it. Otherwise for any non-trivial oldest ancestor lamination with critical Fatou gap U we consider all possible laminations with critical leaves in U or critical quadrilaterals based upon edges of U . We will still characterize (“tag”) our laminations with their minors. Thus, postcritical gaps V of non-trivial oldest ancestor laminations described above and

pinched under the equivalence relation \sim_{QML} in the process of creation of \mathcal{M}_2^c are now completely “unpinched”. Thus, gaps V must be present in the parametric lamination QML^{nr} that we are now constructing. Notice that the resulting lamination QML^{nr} does not have any isolated leaves. As usual, denote by \sim^{nr} the laminational equivalence relation generating QML^{nr} .

Now consider all *countable* laminations \mathcal{L} that are not tunings of non-empty laminations. Recall that we associate with them the empty lamination as their oldest ancestor. As above, we insert all possible critical leaves into them and all possible critical quadrilaterals into them, with one exception. Namely, if U is a critical gap of \mathcal{L} of period n then we do *not* insert a critical quadrilateral in U supported by the edge of U of period n . This corresponds to the fact that all countable laminations \mathcal{L} that are not tunings of non-empty laminations have the same empty oldest ancestor and therefore should be associated with the same gap CA^{nr} in the parameter space.

In QML^{nr} the gap CA^{nr} replaces the Main Cardioid. It plays a special role in QML^{nr} . To describe its edges and vertices, take all edges and vertices of all postcritical Fatou gaps of laminations from the Main Cardioid — except for the edges of the same period as the Fatou gap in question (i.e., except for the edges that are edges of the Main Cardioid itself). Add to this list all vertices of the Main Cardioid associated with Siegel laminations. This collection of edges and vertices forms the boundary of CA^{nr} . Basically this means that to create CA^{nr} we erase all minors of QML contained in postcritical Fatou gaps associated with laminations from the Main Cardioid, and also all edges of the Main Cardioid.

As with limit laminations and their space, alternatively to \sim^{nr} in the situation of Section 3 we can also identify all points of \mathbb{S} on the boundaries of infinite gaps of QML^{nr} . This creates a laminational equivalence relation with possibly infinite classes \sim^{nrd} called the *dendritic version* of \sim^{nrd} . In terms of quotient spaces this will lead to the collapse of all bounded complementary domains of \mathcal{M}_2^{nr} to points yielding a new, dendritic, parameter space denoted by \mathcal{M}_2^{nrd} .

1. PRELIMINARIES

A part of this section is devoted to geodesic laminations, a major tool in studying the dynamics of individual complex polynomials as well as in modeling certain families of complex polynomials. Let $a, b \in \mathbb{S}$. By $[a, b]$, (a, b) , etc., we mean the closed, open, etc., *positively oriented* circle arcs from a to b , and by $|I|$ the normalized length of an arc I in \mathbb{S} (a normalization is made so that the length of \mathbb{S} is 1).

1.1. Laminational equivalence relations. Denote by $\widehat{\mathbb{C}}$ the Riemann sphere. For a compactum $X \subset \mathbb{C}$, let $U^\infty(X)$ be the component of $\widehat{\mathbb{C}} \setminus X$ containing

infinity. If X is connected, there exists a Riemann mapping $\Psi_X : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \rightarrow U^\infty(X)$; we always normalize it so that $\Psi_X(\infty) = \infty$, and $\Psi'_X(z)$ tends to a positive real limit as $z \rightarrow \infty$.

Consider a monic polynomial P of degree $d \geq 2$, i.e., a polynomial of the form $P(z) = z^d +$ lower order terms. Consider the Julia set J_P of P and the filled-in Julia set K_P of P . Extend the map $z \mapsto z^d$ to a map θ_d on $\widehat{\mathbb{C}}$. If J_P is connected, then $\Psi_{J_P} = \Psi : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \rightarrow U^\infty(K_P)$ is such that $\Psi \circ \theta_d = P \circ \Psi$ on the complement of the closed unit disk [DH85, Mil00]. If J_P is locally connected, then Ψ extends to a continuous function

$$\overline{\Psi} : \widehat{\mathbb{C}} \setminus \mathbb{D} \rightarrow \overline{\widehat{\mathbb{C}} \setminus K_P},$$

and $\overline{\Psi} \circ \theta_d = P \circ \overline{\Psi}$ on the complement of the open unit disk. Thus, we obtain a continuous surjection $\overline{\Psi} : \text{Bd}(\mathbb{D}) \rightarrow J_P$ (the *Carathéodory loop*). Throughout the paper, $\text{Bd}(X)$ denotes the boundary of a subset X of a topological space. Identify $\mathbb{S} = \text{Bd}(\mathbb{D})$ with \mathbb{R}/\mathbb{Z} . Set $\psi = \overline{\Psi}|_{\mathbb{S}}$. We will write σ_d for the restriction of θ_d to \mathbb{S} .

Define an equivalence relation \sim_P on \mathbb{S} by $x \sim_P y$ if and only if $\psi(x) = \psi(y)$, and call it the $(\sigma_d$ -invariant) *laminational equivalence relation of P* ; since Ψ defined above conjugates θ_d and P , the map ψ semiconjugates σ_d and $P|_{J(P)}$, which implies that \sim_P is invariant. Equivalence classes of \sim_P have pairwise disjoint convex hulls. The *topological Julia set* $\mathbb{S}/\sim_P = J_{\sim_P}$ is homeomorphic to J_P , and the *topological polynomial* $f_{\sim_P} : J_{\sim_P} \rightarrow J_{\sim_P}$, induced by σ_d , is topologically conjugate to $P|_{J_P}$.

An equivalence relation \sim on the unit circle, with similar properties to those of \sim_P above, can be introduced abstractly without any reference to a complex polynomial.

Definition 1.1 (Laminational equivalence relations). An equivalence relation \sim on the unit circle \mathbb{S} is said to be *laminational* if:

- (E1) the graph of \sim is a closed subset in $\mathbb{S} \times \mathbb{S}$;
- (E2) convex hulls of distinct equivalence classes are disjoint;
- (E3) each equivalence class of \sim is finite.

For a closed set $A \subset \mathbb{S}$ we denote its convex hull by $\text{CH}(A)$. Then by an *edge* of $\text{CH}(A)$ we mean a closed segment I of the straight line connecting two points of the unit circle such that I is contained in the boundary $\text{Bd}(\text{CH}(A))$ of $\text{CH}(A)$. By an *edge* of a \sim -class we mean an edge of the convex hull of that class.

Definition 1.2 (Laminations and dynamics). A laminational equivalence relation \sim is $(\sigma_d$ -invariant) if:

- (D1) \sim is *forward invariant*: for a class \mathbf{g} , the set $\sigma_d(\mathbf{g})$ is a class too;

(D2) for any \sim -class \mathbf{g} , the map $\sigma_d : \mathbf{g} \rightarrow \sigma_d(\mathbf{g})$ extends to \mathbb{S} as an orientation preserving covering map such that \mathbf{g} is the full preimage of $\sigma_d(\mathbf{g})$ under this covering map.

Again, if this does not cause ambiguity, we will simply talk about *invariant* laminational equivalence relations.

Definition 1.2 (D2) has an equivalent version. Given a closed set $Q \subset \mathbb{S}$, a (positively oriented) *hole* (a, b) of Q (or of $\text{CH}(Q)$) is a component of $\mathbb{S} \setminus Q$. Then (D2) is equivalent to the fact that for a \sim -class \mathbf{g} either $\sigma_d(\mathbf{g})$ is a point or for each positively oriented hole (a, b) of \mathbf{g} the positively oriented arc $(\sigma_d(a), \sigma_d(b))$ is a hole of $\sigma_d(\mathbf{g})$. From now on, we assume that, unless stated otherwise, \sim is a σ_d -invariant laminational equivalence relation.

Given \sim , consider the *topological Julia set* $\mathbb{S}/\sim = J_\sim$ and the *topological polynomial* $f_\sim : J_\sim \rightarrow J_\sim$ induced by σ_d . Since $\mathbb{S} \subset \mathbb{C}$, we can use Moore's Theorem to embed J_\sim into \mathbb{C} and then to extend the quotient map $\psi_\sim : \mathbb{S} \rightarrow J_\sim$ to a map $\psi_\sim : \mathbb{C} \rightarrow \mathbb{C}$ with the only non-trivial fibers being the convex hulls of non-degenerate \sim -classes. A *Fatou domain* of J_\sim (or of f_\sim) is a bounded component of $\mathbb{C} \setminus J_\sim$. If U is a periodic Fatou domain of f_\sim of period n , then $f_\sim^n|_{\text{Bd}(U)}$ is either conjugate to an irrational rotation of \mathbb{S} or to σ_k with some $1 < k$, cf. [BL02]. In the case of irrational rotation, U is called a *Siegel domain*. The complement of the unbounded component of $\mathbb{C} \setminus J_\sim$ is called the *filled-in topological Julia set* and is denoted by K_\sim . Equivalently, K_\sim is the union of J_\sim and its bounded Fatou domains. If the laminational equivalence relation \sim is fixed, we may omit \sim from the notation. By default, we consider f_\sim as a self-mapping of J_\sim . For a collection \mathcal{R} of sets, denote the union of all sets from \mathcal{R} by \mathcal{R}^+ .

Definition 1.3 (Leaves). If A is a \sim -class, call an edge \overline{ab} of $\text{CH}(A)$ a *leaf* of \sim . All points of \mathbb{S} are also called (*degenerate*) leaves of \sim .

The family of all leaves of \sim is closed (the limit of a sequence of leaves of \sim is a leaf of \sim); the union of all leaves of \sim is a continuum. For any subset $X \subset \mathbb{D}$ with the property $X = \text{CH}(X \cap \mathbb{S})$, we set $\sigma_d(X) = \text{CH}(\sigma_d(X \cap \mathbb{S}))$. In particular, for any leaf ℓ of \sim , the set $\sigma_d(\ell)$ is a (possibly degenerate) leaf.

1.2. Geodesic laminations. Assume that \sim is a σ_d -invariant laminational equivalence relation.

Definition 1.4. The set \mathcal{L}_\sim of all leaves of \sim is called the *geodesic lamination generated by \sim* .

Thurston studied collections of chords in \mathbb{D} with similar properties to those of \mathcal{L}_\sim .

Definition 1.5 (Geodesic laminations, cf. [Thu85]). Two distinct chords in $\overline{\mathbb{D}}$ are said to be *unlinked* if they meet at most in a common endpoint;

otherwise they are said to be *linked*, or to *cross* each other. A *geodesic pre-lamination* \mathcal{L} is a set of (possibly degenerate) chords in $\overline{\mathbb{D}}$ such that any two distinct chords from \mathcal{L} are unlinked. A geodesic pre-lamination \mathcal{L} is called a *geodesic lamination* if all points of \mathbb{S} are elements of \mathcal{L} , and \mathcal{L}^+ is closed. We will sometimes use the abbreviation *geolamination* for a geodesic lamination. Elements of \mathcal{L} are called *leaves* of \mathcal{L} . By a *degenerate leaf* (chord) we mean a singleton in \mathbb{S} . The continuum $\mathcal{L}^+ \subset \overline{\mathbb{D}}$ is called the *solid* of \mathcal{L} . Let \mathcal{L} be a geodesic lamination. The closure in \mathbb{C} of a non-empty component of $\mathbb{D} \setminus \mathcal{L}^+$ is called a *gap* of \mathcal{L} . If G is a gap or a leaf, call the set $G' = \mathbb{S} \cap G$ the *basis* of G . A gap is said to be *finite* (*infinite*, *countable*, *uncountable*) if its basis is finite (infinite, countable, uncountable). Uncountable gaps are also called *Fatou gaps*. Points of G' are called *vertices* of G . Geolaminations of the form \mathcal{L}_{\sim} , where \sim is a laminational equivalence relation, are called *q-geolaminations* (“q” from “equivalence”).

Let us discuss geodesic laminations in the dynamical context. A chord is called (σ_d) -*critical* if its endpoints have the same image under σ_d . If it does not cause ambiguity, we will simply talk about *critical* chords.

It is essential to associate to each polynomial with connected Julia set a corresponding laminational equivalence relation (we have already done it in the case when the Julia sets is locally connected). By Kiwi [Kiw04], this can also be done if P is a polynomial with connected Julia set and such that all its periodic points are repelling (recall that such polynomials are said to be *dendritic*). Here is an important result proven by Kiwi in [Kiw04].

Theorem 1.6. *Suppose that a polynomial P with connected Julia set $J = J(P)$ has no Siegel or Cremer periodic points. Then there exist a laminational equivalence relation \sim_P , the corresponding topological polynomial $f_{\sim_P} : J_{\sim_P} \rightarrow J_{\sim_P}$ which acts on its topological Julia set, and a monotone semiconjugacy $\varphi_P : J \rightarrow J_{\sim_P}$. The semiconjugacy φ_P is one-to-one on all (pre)periodic points of P belonging to J . If all periodic points of P are repelling, then J_{\sim_P} is a dendrite.*

In what follows, denote by \mathcal{D} the space of all polynomials with connected Julia sets and only repelling periodic points. Let \mathcal{D}_d be the space of all such polynomials of degree d . Then Theorem 1.6 implies that the following diagram is commutative.

$$\begin{array}{ccccc}
 J(P) & \xrightarrow{P|_{J(P)}} & J(P) & & \mathbb{S}^1 & \xrightarrow{\sigma_d} & \mathbb{S}^1 \\
 & \searrow \varphi_P & & \swarrow \varphi_P & & & \\
 & & J_{\sim P} & \xrightarrow{f_{\sim P}|_{J_{\sim P}}} & J_{\sim P} & & \\
 & & & & & \swarrow \psi_{\sim P} & \\
 & & & & & & \mathbb{S}^1
 \end{array}$$

The notion of *sibling invariant laminations* introduced below is slightly different from the original notion of *invariant laminations* in the sense of Thurston. However, sibling invariant laminations form a closed set and include all q -laminations. Thus, for all our purposes, it will suffice to consider sibling invariant laminations only. Some advantage of working with sibling σ_d -invariant geodesic laminations is that they are defined through properties of their leaves; gaps are not involved in the definition. It was shown in [BMOV13] that all sibling invariant laminations are also invariant in the sense of Thurston [Thu85]. In particular for any gap G of a sibling invariant \mathcal{L} the set $\sigma_d(G)$ is either a point, or a leaf of \mathcal{L} , or a gap of \mathcal{L} .

Definition 1.7. A geodesic lamination \mathcal{L} is *sibling σ_d -invariant* provided that:

- (1) for each $\ell \in \mathcal{L}$, we have $\sigma_d(\ell) \in \mathcal{L}$,
- (2) for each $\ell \in \mathcal{L}$ there exists $\ell' \in \mathcal{L}$ so that $\sigma_d(\ell') = \ell$.
- (3) for each $\ell \in \mathcal{L}$ so that $\sigma_d(\ell) = \ell'$ is a non-degenerate leaf, there exist d **disjoint** leaves ℓ_1, \dots, ℓ_d in \mathcal{L} so that $\ell = \ell_1$ and $\sigma_d(\ell_i) = \ell'$ for all $i = 1, \dots, d$.

Let us list a few properties of sibling σ_d -invariant geodesic laminations.

Theorem 1.8 ([BMOV13]). *The space of all sibling σ_d -invariant geodesic laminations is compact. All geodesic laminations generated by σ_d -invariant laminational equivalence relations are sibling σ_d -invariant.*

For brevity, in what follows instead of “sibling σ_d -invariant geodesic laminations” we will say “ σ_d -invariant geodesic laminations”. Below, we talk interchangeably about leaves (gaps) of \sim or of \mathcal{L}_{\sim} . Let us now discuss gaps in the context of σ_d -invariant laminational equivalence relations and geodesic laminations.

Definition 1.9 (Critical gaps). A gap G of a geodesic lamination is called (σ_d -)critical if for each $y \in \sigma_d(G')$ the set $\sigma_d^{-1}(y) \cap G'$ consists of at least 2 points. If it does not cause ambiguity, we talk about *critical gaps*.

Definition 1.10 (Periodic and (pre)periodic gaps). Let G be a gap of an invariant geodesic lamination \mathcal{L} . If the map σ_d restricted to G' extends to

$\text{Bd}(G)$ as a composition of a monotone map and a covering map of some degree m , then m is called the *degree* of $\sigma_d|_G$. A gap/leaf U of \mathcal{L}_\sim is said to be *(pre)periodic* of period k if $\sigma_d^{m+k}(U') = \sigma_d^m(U')$ for some $m \geq 0, k > 0$; if m, k are chosen to be minimal, then if $m > 0$, U is called *preperiodic*, and, if $m = 0$, then U is called *periodic (of period k)*. If the period of G is 1, then G is said to be *invariant*. Define *precritical* and *(pre)critical* objects similarly to (pre)periodic and preperiodic objects defined above.

Consider infinite periodic gaps of σ_d -invariant geodesic laminations. There are three types of such gaps: *caterpillar* gaps, *Siegel* gaps, and *Fatou gaps of degree greater than one*. Observe that, by [Kiw02], infinite gaps are eventually mapped onto periodic infinite gaps. First we state (without a proof) a well-known folklore lemma about the edges of preperiodic (in particular, infinite) gaps (see, e.g., Lemma 2.28 [BOPT17]).

Lemma 1.11. *Any edge of a (pre)periodic gap is either (pre)periodic or (pre)critical.*

Let us now classify infinite gaps.

Definition 1.12. An infinite gap G is said to be a *caterpillar* gap if its basis G' is countable.

As an example, consider a periodic gap Q such that:

- The boundary of Q consists of a periodic leaf $\ell_0 = \overline{xy}$ of period k , a critical leaf $\ell_{-1} = \overline{yz}$ concatenated to it, and a countable concatenation of leaves ℓ_{-n} accumulating at x (the leaf ℓ_{-r-1} is concatenated to the leaf ℓ_{-r} , for every $r = 1, 2, \dots$).
- We have $\sigma^k(x) = x$, $\sigma^k(\{y, z\}) = \{y\}$, and σ^k maps each ℓ_{-r-1} to ℓ_{-r} (all leaves are shifted by one towards ℓ_0 except for ℓ_0 , which maps to itself, and ℓ_{-1} , which collapses to the point y).

The description of σ_3 -invariant caterpillar gaps is in [BOPT16b]. In general, the fact that the basis G' of a caterpillar gap G is countable implies that there are lots of concatenated edges of G . Other properties of caterpillar gaps can be found in Lemma 1.13.

Lemma 1.13 (Lemma 1.15 [BOPT16a]). *Let G be a caterpillar gap of period k . Then the degree of $\sigma_d^k|_{\text{Bd}(G)}$ is one, and G' contains some periodic points of period k .*

Definition 1.14. A periodic Fatou gap G of period n is said to be a periodic *Siegel* gap if the degree of $\sigma_d^n|_G$ is 1, and the basis G' of G is uncountable.

The next lemma is well known. A part of it was actually proven in the proof of [BOPT16a, Lemma 1.13].

Lemma 1.15. *Let G be a Siegel gap of period n . Then the map $\sigma_d^n|_{\text{Bd}(G)}$ is monotonically semiconjugate to an irrational circle rotation and contains no periodic points. A periodic Siegel gap must have at least one image that has a critical edge.*

The following definition completes our series of definitions.

Definition 1.16. A periodic Fatou gap is of degree $k > 1$ if the degree of $\sigma_d^n|_{\text{Bd}(G)}$ is $k > 1$. If the degree of a Fatou gap G is 2, then G is said to be quadratic.

The next lemma is well known.

Lemma 1.17. *Let G be a Fatou gap of period n and of degree $k > 1$. Then the map $\sigma_d^n|_{\text{Bd}(G)}$ is monotonically semiconjugate to σ_k .*

2. LIMIT GEODESIC LAMINATIONS AND THEIR PROPERTIES

In this section, we deal with properties of limits of σ_d -invariant q -laminations. For the most part, the results are obtained in [BOPT16a].

Fix a degree d . In lemmas below, we assume that a sequence of σ_d -invariant q -laminations \mathcal{L}_i converges to a σ_d -invariant geodesic lamination \mathcal{L}_∞ . By a *strip* we mean a (open) part of the unit disk contained between two disjoint chords. By a *strip around a chord* ℓ we mean a strip containing ℓ . In what follows, when talking about convergence of leaves/gaps, closeness of leaves/gaps, and closures of families of geodesic laminations, we always use the Hausdorff metric.

Definition 2.1. Let \mathbb{L}_d^q be the family of all σ_d -invariant geodesic q -laminations. We will write $\overline{\mathbb{L}_d^q}$ for the closure of \mathbb{L}_d^q .

Even though we state below a few general results, we mostly concentrate on periodic objects of limit geodesic laminations.

Lemma 2.2 (Lemma 2.2 [BOPT16a]). *Let ℓ be a periodic leaf of $\mathcal{L} \in \overline{\mathbb{L}_d^q}$. If $\widehat{\mathcal{L}} \in \mathbb{L}_d^q$ is sufficiently close to \mathcal{L} , then any leaf of any $\widehat{\mathcal{L}}$ sufficiently close to ℓ is either equal to ℓ or disjoint from ℓ .*

Definition 2.3 introduces the concept of rigidity.

Definition 2.3. A leaf/gap G of \mathcal{L} is *rigid* if any q -lamination close to \mathcal{L} has G as its leaf/gap.

A series of lemmas proved in [BOPT16a] studies rigidity of periodic leaves/gaps of laminations from $\overline{\mathbb{L}_d^q}$. These are combinatorial counterparts of the fact that repelling periodic points survive under small deformations of complex polynomials. Observe that (periodic) leaves of geodesic laminations are either edges of gaps or limits of other leaves. By a (σ_d) -*collapsing*

polygon we mean a polygon P , whose edges map under σ_d to the same non-degenerate chord. Thus, if a point makes a circuit around P , its σ_d -image moves back and forth along the same non-degenerate chord; as before, if it does not cause ambiguity, we simply talk about *collapsing polygons*. When we say that Q is a *collapsing polygon of a geodesic lamination* \mathcal{L} , we mean that all edges of Q are leaves of \mathcal{L} ; we also say that \mathcal{L} *contains a collapsing polygon* Q . However, this does not imply that Q is a gap of \mathcal{L} as Q might be further subdivided by leaves of \mathcal{L} inside Q . Let us now quote some results of [BOPT16a] concerning (pre)periodic leaves and gaps.

Lemma 2.4 (Lemma 2.5 - 2.10 [BOPT16a]). *Let $\mathcal{L} \in \overline{\mathbb{L}}_d^q$. If $\hat{\ell} \in \mathcal{L}$ is a non-degenerate rigid leaf, a leaf $\ell \in \mathcal{L}$ is such that $\sigma_d^k(\ell) = \hat{\ell}$ for some $k \geq 0$, and no leaf $\ell, \sigma_d(\ell), \dots, \sigma_d^{k-1}(\ell)$ is contained in a collapsing polygon of \mathcal{L} , then ℓ is rigid. Also, the following objects are rigid:*

- (1) *periodic leaves that are not edges of collapsing polygons;*
- (2) *finite periodic gaps;*
- (3) *(pre)periodic leaves of a gap that eventually maps to a periodic gap;*
- (4) *finite gaps that eventually map onto periodic gaps;*
- (5) *periodic Fatou gaps whose images have no critical edges.*

Using these results and other tools, we characterize all σ_2 -invariant limit laminations. Each such lamination \mathcal{L} can be described as a specific modification of an appropriate geodesic lamination \mathcal{L}^q from \mathbb{L}_2^q .

Definition 2.5. Two geodesic laminations *coexist* if their union is a lamination.

This notion was used in [BOPT16b]. If two geodesic laminations coexist, then a leaf of one lamination is either also a leaf of the other lamination or is located in a gap of the other geodesic lamination.

Definition 2.6. A σ_2 -invariant geodesic lamination is called *hyperbolic* if it has a periodic Fatou gap of degree two.

Clearly, if a σ_2 -invariant geodesic lamination \mathcal{L} has a periodic Fatou gap U of period n and of degree greater than one, then the degree of $\sigma_2|_{\text{Bd}(U)}$ is two. By [Thu85], there is a unique edge $M(\mathcal{L})$ of U with $\sigma_2^n(M(\mathcal{L})) = M(\mathcal{L})$; $M(\mathcal{L})$ is such that either all leaves $M(\mathcal{L}), \dots, \sigma_2^{n-1}(M(\mathcal{L}))$ are pairwise disjoint, or their union can be broken down into several gaps permuted by σ_2 , in each of which edges are “rotated” by the appropriate power of σ_2 , or $n = 2k$ and σ_2^k flips $M(\mathcal{L})$ on top of itself while all leaves $M(\mathcal{L}), \dots, \sigma_2^{k-1}(M(\mathcal{L}))$ are pairwise disjoint. In fact, $M(\mathcal{L})$ and its sibling $M'(\mathcal{L})$ are the two *majors* of \mathcal{L} while $\sigma_2(M(\mathcal{L})) = \sigma_2(M'(\mathcal{L})) = m(\mathcal{L})$ is the *minor* of \mathcal{L} [Thu85] (recall that a *major* of a σ_2 -invariant geodesic lamination

is the longest leaf of \mathcal{L}). Any σ_2 -invariant hyperbolic geodesic lamination \mathcal{L} is actually a geodesic lamination \mathcal{L}_\sim generated by the appropriate *hyperbolic σ_2 -invariant laminational equivalence relation* \sim so that the topological polynomial f_\sim considered on the entire complex plane is topologically conjugate to a complex quadratic *hyperbolic* polynomial; this justifies our terminology.

Definition 2.7. A *critical set* $\text{Cr}(\mathcal{L})$ of a σ_2 -invariant geodesic lamination \mathcal{L} is either a critical leaf or a gap G with $\sigma_2|_{\text{Bd}(G)}$ of degree two.

A σ_2 -invariant q -lamination \mathcal{L} either has a finite critical set (a critical leaf, or a finite critical gap) or is hyperbolic. In both cases, the critical set is unique.

Definition 2.8. By a *generalized critical quadrilateral* Q we mean either a 4-gon whose σ_2 -image is a leaf, or a critical leaf (whose image is a point). A *collapsing quadrilateral* is a generalized critical quadrilateral with four distinct vertices.

If $\text{Cr}(\mathcal{L})$ is a generalized critical quadrilateral, then $\sigma_2(\text{Cr}(\mathcal{L})) = m(\mathcal{L})$. The notion of generalized critical quadrilateral was used in [BOPT17] in our study of geodesic laminations of arbitrary degree.

Theorem 2.9 describes all geodesic laminations from $\overline{\mathbb{L}}_2^q$. A periodic leaf \bar{n} such that the period of its endpoints is k and all leaves $\bar{n}, \sigma_2(\bar{n}), \dots, \sigma_2^{k-1}(\bar{n})$ are pairwise disjoint, is said to be a *fixed return periodic leaf*.

Theorem 2.9 (Theorem 3.8 [BOPT16a]). *A geodesic lamination \mathcal{L} belongs to $\overline{\mathbb{L}}_2^q$ if and only if there exists a unique maximal q -lamination \mathcal{L}^q coexisting with \mathcal{L} and such that either $\mathcal{L} = \mathcal{L}^q$ or $\text{Cr}(\mathcal{L}) \subset \text{Cr}(\mathcal{L}^q)$ is a generalized critical quadrilateral, and exactly one of the following holds.*

- (1) *The critical set $\text{Cr}(\mathcal{L}^q)$ is finite, and $\text{Cr}(\mathcal{L})$ is the convex hull of two edges or vertices of $\text{Cr}(\mathcal{L}^q)$ with the same σ_2 -image;*
- (2) *the lamination \mathcal{L}^q is hyperbolic with a critical Fatou gap $\text{Cr}(\mathcal{L})$ of period n , and exactly one of the following holds:*
 - (a) *the set $\text{Cr}(\mathcal{L}) = \overline{ab}$ is a critical leaf with a periodic endpoint of period n , and \mathcal{L} contains exactly two σ_2^n -pullbacks of \overline{ab} that intersect \overline{ab} (one of these pullbacks shares an endpoint a with \overline{ab} , and the other one shares an endpoint b with \overline{ab}).*
 - (b) *the critical set $\text{Cr}(\mathcal{L})$ is a collapsing quadrilateral, and $m(\mathcal{L})$ is a fixed return periodic leaf.*

Thus, any σ_2 -invariant q -lamination corresponds to finitely many geodesic laminations from $\overline{\mathbb{L}}_2^q$, and the union of all of their minors is connected.

Given a geodesic lamination \mathcal{L} , let \mathcal{L}^q denote the σ_2 -invariant q -lamination associated with \mathcal{L} defined in Theorem 2.9.

To interpret the Mandelbrot set as a quotient of $\overline{\mathbb{L}}_2^q$, we define a certain equivalence relation on $\overline{\mathbb{L}}_2^q$, cf. [BOPT16a].

Definition 2.10. Suppose that $\mathcal{L}', \mathcal{L}'' \in \overline{\mathbb{L}}_2^q$. Then the geodesic laminations \mathcal{L}_1 and \mathcal{L}_2 are said to be *minor equivalent* if there exists a finite collection of geodesic laminations $\mathcal{L}_1 = \mathcal{L}', \mathcal{L}_2, \dots, \mathcal{L}_k = \mathcal{L}''$ from $\overline{\mathbb{L}}_2^q$ such that for each i with $1 \leq i \leq k-1$, the minors $m(\mathcal{L}_i)$ and $m(\mathcal{L}_{i+1})$ of the geodesic laminations \mathcal{L}_i and \mathcal{L}_{i+1} are non-disjoint.

The last result of [BOPT16a] is the following theorem. Let $\psi : \overline{\mathbb{L}}_2^q \rightarrow \mathbb{S}/\text{QML}$ be the map which associates to each geodesic lamination $\mathcal{L} \in \overline{\mathbb{L}}_2^q$ the QML-class of the endpoints of the minor $m(\mathcal{L})$ of \mathcal{L} .

Theorem 2.11 (Theorem 3.10 [BOPT16a]). *The map $\psi : \overline{\mathbb{L}}_2^q \rightarrow \mathbb{S}/\text{QML}$ is continuous. The partition of $\overline{\mathbb{L}}_2^q$ into classes of minor equivalence is upper semi-continuous, and the quotient space of $\overline{\mathbb{L}}_2^q$ with respect to the minor equivalence is homeomorphic to \mathbb{S}/QML .*

One can understand Theorem 2.11 as follows. For every geodesic lamination \mathcal{L} define its *minor set* as the image of its critical set unless \mathcal{L} is hyperbolic in which case we call $m(\mathcal{L})$ the *minor set* of \mathcal{L} . Then ψ can be viewed as the map associating to each class A of minor equivalence in $\overline{\mathbb{L}}_2^q$ the minor set of the geodesic lamination \mathcal{L}^q that is the only q -lamination in A . The minor set of \mathcal{L}^q is in fact the convex hull of the union of minors of all laminations in A .

We want to modify this result by considering the maximal perfect subset of $\overline{\mathbb{L}}_2^q$. Observe that this subset consists of all non-isolated laminations. Equivalently, we consider geodesic laminations which are limits of sequences of pairwise distinct σ_2 -invariant geodesic q -laminations. Theorem 2.9 implies Corollary 2.12.

Corollary 2.12. *A geodesic lamination $\mathcal{L} \in \overline{\mathbb{L}}_2^q$ is non-isolated in $\overline{\mathbb{L}}_2^q$ if and only if one of the following holds:*

- (1) *the critical set $\text{Cr}(\mathcal{L}^q)$ is finite, and $\text{Cr}(\mathcal{L})$ is the convex hull of two edges or vertices of $\text{Cr}(\mathcal{L}^q)$ with the same σ_2 -image;*
- (2) *the lamination \mathcal{L}^q is hyperbolic with a critical Fatou gap $\text{Cr}(\mathcal{L})$ of period n , and exactly one of the following holds:*
 - (a) *the set $\text{Cr}(\mathcal{L}) = \overline{ab}$ is a critical leaf with a periodic endpoint of period n , and \mathcal{L} contains exactly two σ_2^n -pullbacks of \overline{ab} that intersect \overline{ab} (one of these pullbacks shares an endpoint a with \overline{ab} and the other one shares an endpoint b with \overline{ab}).*
 - (b) *the set $\text{Cr}(\mathcal{L})$ is a collapsing quadrilateral, and $m(\mathcal{L})$ is a fixed return periodic leaf.*

In order to prove Corollary 2.12, we need the following lemma.

Lemma 2.13. *Suppose that \mathcal{L} is a σ_2 -invariant q -lamination whose critical set is a generalized critical quadrilateral. Then \mathcal{L} is the only σ_2 -invariant geodesic lamination with critical set $\text{Cr}(\mathcal{L})$.*

The proof is based on Thurston's pullback construction [Thu85].

Proof of Lemma 2.13. Indeed, properties of σ_2 -invariant geodesic laminations imply that pullbacks of $\text{Cr}(\mathcal{L})$ are well defined on each finite step; moreover, these pullbacks are all sets from \mathcal{L} . Furthermore, the closure $\widehat{\mathcal{L}}$ of their entire family is a σ_2 -invariant geodesic lamination itself, and since \mathcal{L} is closed it follows that $\widehat{\mathcal{L}} \subset \mathcal{L}$. We claim that $\widehat{\mathcal{L}} = \mathcal{L}$. Indeed, suppose otherwise. Then $\widehat{\mathcal{L}}$ must contain a gap, say, U that itself is the union of $s > 1$ gaps of \mathcal{L} and, therefore, U contains leaves of \mathcal{L} *inside*. If U is finite, it follows that there are non-disjoint finite gaps of \mathcal{L} . The latter is impossible as \mathcal{L} is a q -lamination. Thus, U is infinite. Mapping U forward several times, we may assume without loss of generality that U is periodic of period k (indeed, by [Kiw02], all infinite gaps of geodesic laminations are (pre)periodic).

Consider several cases. First suppose that U is a caterpillar gap. Then the critical leaf of U (or of a gap in the forward orbit of U) must coincide with the critical set of \mathcal{L} . Therefore, \mathcal{L} has a critical leaf with a periodic endpoint, which is impossible for a q -lamination.

Now, suppose that U is a Siegel gap. It is well-known (e.g., it follows from Lemma 1.11) that all edges of U are (pre)critical and that, therefore, some image $\sigma_2^t(U)$ of U has a critical edge ℓ ; it then follows that $\text{Cr}(\mathcal{L}) = \ell$, that all edges of U are pullbacks of ℓ , and that under the map ψ collapsing edges of U to points any chord $\hat{\ell}$ connecting vertices of U projects under ψ to a non-trivial chord $\psi(\hat{\ell})$ of the unit circle. Since ψ semiconjugates $\sigma_2^k|_{\text{Bd}(U)}$ to an irrational rotation, the chord $\psi(\hat{\ell})$ in the unit disk will eventually intersect itself under the irrational rotation in question which implies the same fact for the above chord $\hat{\ell} \subset U$. We see that $\hat{\ell}$ cannot be a leaf of any lamination, a contradiction with the above.

Finally, suppose that $\sigma_2^k|_{\text{Bd}(U)}$ is of degree 2. Then some image of U is an infinite gap V such that $\sigma_2|_{\text{Bd}(V)}$ has degree two. On the other hand, $\text{Cr}(\widehat{\mathcal{L}}) = \text{Cr}(\mathcal{L})$ is a generalized critical quadrilateral, a contradiction with the existence of V . Hence this case is impossible either, and so $\mathcal{L} = \widehat{\mathcal{L}} = \mathcal{L}^q$ is a unique geodesic lamination with critical set $\text{Cr}(\mathcal{L})$. \square

We are now ready to deduce Corollary 2.12 from Lemma 2.13.

Proof of Corollary 2.12. By Theorem 2.9, if \mathcal{L} satisfies the conditions of the corollary, then $\mathcal{L} \in \overline{\mathbb{L}}_2^q$. Clearly, geodesic laminations described in (2)(a)

and (2)(b) above do not belong to \mathbb{L}_2^q . Since they belong to $\overline{\mathbb{L}_2^q}$, it follows that they are limits of sequences of pairwise distinct σ_2 -invariant geodesic q -laminations.

Consider now case (1). Then $\text{Cr}(\mathcal{L}^q)$ is finite and $\text{Cr}(\mathcal{L})$ is the convex hull of two edges or vertices of $\text{Cr}(\mathcal{L}^q)$ with the same σ_2 -image. Suppose that $\text{Cr}(\mathcal{L}^q)$ is a polygon with more than four vertices. Then $\mathcal{L} \neq \mathcal{L}^q$ (in fact, $\mathcal{L} \not\supseteq \mathcal{L}^q$). Hence $\mathcal{L} \notin \mathbb{L}_2^q$, and, as above, \mathcal{L} is the limit of a sequence of pairwise distinct σ_2 -invariant geodesic q -laminations.

Consider now the case when \mathcal{L}^q has a generalized quadrilateral as its critical set $\text{Cr}(\mathcal{L}^q)$. In this case it may happen that \mathcal{L} has a critical leaf that is a diagonal of a quadrilateral $\text{Cr}(\mathcal{L}^q)$ so that $\mathcal{L} \neq \mathcal{L}^q$; as before this implies that \mathcal{L} is the limit of a sequence of pairwise distinct σ_2 -invariant geodesic laminations.

It remains to consider the case when $\mathcal{L} = \mathcal{L}^q$ is generated by an equivalence relation \sim and has a critical set $\text{Cr}(\mathcal{L})$ that is either a critical quadrilateral or a critical leaf. Let us show that then \mathcal{L} is the limit of a non-constant sequence of q -laminations. By Lemma 2.13, the lamination \mathcal{L} is the unique σ_2 -invariant geodesic lamination with critical set $\text{Cr}(\mathcal{L})$. Now, the fact that \mathcal{L} is the limit of a sequence of pairwise distinct q -laminations follows from the uniqueness of \mathcal{L} and the fact that, due to well-known properties of the combinatorial Mandelbrot set, there is a sequence of geodesic laminations \mathcal{L}_i with critical sets $\text{Cr}(\mathcal{L}_i) \rightarrow \text{Cr}(\mathcal{L})$ (recall that we are considering the case when $\text{Cr}(\mathcal{L})$ is a generalized quadrilateral). This completes the proof of the corollary. \square

Thus, isolated geodesic laminations in $\overline{\mathbb{L}_2^q}$ are the following:

- (1) dendritic geodesic laminations with critical sets that have more than four vertices,
- (2) hyperbolic laminations.

If we remove them from $\overline{\mathbb{L}_2^q}$, we will obtain the closed space $\mathbb{L}_2^l \subset \overline{\mathbb{L}_2^q}$ of all σ_2 -invariant geodesic laminations that are non-isolated in $\overline{\mathbb{L}_2^q}$. The minor equivalence on \mathbb{L}_2^l is defined in the same way as the minor equivalence on $\overline{\mathbb{L}_2^q}$: two laminations are *minor equivalent* if their minors can be connected by a chain of non-disjoint minors. However, we only consider minors of laminations from \mathbb{L}_2^l . Therefore, the minor equivalence on \mathbb{L}_2^l is not a restriction of the minor equivalence on $\overline{\mathbb{L}_2^q}$. In particular, some classes of minor equivalence on \mathbb{L}_2^l are slightly different from the restrictions of the corresponding classes of minor equivalence on $\overline{\mathbb{L}_2^q}$. Consider several cases; the analysis below follows from Theorem 2.9 and Corollary 2.12.

(1) Take a dendritic geodesic lamination \mathcal{L} generated by a laminational equivalence relation \sim such that $\text{Cr}(\mathcal{L})$ has more than four vertices. Then

there are several geodesic laminations in \mathbb{L}_2^l with critical sets being generalized critical quadrilaterals in $\text{Cr}(\mathcal{L})$. These laminations form one class A of the minor equivalence in \mathbb{L}_2^l . However, unlike for $\overline{\mathbb{L}_2^q}$, the geodesic lamination \mathcal{L} itself does not belong to \mathbb{L}_2^l and therefore is not included into A . Still, it follows that the convex hull of the union of all minors of laminations in A is the same for \mathbb{L}_2^l and for $\overline{\mathbb{L}_2^q}$.

(2) Now let \mathcal{L} be a dendritic geodesic lamination such that $\text{Cr}(\mathcal{L})$ is either a quadrilateral or a critical leaf. Then, by Corollary 2.12, we have $\mathcal{L} \in \mathbb{L}_2^l$. Hence, in this case, the corresponding class of minor equivalence in \mathbb{L}_2^l consists of \mathcal{L} itself and two geodesic laminations obtained by inserting a critical diagonal in $\text{Cr}(\mathcal{L})$ and then pulling it back. In other words, this class coincides with the corresponding class in $\overline{\mathbb{L}_2^q}$. Of course, as in case (1), the convex hull of the union of minors remains the same as for $\overline{\mathbb{L}_2^q}$.

(3) In the case of a Siegel geodesic lamination \mathcal{L} , we have that \mathcal{L} belongs to both \mathbb{L}_2^l and $\overline{\mathbb{L}_2^q}$. The corresponding class of the minor equivalence then consists of \mathcal{L} only.

(4) Consider a hyperbolic geodesic lamination \mathcal{L} with a critical gap U of period n such that the unique edge M of U of period n is a fixed return leaf. Then \mathcal{L} itself does not belong to \mathbb{L}_2^l . However, three closely related geodesic laminations form a class of minor equivalence. Two of them have critical leaves with endpoints at endpoints of M . The third one has a collapsing quadrilateral based on M . This yields the same convex hull of the union of minors as before in case of $\overline{\mathbb{L}_2^q}$.

(5) Finally, consider a hyperbolic geodesic lamination \mathcal{L} with a critical gap U of period n such that the unique edge $M = \overline{ab}$ of U of period n is *not* a fixed return leaf. It follows that neither \mathcal{L} nor the geodesic lamination with a collapsing quadrilateral based on M belong to \mathbb{L}_2^l . Therefore, there are **two non-equivalent geodesic laminations with critical leaves ℓ_a and ℓ_b with endpoints a and b , respectively** that can be associated with \mathcal{L} , and so there are **two classes of minor equivalence, generated by ℓ_a and ℓ_b , respectively**, that can be associated with \mathcal{L} .

Let A be a class of minor equivalence in \mathbb{L}_2^l . Define $m(A)$ as the convex hull of the union of the corresponding minors. The association $A \mapsto m(A)$ is similar to that made in [BOPT16a] for $\overline{\mathbb{L}_2^q}$. Let A' be the minor equivalence class in $\overline{\mathbb{L}_2^q}$ containing A . The analysis made above implies that, in cases (1) – (4), we have $m(A) = m(A')$. In cases (2) and (3), we have $A = A'$. In cases (1) and (4), the class A' consists of A and the lamination \mathcal{L}^q generated by the corresponding laminational equivalence.

In case (5) the situation is different. The two distinct classes of minor equivalence in \mathbb{L}_2^l correspond to critical leaves ℓ_a and ℓ_b and give rise to sets $\{\sigma_2(a)\}$ and $\{\sigma_2(b)\}$. These singletons replace the minor $m(\mathcal{L}) =$

$\overline{\sigma_2(a)\sigma_2(b)}$ that corresponds to \mathcal{L} in QML. In other words, the leaf $m(\mathcal{L})$ is erased from QML and replaced by its two endpoints. This is what we informally referred to in the beginning of the paper as “unpinching” of the circle. In the end, this yields a new parametric geodesic lamination QML^l , the corresponding laminational equivalence \sim_{QML^l} , and the corresponding quotient space \mathcal{M}_2^l . Let $\psi^l : \mathbb{L}_2^l \rightarrow \mathbb{S}/\text{QML}^l$ be the quotient map.

Theorem 2.14. *The map $\psi^l : \mathbb{L}_2^l \rightarrow \mathbb{S}/\text{QML}^l$ is continuous. Thus, the partition of \mathbb{L}_2^l into classes of minor equivalence is upper semi-continuous, and the quotient space of \mathbb{L}_2^l with respect to the minor equivalence is homeomorphic to \mathbb{S}/QML^l .*

It follows that the corresponding parametric geodesic lamination QML^l , taken as a subset of the space of all chords in $\overline{\mathbb{D}}$, is perfect.

The dendritic versions \sim^{ld} and \mathcal{M}_2^{ld} of \sim^l and \mathcal{M}_2^l were given in the Introduction, so we refer the reader there; notice that the dendritic versions here are well-defined as QML^l is perfect.

3. NON-RENORMALIZABLE GEODESIC LAMINATIONS

In this short section we suggest two spaces of σ_2 -invariant geodesic laminations that can be obtained from \mathcal{M}_2^c through a process of “unpinching” (i.e., replacing some leaves of QML by their endpoints). Both spaces are described by the same parametric geodesic lamination QML^{nr} modifying Thurston’s quadratic minor lamination QML and such that infinite gaps of it are all pairwise disjoint. As before, this geodesic lamination can be interpreted in two different ways resulting in two distinct but related quotient spaces. First, we may consider various vertices and edges of infinite gaps of the QML^{nr} as distinct classes of the corresponding equivalence relation; then infinite gaps of QML^{nr} become closures of bounded complementary domains of the quotient space $\overline{\mathbb{D}}/\text{QML}^{nr} = \mathcal{M}_2^{nr}$. Then we also consider the dendritic version \mathcal{M}_2^{nrd} of \mathcal{M}_2^{nr} . The main ideas were already discussed in the Introduction.

Consider a maximal by inclusion *canonical* copy $X \subsetneq \mathcal{M}_2^c$ of the combinatorial Mandelbrot set \mathcal{M}_2^c . In other words, consider a copy $X \subsetneq \mathcal{M}_2^c$ of \mathcal{M}_2^c that can be described as follows. There exist a special *ancestor laminational equivalence* \sim_X and the associated σ_2 -invariant *ancestor geodesic lamination* \mathcal{L}_X with the following properties. The lamination \mathcal{L}_X is hyperbolic and has a critical Fatou gap U_X of period, say, n . Let $V_X = \sigma_2(U_X)$ be the postcritical gap of \mathcal{L}_X . The boundary $\text{Bd}(V_X)$ can be projected to \mathbb{S} by means of collapsing its edges. This projection will be denoted by φ_X . It semiconjugates $\sigma_2^n|_{\text{Bd}(V_X)}$ and σ_2 . Moreover, if we consider $\text{QML} \subset \overline{\mathbb{D}}$ and

lift it to V_X using φ_X , we will obtain *exactly* the restriction of QML onto V_X ; the quotient space of V_X under this restriction is *exactly* X .

This description applies to any canonical copy of \mathcal{M}_2^c in \mathcal{M}_2^c ; in fact, it can be viewed as the definition of a canonical copy of \mathcal{M}_2^c inside \mathcal{M}_2^c . Since we only want maximal by inclusion canonical copies X of \mathcal{M}_2^c in \mathcal{M}_2^c , we need to specify properties of \mathcal{L}_X . Well-known facts about the combinatorial Mandelbrot set imply that maximal gaps V_X can be of two types.

(1) There is a countable family of such gaps, each of which is based on an edge of the Main Cardioid. The corresponding geodesic lamination \mathcal{L}_X has a finite invariant rotational gap G_X such that $\sigma_2(G_X) = G_X$, and the map $\sigma_2|_{G_X}$ can be viewed as a ‘‘combinatorial rotation’’ (each edge of G_X under σ_2 ‘‘jumps’’ over the same number of edges of G_X in the positive direction). Moreover, there is a cycle of Fatou gaps attached to G_X at its edges. The longest edge M_X of G_X is the major of \mathcal{L}_X at which the critical gap U_X is attached to G_X . The edge $\sigma_2(M_X) = m_X$ is the minor of \mathcal{L}_X . The gap $V_X = \sigma_2(U_X)$ is attached to G_X at m_X . As explained above, X is a copy of \mathcal{M}_2^c that can be viewed as a ‘‘pinched gap’’ V_X . There are countably many such copies X of \mathcal{M}_2^c and, respectively, countably many gaps V_X attached at edges of the Main Cardioid to the Main Cardioid itself. The remaining vertices of the Main Cardioid correspond to laminations with invariant Siegel gap.

(2) Otherwise, there are maximal by inclusion copies X of \mathcal{M}_2^c in \mathcal{M}_2^c that are generated by gaps V_X of different nature. In these cases, \mathcal{L}_X is a hyperbolic lamination such that U_X is a periodic critical Fatou gap of \mathcal{L}_X of some period n with a fixed return edge of period n . Notice that these are only *maximal by inclusion* Fatou gaps U_X of some period n such that U_X has a fixed return edge of period n ; there exist other hyperbolic geodesic laminations with similar periodic Fatou gaps but we consider only maximal ones among them.

Overall, we have countably many sets X and countably many gaps V_X . All these gaps are pairwise disjoint. In order to construct a new parametric geodesic lamination QML^{nr} (‘‘nr’’ stands for ‘‘non-renormalizable’’) we proceed as follows. First, in case (2) above we replace sets X by the corresponding gaps V_X . In other words, we erase all minors inside gaps V_X . In QML, the gap V_X was ‘‘pinched’’ by a geodesic lamination that copies QML itself; now we ‘‘unpinch’’ V_X . In case (1) above we do almost the same with one exception. Indeed, let X be a canonical copy of \mathcal{M}_2^c associated with a lamination \sim_X and a postcritical Fatou gap V_X of \mathcal{L}_X . Then we erase not only all minors of QML inside V_X but also the edge m_X of V_X which is an edge of the Main Cardioid. This removes the ‘‘barrier’’ between V_X and the Main Cardioid. Evidently, it leads to a big gap which unites all gaps V_X from case (1) above and all Siegel vertices of the Main Cardioid into one

new gap CA^{nr} that can be viewed as the central gap of QML^{nr} replacing in this capacity the Main Cardioid.

From the point of view of σ_2 -invariant geodesic laminations, vertices and edges ℓ of V_X represent geodesic laminations that can be constructed as follows. Take, say, an edge ℓ of V_X . Then take the convex hull of its full preimage in $U_X \cap \mathbb{S}$; this will produce a collapsing quadrilateral $Q_\ell \subset U_X$. Using Thurston's pullback construction, we can repeat this process infinitely many times. Then take the closure of the resulting set of leaves. This, according to [Thu85], will be a σ_2 -invariant geodesic lamination whose critical set projects to a critical leaf under the map from $\text{Bd}(U_X)$ to \mathbb{S} collapsing of all edges of U_X .

Thus, again, the idea is to choose a specific class of geodesic laminations (in this case, these are all renormalizable geodesic laminations) and replace them by geodesic laminations that have either critical leaves or collapsing quadrilaterals (projecting to critical leaves under the canonical collapse of edges of the corresponding postcritical Fatou gap). In other words, QML^{nr} , \sim^{nr} , and the corresponding quotient space \mathcal{M}_2^{nr} describe the family of laminations that can be characterized as follows. First, these are all non-renormalizable q -laminations. Second, these are all geodesic laminations obtained by inserting a critical leaf or a quadrilateral projecting to a critical leaf under the collapse of individual edges in a maximal, by inclusion, critical Fatou gap.

Each edge of the gap V_X of period n is an eventual pullback of a unique edge of V_X of period n , which is in fact the minor m_X of \mathcal{L}_X . In \mathcal{M}_2^{nr} , edges of V_X represent cutpoints of \mathcal{M}_2^{nr} belonging to the boundaries of bounded domains inside \mathcal{M}_2^{nr} . These are exactly the places at which V_X connects with the rest of \mathcal{M}_2^{nr} . Since all edges of V_X are preimages of the unique edge of V_X of period n , it follows that \mathcal{M}_2^{nr} grows from V_X (viewed as a parametric gap) at specific places corresponding to specific laminations inside the critical gap U_X . These are laminations that, after renormalization (that is, after projecting U_X to the entire circle), become laminations with a critical leaf eventually mapped to the point of \mathbb{S} with argument 0.

All of the above applies to all gaps V_X from case (2). Moreover, it almost completely applies to the gaps V_X from case (1). The only exception to this is that the laminations from the Main Cardioid itself will not have their minors represented as the edges of CA^{nr} .

Finally, well-known facts about the combinatorial Mandelbrot set imply that QML^{nr} is perfect. Thus the parametric geodesic lamination QML^{nr} can in fact be interpreted in a different fashion giving rise to the dendritic versions \sim^{nrd} and \mathcal{M}_2^{nrd} of \sim^{nr} and \mathcal{M}_2^{nr} . The main idea was explained in the introduction. The space \mathcal{M}_2^{nrd} represents the model for quotient of the space of all q -laminations determined by the equivalence relation under

which two q -laminations $\mathcal{L}_1, \mathcal{L}_2$ are identified in the following two cases: (1) is the oldest ancestor of them is the empty lamination (this means that either lamination is a tuning of a lamination from the Main Cubrioid), or (2) \mathcal{L}_1 and \mathcal{L}_2 have the same oldest ancestor lamination which is not the empty lamination.

REFERENCES

- [BL02] A. Blokh, G. Levin, *Growing trees, laminations and the dynamics on the Julia set*, Ergodic Theory and Dynamical Systems **22** (2002), 63–97.
- [BMOV13] A. Blokh, D. Mimbs, L. Oversteegen, K. Valkenburg, *Laminations in the language of leaves*, Trans. Amer. Math. Soc., **365** (2013), 5367–5391.
- [BOPT16a] A. Blokh, L. Oversteegen, R. Ptacek, V. Timorin, *The combinatorial Mandelbrot set as the quotient of the space of geolaminations*, Contemporary Mathematics, **669** (2016), 37–62
- [BOPT16b] A. Blokh, L. Oversteegen, R. Ptacek, V. Timorin, *Laminations from the Main Cubrioid*, Discrete and Continuous Dynamical Systems - Series A, **36** (2016), 4665–4702
- [BOPT17] A. Blokh, L. Oversteegen, R. Ptacek, V. Timorin, *Laminational models for some spaces of polynomials of arbitrary degree*, preprint arXiv:1401.5123, to appear in Memoirs of the American Mathematical Society (2017)
- [DH85] A. Douady, J. H. Hubbard, *Étude dynamique des polynômes complexes* I, II, Publications Mathématiques d’Orsay **84-02** (1984), **85-04** (1985).
- [Kel00] K. Keller, *Invariant factors, Julia equivalences and the (abstract) Mandelbrot set*, Lecture Notes in Mathematics, **1732**, Springer-Verlag, Berlin, 2000.
- [Kiw02] J. Kiwi, *Wandering orbit portraits*, Trans. Amer. Math. Soc. **354** (2002), 1473–1485.
- [Kiw04] J. Kiwi, *Real laminations and the topological dynamics of complex polynomials*, Advances in Mathematics, **184** (2004), 207–267.
- [Mil00] J. Milnor, *Dynamics in One Complex Variable*, Annals of Mathematical Studies **160**, Princeton (2006).
- [Sch09] D. Schleicher, *Appendix: Laminations, Julia sets, and the Mandelbrot set*, in: “Complex dynamics: Families and Friends”, ed. by D. Schleicher, A K Peters (2009), 111–130.
- [Thu85] W. Thurston. *The combinatorics of iterated rational maps* (1985), with appendix by D. Schleicher, *Laminations, Julia sets, and the Mandelbrot set*, published in: “Complex dynamics: Families and Friends”, ed. by D. Schleicher, A K Peters (2009), 1–137.

(Alexander Blokh) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA AT BIRMINGHAM, BIRMINGHAM, AL 35294

E-mail address: ablokh@math.uab.edu

(Lex Oversteegen) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA AT BIRMINGHAM, BIRMINGHAM, AL 35294

E-mail address: overstee@uab.edu

(V. Timorin) NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, 6 USACHEVA ST., 119048 MOSCOW, RUSSIA