NAME:______________________________

GRADE:______________________________

SCHOOL CODE:______________________

2009-2010 UAB MATH TALENT SEARCH

This is a two hour contest. Answers are to be written in the spaces provided on the test sheet. There will be no credit if the answer is incorrect. You **MUST** justify your answers in order to get full credit; otherwise, partial credit or no credit will be awarded according to the decision made by the judges. Your work (including full justifications) should be shown on the extra paper which is attached. The problems are listed in increasing order of difficulty.

PROBLEM 1 (5 pts) It is given that $u + a + b = 12$ and that u, a and b are non-negative integers. What is the maximal value of $uab + ua + ab + ub$?

YOUR ANSWER:

PROBLEM 2 (10 pts) It is given that $\frac{u}{a-b} = \frac{u+a}{b} = \frac{a}{u}$ where u, a, b are three positive numbers, all different. Find $\frac{a+2b}{u}$.

YOUR ANSWER:

PROBLEM 3 (20 pts) Find all positive integers x such that $x(x+1)(x+7)(x+8)$ is a full square. Sum them up and write the sum below.

YOUR ANSWER:

PROBLEM 4 (30 pts) There are 11 points on the positive x-axis and 8 points on the positive y-axis. The 88 segments connecting the 11 points selected on the positive x-axis and the 8 points selected on the positive y-axis are drawn. What is the maximal number of points of intersection of these segments inside the open first quadrant?

YOUR ANSWER:

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PROBLEM 5 (50 pts) On all 64 squares of a chess board positive integers are written as follows: $1, 2, \ldots, 8$ in the first row from left to right, then $9, 10, \ldots, 16$ in the second row etc. Then 8 rooks ("castles") are put on the board in such a way that they do not capture each other (thus, in every row and every column there stands exactly one rook). The numbers of the squares the rooks are standing upon are summed up. What are the possible values of the sum?

YOUR ANSWER:

PROBLEM 6 (80 pts) Let $UABC$ be a parallelogram such that $UA = 9$ and $AB = 10$. The points E, F and G on segments $\overline{UA}, \overline{AB}$ and $\overline{UC}$ are chosen so that $UE = AF = 3$ and $UG = 8$. The line through G parallel to $\overline{EF}$ intersects $\overline{BC}$ at a point H. Compute out the number

$$72 \times \frac{\text{Area}(EFGH)}{\text{Area}(UABC)}$$

and write it in the space below.

YOUR ANSWER:

PROBLEM 7 (120 pts) How many numbers $m, 1 < m < 20$, are there such that

$$A(m) = (m-1)! \times (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m-1})$$

is divisible by m? **Note that a simple computational solution will not be awarded full credit.** An argument (and NOT a computation) should be provided which shows why certain numbers m are such that m divides $A(m)$, and why certain other numbers do not have this property.

YOUR ANSWER:

2009-2010 UAB MTS: SOLUTIONS

PROBLEM 1 (5 pts) It is given that $u + a + b = 12$ and that u, a and b are non-negative integers. What is the maximal value of $uab + ua + ab + ub$?

Solution: Observe that

$$uab+ua+ab+ub = uab+ua+ab+ub+(u+a+b+1)-(u+a+b+1)$$

which equals $(u + 1)(a + 1)(b + 1) - 13$. Since it is easy to see (and is well-known) that $(u + 1)(a + 1)(b + 1)$ assumes its maximum when $u = a = b = 4$ we see that the maximal value of $uab + ua + ab + ub$ is $5^3 - 13 = 112$.

So the answer is **112**.

PROBLEM 2 (10 pts) It is given that $\frac{u}{a-b} = \frac{u+a}{b} = \frac{a}{u}$ where u, a, b are three positive numbers, all different. Find $\frac{a+2b}{u}$.

Solution: Clearly, if u, a, b satisfy the equations, then so do their multiples by any number. Scale them so that $u = 2$. Then we have

$$\frac{2}{a-b} = \frac{a+2}{b} = \frac{a}{2}$$

which implies that

$$a^2 - 4 = ab = 2a + 4$$

and therefore $a = 4$, then $b = 3$, and since $u = 2$ we see that $\frac{a+2b}{u} = 5$.

So the answer is **5**.

PROBLEM 3 (20 pts) Find all positive integers x such that $x(x + 1)(x+7)(x+8)$ is a full square. Sum them up and write the sum below.

Solution: Let $(x + 4)^2 = A$; since $x \geq 1$ we see that $A \geq 25$. Since $x(x + 8) = (x + 4)^2 - 16$ and $(x + 1)(x + 7) = (x + 4)^2 - 9$ we see that $B = x(x+1)(x+7)(x+8) = (A-16)(A-9)$. Consider now two cases.

(1) Suppose that $A - 16 = 7k$. Then $B = 49k(k + 1)$ with $k \geq 2$ (because $A \geq 25$). Since $k(k + 1)$ cannot be a full square, neither can B.

(2) Suppose that $A - 16$ is not a multiple of 7. Then it follows that $A - 16$ and $A - 9$ have no common factors. If B is a full square, this implies that both $A - 16$ and $A - 9$ are full squares. Clearly, there are only two full squares such that the difference between them is 7, namely 9 and 16 which corresponds to the value 25 of A. Hence the only x solving the problem is 1

So, the answer is **1**.

PROBLEM 4 (30 pts) There are 11 points chosen on the positive x-axis and 8 points chosen on the positive y-axis. The 88 segments connecting the 11 points selected on the positive x-axis and the 8 points selected on the positive y-axis are drawn. What is the maximal number of points of intersection of these segments inside the open first quadrant?

Solution: A point of intersection in the first quadrant can be associated with two intersecting inside it segments from the positive x-axis to the positive y-axis. Hence to each two pairs of points, one on the x-axis and the other one on the y-axis, we can associate one point of intersection of the appropriately chosen segments. On the x-axis such choices can be made in $11 \times 10/2 = 55$ ways, and on the y-axis they can be made in $8 \times 7/2 = 28$ ways.

The maximal number of such points of intersection will be obtained if no three segments intersect at one point which is clearly possible. Then we will have $55 \times 28 = 1540$ such points of intersection.
So, the answer is **1540**.

PROBLEM 5 (50 pts) On all 64 squares of a chess board positive integers are written as follows: $1, 2, \ldots, 8$ in the first row from left to right, then $9, 10, \ldots, 16$ in the second row etc. Then 8 rooks ("castles") are put on the board in such a way that they do not capture each other (thus, in every row and every column there stands exactly one rook). The numbers of the squares the rooks are standing upon are summed up. What are the possible values of the sum?

Solution: Let us write every number on the chess board as sums in the following way: $0{+}1, 0{+}2, \ldots, 0{+}8$ in the first row, $8{+}1, 8{+}2, \ldots, 8{+}8$ in the second etc. In other words, we do it in such a way that the first summand equals the number of the row (numbered from 0 through 7) and the second equals the number of the column (numbered from 1 through 8). Now, suppose that the rooks stand so that none can capture another one. This means that in the sum of the corresponding numbers every possible first summand will appear exactly once, and every possible second summand will appear exactly once. So the entire sum must always be equal to (0+8+16+24+32+40+48+56)+(1+2+3+4+5+6+7+8) = 260.

So, the answer is **260**.

PROBLEM 6 (80 pts) Let $UABC$ be a parallelogram such that $UA = 9$ and $AB = 10$. The points E, F and G on segments $\overline{UA}, \overline{AB}$ and $\overline{UC}$ are chosen so that $UE = AF = 3$ and $UG = 8$. The line through G parallel to $\overline{EF}$ intersects $\overline{BC}$ at a point H. Compute out the number

$$72 \times \frac{\text{Area}(EFGH)}{\text{Area}(UABC)}$$

and write it in the space below.

Solution: It follows that the triangles AEF and GHC are similar. Hence $AE/EF = CH/CG = 2$ which implies that $CH = 4$. Now, suppose that there are two triangles, XYZ and $XY'Z'$ such that the point Y' belongs to the side $\overline{XY}$ and the point Z' belongs to the side XZ. Then it is known that the area of $XY'Z'$ divided by the area of XYZ equals $\dfrac{XZ' \times XY'}{XZ \times XY}$. Hence and since the location of points E, F, H and G on the sides of the parallelogram $UABC$ is known, we can easily see how the areas of the triangles EAF, FBH, HCG and GUE relate to the areas of triangles UAB, ABC, BCU and CUA each of which has the area equal to one half of the area of $UABC$.

Thus, we get that the sum of the areas of triangles EAF, FBH, HCG and GUE forms the following fraction of the area of $UABC$:

$$\frac{1}{2}\left(\frac{3 \cdot 6}{9 \cdot 10} + \frac{7 \cdot 5}{9 \cdot 10} + \frac{4 \cdot 2}{9 \cdot 10} + \frac{8 \cdot 3}{9 \cdot 10}\right) = \frac{17}{36}.$$

Hence the area of the quadrilateral $EFUG$ forms $\dfrac{19}{36}$ of the area of $UABC$. The required number then is $72 \cdot \dfrac{19}{36} = 38$.
So the answer is **38**.

PROBLEM 7 (120 pts) How many numbers $m, 1 < m < 20$, are there such that

$$A(m) = (m-1)! \times (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m-1})$$

is divisible by m?

Solution: Let us make a few observations.

Claim 1. *If m is an odd prime number then $A(m)$ is divisible by m.* Indeed, first observe that no two quotients $(m-1)!/k = a$ and $(m-1)!/l = b$ with $k \neq l$ can have the same remainder with respect to m in this case. Indeed, otherwise $a - b = ms$ is a multiple of m, hence $a = b + ms$ and $ak = bk + bms = bl$. This would imply that $b(k-l) = bms$ is a multiple of m which is impossible because m is a prime number. Since there are $m-1$ such quotients and none of them can be a multiple of m because m is prime, we see that their remainders with respect to m are $1, 2, \ldots, m-1$ (in no particular order) and their sum is $(m-1)m/2$. If m is an odd prime number, this is a multiple of m as claimed.

Claim 2. *If $m = 2p$ is the product of 2 and a prime number p then $A(m)$ is NOT divisible by m.*

Indeed, this is obvious if $m = 4$. If $m > 4$ then in the sum of integers equal to fractions $\frac{(m-1)!}{k}$ all fractions with $k \neq p$ are divisible by $m = 2p$ because they all include at least one 2 and exactly one p not canceled out by the denominator k. However, $\frac{(m-1)!}{p}$ is not divisible even by p, and hence is no divisible by $m = 2p$ either.

It follows from Claims 1 - 2 that for $m = 3, 5, 7, 11, 13, 17, 19$ the sum $A(m)$ is divisible by m and for numbers $4, 6, 10, 14$ the sum $A(m)$ is not divisible by m. The remaining numbers are $2, 8, 9, 12, 15, 16, 18$. They can be treated individually. This shows that for $m = 2$ and $m = 8$ we have that $A(m)$ is not divisible by m and for $m = 9, 12, 15, 16, 18$ the sum $A(m)$ is divisible by m. Thus, the numbers $m, 1 < m < 20$ such that $A(m)$ is divisible by m are $3, 5, 7, 9, 11, 12, 13, 15, 16, 17, 18, 29$.

So the answer is **12**.