NAME:______________________________

GRADE:______________________________

SCHOOL NAME:______________________

2014-2015 UAB MATH TALENT SEARCH

This is a two hour contest. Answers are to be written in the spaces provided on the test sheet. **There will be no credit if the answer is incorrect.** Full credit will be awarded for a correct answer with complete justification. At most half credit will be given for a correct answer without complete justification. Your work (including full justification) should be shown on the extra paper which is attached. Please clearly indicate which problems you are solving.

PROBLEM 1 (10 pts) There are two types of inhabitants of a planet Xynia, xorgs and yorgs. Marriage is only possible between a single xorg and a single yorg. It is known that two thirds of all xorgs are married, and three fifths of all yorgs are married. What proportion of the planet's population is married?

YOUR ANSWER:

PROBLEM 2 (20 pts) Each edge of a cube is either green or yellow. Moreover, each face has at least one green edge. What is the least number of green edges in this cube?

YOUR ANSWER:

PROBLEM 3 (40 pts) One hundred numbers are given. It is known that among their pairwise products there are exactly one thousand negative numbers (strictly less than zero). What is the minimal number of zeros among these one hundred numbers?

YOUR ANSWER:

over, please

PROBLEM 4 (60 pts) A multiple choice test consists of 20 questions. A student gets 5 points for each correct answer, -2 points for each wrong answer, and 0 for each unanswered question. Bob score 48 points on this exam. What is the maximal possible number of questions he could have answered correctly?

YOUR ANSWER:

PROBLEM 5 (70 pts) Numbers $p \neq q$ are such that the equation $x^2 + px + q = 0$ and the equation $x^2 + qx + p = 0$ have a common solution. Find $p + q$.

YOUR ANSWER:

PROBLEM 6 (100 pts) Twelve soccer teams play in a tournament in which each team plays once with every other team. A winner of a game gets 3 points while a loser gets 0 points. If it is a tie, each teams gets one point. The teams in the end got the following number of points:
$0, 3, 6, 9, 12, 15, 17, 19, 21, 23, 25, 27$.
How many ties were there in the tournament?

YOUR ANSWER:

PROBLEM 7 (150 pts) A rectangle is sketched on a page with a grid. Its sides are vertical and horizontal. Its dimensions are 100 cells (height) $\times$ 200 cells (width). The rectangle is being painted. Starting from the upper left corner and moving in a spiral clockwise (first the upper row of cells is painted, then the rightmost column of cells is painted, etc etc etc). Which cell will be painted last? Give its coordinates in the form (the row, the column) counting from the lower left corner. E.g., the bottom right corner cell has coordinates (row 100, column 200).

YOUR ANSWER:

PROBLEM 8 (200 pts) Consider all triples $T = \{p < r < q\}$ of prime numbers such that $p^4 + r^4 + q^4 - 3 = k$ is a prime number. For each such triple, the number set $f(T) = p + q + r$. Let S be the sum of numbers $f(T)$ over all triples described above. What is the value of S?

YOUR ANSWER:

2014-2015 UAB MTS: SOLUTIONS

PROBLEM 1 (10 pts) There are two types of inhabitants of a planet Xynia, xorgs and yorgs. Marriage is only possible between a single xorg and a single yorg. It is known that two thirds of all xorgs are married, and three fifths of all yorgs are married. What proportion of the planet's population is married?

Solution: Let c be the number of couples, x number of xorgs and y the number of yorgs on Xynia. Then $c/x = 2/3$ and $c/y = 3/5$. This implies that $x = (3c)/2$ and $y = (5c)/3$. The entire population of the island equals $p = x + y = (3c)/2 + (5c)/3 = (19c)/6$. Since there are $2c$ married inhabitants on XYnia, we conclude that they form $2c/p = 12/19$ portion of p.

The answer is **12/19**. $\square$

PROBLEM 2 (20 pts) Each edge of a cube is either green or yellow. Moreover, each face has at least one green edge. What is the least number of green edges in this cube?

Solution: The answer is 3. Indeed, each edge serves 2 faces, thus 2 edges can serve at most 4 faces. On the other hand, 3 edges can serve all edges of the cube. E.g., we can paint the following edges green: 1) the edge between front and right faces, 2) the edge between top and left faces, and 3) the edge between bottom and back faces.

The answer is **3**. $\square$

PROBLEM 3 (40 pts) One hundred numbers are given. It is known that among their pairwise products there are exactly one thousand negative numbers. What is the minimal number of zeros among these one hundred numbers?

Solution: Let p be the number of all positive numbers among the given 100 numbers and n be the number of all negative numbers among the given 100 numbers. Then by the assumptions $pn = 1000$ and $p + n \leq 100$. Thus, either p or n is greater than 10 but less than 100. The list of possible values of p and n is then as follows: $p = 20, n = 50; p = 25, n = 40; p = 40, n = 25; p = 50, n = 20$. To minimize the number of zeros among our numbers we need to maximize $p+n$; by the above the maximal value of $p + n$ is 70 which yields that the minimal number of zeros among our numbers is 30.

The answer is **30**. $\square$

PROBLEM 4 (60 pts) A multiple choice test consists of 20 questions. A student gets 5 points for each correct answer, -2 points for each wrong answer, and 0 for each unanswered question. Bob score 48 points on this exam. What is the maximal possible number of questions he could have answered correctly?

Solution: Let c be the number of correct answers given by Bob, i be the number of his incorrect answers. Then $5c-2i=48$ while $c+i \leq 20$ (and so $i \leq 20-c$). It follows that $5c-2(20-c) \leq 5c-2i=48$ which implies that $7c \leq 88$ and hence $c \leq 12$. On the other hand if $c=12$ and $i=7$, Bob's score will be $5 \cdot 12 - 2 \cdot 7 = 48$, thus $c=12$ is possible.

The answer is **12**. $\square$

PROBLEM 5 (70 pts) Numbers $p \neq q$ are such that the equation $x^2+px+q=0$ and the equation $x^2+qx+p=0$ have a common solution. Find $p+q$.

Solution: Since the two given equations have a common solution, say, x_0, the difference between them has the same root x_0. We have $x^2+px+q-(x^2+qx+p)=(x-1)(p-q)=0$ and so $x_0=1$. Hence $1^2+p+q=0$ which yields that $p+q=-1$.

The answer is $\mathbf{p+q=-1}$. $\square$

PROBLEM 6 (100 pts) Twelve soccer teams play in a tournament in which each team plays once with every other team. A winner of a game gets 3 points while a loser gets 0 points. If it is a tie, each teams gets one point. The teams in the end got the following number of points:

$$0, 3, 6, 9, 12, 15, 17, 19, 21, 23, 25, 27.$$

How many ties were there in the tournament?

Unfortunately there were several inaccuracies in the actual distribution of points suggested in the problem. The distribution of the points which was meant is as follows:

$$0, 6, 7, 8, 11, 16, 19, 22, 25, 27, 29.$$

We apologize for this error. Both the students who gave the answer which we meant (21 ties) *and* the students who noticed our error and pointed out that the suggested distribution of points is not possible got full credit for this problem.

Solution: The idea of the solution is described below. Each game brings 2 points overall if it is a tie and 3 points if one team wins. The teams played 66 games overall for the maximum number of points $66 \cdot 3 = 198$. Since they collected 177 points overall, there were $198 - 177 = 21$ ties.

The answer is **21**. □

PROBLEM 7 (150 pts) A rectangle is sketched on a page with a grid. Its sides are vertical and horizontal. Its dimensions are 100 cells (height) $\times$ 200 cells (width). The rectangle is being painted. Starting from the upper left corner and moving in a spiral clockwise (first the upper row of cells is painted, then the rightmost column of cells is painted, etc etc etc). Which cell will be painted last? Give its coordinates in the form (the row, the column) counting from the lower left corner. E.g., the bottom right corner cell has coordinates (row 100, column 200).

Solution: After the first round of painting, the outermost columns and rows will be painted. The remaining rectangle is od dimensions 98×198. The rectangle will be painted in the same fashion as the initial one, etc. Notice, that the vertex of the grid at which the vertical axis of symmetry and the horizontal axes of symmetry of our rectangle intersect remains the points of intersection of the axes of symmetry of the smaller rectangles as well. After 49 rounds we will have unpainted rectangle of dimensions 2×102 located in rows $50 - 51$ and columns $50 - 151$. As we paint it using the same algorithm as before, the last cell painted will be its bottom left cell located in row 51 and column 50.

The answer is $\mathbf{row\,51, column\,50}$. □

PROBLEM 8 (200 pts) Consider all triples $T = \{p < r < q\}$ of prime numbers such that $p^4 + r^4 + q^4 - 3 = k$ is a prime number. For each such triple, the number set $f(T) = p + q + r$. Let S be the sum of numbers $f(T)$ over all triples described above. What is the value of S?

Solution: The only triple satisfying conditions of the problem is $p = 2 < r = 3 < q = 5$. Indeed, One of the numbers p, q, r is even as otherwise k is even and cannot be prime. Hence $p = 2$ and we see that $16 + r^4 + q^4 - 3 = 13 + r^4 + q^4 = k$ is prime. Now, if neither r nor q equals 3 then r and q have remainder 1 or 2 with respect to 3. It follows that r^4 and q^4 have remainder 1 with respect to 3 and that k is actually divisible by 3, a contradiction with k being prime. Thus $r = 3$ and we have that $13 + 3^4 + r^4 = 94 + r^4 = k$ is prime. Suppose that $k \neq 5$. Then the remainder of r with respect to 5 is $1, 2, 3$ or 4. It follows that r^4 has remainder 1 with respect to 5. Hence $k = 94 + r^4$ is divisible by 5, a contradiction. This implies that $k = 5$ as desired. Observe that $2^4 + 3^4 + 5^4 - 3 = 719$ is a prime number, hence $p = 2 < r = 3 < q = 5$ is the unique triple satisfying the conditions of the problem and the answer is $2 + 3 + 5 = 10$.

The answer is **10**. $\square$