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ROTATING AN INTERVAL AND A CIRCLE

ALEXANDER BLOKH AND MICHAŁ MISIUREWICZ

Abstract. We compare periodic orbits of circle rotations with their counterparts for interval maps. We prove that they are conjugate via a map of modality larger by at most 2 than the modality of the interval map. The proof is based on observation of trips of inhabitants of the Green Islands in the Black Sea.

1. Introduction

Most people will agree that it is easier to rotate a circle than an interval. However, the latter was made possible by the introduction of over-rotation numbers (see [BM2]). They allow us to trace how the orbits of an interval map are "rotated".

Let us consider cycles (periodic orbits) of some simple one-dimensional dynamical systems. For a circle map of degree one $f : S^1 \to S^1$ one can measure an average rotation of a cycle P of period n by taking its point x, measuring its displacement in a chosen lifting $F : \mathbb{R} \to \mathbb{R}$ after n iterates, and dividing by n. Thus, the rotation number of P will be $(F^n(x) - x)/n$ (see [P], [NPT], [ALM]). The simplest cycles with a given rotation number are those that can be found in rotations (or in maps conjugate to rotations). They are called *twist cycles* and can be characterized by the property that there is a circle map for which this is the unique cycle with this rotation number (see [ALM]).

We can look at those rotation numbers from the two-dimensional perspective. If the circle whose maps we consider is a subset of a plane, we can imagine connecting x and $f(x)$ by a piece of string and observing how this string rotates when we iterate the map. The number of full rotations after n iterates, divided by n, is the rotation number. When we make a similar construction for an interval map, we get an *over-rotation number* of a cycle (see [BM2]). Then a cycle will be *over-twist* if there is an interval map for which this is the unique cycle with this over-rotation number.

The main aim of this paper is to establish connections between over-twist cycles of interval maps of a given over-rotation number and twist cycles of circle maps of the same rotation number. Since we can think of a cycle as a cyclic permutation of a finite set, all cycles (of all maps) with the same period are conjugate. In our specific case we will show that this conjugacy is quite regular. Namely, it is

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piecewise monotone with the number of pieces larger by at most 2 than the number of pieces of monotonicity of an interval map with our cycle.

The paper is organized as follows. In Section 2 we define the basic notions and state some preliminary results. In Section 3 we characterize a class of cycles that are "twist" for rotation-type theories connected with the dynamics of the map. In Section 4 we characterize over-twist patterns. In Section 5 we prove the main result of the paper about conjugacies of over-twist interval cycles with twist circle cycles. The proof is based on observations of trips of inhabitants of the Green Islands in the Black Sea.

2. Preliminaries

In this section we state preliminary results. We begin with some definitions.

We will work with cycles (that is, periodic orbits) of continuous interval maps. The *pattern* of a cycle is the cyclic permutation we get when we look how the map acts on the points of the cycle, ordered from the left to the right. A cycle of f of pattern A is called a *representative of A in f*. Patterns are partially ordered by the *forcing relation*. A pattern A *forces* pattern B if every continuous interval map having a cycle of pattern A has a cycle of pattern B.

If there is a (non-strictly) increasing semiconjugacy between a pattern A and a pattern B, the pattern is said to have a *block structure* over B. Pre-images of points from B under this semiconjugacy are called *blocks*; each block has the same number of points. Clearly if a pattern A has a block structure over a pattern B then A forces B (see, e.g., [ALM, Lemma 2.10.1]).

If all blocks of A except perhaps one are mapped in a monotone way and on some block the first return map is a cycle with pattern C, then A is called an *extension of B by C*. Note that in such a situation C is unique up to the reversal of orientation. A pattern that is an extension of some pattern by a pattern of period 2 is called its *doubling* (or a 2-*extension*). A cycle P of a map f is called a *Štefan cycle* if its period is an odd number $n \geq 3$ and there exists $x \in P$ such that either

$$f^{n-1}(x) < f^{n-3}(x) < \cdots < f^4(x) < f^2(x) < x < f(x) < f^3(x) < \cdots < f^{n-4}(x) < f^{n-2}(x)$$

or

$$f^{n-2}(x) < f^{n-4}(x) < \cdots < f^3(x) < f(x) < x < f^2(x) < f^2(x) < \cdots < f^{n-3}(x) < f^{n-1}(x).$$

A pattern of a Štefan cycle is called a *Štefan pattern*. An extension of a pattern A by a Štefan pattern is called a *Štefan extension*.

A pattern (or a cycle with such a pattern) that has a block structure over the pattern of period 2 is said to have *division*. Otherwise we say it has *no division*.

For a cycle P there are some special classes of maps with this cycle. A map f is called *P-monotone* if it is monotone between consecutive (in space) elements of P and is constant to the left of the leftmost and to the right of the rightmost element of P. Patterns of all cycles of a P-monotone map are forced by the pattern of P (see [ALM, Theorem 2.7.7]). For every P there are also *P-adjusted* maps. They are P-monotone and have an additional property whereby they do not have other cycles with the same pattern as P.

The *rotation pair* of a cycle is (p, q), where q is the period of the cycle and p is the number of its elements which are mapped to the left of themselves (for cycles of period 1 we can take $p = 1/2$, but it is better to exclude them from our considerations). The number p/q is called the *rotation number* of the cycle. We denote the rotation pair of a cycle P by $\mathrm{rp}(P)$ and the set of the rotation pairs of all cycles of a map f by $\mathrm{RP}(f)$.

Similarly, the *over-rotation pair* of a cycle is $(m/2, q)$, where q is the period of the cycle and m is the number of its elements x such that $f(x)-x$ and $f^2(x)-f(x)$ have different signs (note that m is even). The number $m/(2q)$ is called the *over-rotation number* of the cycle. We denote the over-rotation pair of a cycle P by $\mathrm{orp}(P)$ and the set of the over-rotation pairs of all cycles of a map f by $\mathrm{ORP}(f)$. Note that $\mathrm{orp}(P) \leq 1/2$.

We introduce the following partial ordering among all pairs of integers (p, q) with $0 < p < q$. We will write $(p, q) \gtrdot (r, s)$ if either $1/2 \leq r/s < p/q$, or $p/q < r/s \leq 1/2$, or $p/q = r/s = m/n$ with m and n coprime and $p/m \succ r/m$ (notice that $p/m, r/m \in \mathbb{N}$). Here $k \succ l$ means that k stands to the left of l in the *Sharkovskiĭ ordering*:

$$3,\ 5,\ 7, \dots, 3\cdot 2,\ 5\cdot 2,\ 7\cdot 2, \dots, 3\cdot 2^2,\ 5\cdot 2^2,\ 7\cdot 2^2, \dots, 2^3,\ 2^2,\ 2,\ 1.$$

Completeness Theorem ([B], [BM2]). *If $f : [0, 1] \to [0, 1]$ is continuous, $(p, q) \gtrdot (r, s)$ and $(p, q) \in \mathrm{RP}(f)$ (respectively $\mathrm{ORP}(f)$) then $(r, s) \in \mathrm{RP}(f)$ (respectively $\mathrm{ORP}(f)$).*

A pattern that does not force any other pattern with the same rotation (respectively over-rotation) number is called a *twist pattern* (respectively an *over-twist pattern*). A cycle whose pattern is twist (respectively over-twist) is called a *twist cycle* (respectively an *over-twist cycle*).

A cycle (and the pattern it represents) is *divergent* if it has points $x < y$ such that $f(x) < x$ and $f(y) > y$. A cycle (pattern) that is not divergent will be called *convergent*. The set of all convergent patterns is denoted by CP. There is an equivalent way to define convergent patterns. Namely, let $\mathcal{U}$ be the family of all interval maps with a unique fixed point (we will always denote this fixed point by a). If f is a P-monotone map for a cycle P then P is convergent if and only if $f \in \mathcal{U}$.

The following lemma shows that (over-)twist patterns are convergent.

Lemma 2.1 ([B], [BM2]). *Let A be a divergent pattern. Then A forces patterns of all rotation pairs as well as patterns of all over-rotation pairs.*

When dealing with convergent patterns, we will be using the following terminology. Let P be a convergent cycle of a P-monotone map f. Those points $x \in P$ for which x and $f(x)$ lie on the same side of a are called *green*, and those for which x and $f(x)$ lie on the opposite sides of a are called *black*. A cycle P (and its pattern) is called *green* if it is convergent and f is increasing on the set of green points and decreasing on the set of black points of P. In Figure 2.1, the black points are marked "B" and the green ones "G". The cycle depicted there is green.

Before we state the next lemma, note that by the Sharkovskiĭ Theorem (see e.g. [ALM]) the only (over-)twist pattern of (over-) rotation number 1/2 is that of period 1. Hence, from now on we consider only (over-)twist patterns of (over-)rotation numbers different from 1/2.

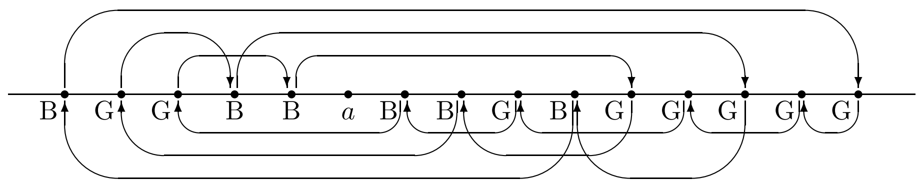

FIGURE 2.1. A green cycle

Lemma 2.2 ([B]). *A twist pattern A of rotation number $\rho \neq 1/2$ is green. Moreover, if P is a representative of A in a P-adjusted map f, then f is monotone on one side of the fixed point a. In particular, if $\rho < 1/2$, then all points mapped to the left are black and if $\rho > 1/2$, then all points mapped to the right are black.*

Note that by reversing the order one can make a pattern of rotation pair (r, s) and rotation number $\rho = r/s$ into a pattern of rotation pair $(s - r, r)$ and rotation number $1 - \rho$. In particular, any twist pattern of rotation number greater than $1/2$ can be obtained from the appropriate twist pattern of rotation number smaller than $1/2$ by reversing the order. Therefore sometimes it makes sense to work with twist patterns of rotation numbers smaller than $1/2$ and then extend the results to all patterns by reversing the orientation.

Now, let A be a convergent pattern, let P be a representative of A in a P-monotone map f and let f be monotone on one side of the fixed point a. This means that all points of P lying on this side of a are black. Then for any cycle Q of f all points lying to the same side of a are black too which is implied by the definition that over-rotation pairs and numbers of these cycles coincide with their rotation pairs and numbers. Together with Lemma 2.2 this proves that twist patterns are also over-twist. On the other hand, if A is an over-twist pattern such that either all its points mapped to the left are black or all its points mapped to the right are black, then the same property holds for all patterns forced by A. So for all these patterns over-rotation pairs and numbers coincide with rotation ones. Therefore by the same arguments, A is a twist pattern. In this way, we obtain the following lemma.

Lemma 2.3. *Twist patterns are those over-twist patterns for which either all points mapped to the right are black or all points mapped to the left are black.*

For a map f with a unique fixed point a it is useful to look at *admissible loops of intervals* $\alpha = J_0 \to J_1 \to \cdots \to J_{q-1} \to J_0$, where $\to$ means *f-covering* (that is, $J \to K$ if $K \subset f(J)$) and every interval J_i has a as one of its endpoints. If p is the number of arrows $J \to K$ in α such that J is to the right of a and K is to the left of a, then we say that the over-rotation number of α is p/q. A cycle Q (other than $\{a\}$) for which there is a point $x \in Q$ with $f^q(x) = x$ and $f^i(x) \in J_i$ for $i = 0, 1, \ldots, q-1$, is *associated* to α. Clearly, its over-rotation number is the same as that of α. It turns out that in most cases associated cycles do exist. Therefore, the following lemma holds.

Lemma 2.4 ([B], [BM2]). *Let $\alpha = J_0 \to J_1 \to \cdots \to J_{q-1} \to J_0$ be an admissible loop. Then there exists a cycle associated to α, unless q is even, all intervals J_i with* *even i are located on one side of a, and all the intervals J_k with odd k are located on the other side of a, in which case there exists a cycle of period* 2*. In particular, a cycle associated to α exists if the over-rotation number of α is not* $1/2$*.*

Let us denote by J_y the interval with endpoints a and y. If P is a cycle of f of period q and $x \in P$, then we call the loop $J_x \to J_{f(x)} \to \cdots \to J_{f^{q-1}(x)} \to J_x$ the *fundamental admissible loop* of P.

3. Green patterns are sequential twist patterns

In the next section we will characterize over-twist patterns as green patterns with some additional properties. It is natural to ask whether all green patterns can be described as a kind of twist patterns. That is, we want to choose some invariant (like an over-rotation number for over-twist patterns or a rotation number for twist patterns) and then prove that green patterns are forcing minimal patterns with the given value of this invariant. To answer this question we need more definitions.

By a *circular sequence* we will understand a finite sequence modulo rotations. That is, sequences

$$(r_0, r_1, \dots, r_{k-1}, r_k, r_{k+1}, \dots, r_{n-1}) \quad \text{and} \quad (r_k, r_{k+1}, \dots, r_{n-1}, r_0, r_1, \dots, r_{k-1})$$

are equal as circular sequences.

Let $f \in \mathcal{U}$ and let $P \neq \{a\}$ be a cycle of f. Define the *rotation sequence* of P as the circular sequence $\mathbf{r}(P) = (r_0, r_1, \dots)$ that is obtained by choosing $x \in P$ and setting $r_i = 0$ if f maps $f^i(x)$ to the right and $r_i = 1$ otherwise. We call a convergent pattern A a *sequential twist* if it does not force any other pattern with the same rotation sequence.

Theorem 3.1. *A pattern $A \in CP$ is a sequential twist if and only if it is green.*

We start with the simpler part of the proof.

Lemma 3.2. *Let a pattern $A \in CP$ be a sequential twist. Then it is green.*

Proof. The proof is similar to the proof of Theorem 2.1 of [BM3].

Let P be a representative of A in a P-adjusted map f. Then clearly $f \in \mathcal{U}$.

If A has a division, then the rotation sequence of P is $010101\dots$. On the other hand, by the Sharkovskiĭ Theorem either A itself is of period 2 or it forces a pattern of period 2. Thus the only sequential twist pattern with division is that of period 2 and from now on we may assume that A has no division.

Let α be the fundamental admissible loop of P. Suppose that f is not increasing on the set of green points of P or not decreasing on the set of black points of P. Then there are points $x, y \in P$ such that $x >_a y$ and $f(x) <_a f(y)$. We can modify α by replacing the interval J_x by J_y. Then the new loop β is admissible and has the same rotation sequence as α. Since we assumed that A has no division then by Lemma 2.4 f has a cycle Q of the same over-rotation number, associated to β.

Let us show that $Q \neq P$. Suppose that P is associated to β. Let z be the point of P closest to a from the right. Since $z \neq x$, the same interval J_z appears in α and β. Moreover, z is the unique point of P belonging to J_z. Therefore for i such that $x = f^i(z)$, by starting at J_z and following the orbit of z and the loop β for i steps we get $x \in J_y$. This is false since $x >_a y$. Hence P is not associated to β, so $Q \neq P$.

Since f is P-adjusted, the pattern of Q is different from A. Thus, A forces a different pattern with the same rotation sequence, so it is not a sequential twist, a contradiction. This proves that f is increasing on the set of green points of P and decreasing on the set of black points of P, that is P (and therefore A) is green. $\square$

Let $f \in \mathcal{U}$. We will say that a cycle P of f *dominates* a cycle Q of f if there are $x \in P$ and $y \in Q$ such that $f^n(x) \geq_a f^n(y)$ for every n. Notice that in such a case the cycles P and Q have the same rotation sequence.

Lemma 3.3. *Assume that P is a green cycle of a P-monotone map f and that it dominates a cycle Q. Then P and Q have the same patterns.*

Proof. By the definition of domination, there are points $x \in P$ and $y \in Q$ such that $f^n(x) \geq_a f^n(y)$ for every n. If there are no points of P between (by this we mean *strictly* between) $f^n(x)$ and $f^n(y)$ for any n then the orderings of the points of both cycles on the interval is the same, that is they have the same pattern.

Suppose that for some i there is a point $z \in P$ between $f^i(x)$ and $f^i(y)$. We claim that there is a point of P between $f^{i+1}(x)$ and $f^{i+1}(y)$. If f is monotone between $f^i(x)$ and $f^i(y)$, then $f(z)$ is such a point. If f is not monotone on $[f^i(x), f^i(y)]$, then there is a point $z' \in P$ between $f^i(x)$ and $f^i(y)$, which has a different color than $f^i(x)$. Let w be the furthest from point a of $f(J_{z'})$ on the same side as $f^{i+1}(x)$. It is an image of a point of P, so it belongs to P. Since $f^i(x) >_a z'$ and P is a green cycle, we have $f^{i+1}(x) >_a w$. Since $f^i(y) \in J_{z'}$, we have $f^{i+1}(y) \in f(J_{z'})$. The points $f^{i+1}(y)$ and w lie on the same side of a, so $w \geq_a f^{i+1}(y)$. The point $f^{i+1}(y)$ does not belong to P, while w does. Therefore they are different, so $w >_a f^{i+1}(y)$. This proves the claim.

By induction it follows that there is a point of P between $f^n(x)$ and $f^n(y)$ for all $n \geq i$. However, for some $n \geq i$ the point $f^n(x)$ is the closest point to a from the left among all points of P, so we have a contradiction. This completes the proof. $\square$

Lemma 3.4. *Assume that P is a green cycle of a P-monotone map f. If $x \in P$, $x \geq_a y$, and $f(x), f(y)$ lie on the same side of a, then $f(x) \geq_a f(y)$. Consequently, if Q is a cycle of f with the same rotation sequence as P, then P dominates Q.*

Proof. Suppose that $x \in P$, $x \geq_a y$, and $f(x) <_a f(y)$. Then, since f is P-monotone, there exists a point $z \in P$ such that $x \geq_a z$ and $f(x) <_a f(z)$. This is impossible since P is green. This proves the first statement of the lemma. The second part follows immediately from the first part by induction (to start induction, notice that all points of Q are between the left-most and the right-most points of P). $\square$

Now Theorem 3.1 follows immediately from Lemmata 3.2, 3.3 and 3.4.

4. Over-twist patterns

We are interested in a characterization of over-twist patterns similar to the characterization of twist patterns (see [BK], [BM1]). For this we have to introduce a code for cycles, similar to the code that was introduced in the case of rotation numbers. Such a code depends on the function that was used to get over-rotation (or rotation) numbers. In case of over-rotation numbers if we restrict our attention to convergent cycles (we may do it) we have a choice of two functions. One of them is $1/2$ at black points and 0 at green ones. The other one is 1 at black points to the right of a and 0 otherwise. Our theorem will hold for any of those functions.

If P is a cycle of $f \in \mathcal{U}$ and φ is any of the above functions then we introduce the *code* for P as follows. The code is a function $L : P \to \mathbb{R}$, defined as $L(x) = 0$ for the left-most point x of P and then by induction as $L(f(y)) = L(y) + \rho - \varphi(y)$, where ρ is the over-rotation number of P. When we get back to x along the orbit P, we add ρ n times (n is the period of P) and subtract the sum of φ along P, which is $n\rho$, so we get again 0. Therefore the definition is correct. Clearly, we can also speak of codes for patterns.

The cycle depicted in Figure 2.1 has period 14 and over-rotation number 3/14. With the second choice of φ we get codes of its points $k/14$, where from left to right the values of k are 0, 2, 4, 5, 7, 15, 13, 12, 11, 10, 9, 8, 6, 3.

If $f \in \mathcal{U}$, we say that the code for P is *monotone* if for any $x, y \in P$, $x >_a y$ implies $L(x) < L(y)$. Our cycle from Figure 2.1 is monotone.

Theorem 4.1. *A pattern is over-twist if and only if it is convergent and has monotone code.*

For clarity, we prove Theorem 4.1 through a sequence of lemmata. First, the easy part.

Lemma 4.2. *Any over-twist pattern is convergent and has monotone code.*

Proof. An over-twist pattern is convergent by Lemma 2.1. Now, let f be a P-admissible map for a cycle P with an over-twist pattern. Suppose that the code for P is not monotone. Then there exist $x, y \in P$ such that $x >_a y$ and $L(x) \geq L(y)$. We have $x = f^m(y)$ for some m smaller than the period of P. The loop $J_y \to J_{f(y)} \to \cdots \to J_{f^{m-1}(y)} \to J_y$ is admissible. Its over-rotation number is equal to $1/m$ times the sum of φ along the orbit of y from y to $f^{m-1}(y)$. Since $L(x) \geq L(y)$, this sum is smaller than or equal to $m\rho$, where ρ is the over-rotation number of P. Thus, the over-rotation number of our loop (and therefore of a cycle Q associated to it) is smaller than or equal to ρ. By the Completeness Theorem for over-rotation numbers it implies that there exists a cycle Q' of over-rotation number ρ. Since f is P-adjusted and P is over-twist, this is a contradiction, unless $Q' = Q = P$. However the period of Q is at most m while the period of P is greater than m, so $Q \neq P$. ☐

Now we start proving the opposite implication.

Lemma 4.3. *Any convergent pattern with monotone code is green.*

Proof. Let f be a P-monotone map for a convergent cycle P with monotone code. Take any points $x, y \in P$ such that $x >_a y$, and $f(x)$ and $f(y)$ are on the same side of a. Then x and y are on the same side of a and have the same color. Therefore $\varphi(x) = \varphi(y)$, and since $L(x) \leq L(y)$, we get $L(f(x)) \leq L(f(y))$. Hence $f(x) \geq_a f(y)$. Since $x \neq y$ and both x and y are in P, we get $f(x) \neq f(y)$. Thus $f(x) >_a f(y)$. This proves that P is green. ☐

We will call an admissible loop a *P-loop* if its elements are intervals of the form J_x with $x \in P$ and $f(x) \geq_a y$ whenever $J_x \to J_y$ is an arrow in the loop.

Lemma 4.4. *Assume that f is a P-adjusted map for a green cycle P. Let Q be a cycle of f, different from P and $\{a\}$. Then there is an admissible P-loop with the same rotation sequence as Q (and therefore with the same over-rotation number as Q), but different from the fundamental loop of P and its repetitions.*

Proof. For any point x with $f(x) \neq a$ we define a point $\xi(x) \in P$ as follows. If $x \in P$, then $\xi(x) = x$. If $x \notin P$, then x belongs to a unique $(P \cup \{a\})$-basic interval J. At least one endpoint of J has the same color as x. If there is only one such endpoint, we choose it as $\xi(x)$. If both endpoints of J have the same color as x, then we choose $\xi(x)$ as the one that is further from a.

Notice that with our choice if also $f^2(x) \neq a$, then $f(\xi(x)) \geq_a \xi(f(x))$. Therefore if $x \in Q$, then $J_{\xi(x)} \to J_{\xi(f(x))} \to J_{\xi(f^2(x))} \to \cdots \to J_{\xi(x)}$ is an admissible P-loop. Call it β. Clearly β has the same rotation sequence as Q.

For each $y \in P$ the set $\{x : \xi(x) = y\}$ is an interval. Those intervals are pairwise disjoint and of course are ordered in the same way as points of P. Therefore, if β is the fundamental loop of P or its repetition, then Q has the same pattern as P or has a block structure over P which due to the remark before the lemma implies that f has a cycle different from P whose pattern coincides with that of P. Since f is P-adjusted, this is impossible. □

Lemma 4.5. *Assume that f is a P-monotone map for a green cycle P with monotone code. Then any admissible P-loop with the same over-rotation number as P is equal to the fundamental loop of P or its repetition.*

Proof. Let $\beta = J_{v_0} \to J_{v_1} \to \cdots \to J_{v_{n-1}} \to J_{v_n}$, where $v_n = v_0$, be an admissible P-loop β with the same over-rotation number as P (call it ρ). Whenever $v_{i+1} \neq f(v_i)$, we have $v_{i+1} = f^j(v_i)$ for some $j > 1$ and we can replace the arrow $J_{v_i} \to J_{v_{i+1}}$ by the block $J_{v_i} \to J_{f(v_i)} \to \cdots \to J_{f^{j-1}(v_i)} \to J_{v_{i+1}}$. With those replacements we get a new admissible P-loop γ. In this loop all arrows are of the form $J_x \to J_{f(x)}$ with $x \in P$, so it has to be either the fundamental loop of P or its repetition. In particular, the over-rotation number of γ is equal to ρ.

Let us compare the over-rotation numbers of β and γ. The over-rotation number of γ is a weighted average of the over-rotation numbers of β and the inserts. Here by the over-rotation number of an insert $\to J_{f(v_i)} \to \cdots \to J_{f^{j-1}(v_i)} \to$ we mean the average of the values of the function φ at the points $f(v_i), \dots, f^{j-1}(v_i)$ (call this average t). Notice that the increment of the code $L(v_{i+1}) - L(f(v_i))$ is equal to j times ρ minus the sum of the values of the function φ at the points $f(v_i), \dots, f^{j-1}(v_i)$, that is to $j(\rho - t)$. Since β is a P-loop and $v_{i+1} \neq f(v_i)$, we have $f(v_i) >_a v_{i+1}$. Since the code for P is monotone, this implies $L(v_{i+1}) - L(f(v_i)) > 0$. Hence $\rho > t$. Therefore the over-rotation numbers of all the inserts are smaller than ρ.

Thus, the only way we can obtain ρ as the weighted average of the over-rotation numbers of β and the inserts, is to have no inserts. This means that $\beta = \gamma$, so β is equal to the fundamental loop of P or its repetition. □

Now Theorem 4.1 follows immediately from Lemmata 4.2-4.5.

We noted in Section 2 (Lemma 2.3) that twist patterns are in fact over-twist patterns such that the map is monotone on one side of the fixed point. Therefore Theorem 4.1 applies to twist patterns as well, and so results of [BK] and [BM1] follow from Theorem 4.1.

Instead of talking about forcing-minimal (i.e., minimal with respect to the forcing relation) patterns of given a over-rotation number we can talk about forcing-minimal patterns of given over-rotation pair. It turns out that by combining Theorem 4.1 and some well known results one can easily characterize such patterns.

We will need the following lemma, easily deducible from [ALM]. Since its proof is a minor variation of the proofs from Section 2.11 of [ALM] (which are rather technical), we will omit it here. Note that the situation here is very similar to the case of lifted patterns for degree one circle maps (see Theorem 3.12.17 of [ALM]).

Lemma 4.6. *Let A be a pattern of period n and let k be a positive integer. Then a pattern B of period kn is forcing-minimal among patterns of period kn that have block structure over A if and only if B can be obtained from A by a finite number (possibly zero) of doublings and then at most one Štefan extension.*

Clearly, if $k = m \cdot 2^r$, where m is odd, then there are r doublings necessary, and an extension by a Štefan pattern of period m is necessary if $m \geq 3$.

Now we can prove the following theorem. Let us note that it holds as well for rotation pairs instead of over-rotation pairs.

Theorem 4.7. *Let k, m, n be positive integers with $2m \leq n$ and m, n coprime. A pattern of period kn is forcing-minimal among all patterns of over-rotation pair (km, kn) if and only if it can be obtained from an over-twist pattern of an over-rotation pair (m, n) by a finite number (possibly zero) of doublings and then at most one Štefan extension.*

Proof. Assume first that a pattern B of period kn is forcing-minimal among all patterns of an over-rotation pair (km, kn). By the Completeness Theorem, B forces an over-twist pattern A with over-rotation pair (m, n). Since m, n are coprime, A is not a doubling. Hence, if B has no block structure over A, then by Theorem 9.12 of [MN] B forces a pattern of an over-rotation pair (km, kn) which has block structure over A, a contradiction. Therefore B has block structure over A. Thus, every pattern of period kn forced by B and forcing A has block structure over A, so it has an over-rotation pair (km, kn). Therefore B is forcing-minimal among patterns with block structure over A. By Lemma 4.6, B can be obtained from A as in the statement of the theorem.

Assume now that B has period kn and can be obtained from an over-twist pattern A as in the statement of the theorem. Then B has block structure over A, so it has an over-rotation pair (km, kn). If B forces a pattern $C \neq B$ of an over-rotation pair (km, kn), then by Theorem 3.7 of [MN] either A forces C or C has block structure over A. In the first case, since A is over-twist and C has an over-rotation number m/n, we get $C = A$. Therefore $k = 1$, so $B = A = C$, a contradiction. The second case is impossible by Lemma 4.6. This completes the proof. □

5. Conjugating with rotations

It is not a coincidence that we are using names that involve the word *rotation*. In particular, in this section we will see how over-twist cycles of the over-rotation number p/q are connected with circle rotations by the angle $2\pi p/q$. Suppose we want to find a conjugacy between such a cycle and a cycle of a circle rotation. Since we are thinking of a conjugacy on finite sets, this poses no problem. Any two cycles of the same period q are conjugate. However, since we are working with one-dimensional dynamical systems, we have an additional structure connected with the ordering of the points. Therefore, we can expect our conjugacy to be order-preserving to some extent. It cannot be order-preserving globally, but it can be piecewise order-preserving with a reasonable number of pieces. The lack of monotonicity for a conjugacy is connected to the lack of monotonicity of the over-twist cycle we are working with. Therefore "reasonable" should mean the number of pieces for the conjugacy that is bounded by a constant depending only on the number of pieces of monotonicity of the over-twist cycle. In this section we prove the existence of such a bound (Theorem 5.9).

This bound is closely connected with a bound for the code of over-twist cycles, and we will work on the latter throughout most of this section. We will be using only some properties of the over-twist patterns. Therefore, we will formulate and prove our results in a more general context, in particular dealing with arbitrary functions rather than with only those which generate the code for over-twist patterns. On the other hand the main application of the results is related to over-twist patterns. Thus, alongside general results we will include particular cases concerning over-twist patterns. We will also show what we get for twist patterns.

Let P be a cycle of an interval map f with the unique fixed point a. To generate the code for P, we used in Section 4 the function $\rho - \varphi$, and we were summing it along the orbit of the left-most point of P. Here, we will replace this function by a more general function ψ for which we make the following assumptions:

(1) $\sum_{x \in P} \psi(x) = 0$.
(2) The cycle P is green and has period larger than 2.
(3) The code L is monotone, that is for $x, y \in P$, $x >_a y$ implies $L(x) < L(y)$.

Here the code $L : P \to \mathbb{R}$ is defined by fixing the value of L at some point of P and then by induction setting $L(f(x)) = L(x) + \psi(x)$ for every $x \in P$. As in Section 4, assumption (1) guarantees that L is well defined. As long as we will be interested only in the differences $L(y) - L(x)$, it does not matter what value and which point we choose initially.

In the special case of over-rotation numbers, we can use (as explained in Section 4) one of two functions: φ_{or}, that is $1/2$ at black points and 0 at green ones; or φ'_{or}, that is 1 at black points to the right of a and 0 otherwise. Thus, instead of ψ we use $\psi_{or} = \rho - \varphi_{or}$ or $\psi'_{or} = \rho - \varphi'_{or}$, respectively, where ρ is the over-rotation number of the pattern (or the cycle). Corresponding codes will be denoted L_{or} and L'_{or}, respectively.

Let us order the cycle P in space: $x_1 < x_2 < \cdots < x_q$. The *modality* of P is the number of its turning points, that is points x_i such that $1 < i < q$ and $(f(x_i) - f(x_{i-1}))(f(x_i) - f(x_{i+1})) > 0$.

Theorem 5.1. *Under assumptions* (1)-(3)*, for every $x, y \in P$ we have*

$$L(y) - L(x) < (2n + 5)M,$$

where n is the modality of P and $M = \max\{|\psi(x)| : x \in P\}$. For the code L_{or} we have

$$L_{or}(y) - L_{or}(x) < (n/2 + 1)(1 - \rho),$$

where ρ is the over-rotation number of P.

To prove this theorem, we will need some geographical notions. The points of P come in blocks of consecutive (in space) points of the same color. We will call the blocks of green points *islands* (it may happen that an island consists of one point only). Thus, we have the Green Islands in the Black Sea. Wherever an island meets the sea, there is a point of $f^{-1}(a)$ between them. There are $n+1$ such points. However, a itself separates two black points, so it does not count. Thus, there are at most $n/2 + 1$ islands.

The point a is the most important point of the region and is referred to as the Center of the World, or the Center for short.

Green Islanders have several means of transport. The two most popular are a green arrow that takes them from a green point x to $f(x)$, and a balloon that takes them from any x to any $y >_a x$ (the winds blow only from the Center outward). Note that this is enough to get from any point on an island to any other point on the same island. However, one can also travel between islands. This divides the set of all islands in a natural way into archipelagos. Two islands I and J belong to the same archipelago if one can get from I to J and from J to I using only green arrows and balloons. One can easily see that an archipelago consists of one or more adjacent islands on the same side of the Center, and that two distinct archipelagos have no common islands.

In our example from Figure 2.1 there is one green island to the left and two to the right of the Center. However, on each side of the Center there is only one archipelago.

Another way of getting around is to use black arrows that take one from a black point x to $f(x)$. They take a traveler to the other side of the Center. They also make interarchipelagian travel (other than by a balloon that carries its passengers further from the Center) possible. However, black arrows cannot replace green ones, as we see in the next lemma.

Lemma 5.2. *If a trip by black arrows and perhaps balloons begins at b and ends at e on the same side of the Center, then $e >_a b$.*

Proof. For such a trip, the number of black arrows taken is even. Let us divide them into consecutive pairs. In such a pair, if the beginning of the first black arrow is x and the beginning of the second one is y, then $y \geq_a f(x)$. If $x \geq_a f(y)$, then one of those inequalities is strict, since we assumed that the period of P is larger than 2. Hence, we may assume that $x >_a f(y)$ (otherwise we switch x and y). Then, since P is green, the set of points of P between x and y (including y, but excluding x) is invariant, a contradiction. Thus $x <_a f(y)$, that is, one can travel from x to $f(y)$ by balloon. In this way we can eliminate all pairs of black arrows, so the whole trip can be made by balloon; hence $e >_a b$. $\square$

The code L indicates how good the living conditions are at a given point. Therefore, it is natural for a Green Islander to try to get closer to the Center. To do this effectively, one has to get to a different archipelago. To get from archipelago A to a different one that is as close to the Center as possible, one should do the following. First, try to get as close to the Center as possible using only green arrows and balloons. Such a trip terminates at a black point that we will denote by $c(A)$. It is the end of the green arrow starting at the closest to the Center point of the closest to the Center island of A. Then use a black arrow. If its end $f(c(A))$ is a black point, use a balloon to get to the nearest island. Such a trip ends at some archipelago $\eta(A)$ on the other side of the Center. On Green Islands this type of trip is commonly referred to as an interarchipelagian jump, or an i.a. jump for short.

It may also happen that there is no $\eta(A)$. A balloon taken at $f(c(A))$ will drift beyond the Black Sea and its despairing passengers will not see any island. We will show that this can happen only if there is no island at all on that side of the Center.

For archipelagos we will use the same notation as for points of P. Namely, $A >_a B$ will mean that A is on the same side of the Center as B, but farther from the Center.

Lemma 5.3. (1) *If for some archipelago A there is no $\eta(A)$, then all islands lie at one side of the Center and they form one archipelago.*

(2) *For every archipelago A we have $A >_a \eta^2(A)$, unless there is no B such that $A >_a B$.*

Proof. Assume that for some archipelago A there is no $\eta(A)$. This means that every point y such that $y \geq_a f(c(A))$ is black. By Lemma 5.2, when returning to the side of the Center where the trip started, one cannot get closer to the Center than $c(A)$. Since P is green, this means that the set of points $t \in P$, such that either $t \geq_a c(A)$ or $t \geq_a f(c(A))$, is invariant. Therefore, the points $c(A)$ and $f(c(A))$ are the closest to the Center points of P from both sides. Thus, all points of P on the same side of the Center as $f(c(A))$ are black.

In such a case, by Lemma 5.2, a trip between islands which includes black arrows can be also done without using them, i.e., only by green arrows and balloons. However, since P is a cycle, one can get from every point of P to any other point of P using only green and black arrows. Therefore, one can get from every island to every other one with green arrows and balloons. This means that there is only one archipelago, so the proof of (1) is complete.

To prove (2), assume that there is an archipelago B such that $A >_a B$. By (1), in such a case η is defined at every archipelago. Suppose $\eta^2(A) \geq_a A$. Let d_1 be closer to the Center of the points $c(\eta(A))$ and $f(c(A))$. Similarly, let d_0 be closer to the Center of the points $c(A)$ and $f(c(\eta(A)))$. If $d_1 = c(\eta(A))$, then it is black (remember that $c(C)$ is black for every archipelago C). If $d_1 = f(c(A))$, then it cannot be green, since, in such a case, we would have $c(\eta(A)) <_a f(c(A))$, contrary to the definition of d_1. Thus, d_1 is always black. In a similar way one can check that d_0 is black (the assumption $\eta^2(A) \geq_a A$ has to be used here).

If $d_1 = c(\eta(A))$, then $f(d_1) \geq_a d_0$ by the definition of d_0 and the assumption $\eta^2(A) \geq_a A$. If $d_1 = f(c(A))$, then we get the same conclusion by Lemma 5.2. Thus, in both cases $f(d_1) \geq_a d_0$. Similarly, $f(d_0) \geq_a d_1$. In other words, if starting at d_0, d_1, or farther from the Center, one cannot get closer to the Center than those points when using black arrows. By the definitions of d_0 and d_1, green arrows and balloons also will not help. This is a contradiction since d_0 is not the closest to the Center point of P from its side. □

The costs of travel are designed to create incentives to move to less desirable places. Namely, the cost of getting from x to y is $L(y) - L(x)$. Thus, one has to pay for getting closer to the Center, but one earns money by moving farther from the Center. Round trips are free. Now we will be estimating (from above) costs of various trips. Note that the cost of a balloon trip is always negative, while each trip by one arrow costs at most M (and at least $-M$). For L_{or} it costs ρ to take a green arrow and $\rho - 1/2$ to take a black arrow.

Lemma 5.4. (1) *Any trip that begins and ends on the same island costs less than $2M$. For L_{or} it costs less than $1 - 2\rho$.*

(2) *Any trip that begins and ends in the same archipelago consisting of m islands costs less than $(3m-1)M$. For L_{or} it costs less than $m(1-\rho) - \rho$.*

(3) *One can travel from any archipelago A to $\eta(A)$ (if $\eta(A)$ exists) for $2M$ or less. For L_{or} one can do it for $2\rho - 1/2$ or less.*

Proof. (1) Let x be the point farthest from the Center point of an island I. The Center is surrounded by sea, so we can take the point y closest to x among black points $t <_a x$. The set of points of P between x and $f(y)$ (excluding x but including $f(y)$) is not invariant, so there is a point $z \leq_a f(y)$ such that $f(z) \geq_a x$. Thus, $L(f(y)) \leq L(z)$ and $L(f(z)) \leq L(x)$, so

$$\begin{aligned} L(y) - L(x) &= [L(f(y)) - \psi(y)] - L(x) + [L(f(z)) - L(z) - \psi(z)] \\ &\leq -\psi(y) - \psi(z) \leq 2M. \end{aligned}$$

For any points $b, e \in I$ we have $x \geq_a b$ and $e >_a y$, so $L(e) - L(b) < L(y) - L(x) \leq 2M$. This proves (1) in the general case. For L_{or} we have $\psi_{or}(y) = \psi_{or}(z) = \rho - 1/2$, so the estimate is $1 - 2\rho$.

(2) One can get from one island to the next island closer to the Center island of the same archipelago by one green arrow and then perhaps a balloon. This costs at most M. The most expensive trip within one archipelago that has m islands consists of $m - 1$ such interisland legs and m trips within islands, so by (1) its cost is less than $2mM + (m-1)M = (3m-1)M$. This proves (2) in the general case. For L_{or} we replace $2mM$ by $m(1-2\rho)$ and $(m-1)M$ by $(m-1)\rho = m\rho - \rho$, so the estimate is $m(1-\rho) - \rho$.

(3) The method of getting from an archipelago A to $\eta(A)$, described earlier, consists of using one green arrow, one black arrow, and perhaps a balloon. Therefore, its cost is at most $2M$. This proves (3) in the general case. For L_{or} the estimate is $\rho + (\rho - 1/2) = 2\rho - 1/2$ (note that this number, and therefore the cost of the trip in question, may be negative). □

Lemma 5.5. *Assume that $x \in P$ is a black point.*

(1) *There is a green point y such that the cost of a trip from x to y is M or less. For L_{or} this cost is negative (and $\rho - 1/2$ or less if x and y lie on opposite sides of the Center).*

(2) *There is a green point z such that the cost of a trip from z to x is $2M$ or less. For L_{or} this cost is ρ or less (and $2\rho - 1/2$ or less if x and z lie on opposite sides of the Center).*

Proof. Suppose there is no green point y such that either $y >_a x$ or $y \geq_a f(x)$. Then $f(x)$ is black, so $f^2(x)$ lies on the same side of the Center as x. By Lemma 5.2, $f^2(x) >_a x$, so the set of those $t \in P$ for which either $t >_a x$ or $t \geq_a f(x)$ is invariant, a contradiction. Therefore, such a green point y has to exist. One can travel from x to y by taking at most one black arrow and perhaps a balloon. The cost of such a trip is at most M. This proves (1) in the general case. For L_{or} it costs $\rho - 1/2 < 0$ to take a black arrow, so the total cost is negative. If x and y lie on opposite sides of the Center, then we have to take a black arrow, so the cost is $\rho - 1/2$ or less.

Assume that $t \in P$ and $f(t)$ is one of the two points of P closest to the Center. If t and $f(t)$ lie on the same side of the Center, then t is green. If they lie on opposite sides of the Center, then t is black, and since P is green, t is the closest to the Center point from its side. In such a case we repeat the above argument with $f(t)$ replaced by t, and since the period of P is larger than 2, we see that t is the image of a green point. Thus, any point of P closest to the Center from its side can be reached from a green point by taking a green arrow and perhaps a black one. By completing our trip by balloon we can reach each point of P from a green point paying at most $2M$. This proves (2) in the general case. For L_{or} the maximal cost of such a trip is the cost of taking a green arrow (note that it costs $\rho - 1/2 < 0$ to take a black arrow), which is ρ. If x and z lie on opposite sides of the Center, then we have to take a black arrow, so the cost is $2\rho - 1/2$ or less. □

Proof of Theorem 5.1. We have to prove that any trip within the Green Islands and the Black Sea costs less than $(2n + 5)M$. Note that the cost of a trip depends only on the origin and destination points, so we can arrange the itinerary in the way that makes the estimates simple.

Suppose that the trip begins and ends at green points b and e, respectively. We can arrange it in such a way that we can divide it into parts involving travel within one archipelago and i.a. jumps. If we get to $x <_a e$, then we take a balloon from x to e. With such an itinerary we visit each archipelago at most once. Hence, if we visit k islands and l archipelagos, the cost is at most the sum of maximal costs of travel within each archipelago visited plus $l - 1$ times the cost of i.a. jumps. By Lemma 5.4 (2), the first part of this sum is less than $(3k-l)M$, and by Lemma 5.4 (3) the second part of this sum does not exceed $2(l-1)M$. Hence, the total cost is less than $(3k + l - 2)M$. Each archipelago contains at least one island, so $l \leq k$. Thus $(3k + l - 2)M \leq (4k - 2)M$. As we noticed after the statement of the theorem, $k \leq n/2 + 1$. Therefore, the cost of any trip beginning and ending at green points is less than $(2n + 2)M$.

For L_{or} the first part of the sum is less than $k(1 - \rho) - l\rho$ and the second part does not exceed $(l - 1)(2\rho - 1/2)$ (this may be negative, but we may include it in our estimate since we count only i.a. jumps actually made). Thus, the total cost is less than $k(1-\rho) - l(1/2-\rho) - 2\rho + 1/2$. We visit at least one archipelago, so this cost is less than $k(1-\rho) - (1/2-\rho) - 2\rho + 1/2 = k(1-\rho) - \rho \leq (n/2+1)(1-\rho) - \rho$.

By Lemma 5.5, to estimate the cost of any trip within the Green Islands and the Black Sea, we have to add $3M$ to the above estimate. This completes the proof in the general case. For L_{or} we add ρ to the estimate we got in this case, and we obtain $(n/2 + 1)(1 - \rho)$. □

Remark 5.6. For L_{or}, suppose that the origin and destination points lie on opposite sides of the Center. If during the standard trip described in the proof of Theorem 5.1 we visit islands on both sides of the Center, then $l \geq 2$, so the estimate becomes $k(1 - \rho) - 2(1/2 - \rho) - 2\rho + 1/2 = k(1 - \rho) - 1/2 \leq (n/2 + 1)(1 - \rho) - 1/2$. Otherwise, we have to go to the other side of the Center at the beginning or at the end of the trip, so when applying Lemma 5.5, we use one of the estimates in parentheses. Thus, in the last step of the proof, instead of adding ρ, we add $2\rho - 1/2$. This gives the final estimate $(n/2 + 1)(1 - \rho) + \rho - 1/2$. Since $(n/2 + 1)(1 - \rho) + \rho - 1/2 > (n/2 + 1)(1 - \rho) - 1/2$, we see that in any case when the origin and destination points lie on opposite sides of the Center, the cost of the trip is less than $(n/2 + 1)(1 - \rho) + \rho - 1/2$.

If we want to get estimates for the code L'_{or}, we can use Theorem 5.1 and the fact that $\varphi'_{or} = \varphi_{or} + \zeta - \zeta \circ f$, where ζ is $1/2$ to the right of a and 0 to the left of a.

Proposition 5.7. *Let P be an over-twist cycle of over-rotation number ρ. Then*

$$L'_{or}(y) - L'_{or}(x) < (n/2+1)(1-\rho) + \rho$$

for every $x, y \in P$.

Proof. We have $\psi'_{or} = \psi_{or} - \zeta + \zeta \circ f$, so $L'_{or}(f(x)) - L'_{or}(x) = L_{or}(f(x)) - L_{or}(x) + \zeta(f(x)) - \zeta(x)$ for $x \in P$. By the "telescopic rule" we get $L'_{or}(y) - L'_{or}(x) = L_{or}(y) - L_{or}(x) + \zeta(y) - \zeta(x)$ for $x, y \in P$. If x and y lie on the same side of a, then $\zeta(y) = \zeta(x)$, so by Theorem 5.1 $L'_{or}(y) - L'_{or}(x) < (n/2+1)(1-\rho) \leq (n/2+1)(1-\rho)+\rho$. If they lie on opposite sides of a, then $\zeta(y) - \zeta(x) \leq 1/2$, so by Remark 5.6 $L'_{or}(y) - L'_{or}(x) < (n/2+1)(1-\rho)+\rho-1/2+1/2 = (n/2+1)(1-\rho)+\rho$. ☐

Let us now investigate another special case—of rotation numbers. As noted in Section 2, we may assume that the rotation number ρ of a twist cycle P is less than $1/2$. Then we use the function φ_r that is 0 to the left of a and 1 to the right of a. Instead of ψ we use $\psi_r = \rho - \varphi_r$, where ρ is the rotation number of P. We denote the corresponding code by L_r.

Proposition 5.8. *Let P be a twist cycle of rotation number ρ. Then*

$$L_r(y) - L_r(x) < (n/2+1/2)(1-\rho) + \rho$$

for every $x, y \in P$.

Proof. By Lemma 2.2, all islands are to the left of the Center. Then $\varphi_r = \varphi'_{or}$, so $\psi_r = \psi'_{or}$, and consequently $L_r = L'_{or}$. It remains to notice that since the period of P is larger than 2, the left-most point of P is green. Thus, in this case the number of islands is $(n+1)/2$. Hence, the estimate is the same as in Proposition 5.7, except that $n/2+1$ (the estimate for the number of islands) can be replaced by $n/2+1/2$. ☐

Let us now turn to the investigation of conjugacies between over-twist cycles and cycles of circle rotations. Let P be an over-twist cycle of rotation number p/q with p and q coprime, and let f be a P-monotone map. We want to conjugate P with a cycle of the circle rotation by the angle $2\pi p/q$. If we cut this circle at one point, and rescale the interval obtained in such a way to $[0,1)$ (this is the usual procedure), we get the map $g : [0,1) \to [0,1)$ given by $g(x) = x + p/q \pmod{1}$. Let Q be the orbit of 0 for this map. Then both P and Q are cycles of period q, so we can define a conjugacy Ψ between them in a natural way. We use the code L'_{or} for P, and we normalize it by setting $L'_{or}(b) = 0$ at the point $b \in P$ with the minimal code. Then we set $\Psi(b) = 0$ and extend it to the rest of P by $\Psi(f(x)) = g(\Psi(x))$. Note that if $\Psi(x) = L'_{or}(x) \pmod{1}$ for some $x \in P$, then

$$L'_{or}(f(x)) \equiv L'_{or}(x) + p/q \equiv \Psi(x) + p/q \equiv g(\Psi(x)) \equiv \Psi(f(x)) \pmod{1}.$$

Moreover, $L'_{or}(b) = 0 = \Psi(b)$. By induction we get $\Psi(z) = L'_{or}(z) \pmod{1}$ for every $z \in P$. This is, perhaps, an explanation of why the role of the code is so important in the investigation of over-twist numbers of cycles of interval maps.

Theorem 5.9. *The conjugacy Ψ between the cycles P and Q defined above is piecewise monotone with at most $n+3$ pieces, where n is the modality of P. It is increasing on the pieces to the left of a and decreasing on the pieces to the right of a.*

Proof. By the monotonicity of the code, Ψ is monotone (increasing to the left of a and decreasing to the right of a) on each block of points of P on which the integer part of L'_{or} is constant. Now we use Proposition 5.7. On each side of a we get at most m blocks on which the integer part of L'_{or} is constant, where m is the smallest integer larger than or equal to $(n/2+1)(1-\rho)+\rho = n/2(1-\rho)+1 < n/2+1$. We have $m \leq (n+3)/2$, so overall there are at most $n+3$ pieces of monotonicity of Ψ. $\square$

Remark 5.10. We can get a better estimate in Theorem 5.9 if we take the rotation number ρ of P into account. Note that if ρ is not too small, then the number of pieces of monotonicity of the conjugacy is smaller than the number of pieces of monotonicity of a P-monotone map.

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Department of Mathematics, University of Alabama in Birmingham, University Station, Birmingham, Alabama 35294-2060

E-mail address: ablokh@math.uab.edu

Department of Mathematical Sciences, IUPUI, 402 N. Blackford Street, Indianapolis, Indiana 46202-3216

E-mail address: mmisiure@math.iupui.edu