

FORCING AMONG PATTERNS WITH NO BLOCK STRUCTURE

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Dedicated to the memory of our colleague and dear friend Sergiy Kolyada

ABSTRACT. Define the following order among all natural numbers except for 2 and 1:

$$4 \succ_s 6 \succ_s 3 \succ_s \dots \succ_s 4n \succ_s 4n + 2 \succ_s 2n + 1 \succ_s 4n + 4 \succ_s \dots$$

Let f be a continuous interval map. We show that if $m \succ_s s$ and f has a cycle with no division (no block structure) of period m then f has also a cycle with no division (no block structure) of period s . We describe possible sets of periods of cycles of f with no division and no block structure.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

The simplest type of limit behavior of a trajectory is periodic; studying *periodic orbits (cycles)* is one of the central topics in one-dimensional dynamics. To some extent this can be explained by a remarkable result, the Sharkovsky Theorem, proved by A. N. Sharkovsky in the 1960s (see [Sha64] and [Sha-tr] for its English translation). To state it, let us first recall the *Sharkovsky order* of the set \mathbb{N} of positive integers:

$$3 \succ_s 5 \succ_s 7 \succ_s \dots \succ_s 2 \cdot 3 \succ_s 2 \cdot 5 \succ_s 2 \cdot 7 \succ_s \dots \succ_s 2^2 \cdot 3 \succ_s 2^2 \cdot 5 \succ_s 2^2 \cdot 7 \succ_s \dots \succ_s 2^2 \succ_s 2 \succ_s 1.$$

Denote by $\text{Sh}(k)$ the set of all integers m such that $k \succ_s m$ or $m = k$, and by $\text{Sh}(2^\infty)$ the set $\{1, 2, 4, 8, \dots\}$; denote by $\text{Per}(f)$ the set of periods of cycles of a map f (by the *period* we mean the *minimal* period). Below I always denotes a closed interval.

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The Sharkovsky Theorem. *If $g : I \rightarrow I$ is continuous, $m \succ_{\mathcal{S}} n$ and $m \in \text{Per}(g)$ then $n \in \text{Per}(g)$ and there exists $k \in \mathbb{N} \cup 2^\infty$ with $\text{Per}(g) = \text{Sh}(k)$. Conversely, if $k \in \mathbb{N} \cup 2^\infty$ then there exists a continuous map $f : I \rightarrow I$ with $\text{Per}(f) = \text{Sh}(k)$.*

The Sharkovsky Theorem is important, in particular, because it introduces a concept of *forcing relation*: it states that if $m \succ_{\mathcal{S}} n$ then the fact that an interval map has a cycle of period m *forces* the presence of a cycle of period n . Thus, it shows how various “types” of cycles (here by “type” one means “period”) force each other. Another interpretation of the Sharkovsky Theorem is that it fully describes all possible sets of periods of cycles of interval maps. This leads to similar problems: (a) how the existence of cycles of certain types forces the existence of cycles of certain other types, and (b) what possible sets of types of cycles an interval map may have.

For example, given a cycle $P = \{x_1 < x_2 < \dots < x_n\}$ of an interval map f , associate with it the (cyclic) permutation π defined by $f(x_i) = x_{\pi(i)}$, $i = 1, 2, \dots, n$. Think of π as the type of P . The family of all cycles associated to π is called an *oriented pattern* (see [ALM00]). If we identify oriented patterns obtained from each other by a flip, we get *patterns* (we denote patterns with capital letters A, B, \dots). Similar to the Sharkovsky Theorem, one can ask for an interval map f (a) how cycles of certain patterns force cycles of other patterns, and (b) what possible sets of patterns of cycles f may have.

A useful way of studying patterns is by decomposing them.

Definition 1.1 (Block structure). Let π be a permutation of the set $X = \{1, \dots, n\}$. Suppose that for some $k > 1$ and $m > 1$ we have $n = km$ and the permutation π maps sets $Y_1 = \{1, \dots, m\}$, $Y_2 = \{m+1, \dots, 2m\}$, \dots , $Y_k = \{n-m+1, \dots, n\}$ to one another. Then sets Y_1, \dots, Y_k are called *blocks* and the permutation π is said to have *block structure*; if blocks are two-point sets, π is said to be a *doubling*. As always, similar terminology is used for patterns and cycles. Otherwise a permutation (a pattern, a cycle) is said to have *no block structure*.

The appropriate power of the map on a block can be viewed as a kind of renormalization of a pattern; patterns with block structure admit a renormalization like that. Consider an important particular case.

Definition 1.2 (No division). Let π be a permutation of $\{1, \dots, 2m\}$ such that $\pi(i) \geq m+1$ for each i , $1 \leq i \leq m$ (and, hence, $\pi(i) \leq m$ for each $i \geq m+1$). Then we say that π (and the corresponding pattern and cycles) *has division*. Otherwise π (and the corresponding pattern and cycles) is said to have *no division*.

Observe that a pattern of period 2 has no block structure but has a division. Therefore we will treat period 2 separately.

Consider the family \mathcal{NBS} of patterns with no block structure and the family \mathcal{ND} of all patterns with no division. A pattern *with* block structure can be studied in two steps: study the factor-pattern obtained if each block is collapsed to a point while the order among blocks is kept, and then study the restriction of the pattern on blocks. On the other hand, no division patterns constitute the “core” in the Sharkovsky Theorem. Thus, both patterns from \mathcal{NBS} and \mathcal{ND} are important. To get uniformity, we consider only patterns of periods larger than 2; patterns of periods 1 and 2 are discussed after the Main Theorem.

Define the following order among all natural numbers larger than 2:

$$4 \gg 6 \gg 3 \gg \dots \gg 4n \gg 4n + 2 \gg 2n + 1 \gg 4n + 4 \gg \dots \quad (*)$$

We get it by writing even numbers in the natural order and inserting odd numbers n after $2n$.

Let \mathcal{N}_p be the set of all integers s with $p \gg s$ and p itself. Given an interval map f , let $ND(f)$ be the set of periods (larger than 2) of all f -cycles with no division, and let $NBS(f)$ be the set of periods (larger than 2) of all f -cycles with no block structure.

Main Theorem. *Let f be a continuous interval map. If $m \gg s$ and f has a cycle with no division (no block structure) of period m then f has also a cycle with no division (no block structure) of period s . The following are the only possible cases (n below can be any positive integer), and all of them occur.*

- (1) $ND(f) = NBS(f) = \emptyset$.
- (2) $ND(f) = NBS(f) = \mathcal{N}_{2n+1}$.
- (3) $ND(f) = NBS(f) = \mathcal{N}_{4n}$.
- (4) $ND(f) = NBS(f) = \mathcal{N}_{4n+2}$.
- (5) $ND(f) = \mathcal{N}_{4n+2}, NBS(f) = \mathcal{N}_{2n+1}$.

Remark 1.3. Consider patterns of period 1 and 2. A continuous interval map always has a fixed point, so 1 should stand at the end of the order $(*)$ both for both types of patterns. The situation with 2 is more complicated. Namely, there is only one pattern of period 2, and it has a division, but not a block structure. Thus, for no division patterns, 2 does not occur in the order. For no block structure patterns, 2 should stand just before 1 because, by the Sharkovsky Theorem, if f has a cycle of period larger than 1, it has also a cycle of period 2.

Remark 1.4. The order $(*)$ is similar to the orders present for the continuous triod map (see [ALM89]) and given by 5, 8, 4, 11, 14, 7, ...

and 7, 10, 5, 13, 16, 8, This makes interesting connections and allows us to look at an interval as a “diod.”

Remark 1.5. In [Mis94] it was proved that (a) patterns from \mathcal{NBS} of period $2n + 1$ force patterns from \mathcal{NBS} of period $4n + 4$, and (b) patterns from \mathcal{NBS} of period $4n$ force patterns from \mathcal{NBS} of period $4n + 2$. However our proofs here are much simpler (because they use the *rotation theory*). The fact that patterns with no block structure of period $4n + 2$ force patterns with no block structure of period $2n + 1$ is new; the order (*) was mentioned in [Mis94] only for unimodal maps. Finally, in our Main Theorem we take into account not only patterns with no block structure but also patterns with no division.

2. PRELIMINARIES

We will be using standard tools of combinatorial dynamics. The reader that is not acquainted with them can find details for instance in [ALM00], [BC92] or [MiNi91].

In particular, we will consider forcing among patterns. Given a pattern A we will often consider a cycle P and the P -linear (“connect the dots”) map f . Patterns forced by A are then exactly the patterns of cycles of f . They can be found by looking at the Markov graph of (f, P) , where vertices are the P -basic intervals (intervals between consecutive points of P) and arrows correspond to f -covering (there is an arrow from J to K if $K \subset f(J)$). The loops in this graph correspond to cycles of f (and therefore they determine which patterns are forced by A).

We will also use extensively rotation theory for interval maps. Since it is less known, we will present its basic notions and results (see [Blo95, BM97, BM97a, BM99, BS13]). We also prove some simple lemmas that will be necessary later.

Let $f : I \rightarrow I$ be a continuous map with a cycle P of period $q > 1$. Let m be the number of points $x \in P$ with $f(x) - x$ and $f^2(x) - f(x)$ of different signs. Then the pair $(m/2, q)$ is called the *over-rotation pair* of P and is denoted by $\text{orp}(P)$; the number $m/(2q)$ is called the *over-rotation number* of the cycle P and is denoted by $\varrho(P)$. The set of the over-rotation pairs of all cycles of f is denoted by $\text{ORP}(f)$. Note that the number m above is even, positive, and does not exceed $q/2$. Therefore in an over-rotation pair (p, q) both p and q are integers and $0 < p/q \leq 1/2$. We call over-rotation pairs (p, q) *coprime* if p and q are coprime. Clearly, we can speak of over-rotation pairs and over-rotation numbers of patterns and permutations.

Definition 2.1. We write $(p, q) \succ (r, s)$ if $p/q < r/s$, or $p/q = r/s = m/n$ with m and n coprime and $p/m \succ_r r/m$ (clearly, $p/m, r/m \in \mathbb{N}$).

The next lemma relates the fact that a pattern has a block structure to the properties of the pattern's over-rotation pair.

Lemma 2.2. *If a cycle P with the over-rotation pair (k, m) has block structure with q points in every block, then q divides both k and m . In particular, if k and m are coprime then P has no block structure.*

Proof. Clearly, q divides m . To see that q divides k , observe that if we identify each block to a point to get a cycle Q of period m/q , the over-rotation number of Q will be the same as for P , i.e., k/m . If $\text{orp}(Q) = (k', m')$, then $k'/m' = k/m$ and $m' = m/q$, so $k = k'q$. However, k' is an integer, so q divides k . \square

Definition 2.3. Let \mathbb{M} be the set consisting of 0, all irrational numbers between 0 and $1/2$, and all pairs (α, n) , where α is a rational number from $(0, 1/2]$ and $n \in \mathbb{N} \cup \{2^\infty\}$. Then for $\eta \in \mathbb{M}$ the set $\text{Ovr}(\eta)$ is equal to the following.

- (1) If η is an irrational number or 0, then $\text{Ovr}(\eta)$ is the set of all integer pairs (p, q) with $\eta < p/q \leq 1/2$.
- (2) If $\eta = (r/s, n)$ with r, s coprime, then $\text{Ovr}(\eta)$ is the union of the set of all integer pairs (p, q) with $r/s < p/q \leq 1/2$ and the set of all integer pairs (mr, ms) with $m \in \text{Sh}(n)$.

In case (2) of Definition 2.3 if $n \neq 2^\infty$ then $\text{Ovr}(\eta)$ is the set of all over-rotation pairs (p, q) with $(nr, ns) \succ (p, q)$, plus (nr, ns) itself.

Theorem 2.4 (Theorem 3.1 of [BM97]). *If $f : [0, 1] \rightarrow [0, 1]$ is continuous, $(p, q) \succ (r, s)$, and $(p, q) \in \text{ORP}(f)$, then $(r, s) \in \text{ORP}(f)$. Thus, $\text{ORP}(f) = \text{Ovr}(\eta)$ for some $\eta \in \mathbb{M}$. Conversely, if $\eta \in \mathbb{M}$ then there exists a continuous map $f : [0, 1] \rightarrow [0, 1]$ such that $\text{ORP}(f) = \text{Ovr}(\eta)$.*

For some patterns automatically we get cycles of all periods.

Definition 2.5 (Convergent/divergent patterns). A pattern (cycle) of period n is *convergent* if for the corresponding permutation π there is a number $m < n$ such that $\pi(i) > i$ for $i \leq m$ and $\pi(i) < i$ for $i > m$; otherwise a pattern (cycle) is *divergent*.

Observe that if P is a convergent cycle of a P -linear map f then f has only one fixed point.

Lemma 2.6. *Any divergent pattern forces a pattern with no block structure of period n for every $n > 1$. Moreover, if f is an interval map with a periodic orbit of divergent pattern then $\text{ORP}(f) = \text{Ovr}(0)$.*

Proof. Let P be a cycle of divergent pattern A . By Lemma 3.2 of [BM97], if f is an interval map with a cycle of divergent pattern then $\text{ORP}(f) = \text{Ovr}(0)$, which is exactly the second claim of the lemma. It follows that f has cycles of over-rotation pair $(1, n)$ for every n . By Lemma 2.2 these cycles have no block structure. Considering a P -linear map f we see that A forces patterns with no block structure of any period n as desired. \square

From now on we consider only convergent patterns. Then we can use an alternative way of computing over-rotation pairs. Let P be a cycle of a convergent pattern A with $\text{orp}(P) = (p, q)$. We will always denote by $a_P = a$ the fixed point of the P -linear map f (we may omit the subscript P if no ambiguity is possible). Then $(x - f(x))(f(x) - f^2(x)) < 0$ if and only if x is mapped to the other side of a under f . Thus, p equals the number of times when a point in P is mapped from the left of a to the right of a (alternatively, from the right of a to the left of a); p can also be computed if we count the number of times in P when a point maps from one side of a to the other side of a , and divide this number by 2. We can think of p as a cumulative rotation of P about a . This interpretation helps, in particular, in the proof of the next lemma.

Lemma 2.7. *If A is a convergent pattern, $\varrho(A) = 1/2$ if and only if P has division.*

Proof. If A has division then $\varrho(A) = 1/2$. Now, if $\varrho(A) = 1/2$, then $\text{orp}(A) = (n, 2n)$ for some n . Let P be a cycle of pattern A , and let f be a P -linear map. Then P has $2n$ points and all of them are mapped from one side of a to the other side. Therefore, P has a division. \square

Another concept related to Theorem 2.4 is that of a *twist pattern*.

Definition 2.8 (Twist patterns). A pattern of over-rotation number ϱ is *twist* if it does not force any other pattern of over-rotation number ϱ ; we use the same terminology for cycles and permutations.

By Lemma 2.6 a twist cycle must be convergent. In particular, if P is a twist cycle then the P -linear map has a unique fixed point.

Lemma 2.9 ([BM97a, BM99]). *Let P be a twist cycle P of the P -linear map f . Then, if points $u, v \in P$ lie on the same side of a , map to the same side of a , and u is farther away from a than v , then $f(u)$ is farther away from a than $f(v)$.*

3. PROOF OF MAIN THEOREM

We start by recalling the definition of a well known Štefan pattern.

Definition 3.1 (Štefan pattern). Consider the cyclic permutation $\sigma : \{1, 2, \dots, 2n + 1\} \rightarrow \{1, 2, \dots, 2n + 1\}$ defined as follows:

- $\sigma(1) = n + 1$;
- $\sigma(i) = 2n + 3 - i$, if $2 \leq i \leq n + 1$;
- $\sigma(i) = 2n + 2 - i$, if $n + 2 \leq i \leq 2n + 2$.

Then the pattern of this cyclic permutation is called the *Štefan pattern*, and any cycle of this pattern is said to be a *Štefan cycle*.

The importance of those patterns is due to the following fact.

Theorem 3.2 ([Šte77]). *Any pattern of period $2n + 1$ forces the Štefan pattern of period $2n + 1$.*

Now we prove some preliminary results. If P is a cycle of period $n > 1$ then for each point $x \in P$ we consider *germs* at x , i.e., small intervals with x as one of the endpoints. Each point of P has two germs, except the leftmost and rightmost points, which have one germ each. There is a natural map induced on the set of germs by the P -linear map f , and if we start at the germ of the leftmost point (or the rightmost point), we get back exactly after n applications of this map. Each germ is contained in a P -basic interval, so this loop of germs gives us a loop of P -basic intervals. These loops are called the *fundamental loop of germs* and the *fundamental loop of intervals*. Both loops correspond to the original periodic orbit P . Thus, the fundamental loop of intervals contains both the leftmost and the rightmost P -basic intervals. Observe that, by Lemma 2.9, if P is a twist cycle then any germ at $x \in P$ that points toward a maps to the germ at $f(x) \in P$ that points toward a too. Thus, if P is a twist cycle, then the vertices of the fundamental loop of germs form the set of germs pointing toward a .

In what follows we use the following notation. Denote by $I = [b_l, b_r]$ the P -basic interval containing the point a . Observe that the arrow $I \rightarrow I$ is a part of the Markov graph G of P . It follows that I is repeated in the fundamental loop of intervals of P twice while all other P -basic intervals are repeated there once. Consider the set $P' = P \cup \{a\}$. Though the germs at points of P stay the same whether we consider P or P' , there is a change concerning P' -basic intervals versus P -basic intervals: I is now replaced by two P' -basic intervals, $I_l = [b_l, a]$ and $I_r = [a, b_r]$. Notice that the arrows $I_l \rightarrow I_r$ and $I_r \rightarrow I_l$ are in the Markov graph of P' . Clearly, a germ at a point of P' is contained in a well-defined P' -basic interval. Hence the fundamental loop of germs of P gives rise to the *fundamental loop of P' -basic intervals* and the *fundamental loop of germs of P'* . Thus, we get the following lemma.

Lemma 3.3. *If P is a twist cycle of period larger than 1 then the fundamental loop of intervals of P' passes exactly once through every P' -basic interval.*

Now we investigate twist cycles close to the fixed point.

Lemma 3.4. *If P is a twist cycle of period $n > 2$ of a P -linear map f then at least one of the points b_l, b_r is the image of a point of P that lies on the same side of a .*

Proof. Suppose that $b_l = f_P(c_l)$, $b_r = f(c_r)$, where $c_l, c_r \in P$, and $c_r \leq b_l < b_r \leq c_l$. Since $n > 2$, either $c_r < b_l$ or $b_r < c_l$. We may assume that $c_r < b_l$. However, then $c_r < b_l < a < b_r = f(c_r) < f(b_l)$, which contradicts Lemma 2.9. \square

Twist patterns force other patterns with specific properties.

Proposition 3.5. *If P is a twist cycle of the P -linear map f and P has over-rotation pair (k, m) and over-rotation number $\rho(P) < \frac{1}{2}$, then f has a cycle of over-rotation pair $(k+1, m+2)$, which is not a doubling.*

Proof. By Lemma 3.4, we may assume that $b_l = f(c_l)$ for some $c_l \in P$ with $c_l < b_l$. Let L be the fundamental loop of intervals of P' . By Lemma 3.3 it passes through I_l exactly once, so we can insert into L the two arrows, $I_l \rightarrow I_r \rightarrow I_l$, at that place. Denote by M the loop of length $m+2$ obtained in this way, and by Q a corresponding periodic orbit of f . By the construction, each P' -basic interval contains one element of Q , except I_l and I_r , which contain two elements each. This, in particular, shows that the period of Q is $m+2$.

Let $x \in Q$ be the point that belongs to the P' -basic interval whose left endpoint is c_l . Then $f(x) \in I_l$, $f^2(x) \in I_r$, and $f^3(x) \in I_l$. Since the fixed point a is repelling (because the interval $[b_l, b_r]$ is mapped linearly onto a larger interval), we get $x < f^3(x) < f(x) < a < f^2(x)$. The other point of Q which is in I_r , is to the right of $f^2(x)$, because otherwise its image would be the third point of Q in I_l (and by the construction there are two points of Q in I_l and two points of Q in I_r).

If Q is a doubling, then $f(x)$ is paired with $f^3(x)$ or $f^2(x)$. The first option is impossible because if it holds then the pair of points mapped to the pair $\{f^3(x), f(x)\}$ must be the pair $\{x, f^2(x)\}$ and the points x and $f^2(x)$ are not consecutive in space. The second option is impossible because then the image pair $\{f^2(x), f^3(x)\}$ consists of points that are not consecutive in space. Thus, Q is not a doubling.

Finally, since we added two points that are mapped onto the opposite side of a , and the rest of the points of Q are mapped like the analogous points of P , the over-rotation pair of Q is $(k+1, m+2)$. \square

From Lemma 2.2 and Proposition 3.5 we get the following lemma.

Lemma 3.6. *Any pattern A with $\text{orp}(A) = (ks, ms)$ where k and m are coprime, and $\varrho(A) = k/m < 1/2$ forces a pattern of over-rotation pair $(k + 1, m + 2)$, which is not a doubling. In particular, a pattern A of over-rotation pair $(2n - 1, 4n)$ forces a pattern of over-rotation pair $(2n, 4n + 2)$ which has no block structure.*

Proof. Let A be a pattern with $\text{orp}(A) = (ks, ms)$ where k and m are coprime, and $\varrho(A) = k/m < 1/2$. By Theorem 2.4 A forces a twist pattern A' of over-rotation pair (k, m) . By Lemma 3.5, A' forces a pattern of over-rotation pair $(k + 1, m + 2)$ as desired. Now, let the over-rotation pair of A be $(2n - 1, 4n)$. By the above A forces a pattern B of over-rotation pair $(2n, 4n + 2)$ that is not a doubling. Let us show that B has no block structure. Indeed, the only common divisor of $2n$ and $4n + 2$ is 2. Hence by Lemma 2.2 the only way B can have block structure is when B is a doubling, a contradiction. \square

We will also need the following lemma.

Lemma 3.7. *Let A be a unimodal pattern. Then there exists a unique unimodal doubling of A .*

Proof. Let m be the period of A and let $\pi : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, m\}$ be a permutation with pattern A . We may assume that there is i such that π is increasing to the left of i and decreasing to the right of i . To construct the permutation of the unimodal doubling of π we have to replace each point $j \in \{1, 2, \dots, m\}$ by two points $2j - 1$ and $2j$. The new permutation has to map $2j - 1$ to $2\pi(j) - 1$ and $2j$ to $2\pi(j)$ if $j < i$ and $2j - 1$ to $2\pi(j)$ and $2j$ to $2\pi(j) - 1$ if $j > i$. If $j = i$ then we have two choices, but while one of them produces a cyclic permutation, the other one produces a permutation consisting of two cycles. Thus, we get a unique permutation with a unimodal pattern which is the doubling of A . \square

In what follows we will use the notation below: for every $m > 2$ set

$$\eta(m) = \begin{cases} (s - 1, 2s) & \text{if } m = 2s, \\ (n, 2n + 1) & \text{if } m = 2n + 1, \end{cases}$$

In particular, $\eta(4n) = (2n - 1, 4n)$ and $\eta(4n + 2) = (2n, 4n + 2)$.

We are ready to prove our Main Theorem. By Lemma 2.7, in the proof we can consider only convergent patterns.

Proof of Main Theorem. Recall that because we are excluding the pattern of period 2, each pattern with no block structure has no division.

By Lemma 2.7, patterns with no division have over-rotation numbers less than $1/2$. Each integer larger than 2 is of one of the three forms: $2n+1$, $4n$, $4n+2$, with $n \geq 1$. The largest possible over-rotation numbers smaller than $1/2$ for patterns of those periods are, respectively, $\frac{n}{2n+1}$, $\frac{2n-1}{4n}$, $\frac{2n}{4n+2}$. Those numbers are ordered as follows:

$$\dots < \frac{2n-1}{4n} < \frac{2n}{4n+2} = \frac{n}{2n+1} < \frac{2n+1}{4n+4} < \dots$$

Thus, by the definition of the order \succ , we get the following order among over-rotation pairs associated with these over-rotation numbers:

$$\dots \succ \eta(4n) \succ \eta(4n+2) \succ \eta(2n+1) \succ \eta(4n+4) \succ \dots \quad (**)$$

Observe that the over-rotation pairs $(2n-1, 4n)$ and $(n, 2n+1)$ are coprime; on the other hand, the over-rotation pair $(2n, 4n+2)$ is not coprime as the greatest common divisor of $2n$ and $4n+2$ is 2.

Let f be a continuous interval map. If all cycles of f have division then all cycles of f have block structure as division is a particular case of block structure. This means that case (1) of Main Theorem takes place. To proceed with less trivial cases of Main Theorem, fix two integers, $m > 2$ and s such that $m \gg s$.

Consider first the case of cycles with no division. Assume that f has a cycle P of period $m > 2$ with no division. This cycle has over-rotation number less than $1/2$, so by Theorem 2.4 the map f has a cycle of over-rotation pair $\eta(m)$. If $m \gg s$ then $\eta(m) \succ \eta(s)$, so again by Theorem 2.4, f has a cycle Q of over-rotation pair $\eta(s)$. Since its over-rotation number is smaller than $1/2$, Q has no division. In other words, if f has a cycle of period m with no division and $m \gg s$ then f must have a cycle of period s with no division. This proves, for cycles with no division, the first statement of Main Theorem.

Now, assume that f has a cycle P of period $m > 2$ with no block structure. Then, in particular, P has no division. As before, this implies that f has a cycle of over-rotation pair $\eta(m)$ and, again, f has some cycles of over-rotation pair $\eta(s)$. To prove the first statement of Main Theorem for cycles with no block structure we need to show that a cycle of over-rotation pair $\eta(s)$, forced by P , can be chosen to be with no block structure. By Lemma 2.2 and by the analysis of over-rotation pairs $\eta(4n)$, $\eta(4n+2)$, and $\eta(2n+1)$, any cycle Q of over-rotation pair $\eta(s)$ with $s = 4n$ or $s = 2n+1$ automatically has no block structure. If $s = 4n+2$, then either $m = 4n$ or $m \gg 4n$, so f must have a cycle of over-rotation pair $\eta(4n) = (2n-1, 4n)$. Then by Lemma 3.6 the map f must have a cycle Q of over-rotation pair $(2n, 4n+2)$ and

no block structure. This completes the proof of the first statement of Main Theorem for cycles with no block structure.

This also proves that $ND(f)$ and $NBS(f)$ are either empty or of the form \mathcal{N}_r , i.e., $ND(f) = \mathcal{N}_{r_{nd}}$ and $NBS(f) = \mathcal{N}_{r_{nbs}}$ for some numbers r_{nd} and r_{nbs} . Consider all the cases in more detail. Any cycle with no block structure has no division. Hence in general $ND(f) \supset NBS(f)$, so $r_{nd} \gg r_{nbs}$. If $r_{nd} = 2n + 1$ then, as before, f must have a cycle of over-rotation pair $\eta(2n + 1)$ which is coprime. It follows that this cycle has no block structure and, hence, in this case $ND(f) = NBS(f) = \mathcal{N}_{2n+1}$. This covers case (2) of Main Theorem. If $r_{nd} = 4n$, then, again, f must have a cycle of over-rotation pair $\eta(4n)$ which is coprime, this cycle has no block structure, and $ND(f) = NBS(f) = \mathcal{N}_{4n}$. This covers case (3) of Main Theorem.

Suppose now that $r_{nd} = 4n + 2$. Then f must have a cycle of over-rotation pair $\eta(4n + 2)$. If f has a cycle of over-rotation pair $\eta(4n + 2)$ with no block structure, then $ND(f) = NBS(f) = \mathcal{N}_{4n+2}$, which corresponds to case (4) of Main Theorem. Suppose now that all cycles of over-rotation pair $\eta(4n + 2)$ have block structure. Then, while $ND(f) = \mathcal{N}_{4n+2}$, the set $NBS(f)$ is strictly smaller than $ND(f)$. The first statement of Main Theorem implies that f has a point of period $2n + 1$; we may assume that its over-rotation pair is $\eta(2n + 1)$, which is coprime, so the corresponding periodic orbit has no block structure. We conclude that in this case $ND(f) = \mathcal{N}_{4n+2}$ while $NBS(f) = \mathcal{N}_{2n+1}$. This covers case (5) of Main Theorem.

To prove that all cases (1)-(5) can occur, we rely upon the method used in [ALM00] in the proof of the Sharkovsky Theorem. Consider the *full tent map* $T : [0, 1] \rightarrow [0, 1]$, given by $T(x) = 1 - |2x - 1|$, and the *truncated tent maps* $T_\alpha : [0, 1] \rightarrow [0, 1]$, given by $T_\alpha(x) = \min(T(x), \alpha)$. For any periodic orbit P of T the map T_α , where $\alpha = \max(P)$, has only cycles of patterns forced by the pattern of P .

As we have already seen, if $m > 2$ then the T -orbit of $1 - \frac{1}{2^{m+1}}$ is of period m and has no block structure (and, hence, no division). Also, for every period the map T has only finitely many periodic orbits of that period. Therefore we may define P_m as the orbit of T with the smallest maximum among the cycles of period m with no division, and Q_m as the orbit of T with the smallest maximum among the cycles of period m with no block structure. Evidently, these are forcing-weakest orbits of T of period m that have not division or no block structure, respectively. Recall that by Lemma 2.7 the over-rotation numbers of cycles Q_m and P_m are less than $1/2$. Since a cycle with no block structure has no division, then Q_m forces P_m .

By Theorem 2.4, any cycle of T of period m that has no division forces a cycle of over-rotation pair $\eta(m)$. Consider two cases depending on the period m .

If $m = 4n$ or $m = 2n + 1$, then $\eta(m)$ is coprime. By Lemma 2.2, any periodic orbit of over-rotation pair $\eta(m)$ has no block structure. Hence $P_m = Q_m$ and the corresponding map T_α , with $\alpha = \max(P_m) = \max(Q_m)$ is a map with $ND(f) = NBS(f) = \mathcal{N}_m$. This shows that cases (2) and (3) of Main Theorem are possible.

Suppose now that $m = 4n + 2$. By the theorems on forcing extensions of patterns (see [ALM00]), if A is a pattern and B is a doubling of A , then B forces A and the only pattern forced by B but not by A is B itself. By Lemma 3.7, if T has a cycle R then it also has a cycle which is a doubling of R . Therefore, P_{4n+2} is a doubling of P_{2n+1} . However, P_{4n+2} has a block structure and hence $\max(Q_{4n+2}) > \max(P_{4n+2})$. A map T_α with $\max(Q_{4n+2}) > \alpha \geq \max(P_{4n+2})$ is an example for case (5), while if $\alpha = \max(Q_{4n+2})$, we get an example for case (4).

Clearly, a constant map is an example for case (1). \square

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