NAME:______________________________

GRADE:______________________________

SCHOOL CODE:______________________

2007-2008 UAB MATH TALENT SEARCH

This is a two hour contest. Answers are to be written in the spaces provided on the test sheet. You **MUST** justify your answers in order to get full credit; otherwise, partial credit or no credit will be awarded according to the decision made by the judges. Your work (including full justifications) should be shown on the extra paper which is attached. The problems are listed in increasing order of difficulty.

PROBLEM 1 (10 pts) Jenny wants to buy a lollipop, but she is lacking 12 pennies to do so. Jack wants to buy a lollipop too, but he is lacking one penny to do so. They put their money together, but still could not buy a lollipop. What is the price of one lollipop?

YOUR ANSWER:

PROBLEM 2 (20 pts) Bill has a few sisters and brothers. The number of his sisters is by 2 greater than the number of his brothers. How many more daughters than sons do Bill's parents have?

YOUR ANSWER:

PROBLEM 3 (30 pts) A box is filled with cubes and balls, all of which are either blue or red. Thirty percent of the objects in the box are cubes. Thirty percent of the balls in the box are blue. What percent of the objects in the box are red balls?

YOUR ANSWER:

PROBLEM 4 (50 pts) A square hall's floor is tiled with congruent square tiles each of which is 1 square foot. There are 2009 tiles on the two diagonals of the floor. How long is a wall of the hall?

YOUR ANSWER:

PROBLEM 5 (70 pts) It is given that $t = \frac{y}{x-z} = \frac{x+y}{z} = \frac{x}{y}$ for three positive numbers x, y, z. What is then the value of t?

YOUR ANSWER:

PROBLEM 6 (90 pts) Early in the morning a camel started walking from an oasis A to an oasis B. Simultaneously, a donkey started walking from the oasis B to the oasis A. They met at noon and continued on their paths with the same speed. The camel came to the oasis B at 4pm while the donkey came to the oasis A at 9 pm. When did they start their journeys?

YOUR ANSWER:

PROBLEM 7 (120 pts) What is the size of the largest subset, S, of $\{1, 2, \ldots, 2008\}$ such that no pair of distinct elements of S has a sum divisible by 9?

YOUR ANSWER:

PROBLEM 8 (160 pts) The numbers 1, 2, 3, 5, 7, 11, 13 are written on a board. We may erase any two numbers a and b and replace them by $ab+a+b$. After repeating this process several times, only one number remains on the board. What can be this number?

YOUR ANSWER:

PROBLEM 9 (220 pts) Find formulas for all real roots of the equation $\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} = x$; determine for what values of the real parameter p real roots of the equation exist.

YOUR ANSWER:

2006-2007 UAB MTS: SOLUTIONS

PROBLEM 1 (10 pts) Jenny wants to buy a lollipop, but she is lacking 12 pennies to do so. Jack wants to buy a lollipop too, but he is lacking one penny to do so. They put their money together but still could not buy a lollipop. What is a price of one lollipop?

Solution: If Jenny had any money, they would be able to buy a lollipop because Jack is lacking only one penny. Hence Jenny has no money. Since she is lacking 12 pennies to buy a lollipop, the price of a lollipop is 12 pennies.

So the answer is **12 pennies**.

PROBLEM 2 (20 pts) Bill has a few sisters and brothers. The number of his sisters is by 2 greater than the number of his brothers. How many more daughters than sons do Bill's parents have?

Solution: If the number of Bill's brothers is b and the number of his sisters is s then $s = b + 2$. His parents have s daughters and $b + 1$ sons because Bill himself is a boy. Since $s = (b + 1) + 1$ we see that Bill's parents have one more daughters than sons.

So the answer is **1**.

PROBLEM 3 (30 pts) A box is filled with cubes and balls, all of which are either blue or red. Thirty percent of the objects in the box are cubes. Thirty percent of the balls in the box are blue. What percent of the objects in the box are red balls?

Solution: Denote the relative number of blue (red) balls (cubes) by b_b, b_r, c_b, c_r respectively. It is given that $c_b + c_r = .3$, hence $b_b + b_r = .7$. On the other hand, it is given that $\frac{b_b}{b_b+b_r} = .3$, hence $\frac{b_r}{b_b+b_r} = \frac{b_r}{.7} = .7$. Hence $b_r = .49$, so that the answer is 49%.

So, the answer is **49 percent**.

PROBLEM 4 (50 pts) A square hall's floor is tiled with congruent square tiles each of which is 1 square foot. There are 2009 tiles on the two diagonals of the floor. How long is a wall of the hall?

Solution: Suppose that the floor has the size $2n \times 2n$. Then the two diagonals of the floor do not meet over a tile and the number of tiles in them is equal to $2n$ in each. In this case the overall number of tiles in the diagonals of the floor is $4n$ which cannot be equal to 2009.

Suppose that each side has the size $(2n+1) \times (2n+1)$. Then each diagonal has $2n+1$ tiles and they meet over exactly one central tile. Hence the number of tiles in both diagonals is $2(2n+1)-1$. Clearly, this number may be equal to 2009, and if $2(2n+1)-1 = 2009$ then $2n+1 = 1005$. Hence any wall of the hall is 1005 feet long.

So the answer is **1005 feet**.

PROBLEM 5 (70 pts) It is given that $t = \frac{y}{x-z} = \frac{x+y}{z} = \frac{x}{y}$ for three positive numbers. What is then the value of t?

Solution: It follows that $x = ty$. Plugging this into the equality $x + y = zt$ we see that $yt + y = zt$. Since $y = xt - zt$ we conclude that $y = yt^2 - yt - y$. Hence $t^2 - t - 2 = (t-2)(t+1) = 0$. This equation has two roots, -1 and 2, however $t = \frac{x}{y}$ must be positive, hence $t = 2$. For example, we can choose $x = 4, y = 2, z = 3$.

So the answer is **2**.

PROBLEM 6 (90 pts) Early in the morning a camel started walking from an oasis A to an oasis B. Simultaneously, a donkey started walking from the oasis B to the oasis A. They met at noon and continued on their paths with the same speed. The camel came to the oasis B at 4pm while the donkey came to the oasis A at 9 pm. When did they start their journeys?

Solution: Let us denote the camel's speed by c and the donkey's speed by d. Also, denote by x the amount of time they walked from the beginning until noon. Then we see that $cx = 9d$ and $dx = 4c$. If we multiply these equalities we get $cdx^2 = 36cd$ which implies that $x = 6$. Since the question is about the time when they started we conclude that the answer is 6 in the morning.

So the answer is **6 in the morning**.

PROBLEM 7 (120 pts) What is the size of the largest subset, S, of $\{1, 2, \ldots, 2008\}$ such that no pair of distinct elements of S has a sum divisible by 9?

Solution: Denote by A_r the set of all positive integers less than 2009 which have the remainder r when we divide them by 9. Since $2007 = 223 \cdot 9$ then $A_0 = \{9, 18, \dots, 2007\}$ consists of 223 elements, $A_1 = \{1, 10, \dots, 2008\}$ consists of 224 elements, and all other sets $A_2, A_3, \dots, A_8$ consist of 223 elements. As a set S we suggest the following set: $\{9\} \cup A_1 \cup A_2 \cup A_3 \cup A_4$. This set consists of $1 + 223 \cdot 3 + 224 = 894$ elements. On the other hand, no two elements of S add up to a multiple of 9 because no remainders of two numbers from S add up to 9 or 0.

Let us show that if a set S' has more than 894 elements then it contains two numbers whose sum is divisible by 9. Indeed, if there are two elements of S' each of which is a multiple of 9 then their sum is a multiple of 9 too. Hence we may assume that there is no more than one multiple of 9 in S'. All other elements of S' are not multiples of 9. They can give remainders $1, 2, \dots, 8$ when divided by 9. If there are 5 non-zero remainders among those given by elements of S' then it is easy to see that two of them must add up to 9 which implies that elements of S' with these remainders will add up to a multiple of 9, a contradiction. Hence there are only 4 non-zero remainders among those given by elements of S'. Thus, the set S' can contain no more than $1 + 223 \cdot 3 + 224 = 894$ elements as desired.

So the answer is **894**.

PROBLEM 8 (160 pts) The numbers 1, 2, 3, 5, 7, 11, 13 are written on a board. We may erase any two numbers a and b and replace them by $ab+a+b$. After repeating this process several times, only one number remains on the board. What can be this number?

Solution: Clearly, $a+b+ab = (a+1)(b+1)-1$. It is easy to see now that if we apply the same operation to numbers $(a_1+1)\dots(a_n+1)-1$ and a_{n+1}, we will get $(a_1+1)\dots(a_{n+1}+1)-1$. Therefore regardless of the order in which the procedure is applied the result is the product of all the listed numbers increased by 1 from which then 1 is subtracted. That is, the answer is $(1+1)(2+1)(3+1)(5+1)(7+1)(11+1)(13+1)-1 = 193535$

So, the answer is **193535**.

PROBLEM 9 (220 pts) Find formulas for all real roots of the equation $\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} = x$; determine for what values of the real parameter p real roots of the equation exist.

Solution: If $p < 0$ then

$$\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} \geq \sqrt{x^2 - p} > x$$

which implies that in order for the equation to have a solution, we must have $p \geq 0$. Now, rewrite the equation in the form

$$2\sqrt{x^2 - 1} = x - \sqrt{x^2 - p}$$

and square both sides, obtaining

$$2x^2 + p - 4 = -2x\sqrt{x^2 - p}.$$

Squaring again, and solving for x^2, we get

$$x^2 = \frac{(p-4)^2}{8(2-p)}.$$

Hence in order for a solution to exist, we must have $0 \leq p < 2$, and then the only possible solution is

$$x = \frac{4-p}{\sqrt{8(2-p)}}$$

which we can substitute into the original equation. After multiplying all terms by $\sqrt{8(2-p)}$, we obtain

$$|3p - 4| + 2p = 4 - p, \text{which implies} |3p - 4| = -(3p - 4).$$

Clearly this holds if and only if $3p - 4 \leq 0$, i.e. $p \leq 4/3$. Therefore, the equation has solutions only when $0 \leq p \leq 4/3$, and then $x = \dfrac{4-p}{\sqrt{8(2-p)}}$.

So, the answer is $\mathbf{0 \leq p \leq 4/3}$.