NAME:______________________________

GRADE:______________________________

SCHOOL CODE:______________________

2005-2006 UAB MATH TALENT SEARCH

This is a two hour contest. Answers are to be written in the spaces provided on the test sheet. You **MUST** justify your answers in order to get full credit. Your work (including full justification) should be shown on the extra paper which we supply.

PROBLEM 1 (10 pts) A group of N students arrives to participate in a mathematics tournament. There are actually four different tests given out, one to each student at random. After they are collected, it is found that 1/3 of the N students took test 1, 1/4 took test 2, 1/5 took the test 3, and 26 students took test 4. Determine the total number N of the students.

YOUR ANSWER:

PROBLEM 2 (20 pts) Recall that a chord of a circle is a straight line segment having its endpoints on the circle. Now suppose two circles are drawn with the same center. The region consisting of the points on the circles, together with all those between the two circles, is referred to as the "ring". Suppose that the ring has the area 1. Find the length of the longest chord that lies entirely within the ring.

YOUR ANSWER:

PROBLEM 3 (30 pts) ID numbers are to be assigned, having 3 digits (each any integer from 1 through 9 inclusive) and three letters (each a lower case letter of the English alphabet taken from the list a, b, c, d, e, f, g). How many different codes can be formed this way if the digits must all be distinct, and the the three letters must be all different?

YOUR ANSWER:

PROBLEM 4 (70 pts) Find all pairs x, y of positive integers which solve the equation $2^x + 1 = y^2$. Sum up all x's and y's and write the answer in the space provided.

YOUR ANSWER:

PROBLEM 5 The problem consists of two parts.

a) (90 pts) A triangle ABC of area 90 is given. The point X on AB is such that $\frac{|AB|}{|XB|} = 5$. The point Y on BC is such that $\frac{|CB|}{|YB|} = 5$. The segments AY and CX intersect each other at a point Z. Find the area of the quadrilateral BXZY.

YOUR ANSWER:

b) (90 pts) Fully justify your answer on your scratch paper (if you did it, write JUSTIFIED below, otherwise write NOT JUSITIFIED below).

YOUR ANSWER:

PROBLEM 6 This problem deals with the plane.

a) (90 pts) The closed disk of radius 1 consists of all points on or enclosed by a circle of radius 1. Determine the largest number of points that can be put in the closed disk of radius 1 so that the distance between any two of these points is at least 1 and describe the actual placement of this largest number of points in the closest disk.

YOUR ANSWER:

b) (110 pts) Fully justify your answer on your scratch paper (if you did it, write JUSTIFIED below, otherwise write NOT JUSITIFIED below).

YOUR ANSWER:

2005-2006 UAB MTS: SOLUTIONS

PROBLEM 1 (10 pts) A group of N students arrives to participate in a mathematics tournament. There are actually four different tests given out, one to each student at random. After they are collected, it is found that 1/3 of the N students took test 1, 1/4 took test 2, 1/5 took the test 3, and 26 students took test 4. Determine the total number N of the students.

Solution: By the assumptions, 26 students who took test 4 form $1 - 1/3 - 1/4 - 1/5 = 13/60$ part of the entire group of N students. Hence, $N = (26 \cdot 60)/13 = 120$.

So the answer is **120**.

PROBLEM 2 (20 pts) Recall that a chord of a circle is a straight line segment having its endpoints on the circle. Now suppose two circles are drawn with the same center. The region consisting of the points on the circles, together with all those between the two circles, is referred to as the "ring". Suppose that the ring has the area π. Find the length of the longest chord that lies entirely within the ring.

Solution: Denote the radii of the circles r and R respectively. If a chord l inside the circle of radius R is given, then we can denote the segment connecting the center of the circle with the midpoint of the chord by m and see that by Pythagorean Theorem $l = 2\sqrt{R^2 - m^2}$. Hence the longest chord in the ring will be the one which is the closest to the center and still is inside the ring - thus, it will be the chord which is tangent to the inner circle and for which $m = r$. We conclude that the length of the longest chord in the ring is $2\sqrt{R^2 - r^2}$. On the other hand, by the assumptions the area $\pi(R^2 - r^2)$ of the ring is equal to π. Hence $R^2 - r^2 = 1$ and the length of the longest chord inside the ring is 2.

So the answer is **2**.

PROBLEM 3 (30 pts) ID numbers are to be assigned, having 3 digits (each any integer from 1 through 9 inclusive) and three letters (each a lower case letter of the English alphabet taken from the list a, b, c, d, e, f, g). How many different codes can be formed this way if the digits must all be distinct, and the the three letters must be all different?

Solution: First let us count the number of ways the places for the digits can be chosen. It is well known that you can choose 3 places out of 6 in $\frac{6\cdot5\cdot4}{1\cdot2\cdot3\cdot3} = 20$ ways. Now, you can put digits in their 3 places in $9\cdot8\cdot7 = 504$ ways, and you can put letters in their 3 places in $7\cdot6\cdot5 = 210$ ways. Hence there are $20\cdot504\cdot210 = 2116800$ ID numbers.

So the answer is **2116800**.

PROBLEM 4 (70 pts) Find all pairs x, y of positive integers which solve the equation $2^x + 1 = y^2$. Sum up all x's and y's and write the answer in the space provided.

YOUR ANSWER:

Solution: It follows that $2^x = (y-1)(y+1)$. Since divisors of a power of 2 are all powers of 2 we see that for some $u < v$ we have $y - 1 = 2^u, y + 1 = 2^v$ and $v > u \geq 1$. Hence $2^u + 2 = 2^v$ and so $2^{u-1} + 1 = 2^{v-1}$. Since 2^{v-1} is even (as follows from above, $v \geq 2$), this is only possible if $2^{u-1} = 1$ and $u = 1$. Then $v = 2$. Thus, $y = 2^u + 1 = 3, x = 3$ and $x + y = 6$.

So the answer is **6**.

PROBLEM 5 The problem consists of two parts.

a) (90 pts) A triangle ABC of area 90 is given. The point X on AB is such that $\frac{|AB|}{|XB|} = 5$. The point Y on BC is such that $\frac{|CB|}{|YB|} = 5$. The segments AY and CX intersect each other at a point Z. Find the area of the quadrilateral BXZY.

YOUR ANSWER:

b) (90 pts) Fully justify your answer on your scratch paper (if you did it, write JUSTIFIED below, otherwise write NOT JUSITIFIED below).

YOUR ANSWER:

Solution: The area of a triangle is the product of the height and the base divided by 2. Since in the triangle XBY the height and the base are shorter than those of the triangle ABC by the factor of 5, we see that the area of the triangle XBY is $\frac{1}{25}$ of the area of the triangle ABC. It remains to see what part of the area of the triangle ABC is the area of the triangle XYZ. As before, we know that $|XY| = \frac{1}{5}|AC|$. Hence we need to figure the fraction between the height of the triangle XYZ and the height of the triangle ABC.

Denote the height BD of ABC by h. Observe that the height h' of AXC is such that $h'/h = 4/5$. Denote the length of the height of the triangle XZY passing through Z by h''. Now, draw a line through X parallel to AY until it crosses BC at a point E. Then $BE/EY = BX/XA = 1/4$. This implies that $EY/YC = XZ/ZC = 1/5$. Hence $XZ/XC = h''/h' = 1/6$. So, $h''/h = 4/5 \cdot 1/6 = 2/15$. Thus, the area of the triangle XYZ is $1/5 \cdot 2/15 = 2/75$ of the area of the triangle ABC.

As we saw before, the area of the triangle XBY is $1/25$ of the area of the triangle of ABC. Hence the area of the quadrilateral BXZY is $1/25 + 2/75 = 1/15$ of 90 which equals 6.

So the answer is **6**.

PROBLEM 6 This problem deals with the plane.

a) (90 pts) The closed disk of radius 1 consists of all points on or enclosed by a circle of radius 1. Determine the largest number of points that can be put in the closed disk of radius 1 so that the distance between any two of these points is at least 1 and describe the actual placement of this largest number of points in the closest disk.

YOUR ANSWER:

b) (110 pts) Fully justify your answer on your scratch paper (if you did it, write JUSTIFIED below, otherwise write NOT JUSITIFIED below).

YOUR ANSWER:

Solution: Denote by D the closed disk of radius 1, and by $\mathcal{A}$ a collection of points in D satisfying the conditions of Problem 2. Connect each of them with the center of O of D (O may well belong to $\mathcal{A}$). Then the angle between any two such segments cannot be less than 60 degrees. Indeed, suppose that points B, C belong to $\mathcal{A}$ and that the angle BOC is less than 60 degrees. Then it is easy to see that

$|BC| < 1$. Since the full angle is 360 degrees, we see that except for the center we can have no more than 6 points in the set $\mathcal{A}$, and with the center $\mathcal{A}$ may contain no more than 7 points. The example of the set $\mathcal{A}$ with 7 points is the center and the vertices of the perfect hexagon.

So, the answer is **7**.