UAB MATH-BY-MAIL COMPETITION

ELIGIBILITY. Math-by-Mail competition is an individual contest run by the UAB Department of Mathematics and designed to test logical thinking and depth of understanding of mathematics by high school students. The contest is open to all high school students from participating schools.

PRIZES AND CEREMONY. One first prize ($60 and trophy), one-two second prizes ($40 and a trophy) and some third prizes ($20 and an award diploma) will be handed out (since awards/prizes to individuals are reportable annually by UAB to the IRS, cash prize-winners will have to have a valid U.S. federal tax identification number, will need to fill out W-9/W-8 form prior to receiving cash prizes, and will need to report prizes on their tax returns). All other participants will receive certificates honoring their participation. The school with the best overall performance will be awarded the UAB traveling Math trophy. Prize winners will be invited to an awards ceremony at UAB on Friday, April 11. Parents and teachers will also be invited. The program will feature awards presentation and refreshments.

RULES AND JUDGING. Math-by-Mail contest is an individual contest. Students must solve problems on their own. All solutions must be self-contained; no references to facts and mathematical results beyond the high school standard curriculum are permitted (nor will they be necessary). The explanations must be detailed and thorough, all answers must be explained and justified! Professors of the Department of Mathematics at UAB will judge the entries and evaluate your reasoning process and method of finding the solution. Criteria will include correctness and elegance of solutions. The decision of the judges is final. The entries will not be returned.

FORMAT AND CONTACTS. The participating schools will receive problems and cover sheets before or on Monday, March 10. On Monday, March 10 the problems and cover sheets should be made available to the students. Every participant must submit a filled copy of the cover sheet; each solution must be written on a separate sheet of paper with the student name and the school name on it. The entries must be postmarked no later than Monday, March 24, and mailed to:

UAB Math-by-Mail Contest
c/o Dr Alexander Blokh
Department of Mathematics, UAB
1300 University Blvd, CH 452
Birmingham AL 35294-1170

For additional information you can contact us at (205)934-2154 or by electronic mail at ablokh@math.uab.edu

UAB MATH-BY-MAIL CONTEST COVER SHEET

(1) YOUR NAME

(2) YOUR SCHOOL

(3) YOUR GRADE

(4) YOUR ADDRESS

(5) YOUR PHONE NUMBER

(6) YOUR E-MAIL ADDRESS (IF ANY)

UAB MATH-BY-MAIL CONTEST 2003, PROBLEMS

Solve the problem(s) and write each solution on a separate sheet of paper with your name and your school's name on it. If you can (and only after you have done as much as you could solving problems), try to see if you can solve more general problems using your methods. Some third prizes will be awarded for particularly good solutions of individual problems, so **you can win a third prize for an excellent job in just one problem!**

Complete your cover sheet (see your teacher to get one) and include it with your submission. Submit your entries (postmarked by **March 24, 2003**) to:

UAB Math-by-Mail Contest
c/o Dr Alexander Blokh
Department of Mathematics, UAB
1300 University Blvd, CH 452
Birmingham AL 35294-1170

PROBLEMS

PROBLEM 1. *Which of the two numbers is greater,*

$$\frac{1+13^{2002}}{1+13^{2003}}$$

or

$$\frac{1+13^{2003}}{1+13^{2004}}?$$

Could you generalize this replacing 13 *by a positive* x*? replacing* 2002 *by an integer* n*? For what positive value of* x *can the two fractions in question coincide?*

PROBLEM 2. *Can you divide any triangle into several isosceles triangles? If yes then how? If not, give an example when it is impossible.*

PROBLEM 3. *The sum of three positive numbers equals* 6*. What is the minimal value of the sum of their squares? Fully justify your answer.*

PROBLEM 4. *A box contains 300 matches. Players take turns removing no more than half the matches from the box. The player after whose move only one match remains in the box wins. Which of the two players has the winning strategy, and what is it?*

PROBLEM 5. *In a chess tournament, the winner of each game gets 1 point and the looser 0 points. In the case of a tie each gets* $\frac{1}{2}$ *points. An important tournament was held. The observation is made that exactly half of each player's total points was earned in games played with the 3 players who finished in the last 3 positions in the tournament. How many players took part in the tournament?*

UAB MATH-BY-MAIL CONTEST 2003, PROBLEMS AND THEIR SOLUTIONS

PROBLEM 1. *Which of the two numbers is greater,*

$$\frac{1+13^{2002}}{1+13^{2003}}$$

or

$$\frac{1+13^{2003}}{1+13^{2004}}\,?$$

Could you generalize this replacing 13 *by a positive x? replacing* 2002 *by an integer n? For what positive value of x can the two fractions in question coincide?*

Solution: Replacing 13 by x and 2002 by n we get to compare two numbers,

$$\frac{1+x^n}{1+x^{n+1}} \text{ and } \frac{1+x^{n+1}}{1+x^{n+2}}.$$

Cross-multiplying and using the fact that x is positive we see that this is equivalent to comparing $1+x^n+x^{n+2}+x^{2n+2}$ and $1+2x^{n+1}+x^{2n+2}$, or, after subtracting common terms and factoring, $x^n(1+x^2)$ and $x^n \cdot 2x$. Dividing by x^n we then get to compare x^2+1 and $2x$. However it is easy to see that since $x^2+1-2x=(x-1)^2 \geq 0$ then in fact $x^2+1 \geq 2x$ with equality possible if and only if $x=1$. This proves that in fact

$$\frac{1+x^n}{1+x^{n+1}} \geq \frac{1+x^{n+1}}{1+x^{n+2}}$$

and the equality holds only for $x=1$.

So the answer is

$$\frac{1+13^{2002}}{1+13^{2003}} > \frac{1+13^{2003}}{1+13^{2004}}$$

and the problem is solved. □

PROBLEM 2. *Can you divide any triangle into several isosceles triangles? If yes then how? If not, give an example when it is impossible.*

Solution: Let us consider first a right triangle. It is known that the distance from the mid-point of its hypotenuse to the three vertices of the triangle is the same. Hence a right triangle can be divided into two isosceles triangles. Now, if a triangle is not iscoseles, we can find a vertex of the triangle such that the height of the triangle through this vertex intersects the opposite side. This divides the triangle into two right triangles each of which can be then divided into two isosceles triangles. Thus any triangle can be divided into at most 4 isosceles triangles.

So the answer is "Yes, any triangle can be divided into several isosceles triangles". □

PROBLEM 3. *The sum of three positive numbers equals* 6*. What is the minimal value of the sum of their squares? Fully justify your answer.*

Solution: First of all, observe that for any two real numbers a, b we have $(a-b)^2 = a^2 - 2ab + b^2 \geq 0$ which implies that $a^2 + b^2 \geq 2ab$ with equality taking place if and only if $a = b$. Now, we are given three numbers a, b, c such that $a+b+c = 6$. This implies that $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ac) = 36$. However we saw that $a^2 + b^2 \geq 2ab$, and similarly $b^2 + c^2 \geq 2bc$ and $c^2 + a^2 \geq 2ac$. Summing up these inequalities, we see that $2a^2 + 2b^2 + 2c^2 \geq 2(ab + bc + ac)$. Hence, $3(a^2 + b^2 + c^2) \geq a^2 + b^2 + c^2 + 2(ab + bc + ac) = 36$ and so $a^2 + b^2 + c^2 \geq 12$. On the other hand, choosing $a = b = c = 2$ we see that this lower bound is sharp which completes the solution of the problem.

So the answer is that the minimal value of the sum of the squares of the numbers in question is 6. □

PROBLEM 4. *A box contains 300 matches. Players take turns removing no more than half the matches from the box. The player after whose move only one match remains in the box wins. Which of the two players has the winning strategy, and what is it?*

Solution: Let us begin by considering the cases when the number of matches is much less. If there are 2 matches to begin with, then the first player clearly wins. If there are 3 matches in the beginning of the game, then the first player can only remove one match, so he/she loses. This is an important moment, because now we see that if there are $4, 5, 6$ matches in the beginning of the game, then the first player can remove several matches so that exactly 3 matches remain, after which the second player must remove exactly one match, after which the first one removes one match and wins. However, if there are 7 matches in the beginning of the game then the first player loses because no matter how many matches he/she removes, there will be $4, 5$ or 6 left and since now it is time for the second player to make a move, we know that this time he/she will win.

Clearly, we see now what the pattern is and can design the strategy for the winning player. There are 2 cases which we now consider step by step.

Case 1. Suppose that the initial number k_0 of matches is not equal to $2^n - 1$ for any n (in particular this is the case in the problem in question); in other words, there exists an integer m such that $2^m - 1 < k_0 < 2^{m+1} - 1$. Then the first player wins. He/she can proceed as follows. First, remove several matches (in fact, exactly $m - 2^n + 1$) so that the number of remaining matches is now $k_1 = 2^m - 1$. Then no matter what the other player does, after his/her next move the number of matches will be k_2 such that $2^{m-1} - 1 < k_2 < 2^m - 1$, and the same kind of move can be repeated by the first player reducing the number of matches to $k_3 = 2^{m-1} - 1$. Keeping it up, the first player always gets to remove matches in such a way that there are $2^{m-2} - 1, 2^{m-3} - 1, \ldots$

matches after his/her move. In particular, there will be $2^1 - 1 = 1$ match after his/her final move which proves that with the right strategy the first player always wins if the number of matches is not $2^n - 1$ for any integer n.

Case 2. If there are $2^n - 1$ matches to begin with, then after any move by the first player the situation is such that Case 1 is applicable. This shows that in this case with the right strategy the second payer always wins.

So the answer is that in the case when to begin with there are 300 matches the first player wins (by first reducing the number of matches to 255, and then responding to the second player by reducing the number of matches on the next step to 127, 63, . . .). □

PROBLEM 5. *In a chess tournament, the winner of each game gets 1 point and the looser 0 points. In the case of a tie each gets $\frac{1}{2}$ points. An important tournament was held. The observation is made that exactly half of each player's total points was earned in games played with the 3 players who finished in the last 3 positions in the tournament. How many players took part in the tournament?*

Solution: Clearly, the number of participants is greater than 3, so the participants can be divided into two groups: the bottom three players form the group B and the rest form the group T (for brevity, we will talk of T-players and B-players and will also use the terms like BT-game meaning a game played by a B-player and a T-player).

The number of points added to the overall number of points after one game is one. Hence the number of points equals the number of games. The number of games played within the group made up of k players equals $.5k(k-1)$. Hence there are 3 points gained in all BB-games and $.5(n-3)(n-4)$ points gained in all TT-games. Since B-players gained the same number of points playing in BT-games as in BB-games, then they gained 3 points in BT-games. The overall number of points gained in BT-games was 3(n-3). Hence T-players gained 3(n-3)-3=3n-12 points in BT-games. These points form one half of all points gained by T-players, hence $.5(n-3)(n-4) = 3n-12$ which yields that $n = 4$ or $n = 9$. However if $n = 4$ then the number of points gained by the only T-player is zero, a contradiction.

So the answer is $n = 9$. □

UAB MATH-BY-MAIL 2002 AWARD CEREMONY

1. Tea

2. Speech: Dr Mayer (overall 97 students participated in the contest!).

3. Handing out awards.

a) Schools Honorable Mentions with Certificates of Participation for their students:

i) ASFA, 13 participants

ii) Grissom High, 2 participants

iii) Hoover High School, 22 participants

iv) Minor High, 1 participant

v) Spain Park High, 13 participant

vi) Vestavia High, 46 participants

b) Traveling Trophy: Vestavia High School

c) Individual Trophies:

Honorable Mentions:

Faraz Chowdhury, Grissom High

Sergey Sarkisov, Grissom High

Adam Roth, Hoover High

Masha Blokh, ASFA

Yang Su, Hoover High

Third prizes:

i) Anton Fedorov, Vestavia High

ii) Kyle Fritz, Hoover High

Second prize:

i) Weichen Zhu, Hoover High

ii) Fan Yang, ASFA

First Prize:

Avinash Murthy, Hoover High