UAB MATH-BY-MAIL COMPETITION

ELIGIBILITY. Math-by-Mail competition is an individual contest run by the UAB Department of Mathematics and designed to test logical thinking and depth of understanding of mathematics by high school students. The contest is open to all high school students from participating schools.

PRIZES AND CEREMONY. One first prize ($60 and trophy), one-two second prizes ($40 and a trophy) and some third prizes ($20 and an award diploma) will be handed out (since awards/prizes to individuals are reportable annually by UAB to the IRS, cash prize-winners will have to have a valid U.S. federal tax identification number, will need to fill out W-9/W-8 form prior to receiving cash prizes, and will need to report prizes on their tax returns). All other participants will receive certificates honoring their participation. The school with the best overall performance will be awarded the UAB traveling Math trophy. Prize winners will be invited to an awards ceremony at UAB on Friday April 19th. Parents and teachers will also be invited. The program will feature awards presentation and refreshments.

RULES AND JUDGING. Math-by-Mail contest is an individual contest. Students must solve problems on their own. All solutions must be self-contained; no references to facts and mathematical results beyond the high school standard curriculum are permitted (nor will they be necessary). The explanations must be detailed and thorough, all answers must be explained and justified! Professors of the Department of Mathematics at UAB will judge the entries and evaluate your reasoning process and method of finding the solution. Criteria will include correctness and elegance of solutions. The decision of the judges is final. The entries will not be returned.

FORMAT AND CONTACTS. The participating schools will receive problems and cover sheets before **Thursday March 21st**. On **Thursday March 21st** the problems and cover sheets should be made available to the students. Every participant must submit a filled copy of the cover sheet; each solution must be written on a separate sheet of paper with the student name and the school name on it. The entries must be postmarked no later than **Wednesday, April 3rd**, and mailed to:

UAB Math-by-Mail Contest
c/o Dr Alexander Blokh
Department of Mathematics, UAB
1300 University Blvd, CH 452
Birmingham AL 35294-1170

For additional information you can contact us at (205)934-2154 or by electronic mail at ablokh@math.uab.edu

UAB MATH-BY-MAIL CONTEST COVER SHEET

(1) YOUR NAME

(2) YOUR SCHOOL

(3) YOUR GRADE

(4) YOUR ADDRESS

(5) YOUR PHONE NUMBER

(6) YOUR E-MAIL ADDRESS (IF ANY)

UAB MATH-BY-MAIL CONTEST 2002, PROBLEMS

Solve the problem(s) and write each solution on a separate sheet of paper with your name and your school's name on it. If you can (and only after you have done as much as you could solving problems), try to see if you can solve more general problems using your methods. Some third prizes will be awarded for particularly good solutions of individual problems, so **you can win a third prize for an excellent job in just one problem!**

Complete your cover sheet (see your teacher to get one) and include it with your submission. Submit your entries (postmarked by **April 3rd**) to:

UAB Math-by-Mail Contest
c/o Dr Alexander Blokh
Department of Mathematics, UAB
1300 University Blvd, CH 452
Birmingham AL 35294-1170

PROBLEMS

PROBLEM 1. *Find the angle of* 1° *if the angle of* 19° *is given. You are allowed to use only a compass and a ruler.*

PROBLEM 2. *Find all real solutions of the equation*

$$x^2 + y^2 + z^2 + t^2 = x(y + z + t)$$

PROBLEM 3. *There is a blot on a piece of paper. For every point of the blot we measure the least and the greatest distance from this point to the boundary of the blot. Then we choose the greatest of all the least distances and denote it by* r*. We also choose the least of the greatest distances and denote it by* R*. It turns out that* $r = R$*. What then is the shape of the blot?*

PROBLEM 4. *In a soccer tournament every team played with every team exactly once. The winner got 7 points, the second place winner got 5 points, and the third place winner got 3 points (in a game the winner gets 2 points, tie gives both teams 1 point, and the loser gets 0 points; if two teams have the same number of points, the places are decided by drawing lots). How many teams participated and how many points did the last team get?*

PROBLEM 5. *A positive integer* $n > 1$ *is* **prime** *if it has no positive integer divisors other than* 1 *and* n*. The function* $f(n)$ *is defined as the product of all prime numbers* $i \le n$*. Find all* n *such that* $f(n) \le n$*.*

PROBLEM 6. *On an infinite chess board a closed (circular) non-self-intersecting broken line* L *is sketched so that it goes along the boundaries of squares. There are* k *dark squares inside the region* A *bounded by* L*. What is the largest possible area of* A*? A single square has area one.*

UAB MATH-BY-MAIL CONTEST 2002, PROBLEMS AND THEIR SOLUTIONS

PROBLEM 1. *Find the angle of* $1^{\rm o}$ *if the angle of* $19^{\rm o}$ *is given. You are allowed to use only a compass and a ruler.*

Solution. Since $19^2 = 361$, one can copy the given angle 19 times so that every new copy has the common vertex and one common side with the previous one and is located to the counterclockwise direction of the previous one. After this is done 19 times, there will be overall an angle of $361^{\rm o}$ constructed, so the last side of the last angle forms the angle of $1^{\rm o}$ with the first side of the first angle. $\square$

PROBLEM 2. *Find all real solutions of the equation*

$$x^2 + y^2 + z^2 + t^2 = x(y + z + t)$$

Solution. Let us replace the given equation by the following equivalent one: $.25x^2 + (.5x - y)^2 + (.5x - z)^2 + (.5x - t)^2 = 0$. This is only possible if $x = .5x - y = .5x - z = .5x - t = 0$ which in turn implies that $x = y = z = t = 0$. $\square$

PROBLEM 3. *There is a blot on a piece of paper. For every point of the blot we measure the least and the greatest distance from this point to the boundary of the blot. Then we choose the greatest of all the least distances and denote it by* r*. We also choose the least of the greatest distances and denote it by* R*. It turns out that* $r = R$*. What then is the shape of the blot?*

Solution. Suppose that the least distance r to the boundary of the blot is assumed at a point A. Then the entire disk D_1 of radius r centered at A is contained in the blot. Suppose that the greatest distance R to the boundary of the blot is assumed at a point B. Then the entire disk D_2 of radius R centered at B contains the blot. So, D_2 contains the blot which in turn contains D_1. Since by the assumption $r = R$ we conclude that $A = B$, $D_1 = D_2$ and the blot has the shape of the disk of radius r. $\square$

PROBLEM 4. *In a soccer tournament every team played with every team exactly once. The winner got 7 points, the second place winner got 5 points, and the third place winner got 3 points (in a game the winner gets 2 points, tie gives both teams 1 point, and the loser gets 0 points; if two teams have the same number of points, the places are decided by drawing lots). How many teams participated and how many points did the last team get?*

Solution. Suppose there are n teams playing in the tournament. First let us show that there have been $\dfrac{n(n-1)}{2}$ games and the overall number of points is $n(n-1)$. Indeed, let us count the games registering every game for both teams who played. Since every team plays with $n-1$ teams, there will be $n(n-1)$ teams registered. However then every game will be registered twice, so to count the number of games we need to divide $n(n-1)$ by 2. Since every game brings up the overall number of points by 2 we see that indeed the overall number of points after the tournament is $n(n-1)$.

Now, the first, second and third teams finished the tournament with $7+5+3=15$ points. Hence $n(n-1) \geq 15$ and $n \geq 5$. On the other hand all teams starting with the one who placed the third finished with no more than 3 points each, and there were $n-2$ such teams. Hence for the overall number of points we have an estimate $n(n-1) \leq 7+5+3(n-2) = 3n+6$. Solving this quadratic inequality we see that $n \leq 5$. Thus $n=5$.

It remains to notice that the teams which placed the fourth and the fifth finished with 5 points overall while none of them got more than 3 points (otherwise they would have placed higher). The only way it can happen is when the fourth place team got 3 points while the fifth place team got 2 points. $\square$

PROBLEM 5. *A positive integer $n > 1$ is* **prime** *if it has no positive integer divisors other than* 1 *and n. The function $f(n)$ is defined as the product of all prime numbers $i \leq n$. Find all n such that $f(n) \leq n$.*

Solution. Observe that $f(2) = 2$. Let us show that for any $n > 2$ we have $f(n) > n$. Indeed, consider the number $f(n) - 1 = 2 \cdot 3 \cdot \dots \cdot p - 1$ where p is the greatest prime number not exceeding n. Then the number $f(n) - 1$ cannot have prime divisors less than or equal to n which implies that $f(n) - 1$ cannot be less than or equal to n as desired. $\square$

PROBLEM 6. *On an infinite chess board a closed (circular) non-self-intersecting broken line L is sketched so that it goes along the boundaries of squares. There are k dark squares inside the region A bounded by L. What is the maximal area of A? A single square has area one.*

Solution. The answer is $4k+1$. First let us show that it is possible to have a region A described in the problem whose area is $4k+1$. Consider a line of squares consisting of $2k+1$ squares which starts and ends on a white square. This line will contain exactly k dark squares. Let us also add to this $2k$ white squares located right above and right below the dark squares in the aforementioned line of squares. This creates the region A of area $4k+1$ bounded by a closed non-self intersecting broken line L as desired.

Let us now show that this is the maximal possible area for a region like that. Consider a region A as described in the problem. Consider centers of squares in A. A segment is said to be **legal** if it connects centers of two adjacent squares in A. A broken line which is formed by legal segments is said to be **legal** too. Also, a broken line (perhaps with self-intersections) is said to be **connected** if it can be drawn without lifting a pen from the paper.

Clearly, there exist connected legal broken lines. Moreover, because of how A is defined it is possible to have a connected legal broken line which passes through all vertices of all squares in A. Of all connected legal broken lines passing through all centers of squares of A we can choose a line with the least number of legal segments in it. Choose one such broken line and denote it by M.

As we know there are k dark squares inside A. Since each dark segment may have at most 4 legal segments coming out of it, we conclude that there are at most $4k$ legal segment in M. On the other hand, the number n of vertices in A equals the number of squares in A, that is it equals the area of A. Let us see how n and $4k$ are related. In fact we will show now that $4k \geq n - 1$.

First observe that in M there are no loops of legal segments. Indeed, otherwise we could remove one segment from the loop and the resulting broken line will be still a legal connected broken line having vertices at centers of all squares in A. This contradicts the minimality of M and proves that there are no loops of legal segments in M.

Choose a vertex R in L and call it a **root** vertex. Then it is easy to see that for every other vertex D in M there exists a unique path connecting R and D (by **path** we mean a sequence of legal segments concatenated at their ends such that no legal segment is repeated twice). Indeed, a path like that must exist because M is connected; also, if it not unique then there would exist a loop of segments, a contradiction with the proven above.

Now, to every vertex $D \neq R$ we associate the last legal segment in the path connecting R with D. Clearly, different vertices D will be then associated with different legal segments in M, and this association can be done for all vertices of M but R. We conclude that the number of vertices in M is less than or equal to the number of legal segments in M plus 1; in other words, we show that $n \leq 4k + 1$ as desired. $\square$

UAB MATH-BY-MAIL 2002 AWARD CEREMONY

1. Tea

2. Speech: Dr Mayer (overall 30 students participated).

3. Handing out awards.

a) Schools Honorable Mentions with Certificates of Participation for their students:

i) Hoover High School, 19 participants

ii) Indian Springs School, 1 participant

iii) Muscle Shoals High School, 1 participant

iv) ASFA, 8 participants

v) Loveless Magnet High School, 1 participant

b) Traveling Trophy: Hoover High School

c) Individual Trophies:

Third prizes:

i) Avinash Murthy, Hoover

ii) Kyle Fritz, Hoover

iii) Weichen Zhu, Hoover

iv) Fan Yang, ASFA

Second Prize:

Adam Roth, Hoover

First Prize:

Drew Newman, Muscle Shoals