NAME:______________________________

GRADE:______________________________

SCHOOL NAME:______________________

2012-2013 UAB MATH TALENT SEARCH

This is a two hour contest. Answers are to be written in the spaces provided on the test sheet. There will be no credit if the answer is incorrect. You **MUST** justify your answers in order to get full credit; otherwise, partial credit or no credit will be awarded according to the decision made by the judges. Your work (including full justifications) should be shown on the extra paper which is attached. The problems are listed in increasing order of difficulty.

PROBLEM 1 (30 pts) In a rural town in Alabama, $\frac{2}{3}$ of men are married to $\frac{3}{4}$ of women. The entire adult population of the town is 612 men and women. How many of them are married?

YOUR ANSWER:

PROBLEM 2 (50 pts) Find all integer solutions to the equation $x^3 = 6y^3 + 20z^3$.

YOUR ANSWER:

PROBLEM 3 (70 pts) For a set S, let $|S|$ be the number of elements in S, and let $n(S)$ be the number of all subsets of S, including the empty set and the set S itself. If A, B and C are sets for which

$$n(A) + n(B) + n(C) = n(A \cup B \cup C) \text{ and } |A| = |B| = 100,$$

then what is the minimum possible value of $|A \cap B \cap C|$?

YOUR ANSWER:

over, please

PROBLEM 4 (110 pts) Three 1×1 squares on the plane, ABCD, DCEF and FEGH, are attached to each other to form a rectangle ABGH of size 1×3. This is done so that the collinear points B, C, E and G are on one side of ABGH (in this order), and the collinear points A, D, F, H are on the other side of ABGH (in this order). Find the sum of angles $\angle DBG, \angle FBG$ and $\angle HBG$. (To solve the problem you do not have to use trigonometry. Give the answer in radians or degrees).

YOUR ANSWER:

PROBLEM 5 (160 pts) An integer is said to be *symmetric* if it reads the same forwards as backwards. For example, 737 and 2442 are symmetric. (a) (81 pts) Consider the set of all integers from 1 through 999999. How many symmetric integers are among them? (b) (81 pts) More generally, consider the set of all integers from 1 through $10^{2n} - 1$. How many symmetric integers are among them?

YOUR ANSWER:

PROBLEM 6 (240 pts) Point P is inside ΔABC. Line segments $\overline{APD}, \overline{BPE}$ and $\overline{CPF}$ are drawn with D on $\overline{BC}$, E on $\overline{CA}$, and F on $\overline{AB}$. Given that $AP = 6, BP = 9, PD = 6, CP = 20$ and $PE = 3$, find the length of the segment CX where X is the point of intersection of CF and ED.

YOUR ANSWER:

2011-2012 UAB MTS: SOLUTIONS

PROBLEM 1 (30 pts) In a rural town in Alabama, $\frac{2}{3}$ of men are married to $\frac{3}{4}$ of women. The entire adult population of the town is 612 men and women. How many of them are married?

Solution: Let m be the number of men and let w be the number of women in the town. Then $\frac{2}{3}m = \frac{3}{4}w = x$ is the number of married couples so that the overall number of married men and women is $2x$. It follows that $m = \frac{3}{2}x$ and $w = \frac{4}{3}x$. Hence $(\frac{3}{2} + \frac{4}{3})x = 612 = \frac{17}{6}x$ and so $x = \frac{6 \cdot 612}{17} = 216$ and the overall number of married men and women is 432.

So the answer is **432**. $\square$

PROBLEM 2 (50 pts) Find all integer solutions to the equation $x^3 = 6y^3 + 20z^3$.

Solution: Observe that x is even because x^3 is even. Hence $x = 2x_1$ and the equation becomes $8x_1^3 = 6y^3 + 20z^3$. This implies that $4x_1^3 = 3y^3 + 10z^3$. Clearly, $y = 2y_1$ is even because $3y^3 = 4x^3 - 10z^3$ is even. Thus, $4x_1^3 = 24y_1^3 + 10z^3$, and so $2x_1^3 = 12y_1^3 + 5z^3$. Similarly to the above, $z = 2z_1$ is even because $5z^3 = 2x_1^3 - 12y_1^3$. We conclude that $2x_1^3 = 12y_1^3 + 40z_1^3$, or $x_1^3 = 6y_1^3 + 20z_1^3$. It follows that we can continue by induction infinitely many times which implies that $x = y = z = 0$ is the only solution.

So the answer is $\mathbf{x = 0, y = 0, z = 0}$. $\square$

PROBLEM 3 (70 pts) For a set S, let $|S|$ be the number of elements in S, and let $n(S)$ be the number of all subsets of S, including the empty set and the set S itself. If A, B and C are sets for which

$$n(A) + n(B) + n(C) = n(A \cup B \cup C) \text{ and } |A| = |B| = 100,$$

then what is the minimum possible value of $|A \cap B \cap C|$ if it is known that C is non-empty?

Solution: It is well known that $n(S) = 2^{|S|}$. Set $a = |A| = 100, b = |B| = 100$ and $c = |C|$. We are given that $2^{100}+2^{100}+2^c = 2^{101}+2^c = 2^d$ where $A \cup B \cup C = D$ and $|D| = d$. Since $c > 0$ it follows that $d \geq 102$.

Hence $2^c \geq 2^{102} - 2^{101} = 2^{101}$ and so $c \geq 101$. On the other hand, $2^{101} + 2^c \geq 2^{c+1}$ and so $2^{101} \geq 2^{c+1} - 2^c = 2^c$ which implies that $c \leq 101$. We conclude that $c = 101$ and $d = 102$. In other words, the sets A, B and C are such that $|A| = |B| = 100, |C| = 101$ and $|A \cup B \cup C| = |D| = 102$. Without loss of generality we may assume that $D = \{1, 2, \ldots, 102\}$ and $C = \{1, \ldots, 101\}$. Let $E = A \cap B \cap C$. If $A = B$ then either both sets A, B are contained in C and $|E| = 100$, or both sets contain 102 and $|E| = 99$. Suppose that $A \neq B$. If $A \subset C, B \subset C$ then it follows that $|E| = 99$. If, say, $A \subset C$ and $B \not\subset C$ then there are 99 elements of B inside C and 100 elements of A inside C. It follows that $|E|$ can be made 98 in this case but not less. Finally, suppose that 102 belongs to both A and B and $A \neq B$. Then A and B have 99 of their elements inside C and can be chosen so that $|E| = 97$ (e.g., set $A = \{1, \ldots, 99, 102\}$ and $B = \{3, 4, \ldots, 101, 102\}$.

So the answer is **97**. $\square$

PROBLEM 4 (110 pts) Three 1×1 squares on the plane, ABCD, DCEF and FEGH, are attached to each other to form a rectangle ABGH of size 1×3. This is done so that the collinear points B, C, E and G are on one side of ABGH (in this order), and the collinear points A, D, F, H are on the other side of ABGH (in this order). Find the sum of angles $\angle DBG, \angle FBG$ and $\angle HBG$. (To solve the problem you do not have to use trigonometry. Give the answer in radians or degrees).

Solution: Clearly, $\angle DBG = \pi/4$. To figure $\angle FBG + \angle HBG$, construct the square grid on the plane so that our squares are a part of it. Assume that the straight line with points B, C, E and G is horizontal and is located right under the line with points A, D, F and H. Let I be the vertex of the grid right above the point A, and let J be the vertex of the grid right above the point D. Then the triangles ΔJBI and ΔFBE are equal. Thus, $\angle JBI = \angle FBG$. Therefore $\angle FBG + \angle HBG = \angle JBI + \angle HBG = \pi/2 - \angle JBH$. Consider ΔJBH. It is easy to see that this is a right isosceles triangle hence $\angle JBH = \pi/4$. Thus, $\angle FBG + \angle HBG = \pi/4$ and so $\angle DBG + \angle FBG + \angle HBG = \pi/2$.

Thus, $\angle \mathbf{DBG} + \angle \mathbf{FBG} + \angle \mathbf{HBG} = \pi/\mathbf{2}$. $\square$

PROBLEM 5 (160pts) An integer is said to be *symmetric* if it reads the same forwards as backwards. For example, 737 and 2442 are symmetric. (a) (80 pts) Consider the set of all integers from 1 through 999999. How many symmetric integers are among them? (b) (80 pts) More generally, consider the set of all integers from 1 through $10^{2n}-1$. How many symmetric integers are among them?

Solution: Clearly, numbers $1, 2, \ldots, 9$ are symmetric. Consider all integers from 10 through 99. The only symmetric integers among them are $11, 22, 33, \ldots, 99$. So there are 9 more symmetric integers. Consider integers from 100 through 999. A 3-digit symmetric integer is completely defined by its first 2 digits. Hence there are as many 3-digit symmetric integers as 2-digit integers, i.e. there are 90 3-digit symmetric integers. similarly, there are as many 4-digit symmetric integers as there are 2-digit integers, i.e. there are 90 4-digit symmetric integers. In the same fashion you can show that there are 900 symmetric integers with 5 digits and 900 symmetric integers with 6 digits. Hence overall there are $18 + 180 + 1800 = 1998$ symmetric integers between 1 and 999999.

In general, k-digit integers are integers between 10^{k-1} and $10^k - 1$. A symmetric k-digit integer is determined by its first $k/2$-digits (if k is even) or its first $k/2 + 1$ digits (if k is odd). Consider all integers with either 2i+1 digits or 2i+2 digits. Then in either group there are as many symmetric numbers as there are integers with i+1 digits, i.e. $9 \cdot 10^i$. Hence overall there are $18 \cdot 10^i$ symmetric integers between 10^{2i+1} and 10^{2i+2}. Hence among integers from 1 through $10^{2n}-1$ there are $18 + 180 + 1800 + \cdots + 18 \cdot 10^{n-1} = 18 \cdot (10^n - 1)/9 = 2 \cdot (10^n - 1)$ symmetric integers.

So the answers are: (a) there are **1998** symmetric integers between 1 and 999999; (b) there are $\mathbf{2 \cdot (10^n - 1)}$ symmetric integers between 1 and $10^{2n} - 1$. $\square$

PROBLEM 6 (240 pts) Point P is inside ΔABC. Line segments $\overline{APD}, \overline{BPE}$ and $\overline{CPF}$ are drawn with D on $\overline{BC}$, E on $\overline{CA}$, and F on $\overline{AB}$. Given that $AP = 6, BP = 9, PD = 6, CP = 20$ and $PE = 3$, find the length of the segment CX where X is the point of intersection of CF and ED.

Solution: Choose the point G on PB so that $PG = 3$. Then $AP = PD$ and $EP = PG$ so that $AGDE$ is a parallelogram. Hence AG is parallel to ED and GD is parallel to AC. This implies that $GD = AE$. On the other hand, $EG = GB = 6$, hence the fact that GD is parallel to AC implies that D is the midpoint of CB. We conclude that $EC = 2 \cdot GD$ and so $EC = 2 \cdot AE$. Denote the point of intersection of CF and AG by Q. Then the fact that $EC = 2 \cdot AE$ and the fact that ED is parallel to AG implies that $CX = 2 \cdot XQ$. Since P is the center of the parallelogram $AGDE$, $PQ = XP$. This implies that $CX = 4 \cdot XP$ and hence that $CP = 20 = 5 \cdot XP$. Thus, $XP = 4$ and $CX = 16$.

So, the answer is $\mathbf{CX} = \mathbf{16}$. $\square$