NAME:______________________________

GRADE:______________________________

SCHOOL CODE:_______________________

2010-2011 UAB MATH TALENT SEARCH

This is a two hour contest. Answers are to be written in the spaces provided on the test sheet. There will be no credit if the answer is incorrect. You **MUST** justify your answers in order to get full credit; otherwise, partial credit or no credit will be awarded according to the decision made by the judges. Your work (including full justifications) should be shown on the extra paper which is attached. The problems are listed in increasing order of difficulty.

PROBLEM 1 (10 pts) If 100 chickens eat 100 bushels of grain in 100 days, how many bushels will 10 chickens eat in 10 days?

YOUR ANSWER:

PROBLEM 2 (30 pts) X, Y, and Z, eat bread; X brings 3 loaves, Y brings 4 loaves. After the seven loaves are equally divided and eaten, Z produces 7 dollars to pay for his/her share and asks X and Y to divide the money equitably between them. How much should X and Y receive?

YOUR ANSWER: X receives \$ and Y receives \$.

PROBLEM 3 (70 pts) Five integers $x_1, \dots, x_5$ are given. It is known that the numbers

$$x_1, \frac{x_1+x_2}{2}, \frac{x_1+x_2+x_3}{3}, \frac{x_1+x_2+x_3+x_4}{4}, \frac{x_1+x_2+x_3+x_4+x_5}{5}$$

are all integers. The numbers $x_1, x_2, \dots, x_5$ written in random order are $5, 26, 13, 10, 16$. What are x_3, x_4, x_5 equal to?

YOUR ANSWER: $x_3 =$ $, x_4 =$ $, x_5 =$

over, please

PROBLEM 4 (150 pts) Solve the riddle: $SEND + MORE = MONEY$ in which distinct letters represent distinct digits. Please specify the number for $SEND$, the number for $MORE$ and the number for $MONEY$, and give your answer in the form where the first two numbers are added giving the number for $MONEY$.

YOUR ANSWER:

PROBLEM 5 (210 pts) Through a point on the hypotenuse of a right triangle, lines are drawn parallel to its sides. This divides the triangle into a square S and two smaller triangles T_1 and T_2. It is given that $\frac{Area(T_1)}{Area(S)} = m$. What is $\frac{Area(T_2)}{Area(S)}$?

YOUR ANSWER:

PROBLEM 6 (270 pts) Let 9^N have M digits. How many numbers from the set $\{9, 9^2, \ldots, 9^N\}$ have 9 as their first (leftmost) digit?

YOUR ANSWER:

2010-2011 UAB MTS: SOLUTIONS

PROBLEM 1 (10 pts) If 100 chickens eat 100 bushels of grain in 100 days, how many bushels will 10 chickens eat in 10 days?

Solution: It follows that 100 chickens eat 1 bushel of grain in 1 day. Hence 1 chicken eats $\frac{1}{100}$-th of 1 bushel in 1 day. Hence 10 chickens eat $\frac{1}{10}$-th of 1 bushel in 1 day. Hence in 10 days they will eat 1 bushel of grain.

So the answer is **1**.

PROBLEM 2 (30 pts) X, Y, and Z, eat bread; X brings 3 loaves, Y brings 4 loaves. After the seven loaves are equally divided and eaten, Z produces 7 dollars to pay for his/her share and asks X and Y to divide the money equitably between them. How much should X and Y receive?

Solution: Each person ate $\frac{7}{3}$ loaves of bread. Hence X contributed $3-\frac{7}{3}=\frac{2}{3}$ loaves of bread towards Z's meal and Y contributed $4-\frac{7}{3}=\frac{5}{3}$ loaves of bread towards Z's meal.

So, X receives **\$2** and Y receives **\$5**.

PROBLEM 3 (70 pts) Five integers $x_1, \dots, x_5$ are given. It is known that the numbers

$$x_1, \frac{x_1+x_2}{2}, \frac{x_1+x_2+x_3}{3}, \frac{x_1+x_2+x_3+x_4}{4}, \frac{x_1+x_2+x_3+x_4+x_5}{5}$$

are all integers. The numbers $x_1, x_2, \dots, x_5$ written in random order are $5, 26, 13, 10, 16$. What are x_3, x_4, x_5 equal to?

Solution: As all the averages are integers, it follows that the corresponding sums are multiples of 2, 3, 4 and 5 respectively. The sum $x_1+x_2+x_3$ must be a multiple of 3. Since the remainders of given numbers when divided by 3 are 2, 1, 1, 2, 1 then the first three numbers are 10, 13, and 16. Since x_1+x_2 is even, it follows that x_1, x_2 are 10 and 16 (in some order) and $x_3=13$. The remaining two cases are $x_4=5, x_5=26$ or $x_4=26, x_5=5$. It is easy to check that $x_4=5, x_5=26$.

So, the answer is $\mathbf{x_3=13, x_4=5, x_5=26}$.

PROBLEM 4 (150 pts) Solve the riddle: $SEND+MORE = MONEY$ in which distinct letters represent distinct digits. Please specify the number for $SEND$, the number for $MORE$ and the number for $MONEY$, and give your answer in the form where the first two numbers are added giving the number for $MONEY$.

Solution: Clearly, M=1. Hence O is either 1 or 0 (zero). However 1 is already taken by M, hence O=0 (zero). It follows that S is either 8 or 9. If S is 8 then there must be a carryover from the previous column, i.e. from E+0-column. However N is at least 2, hence this is impossible and S=9. Them E+0-column implies that N=E+1. Hence N+R-column must produce a carryover. Since N=E+1, to have N+R plus, possibly, a carryover from D+E-column reach out to 10+E we have to have a big R, equal at least to 8 or possibly 9. Since S=9, we see that R=8.

Now, it follows that D+E produces a carryover. On the other hand, D is at most 7 as 8 and 9 are already taken. Moreover, D+E is at least 12 as Y is greater than 1 (recall that M=1). Hence, E is at least 5. It remains to consider the cases when E=5, 6 and 7 separately. This yields the answer.

So, the answer is **9567+1085=10652**.

PROBLEM 5 (210 pts) Through a point on the hypotenuse of a right triangle, lines are drawn parallel to its sides. This divides the triangle into a square S and two smaller triangles T_1 and T_2. It is given that $\frac{Area(T_1)}{Area(S)} = m$. What is $\frac{Area(T_2)}{Area(S)}$?

Solution: Let the big triangle be ABC with the right angle at B. Let the point on the hypotenuse which is considered in the problem be D and the square in question be $BUDV = S$ with U belonging to AB and V belonging to BC. Suppose that the area of the triangle ABC is 1. Let $|BC| = x, |AB| = y$.

It follows that BD is a bisectrix of the angle ABC. Hence D partitions AC in two segments, CD proportional to $|AB| = x$ and DA proportional to $|CB| = y$. Since the areas of similar triangles are related as the square of their coefficients of similarity, we conclude that $Area(CDV) = (\frac{x}{x+y})^2$ and $Area(DUA) = (\frac{y}{x+y})^2$ (recall that we assume that $Area(ABC) = 1$). This implies that $Area(BUDV) = \frac{2xy}{(x+y)^2}$.

We may assume that $T_1 = CDV$ and $T_2 = DUA$. Then $m = \frac{Area(T_1)}{Area(S)} = \frac{x^2}{2xy} = \frac{x}{2y}$. On the other hand, $\frac{Area(T_2)}{Area(S)} = \frac{y^2}{2xy} = \frac{y}{2x}$. Hence

$\frac{Area(T_2)}{Area(S)} = \frac{1}{4m}$.

So, the answer is $\mathbf{\frac{1}{4m}}$.

PROBLEM 6 (270 pts) Let 9^N have M digits. How many numbers from the set $\{9, 9^2, \ldots, 9^N\}$ have 9 as their first (leftmost) digit?

Solution: If the first digit of 9^{k+1} is 9, then the first digit of 9^k is 1 and the number of digits in 9^{k+1} and in 9^k are the same. Otherwise, i.e. if the first digit of 9^{k+1} is not 9, then the number of digits in 9^{k+1} is by one greater than the number of digits in 9^k. Denote by n_k the number of digits in 9^k (e.g., $n_1 = 1, n_2 = 2, n_N = M$ etc). It follows that the number of times n_k increases by 1 equals the number of times the first digit of 9^l is *not* equal to 9. Observe that as we move from k to $k+1$, n_k either grows by 1 under multiplication by 9, or stays the same. In other words, n_{k+1} either equals n_k or $n_k + 1$. As we move along the sequence $9^1, 9^2, \ldots$ to 9^N, the number of times n_k increases by 1 is $M - 1$. Hence the number of times n_k does not increase is $N - (M - 1) = N - M + 1$. By the above this is how many times the first digit of 9^k is 9.

So the answer is **N-M+1**.