DENSITY OF ORBITS IN LAMINATIONS AND THE SPACE OF CRITICAL PORTRAITS

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ABSTRACT. Thurston introduced σ_d-invariant laminations (where $\sigma_d(z)$ coincides with $z^d : \mathbb{S} \to \mathbb{S}$, $d \geq 2$). He defined *wandering k-gons* as sets $\mathbf{T} \subset \mathbb{S}$ such that $\sigma_d^n(\mathbf{T})$ consists of $k \geq 3$ distinct points for all $n \geq 0$ and the convex hulls of all the sets $\sigma_d^n(\mathbf{T})$ in the plane are pairwise disjoint. Thurston proved that σ_2 has no wandering k-gons and posed the problem of their existence for σ_d, $d \geq 3$.

Call a lamination with wandering k-gons a *WT-lamination*. Denote the set of cubic critical portraits by $\mathcal{A}_3$. A critical portrait, compatible with a WT-lamination, is called a *WT-critical portrait*; let $\mathcal{WT}_3$ be the set of all of them. It was recently shown by the authors that cubic WT-laminations exist and cubic WT-critical portraits, defining polynomials with *condense* orbits of vertices of order three in their dendritic Julia sets, are dense and locally uncountable in $\mathcal{A}_3$ ($D \subset X$ is *condense in X* if D intersects every subcontinuum of X). Here we show that $\mathcal{WT}_3$ is a dense first category subset of $\mathcal{A}_3$. We also show that (a) critical portraits, whose laminations have a condense orbit in the topological Julia set, form a residual subset of $\mathcal{A}_3$, (b) the existence of a condense orbit in the Julia set J implies that J is locally connected.

1. INTRODUCTION

Let $\mathbb{C}$ be the complex plane and $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the complex sphere. The following result is a special case of a theorem due to Thurston [Thu08].

Theorem 1.1 (No Wandering Vertices for Quadratics). *Let $P(z) = z^2 + c$ be a polynomial with connected Julia set J_P. If $z_0 \in J_P$ is a point such that $J_P \setminus \{z_0\}$ has at least three components, then z_0 is either pre-periodic or eventually maps to the critical point 0.*

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In [BO08] we construct an uncountable family of cubic polynomials P with $z_0 \in J_P$ such that $J_P \setminus \{z_0\}$ has three components and z_0 is neither pre-periodic nor precritical; such a point is called a *wandering vertex*. In [BCO10], we improve on these results by finding a collection of polynomials, dense in the appropriate parameter space, with wandering vertices whose orbits have a property that we call *condensity*.

Definition 1.2. For a topological space X a set $A \subset X$ is *continuumwise dense* (abbreviated *condense*) in X if $A \cap Z \neq \emptyset$ for each non-degenerate continuum $Z \subset X$. A map $f : X \to X$ is also called condense if there exists $x_0 \in X$ such that $\{f^n(x_0) \mid n \geq 0\}$ is condense in X.

It is not hard to see that condensity is much stronger than density. For example, if J is a Julia set from the real quadratic family which is not homeomorphic to an interval, the set of endpoints is dense in J, but not condense. Moreover, in this case the set of transitive points (i.e., points with dense orbit in J) is a subset of the endpoints of J, so such maps are not condense.

To state the results of [BCO11] precisely, we must indicate in which parameter space we are working. Polynomials are naturally associated to *critical portraits*, introduced by Yuval Fisher in his Ph.D. thesis [Fis89]. Let $\sigma_d : \mathbb{S} \to \mathbb{S}$ be the angle d-tupling map $\sigma_d(z) = z^d$. A degree d critical portrait, loosely speaking, is a *maximal* collection $\Theta = \{\Theta_1, \ldots, \Theta_n\}$ of sets of angles in $\mathbb{S}$ which are pairwise disjoint, pairwise unlinked (i.e., having disjoint convex hulls in $\overline{\mathbb{D}}$ when angles are interpreted as points in $\mathbb{S}$), and such that $\sigma_d(\Theta_i)$ is a point for each $\Theta_i \in \Theta$ (it is easy to see that $\sum(|\Theta_i| - 1) = d - 1$).

This notion is used to capture the location of critical points. The set of all critical portraits of degree d is denoted $\mathcal{A}_d$, and is naturally endowed with a topology (see Definition 2.4 for details). We say that a *critical portrait* Θ *corresponds to a polynomial* P with dendritic Julia set if for each $\Theta_i \in \Theta$ there is a distinct critical point $c_i \in J_P$ such that the external rays whose angles are in Θ_i land at c_i (see Section 2.3 for more information). Now we state the main theorem of [BCO10].

Theorem 1.3 ([BCO10]). *$\mathcal{A}_3$ contains a dense locally uncountable set $\{\Theta_\alpha \mid \alpha \in A\}$ of critical portraits such that for each $\alpha \in A$ the following holds:*

- *Θ_α corresponds to a polynomial P_α with dendritic Julia set J_{P_α},*
- *$\{P_\alpha|_{J_{P_\alpha}}\}$ are pairwise non-conjugate, and*
- *J_{P_α} contains a wandering vertex with condense orbit.*

The aim of this paper is to further investigate the notions and objects studied in Theorem 1.3, such as condensity and the set of critical portraits which correspond to polynomials with wandering vertices. To explain our results, we recall constructions from [Kiw04, BCO11]: given a polynomial

P with connected Julia set J_P, one can construct a corresponding locally connected continuum $J \subset \mathbb{C}$ (called a *topological Julia set*) and branched covering map $f : \mathbb{C} \to \mathbb{C}$ (called a *topological polynomial*) so that P is *monotonically semiconjugate* to f (i.e., there exists a monotone map $m : \mathbb{C} \to \mathbb{C}$ such that $m \circ P = f \circ m$) and $J = m(J_P)$. We refer to $f|_J$ as the *locally connected model* of P. It is known [BO10] that in some cases J is a single point.

Let us describe the organization of the paper and the main results. After discussing preliminary notions and history in Section 2, we study properties of condense maps in Section 3. In particular we show in Theorem 3.6 that polynomials which admit condense orbits either in their Julia sets (or in some circumstances their locally connected models) have locally connected Julia sets. In Section 4 we prove that the set of cubic critical portraits corresponding to polynomials with condense orbits in their Julia sets is residual in $\mathcal{A}_3$ (Theorem 4.1), while the set of critical portraits which correspond to polynomials with wandering vertices is meager (Theorem 4.4).

2. Preliminaries

2.1. **Laminations.** In what follows, we parameterize the circle as $\mathbb{S} = \mathbb{R}/\mathbb{Z}$, so the total arclength of $\mathbb{S}$ is 1. The *positive* direction on $\mathbb{S}$ is the counterclockwise direction, and by the arc (p, q) in the circle we mean the positively oriented arc from p to q. A *(strictly) monotone* map $g : (p, q) \to \mathbb{S}$ is a map (strictly) monotone at each point of (p, q) in the sense of positive direction on $\mathbb{S}$. By $\mathrm{Ch}(A)$ we denote the *convex hull* of a set $A \subset \mathbb{C}$ and by $|B|$ we denote the cardinality of the set B.

Laminations are combinatorial structures on the unit circle, introduced by Thurston [Thu08] as a tool for studying individual complex polynomials $P : \mathbb{C}_\infty \to \mathbb{C}_\infty$ and the space of all of them. Let P be a degree d polynomial with a locally connected (and hence connected) Julia set J_P; we will recall how to associate an equivalence relation $\sim_P$ on $\mathbb{S}$ to P, called the d*-invariant lamination generated by* P.

The filled-in Julia set K_P is compact, connected, and full, so its complement $\mathbb{C}_\infty \setminus K_P$ is conformally isomorphic to the open unit disk $\mathbb{D}$. By [Mil06, Theorem 9.5], there is a particular conformal isomorphism $\Psi : \mathbb{D} \to \mathbb{C}_\infty \setminus K_P$ so that Ψ conjugates $\sigma_d(z) = z^d$ on $\mathbb{D}$ to $P|_{\mathbb{C}_\infty \setminus K_P}$ (i.e., $\Psi(z^d) = (P|_{\mathbb{C}_\infty \setminus K_P} \circ \Psi)(z)$ for $z \in \mathbb{D}$). When J_P is locally connected, Ψ extends to a continuous map $\overline{\Psi} : \overline{\mathbb{D}} \to \overline{\mathbb{C}_\infty \setminus K_P}$ which semiconjugates $z \mapsto z^d$ on $\overline{\mathbb{D}}$ to $P|_{\overline{\mathbb{C}_\infty \setminus K_P}}$. Let $\psi : \mathbb{S} \to J_P$ denote the restriction $\overline{\Psi}|_{\mathbb{S}}$. Define the equivalence $\sim_P$ on $\mathbb{S}$ so that $x \sim_P y$ if and only if $\psi(x) = \psi(y)$; this equivalence relation is the aforementioned d-invariant lamination generated by P. The quotient space $\mathbb{S}/\sim_P = J_{\sim_P}$ is homeomorphic to J_P and the *induced map* $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ defined by $f_{\sim_P} = \psi \circ \sigma_d \circ \psi^{-1}$ is conjugate to $P|_{J_P}$.

Kiwi [Kiw04] extended this construction to polynomials P with no irrationally neutral cycles and introduced a similar d-invariant lamination $\sim_P$. Then $J_{\sim_P} = \mathbb{S}/\sim_P$ is locally connected and $P|_{J_P}$ is semi-conjugate to $f_{\sim_P}$ by a *monotone* map $m : J_p \to J_{\sim_P}$, i.e., a map m whose point preimages are connected. This was extended in [BCO11] to all polynomials P with connected J_P. The lamination $\sim_P$ combinatorially describes the dynamics of $P|_{J_P}$.

One can introduce abstract laminations (frequently denoted by $\sim$) as equivalence relations on $\mathbb{S}$ having properties in common with laminations generated by polynomials as above. Consider an equivalence relation $\sim$ on the unit circle $\mathbb{S}$. Equivalence classes of $\sim$ will be called *($\sim$-)classes* and will be denoted by boldface letters. A $\sim$-class consisting of two points is called a *leaf*; a class consisting of at least three points is called a *gap* (this is more restrictive than Thurston's definition in [Thu08]). Fix an integer $d > 1$. Then $\sim$ is said to be a *d-invariant lamination* if:

(E1) $\sim$ is *closed*: the graph of $\sim$ is a closed set in $\mathbb{S} \times \mathbb{S}$;
(E2) $\sim$-classes are *pairwise unlinked*: if $\mathbf{g}_1$ and $\mathbf{g}_2$ are distinct $\sim$-classes, then their convex hulls $\mathrm{Ch}(\mathbf{g}_1), \mathrm{Ch}(\mathbf{g}_2)$ in the unit disk $\mathbb{D}$ are disjoint;
(E3) $\sim$-classes are either *totally disconnected* (and hence $\sim$ has uncountably many classes) or the entire circle $\mathbb{S}$ is one class;
(D1) $\sim$ is *forward invariant*: for a class $\mathbf{g}$, the set $\sigma_d(\mathbf{g})$ is also a class;
(D2) $\sim$ is *backward invariant*: for a class $\mathbf{g}$, its preimage $\sigma_d^{-1}(\mathbf{g}) = \{x \in \mathbb{S} : \sigma_d(x) \in \mathbf{g}\}$ is a union of classes; and
(D3) for any gap $\mathbf{g}$, the map $\sigma_d|_{\mathbf{g}} : \mathbf{g} \to \sigma_d(\mathbf{g})$ is a *covering map with positive orientation*, i.e., for every connected component (s, t) of $\mathbb{S} \setminus \mathbf{g}$ the arc $(\sigma_d(s), \sigma_d(t))$ is a connected component of $\mathbb{S} \setminus \sigma_d(\mathbf{g})$.

Notice that (D2) and (E3) follow from (D1).

Call a class $\mathbf{g}$ *critical* if $\sigma_d|_{\mathbf{g}} : \mathbf{g} \to \sigma_d(\mathbf{g})$ is not one-to-one, and *precritical* if $\sigma_d^j(\mathbf{g})$ is critical for some $j \geq 0$. Call $\mathbf{g}$ *preperiodic* if $\sigma_d^i(\mathbf{g}) = \sigma_d^j(\mathbf{g})$ for some $0 \leq i < j$. A gap $\mathbf{g}$ is *wandering* if $\mathbf{g}$ is neither preperiodic nor precritical. Let $J_\sim = \mathbb{S}/\sim$, and let $\pi_\sim : \mathbb{S} \to J_\sim$ be the corresponding quotient map. The map $f_\sim : J_\sim \to J_\sim$ defined by $f_\sim = \pi_\sim \circ \sigma_d \circ \pi_\sim^{-1}$ is the map *induced* on $J_\sim$ by σ_d. Then we call $f_\sim$ a *topological polynomial*, and $J_\sim$ a *topological Julia set*.

2.2. **Bounds for wandering classes.** J. Kiwi [Kiw02] extended the No Wandering Triangles Theorem by showing that a wandering gap in a d-invariant lamination is at most a d-gon. Thus all infinite $\sim$-classes (and

Jordan curves in $J_\sim$) are preperiodic. In [Lev98] G. Levin showed that laminations with one critical class have no wandering gaps. For a lamination $\sim$, let $k_\sim$ be the size of a maximal collection of non-degenerate $\sim$-classes whose σ_d-images are points and whose orbits are infinite and pairwise disjoint. Also, let $N_\sim$ be the number of cycles of infinite $\sim$-classes plus the number of cycles of Jordan curves in $J_\sim$.

Theorem 2.1 ([BL02]). *Let $\sim$ be a d-invariant lamination and let Γ be a* non-empty *collection of wandering d_j-gons ($j = 1, 2, \dots$) with distinct grand orbits. Then $\sum_j (d_j - 2) \le k_\sim - 1$ and $\sum_j (d_j - 2) + N_\sim \le d - 2$. In particular, in the cubic case if Γ is non-empty, then it must consist of one non-precritical $\sim$-class with three elements, all $\sim$-classes are finite, $J_\sim$ is a dendrite, and both critical classes are leaves with disjoint forward orbits.*

2.3. **Critical portraits.** Following [Thu08] and [DH84] we look at the set $\mathcal{C}_d$ from infinity and consider the *shift locus*, which is the set $\mathcal{S}_d$ of polynomials whose critical points escape to infinity. The set $\mathcal{S}_d$ is the unique hyperbolic component of $\mathcal{P}_d$ consisting of polynomials with all cycles repelling. It is not known if all polynomials with all cycles repelling belong to the set $\overline{\mathcal{S}_d}$. Looking at $\mathcal{C}_d$ from infinity means studying locations of polynomials in $\mathcal{S}_d$ depending on their dynamics and using this to describe the polynomials belonging to $\overline{\mathcal{S}_d} \cap \mathcal{C}_d$. A key tool in studying $\mathcal{C}_d$ is *critical portraits*, introduced in [Fis89], and widely used afterward (see, e.g., [BFH92, Poi93, GM93] and [Kiw05]). We now recall some standard material; here we closely follow [Kiw05, Section 3]. Call a chord with endpoints $a, b \in \mathbb{S}$ *critical* if $\sigma_d(a) = \sigma_d(b)$.

Definition 2.2. A **critical portrait** is a collection $\Theta = \{\Theta_1, \dots, \Theta_n\}$ of finite subsets of $\mathbb{S}$ such that the following hold:

(1) the boundary of the convex hull $\mathrm{Ch}(\Theta_i)$ of every set Θ_i consists of critical chords (under σ_d);
(2) the sets $\Theta_1, \dots, \Theta_n$ are **pairwise unlinked** (that is, convex hulls of the sets Θ_i are pairwise disjoint); and
(3) $\sum(|\Theta_i| - 1) = d - 1$.

The sets $\Theta_1, \dots, \Theta_n$ are called the *initial sets of* Θ (or Θ-*initial sets*). Denote by $A(\Theta)$ the union of all angles from the initial sets of Θ. The convex hulls of the Θ-initial sets divide the rest of the open unit disk into components. In Definition 2.3, points of $\mathbb{S} \setminus A(\Theta)$ are declared equivalent if they belong to the boundary of one such component; we do not assume that Θ is a critical portrait because we need this equivalence later in a more general situation.

Definition 2.3. Let Θ be a finite collection of pairwise unlinked finite subsets of $\mathbb{S}$. Angles $\alpha, \beta \in \mathbb{S} \setminus A(\Theta)$ are **Θ-unlinked equivalent** if $\{\alpha, \beta\}$, Θ_1,

$\dots$, Θ_n are pairwise unlinked. The equivalence classes $L_1(\Theta), \dots, L_d(\Theta)$ are called **Θ-unlinked classes**. Each Θ-unlinked class L is the intersection of $\mathbb{S} \setminus A(\Theta)$ with the boundary of a component of $\mathbb{D} \setminus \bigcup \mathrm{Ch}(\Theta_i)$. In the degree d case, each Θ-unlinked class of a critical portrait Θ is the union of finitely many *open* arcs of total length $1/d$. Thus, there are d Θ-unlinked classes.

Definition 2.4 (compact-unlinked topology [Kiw05])**.** Define the space $\mathcal{A}_d$ as the set of all critical portraits endowed with the **compact-unlinked** topology generated by the subbasis $V_X = \{\Theta \in \mathcal{A}_d : X \subset L_\Theta\}$ where $X \subset \mathbb{S}$ is closed and L_Θ is a Θ-unlinked class.

Note for example that $\mathcal{A}_2$ is the quotient of $\mathbb{S}$ with antipodal points identified, so it is homeomorphic to the unit circle. For a critical portrait Θ, a lamination $\sim$ is called Θ-*compatible* if all Θ-initial sets are contained in $\sim$-classes; if there is a Θ-compatible WT-lamination, Θ is said to be a *WT-critical portrait*. The trivial lamination, identifying all points of $\mathbb{S}$, is compatible with any critical portrait.

To define *critical portraits with aperiodic kneading*, let us introduce the notion of a *one-sided itinerary* for $t \in \mathbb{S}$ (see [Kiw05]). Given a critical portrait $\Theta = \{\Theta_1, \dots, \Theta_d\}$ with Θ-unlinked classes $L_1(\Theta), \dots, L_d(\Theta)$ and $\theta \in \mathbb{S}$, define $i^+(\theta)$ (respectively, $i^-(\theta)$) as the sequence $(i_0, i_1, \dots)$, with $i_j \in \{1, \dots, d\}$ such that there are $y_n \searrow \theta$ (respectively, $y_n \nearrow \theta$) with $\sigma_d^j(y_n) \in L_{i_j}(\Theta)$ for n sufficiently large. Also, define the itinerary $i(\theta)$ as a sequence $I_0 I_1 \dots$ such that each I_j is the set from $\Theta \cup \{L_1(\Theta), \dots, L_d(\Theta)\}$ to which $\sigma^j(\theta)$ belongs. An angle $\theta \in \mathbb{S}$ is said to have a *periodic kneading* if $i^+(\theta)$ or $i^-(\theta)$ is periodic. A critical portrait Θ is said to have *aperiodic kneading* if no angle from $A(\Theta)$ has periodic kneading. The family of all degree d critical portraits with aperiodic kneading is denoted by $\mathcal{AP}_d$.

Definition 2.5 ([Kiw04, Kiw05])**.** The lamination $\sim_\Theta$ is the smallest closed equivalence relation identifying any pair of points $x, y \in \mathbb{S}$ where $i^+(x) = i^-(y)$. By Kiwi [Kiw04, Kiw05], for any critical portrait Θ the relation $\sim_\Theta$ is a Θ-compatible lamination; it is said to be *generated* by Θ.

Critical portraits reflect the landing patterns of the external rays at the critical points. By Kiwi [Kiw05], a nice correspondence between critical portraits of degree d and the set $\overline{\mathcal{S}_d} \cap \mathcal{C}_d$ associates to each critical portrait $\Theta \in \mathcal{A}_d$ a connected set $I(\Theta) \subset \overline{\mathcal{S}_d} \cap \mathcal{C}_d$, called the *impression of* Θ, such that the dynamics of a polynomial in $I(\Theta)$ is closely related to the properties of Θ. The relation is especially nice when Θ has aperiodic kneading. The following fundamental result of Kiwi [Kiw04, Kiw05] explicitly lists properties of critical portraits with aperiodic kneading and their connections to polynomials.

Theorem 2.6. *Let $\Theta \in \mathcal{AP}_d$. Then $\sim_\Theta$ is the unique Θ-compatible invariant lamination. The quotient $J_{\sim_\Theta}$ is a non-degenerate dendrite, and all $\sim$-classes are finite. Furthermore, there exists a polynomial P whose Julia set J_P is a non-separating continuum in the plane and $P|_{J_P}$ is monotonically semiconjugate to $f_{\sim_\Theta}|_{J_{\sim_\Theta}}$. The semiconjugating map $m_{\Theta,P} = m : J_P \to J_{\sim_\Theta}$ maps impressions of external angles to points and maps the set of P-preperiodic points in J_P bijectively to the set of $f_{\sim_\Theta}$-preperiodic points. Moreover, J_P is locally connected at all P-preperiodic points.*

In the situation of Theorem 2.6 polynomials P such that $P|_{J_P}$ is monotonically semiconjugate to $f_{\sim_\Theta}|_{J_{\sim_\Theta}}$ are said to be *associated to the critical portrait* Θ.

2.4. **Monotone models for connected Julia sets.** As was explained in Section 1, the main results of [Kiw04, BCO11] yield a locally connected model for the restriction of a polynomial to its connected Julia set. We will need a detailed version of these results stated below in Theorem 2.7.

Theorem 2.7 ([Kiw04, BCO11])**.** *Let P be a degree d polynomial with connected Julia set J_P. Then there exists a d-invariant lamination $\sim$ and a monotone onto map $M_P : \mathbb{C} \to \mathbb{C}$ with the following properties.*

(1) *$J_\sim = M_P(J_P)$ and $J_P \subset M_P^{-1}(J_\sim) \subset K_P$.*
(2) *M_P sends impressions of J_P to points.*
(3) *$m_P = M_P|_{J_P}$ is the finest monotone map of J_P onto a locally connected continuum (i.e., if $\psi : J_P \to T$ is a monotone map onto a locally connected continuum T, then there is a monotone map $\psi' : J_\sim \to T$ such that $\psi = \psi' \circ m_P$).*
(4) *M_P semiconjugates P to a branched covering map $g_P : \mathbb{C} \to \mathbb{C}$ under which $J_\sim$ is fully invariant so that $g_P|_{J_\sim}$ is conjugate to the topological polynomial $f_\sim$.*

Remark 2.8. Suppose that $\Theta \in \mathcal{AP}_d$ is associated to the polynomial P; let us show that the lamination $\sim_\Theta$ defined in Theorem 2.6 and the lamination $\sim_P$ defined in Theorem 2.7 coincide. Indeed, by Theorem 2.7 there exists a monotone map $\psi' : J_{\sim_P} \to J_{\sim_\Theta}$. If this map is not a homeomorphism, it will collapse a non-degenerate subcontinuum $Q \subset J_{\sim_P}$ to a point $x \in J_{\sim_\Theta}$. Since impressions map to points of $J_{\sim_P}$, infinitely many distinct impressions of external rays are contained in the fiber $m_{\Theta,P}^{-1}(x)$ which by Theorem 2.6 implies that the $\sim_\Theta$-class corresponding to x is infinite. This contradicts Theorem 2.6, which states that $\sim_\Theta$-classes are finite.

Theorem 2.7 establishes the semiconjugacy m_P on the *entire complex plane*, so that m_P-images of external rays to J_P are curves in $\mathbb{C}$ accumulating on points of $J_{\sim_P}$. For $x \in J_{\sim_P}$, the set $m_P^{-1}(x) \cap J_P$ is the union of impressions of angles α such that $m_P(R_\alpha)$ lands on x . The *order* of x in $J_{\sim_P}$ is the number of components of $J_{\sim_P} \setminus \{x\}$ and can be either a finite number or infinity. By Theorem 2.7 if the order of x in $J_{\sim_P}$ is finite then it equals the number of angles with impressions in $m_P^{-1}(x)$ (or equivalently the number of angles whose impressions intersect $m_P^{-1}(x)$). If the order of x in $J_{\sim_P}$ is infinite, then there are infinitely many angles with impressions in $m_P^{-1}(x)$.

3. CONDENSITY

We begin with a few lemmas concerning the dynamics of a condense topological polynomial. If J is a dendrite, by $[a, b]_J$ we mean the unique arc in J connecting the points $a, b \in J$. A continuum $X \subset \mathbb{C}$ is called *unshielded* if it coincides with the boundary of the unique unbounded component of $\mathbb{C} \setminus X$. Note that all connected Julia sets of polynomials and all topological Julia sets are unshielded continua. A point $x \in X$ is called a *cutpoint* of X if $X \setminus \{x\}$ is not connected. In what follows a lamination $\sim$ such that $f_\sim$ is condense is called *condense*; also, a critical portrait compatible with a condense lamination is said to be *condense*.

Lemma 3.1. *If $X \subset \mathbb{C}$ is an unshielded locally connected continuum and $A \subset X$ is connected and dense in X, then A is condense in X and contains all cutpoints of X.*

Proof. If $Z \subset X$ is a closed set with $X \setminus Z$ disconnected, then all components of $X \setminus Z$ are open. Hence all such components intersect A. Since A is connected, this implies that $A \cap Z \neq \emptyset$. Suppose that A is not condense in X. Then there exists an arc $I \subset X$ disjoint from A. Note that $X \setminus I$ is open and connected (by virtue of containing A). Therefore $X \setminus I$ is path connected and locally path connected. It follows that there exists a simple closed curve $S \subset X$ which contains a non-degenerate subsegment I' with endpoint a', b' of I. The curve S encloses a topological disk U. Clearly, any two-point set $\{a, b\} \subset S$ separates X (two external rays landing at a and b and an arc inside U from a to b disconnect $\mathbb{C}$). Hence $A \cap \{a', b'\} \neq \emptyset$, a contradiction.

□

Let us now study condensity in the context of laminations. We call a lamination $\sim$ *degenerate* if the whole $\mathbb{S}$ forms a $\sim$-class (and so $J_\sim$ is a point); we call $\sim$ *trivial* if all $\sim$-classes are singletons (and $J_\sim = \mathbb{S}$).

Lemma 3.2. *Let $\sim$ be a condense lamination. Then either $\sim$ is degenerate, or $\sim$ is trivial, or $J_\sim$ is a dendrite.*

Proof. Suppose $J_\sim$ is non-degenerate and let $x \in J_\sim$ be a point with condense orbit. If $J_\sim$ is not a dendrite, then it contains a Jordan curve. By [BL02] it follows that $J_\sim$ contains a periodic Jordan curve B of period, say, k. Since x must enter B, it follows that the union of $\bigcup_{i=1}^{k} f_\sim^i(B) = J_\sim$. Since $J_\sim$ is a topological Julia set, it is easy to see that then $J_\sim$ is the unit circle and the lamination $\sim$ is trivial. □

Lemma 3.3. *Suppose that $K \subset J_\sim$ is a continuum with dense orbit and that $f^n(K) \cap K \neq \emptyset$. If $t \geq 0$ is an integer, the union $\bigcup_{j=0}^{\infty} f_\sim^{nj+t}(K)$ is a condense connected subset of J containing all cutpoints of $J_\sim$. Further, if $f^n(K) \subset K$, then $K = J_\sim$.*

Observe that in this lemma we do not assume that f is condense.

Proof. Under the hypotheses, $A_0 = \bigcup f_\sim^{nk}(K)$ is a connected subset of $J_\sim$, and so are the sets $A_l = \bigcup f_\sim^{nk+l}(K)$ where $1 \leq l \leq n-1$. By the assumption, the union $A = \cup_{l=0}^{n-1} A_l$ is dense in J. Observe that $f_\sim(A_l) \subset A_{l+1}$, where indices are interpreted modulo n.

Since $\bigcup_{l=0}^{n-1} \overline{A_l} = J_\sim$ it follows from the Baire Category Theorem that some $\overline{A_s}$ contains an open subset of $J_\sim$. Since $f_\sim$ eventually maps any open set onto $J_\sim$, it follows that $f_\sim^r(\overline{A_s}) = J_\sim$ for some $r \geq 0$. Hence, for all $i \geq 0$, $f_\sim^{r+i}(\overline{A_s}) = \overline{A_{s+i}} = J_\sim$, and so for any t the set A_t is connected and dense in $J_\sim$. Then Lemma 3.1 implies that A_t is condense and contains all cutpoints of $J_\sim$.

In the case that $f_\sim^n(K) \subset K$, it follows that $A_0 \subset K$; that K is closed and A_0 is dense implies that $K = J_\sim$. □

The next lemma shows that condense maps resemble transitive maps. Recall that any topological polynomial on a dendrite must have fixed cutpoints (see, e.g., [Thu08, BFMOT11]).

Lemma 3.4. *For any topological polynomial $f_\sim$, the following claims are equivalent.*

1. *$f_\sim$ is condense.*
2. *The orbit of every continuum $K \subset J_\sim$ is dense.*
3. *The orbit of every interval $I \subset J_\sim$ is dense.*
4. *There are no proper periodic continua in $J_\sim$.*

Moreover, if these conditions are satisfied, then the set of all points with condense orbits is residual in every interval $I \subset J_\sim$.

Proof. Since every subcontinuum of $J_\sim$ contains an interval, it is clear that (3) and (2) are equivalent. If a point $x \in J_\sim$ has condense orbit and $K \subset J_\sim$ is a continuum, then x must enter K, and the orbit of K is dense. This shows that (1) implies (2). Moreover, by Lemma 3.3, (1) implies (4).

Let us show that (2) and (4) are equivalent. Suppose that (2) holds and let K be a periodic continuum K. Then K has to have a dense orbit which by Lemma 3.3 implies that $K = J_\sim$. Suppose that (4) holds and let $L \subset J_\sim$ be a continuum. By [BL02] there exist m and $n > 0$ such that $f_\sim^m(L) \cap f_\sim^{m+n}(L) \neq \emptyset$. Then the set $\overline{\bigcup_{i=0}^\infty f_\sim^{m+ni}(L)} = T$ is a periodic continuum which by the assumption coincides with $J_\sim$. Hence L has a dense orbit as desired.

Let us show that (2) implies (1). If $J_\sim$ has a bounded complementary domain U, then we may assume that $\mathrm{Bd}(U)$ is periodic. By Lemma 3.3 we conclude that $\mathrm{Bd}(U) = J_\sim$, so $f_\sim$ is conjugate to $z \mapsto z^d$ and condense. Therefore we may assume that $J_\sim$ is a dendrite. Let $\{A_i \mid i \geq 0\}$ be a countable collection of closed arcs such that any continuum $K \subset J_\sim$ contains some A_s. For convenience, we choose the sequence $\{A_i\}$ so that no element of the sequence contains an endpoint of $J_\sim$.

Let $I \subset J_\sim$ be an arc; we will show for each $s \geq 0$ that $B_s = \{x \in I \mid f_\sim^k(x) \in A_s \text{ for some } k\}$ is an open and dense subset of I. Let α denote a fixed cutpoint of $J_\sim$. It follows that, for i sufficiently large, $\alpha \in f^i(I)$. This is because no subcontinuum of $J_\sim$ is wandering, i.e., there exists s, n such that $f_\sim^s(I) \cap f_\sim^{s+n}(I) \neq \emptyset$ [BL02]. By Lemma 3.3, for some $M \geq 0$ we have $\alpha \in f^{s+Mn}(I)$; since α is fixed, $\alpha \in f^i(I)$ for all $i \geq s + Mn$.

There exist components K of $J_\sim \setminus A_s$ such that every arc intersecting K and containing α also contains a subinterval of A_s. Since every continuum in J has a dense orbit, there exists $k \geq 0$ such that $\alpha \in f_\sim^k(I)$ and $f_\sim^k(I) \cap K \neq \emptyset$. Hence, $f_\sim^k(I)$ intersects A_s in an open subset. Since $f_\sim^k$ is finite-to-one, this implies that an open subset of I maps into A_s. Since we can repeat this argument on any subinterval of I, B_s is a dense open subset of I.

By the Baire Category Theorem, $\bigcap_{s \geq 0} B_s$ is then a residual (and hence non-empty) subset of I; this is the set of points in I which eventually map into each A_s, and therefore into every subcontinuum of $J_\sim$ as desired. □

Powers of condense maps are condense, too.

Lemma 3.5. *If $f_\sim$ is condense and $s \geq 1$, then $f_\sim^s$ is condense.*

Proof. By Lemma 3.4 we need to show that any continuum $K \subset J_\sim$ has dense $f_\sim^s$-orbit in $J_\sim$. By Lemma 3.2 we only need to consider the case that $J_\sim$ is a dendrite. Let $\alpha \in J_\sim$ be a fixed cutpoint. By Lemma 3.3 there exists $i \geq 0$ such that $\alpha \in f_\sim^i(K)$; since α is fixed we may assume that $i = ks$ for some integer k. Clearly, $(f_\sim^s)^{k+1}(K) \cap (f_\sim^s)^k(K) \neq \emptyset$, since it contains α. By Lemma 3.3, $\bigcup_{j=0}^\infty f_\sim^{js}(f_\sim^{ks}(K))$ is a connected condense subset of J, so the $f_\sim^{ks}$-orbit of K is condense. Since K was an arbitrary continuum in $J_\sim$, $f_\sim^s$ is condense by Lemma 3.4. □

Theorem 3.6. *Let P be a polynomial with connected Julia set. Then the following claims hold.*

(1) *Suppose that the finest model $J_\sim$ of J_P, given by a lamination $\sim$, is non-degenerate, all points of $J_\sim$ are of finite order, and $f_\sim$ is condense. Then J_P is locally connected and $P|_{J_P}$ is conjugate to $f_\sim$.*

(2) *Suppose that $P|_{J_P}$ is condense. Then P has no proper periodic subcontinua (in particular, P is non-renormalizable), J_P is locally connected and P is conjugate to g_P from Theorem 2.7.*

Observe, that by this theorem $P|_{J_P}$ satisfies Lemmas 3.2 - 3.5. Observe also, that by Theorem 2.6 (1) holds for polynomials associated with condense critical portraits having aperiodic kneading.

Proof. (1) Let $m : J_P \to J_\sim$ be the finest monotone map to a locally connected continuum defined in Theorem 2.7. Since the order of any periodic point $p \in J_\sim$ is finite, by [BCO11, Lemma 37] the set $m^{-1}(p)$ is a repelling or parabolic periodic point. Hence, P has no Cremer points: if U were a periodic Siegel domain of P, then $m(\mathrm{Bd}(U))$ would be a periodic subcontinuum of $J_\sim$ homeomorphic to a circle on which the appropriate power of the map is an irrational rotation, and hence a proper subcontinuum.

Now we show that P is non-renormalizable. Indeed, if P is renormalizable, then there exists a polynomial-like connected Julia set $J' \subsetneqq J_P$ which is a periodic continuum. If $m(J')$ is a point, then it is periodic and again by [BCO11, Lemma 37] the set $m^{-1}(m(J'))$ is a point, a contradiction. Hence $m(J')$ is a periodic continuum in $J_\sim$. Clearly, $m(J') \neq J_\sim$. This contradicts Lemma 3.3 and Lemma 3.4 and shows that P is non-renormalizable. Hence J_P is locally connected [KvS09]. By Theorem 2.7, $P|_{J_P}$ and $f_\sim$ are conjugate as required.

(2) Assume now that $P|_{J_P}$ is condense. Let us show that J_P has no proper periodic subcontinua. Indeed, let $A \subset J_P$ be a periodic continuum. Then the (finite) union B of its images must coincide with J_P (because $P|_{J_P}$ is condense). As at least one of these images must have non-empty interior, A must coincide with J_P.

This fact has several consequences. To begin with, let us show that J_P cannot have Cremer points. Indeed, suppose that $z_0 \in J_P$ is a periodic Cremer point of period p. Then, for any small neighborhood U of z_0, the component of the set $\{z \mid P^{kp}(z) \in \overline{U} \text{ for all } k\}$ containing z_0, called a *hedgehog*, is a proper periodic subcontinuum of J_P [PM97], contradicting that $P|_{J_P}$ has no proper periodic subcontinuum.

Now let us show that P cannot have Siegel domains either. Since J_P contains no proper periodic subcontinua, then any periodic Siegel domain U of P must be such that $\mathrm{Bd}(U) = J_P$. By J. Rogers' result [Rog92], there are two cases. In the first case, $P|_{\mathrm{Bd}(U)}$ is monotonically semiconjugate to an irrational rotation which contradicts the fact that $\mathrm{Bd}(U) = J_P$. In the second case, $\mathrm{Bd}(U)$ is an *indecomposable* continuum (i.e., cannot be represented as $A \cup B$ where A and B are proper subcontinua of $\mathrm{Bd}(U)$). Then, given a point $x \in \mathrm{Bd}(U)$, one can define the *composant* of x in $\mathrm{Bd}(U)$, that is the union of all proper subcontinua of $\mathrm{Bd}(U)$ containing x. Then it is known [Nad92, Theorem 11.15] that distinct non-degenerate composants of $\mathrm{Bd}(U)$ are pairwise disjoint and there are uncountably many of them. Since the orbit of x can only enter countably many composants of $\mathrm{Bd}(U)$, we have a contradiction with the assumption that $P|_{J_P}$ is condense. Hence, P does not have Siegel domains.

Since J_P has no proper periodic subcontinua, P is non-renormalizable. Thus, as before, all this implies that J_P is locally connected [KvS09]. The rest follows from Theorem 2.7. □

4. Family of critical WT-portraits

First we show that condense laminations are residual in $\mathcal{A}_3$.

Theorem 4.1. *Let $\Theta \in \mathcal{A}_d$ be a critical portrait which consists of $d-1$ critical chords whose orbits are dense in $\mathbb{S}$. Then Θ has aperiodic kneading, is condense, and any polynomial P associated to Θ has locally connected Julia set J_P so that $P|_{J_P}$ is conjugate to $f_{\sim_\Theta}|_{J_{\sim_\Theta}}$.*

Proof. Let us show that Θ has aperiodic kneading. Indeed, the orbit of any critical leaf ℓ comes arbitrarily close to the fixed point 0. Hence, if C is the Θ-unlinked class of 0, then the itinerary of ℓ includes arbitrarily long segments consisting of C. This implies that Θ has aperiodic kneading, and Theorem 2.6 applies. Let $\sim$ denote the lamination generated by Θ.

Let us show that Θ is condense. Take an arc $I \subset J_\sim$ and consider its orbit. By [BL02] there are positive numbers m, k with $f^m_\sim(I) \cap f^{m+k}_\sim(I) \neq 0$. Consider the connected set $A_0 = \bigcup_{i=0}^{\infty} f^{m+ki}_\sim(I)$. Clearly, $\overline{A_0} = B \subset J_\sim$ is a subdendrite of $J_\sim$ and $f^k_\sim(B) \subset B$. Let us show that $f^k_\sim|_B$ has a critical point c. Indeed, by Theorem 7.2.6 of [BFMOT11] there are infinitely many periodic cutpoints of $f^k_\sim|_B$; let $Q \subset B$ be an arc joining some pair x and y of such periodic cutpoints. If $f^k_\sim|_B$ has no critical points, then some power of $f^k_\sim|_Q$ is a homeomorphism and there must exist a point $z \in Q$ attracting for g from at least one side, which is impossible. Hence, B contains a critical point of $f^k_\sim$. By the assumptions on Θ, B contains a point with dense orbit, so I has a dense orbit. Since I was arbitrary, we conclude by Lemma 3.4 that Θ is condense.

Since Θ satisfies Theorem 2.6, and since $\sim = \sim_P$ by Remark 2.8, it follows from Theorem 3.6 (1) that $J_\sim$ is locally connected and that $P|_{J_P}$ is conjugate to $f_\sim$.

□

Since the set of critical portraits consisting of $d-1$ critical leaves with dense orbits in $\mathbb{S}$ is residual in $\mathcal{A}_d$, we obtain the following corollary.

Corollary 4.2. *A residual subset of critical portraits in $\mathcal{A}_d$ correspond to polynomials whose restrictions to their Julia sets are condense.*

Recall that a lamination with wandering k-gons ($k \geq 3$) is called a *WT-lamination*. A critical portrait, compatible with a WT-lamination, is called a *WT-critical portrait*; $\mathcal{WT}_3$ is the set of all cubic WT-critical portraits. By Theorem 1.3, $\mathcal{WT}_3$ is a dense and locally uncountable subset of $\mathcal{A}_3$.

Now we show that $\mathcal{WT}_3$ is a meager subset of $\mathcal{A}_3$. We will do so by showing that the set of critical portraits in $\mathcal{WT}_3$ compatible with a wandering triangle of area at least $\frac{1}{n}$ is disjoint from a particular dense subset of critical portraits. The dense subset we consider, called $\mathcal{K}$, is the set of critical portraits consisting of two leaves $\{\mathbf{c}, \mathbf{d}\} \in \mathcal{A}_3$ such that the orbits of $\mathbf{c}$ and $\mathbf{d}$ are dense, neither $\mathbf{c}$ nor $\mathbf{d}$ maps to an endpoint of the other, and $\mathbf{c}$ and $\mathbf{d}$ eventually map to the same point.

Lemma 4.3. *The set $\mathcal{K}$ is dense in $\mathcal{A}_3$. All orbit portraits $\Theta \in \mathcal{K}$ have aperiodic kneading. The critical classes of the lamination $\sim_\Theta$ generated by Θ are leaves.*

Proof. The fact that $\mathcal{K}$ is dense in $\mathcal{A}_3$ is easy and left to the reader. Consider some $\Theta = \{\mathbf{c}, \mathbf{d}\} \in \mathcal{K}$. By Theorem 4.1, Θ has aperiodic kneading. Let $\mathbf{g}$ be the critical $\sim_\Theta$-class containing $\mathbf{c}$, and $\mathbf{h}$ the critical class containing $\mathbf{d}$. It is easy to see that if $\mathbf{g}$ contains at least three points, then $|\sigma(\mathbf{g})| \geq 2$. Indeed, consider two cases. If $\mathbf{g}$ maps to its image in the two-to-one fashion, then $|\sigma(\mathbf{g})| \geq 2$ is obvious. If $\mathbf{g}$ maps to its image in the three-to-one fashion then $\mathbf{g} = \mathbf{h}$ contains four endpoints of the leaves $\mathbf{c}$ and $\mathbf{d}$, so again $|\sigma(\mathbf{g})| \geq 2$. Similarly, if $|\mathbf{h}| \geq 3$ then $|\sigma(\mathbf{h})| \geq 2$.

Suppose for contradiction that $\mathbf{g}$ contains at least three points. We will first show that then all forward images of all critical classes of $\sim_\Theta$ are non-degenerate. Indeed, note that neither $\mathbf{g}$ nor $\mathbf{h}$ may eventually map onto itself, since the orbits of $\mathbf{c}$ and $\mathbf{d}$ are dense in $\mathbb{S}$. This further implies that, if $\mathbf{g}$ maps onto $\mathbf{h}$, then $\mathbf{h}$ cannot map onto $\mathbf{g}$. We consider three cases.

(1) Suppose that $\mathbf{g} = \mathbf{h}$. Since $|\sigma(\mathbf{g})| \geq 2$ and $\mathbf{g}$ is not periodic, it is not pre-critical, so $|\sigma^l(\mathbf{g})| = |\sigma^l(\mathbf{h})| \geq 2$ for all $l \geq 0$.
(2) Suppose that $\sigma^k(\mathbf{g}) = \mathbf{h}$ for some $k \geq 1$. Since $|\sigma(\mathbf{g})| \geq 2$ and $\mathbf{c}$ never maps into $\mathbf{d}$, we see that $\mathbf{h}$ contains at least three points (the endpoints of $\mathbf{d}$ and the point $\sigma^k(\mathbf{c})$). Therefore by the above $|\sigma(\mathbf{h})| \geq 2$. As noted before, $\mathbf{h}$ is not pre-critical, so $|\sigma^k(\mathbf{g})|$ and $|\sigma^k(\mathbf{h})|$ are both at least two for all k.

(3) If $\mathbf{g}$ never maps onto $\mathbf{h}$, then $|\sigma^k(\mathbf{g})| \geq 2$ for all k, since $\mathbf{g}$ is not pre-critical and contains at least three points. Since $\mathbf{c}$ and $\mathbf{d}$ have a common image, so do g and $\mathbf{h}$, and $|\sigma^k(\mathbf{h})| \geq 2$ for all k.

We will use the metric where the distance between two points on $\mathbb{S}$ is the length of the shortest arc in $\mathbb{S}$ joining them. By the diameter of a chord we will mean the distance between its endpoints. Let us show that $\operatorname{diam}(\sigma^k(\mathbf{g}))$ is bounded away from 0. It is easy to see that, for any chord ℓ',

$$\operatorname{diam}(\sigma(\ell')) = \begin{cases} 3\operatorname{diam}(\ell') & \text{if } \operatorname{diam}(\ell') \leq 1/6 \\ 3|\operatorname{diam}(\ell') - 1/3| & \text{if } 1/6 \leq \operatorname{diam}(\ell'). \end{cases} \tag{4.1}$$

This implies that $\operatorname{diam}(\sigma(\ell')) \geq \operatorname{diam}(\ell')$ if and only if $\operatorname{diam}(\ell') \leq 1/4$. Hence, every class of diameter less than $1/4$ maps to a class of larger diameter. Let ℓ be the chord on $\operatorname{Bd}(\operatorname{Ch}(\mathbf{g})) \cup \operatorname{Bd}(\operatorname{Ch}(\mathbf{h}))$ of length closest to $1/3$; since $\sigma(\mathbf{g})$ and $\sigma(\mathbf{h})$ are non-degenerate, $\varepsilon = |\operatorname{diam}(\ell) - 1/3|$ is positive. Since $\sim$-classes are unlinked, $|\operatorname{diam}(\ell') - 1/3| \geq \varepsilon$ for any chord ℓ' from the boundary of the convex hull of a $\sim$-class. Hence, by Equation 4.1 no class of diameter at least $1/4$ has an image of diameter less than 3ε. In particular, $\operatorname{diam}(\sigma^k(\mathbf{g})) \geq 3\varepsilon$ for all k.

Since the convex hulls of classes are dense in $\mathbb{D}$, we can choose a class $\mathbf{k}$ so that there exists a component A of $\mathbb{S} \setminus \mathbf{k}$ of diameter less than ε. Since $\operatorname{diam}(\sigma^k(\mathbf{g})) \geq 3\varepsilon$, the orbit of g can never enter A. This contradicts that the orbit of $\mathbf{c}$ is dense. We conclude that the classes $\mathbf{g}$ and $\mathbf{h}$ are leaves. □

Theorem 4.4. *The set $\mathcal{WT}_3$ is of first category in $\mathcal{A}_3$.*

Proof. Let $\mathcal{W}_n$ be the set of critical portraits $\Theta \in \mathcal{WT}_3$ such that there is a $\sim_\Theta$-class $\mathbf{T}$ which is a wandering triangle and $\operatorname{Ch}(\mathbf{T})$ has area at least $1/n$. We will show that $\mathcal{W}_n$ is nowhere dense by showing that $\overline{\mathcal{W}_n} \cap \mathcal{K} = \emptyset$.

By Theorem 2.1, $\mathcal{W}_n$ is disjoint from $\mathcal{K}$. Suppose that there is a sequence $(\Theta_i)_{i=1}^\infty$ of elements of $\mathcal{W}_n$ which converges to a critical portrait $\Theta = \{\mathbf{c}, \mathbf{d}\} \in \mathcal{K}$. For each i set $\sim_i = \sim_{\Theta_i}$ and let $\mathbf{T}_i$ be a wandering triangle in $\sim_i$ such that $\operatorname{Ch}(\mathbf{T}_i)$ has area at least $1/n$. We may assume that $(\mathbf{T}_i)_{i=1}^\infty$ converges to a triangle $\mathbf{T} = \{a, b, c\}$, with area of $\operatorname{Ch}(\mathbf{T})$ at least $1/n$.

Let us prove that $\mathbf{T}$ is contained in some $\sim_\Theta$-class $\mathbf{T}'$; it is enough to show that $a \sim_\Theta b$, i.e., that one-sided itineraries of $\sigma_3^m(a)$ and $\sigma_3^m(b)$ coincide (see Definition 2.5). Since $\sigma_3^n(\mathbf{T}_i)$ is contained in a Θ_i-unlinked class for each i and Θ_i-unlinked classes converge to Θ-unlinked classes, the points $\sigma_3^m(a)$ and $\sigma_3^m(b)$ belong to the *closure* of the same Θ-unlinked class. Since $\Theta \in \mathcal{K}$, the orbit of a (or b) intersects $\mathbf{c} \cup \mathbf{d}$ no more than once. It is now evident that $a \sim_\Theta b$.

Thus for some $\sim_\Theta$-class $\mathbf{T}'$ we have $\mathbf{T} \subset \mathbf{T}'$. By Theorem 2.6, $\mathbf{T}'$ is finite. Since $\Theta \in \mathcal{K}$, $\mathbf{T}'$ is not wandering by Theorem 2.1, and $\mathbf{T}'$ is not precritical by Lemma 4.3. Hence, $\mathbf{T}'$ is preperiodic. This implies that either $\mathbf{T}$ itself is preperiodic or its future images cross each other inside $\mathbb{D}$. As the latter is impossible by continuity, we may assume that there exist powers n and $m > 0$ such that $\sigma_3^n(\mathbf{T}) = \sigma_3^{n+m}(\mathbf{T})$. Again by continuity $\sigma_3^m(\mathbf{T}_i)$ and $\sigma_3^{n+m}(\mathbf{T}_i)$ approach $\sigma_3^m(\mathbf{T})$ in the Hausdorff metric while the area of $\mathrm{Ch}(\mathbf{T})$ is at least $1/n$. For geometric reasons this contradicts that $\sigma_3^m(\mathbf{T}_i)$ and $\sigma_3^{n+m}(\mathbf{T}_i)$ are disjoint for all i. Therefore, $\Theta \notin \mathcal{K}$.

We have established that $\mathcal{W}_n$ is nowhere dense in $\mathcal{A}_3$, so $\bigcup_{n=1}^{\infty} \mathcal{W}_n = \mathcal{WT}_3$ is a first category subset of $\mathcal{A}_3$. □

REFERENCES

[BFH92] B. Bielefield, Y. Fisher, J. H. Hubbard, *The classification of critically preperiodic polynomials as dynamical systems*, Journal AMS, **5** (1992), 721–762.

[BCO10] A. Blokh, C. Curry, L. Oversteegen, *Cubic Critical Portraits and Polynomials with Wandering Gaps*, preprint arxiv:1003.4467 (2010).

[BCO11] A. Blokh, C. Curry, L. Oversteegen, *Locally connected models for Julia sets*, Advances in Mathematics, **226** (2011), 1621–1661.

[BFMOT11] A. Blokh, R. Fokkink, J. Mayer, L. Oversteegen, E. Tymchatyn, *Fixed point theorems for plane continua with applications*, preprint arxiv:1004.0214 (2010).

[BL02] A. Blokh, G. Levin, *An inequality for laminations, Julia sets and "growing trees"*, Erg. Th. and Dyn. Sys., **22** (2002), 63–97.

[BO10] A. Blokh and L. Oversteegen, *Monotone images of Cremer Julia sets*, Houston J. Math. **36** (2010), 469–476.

[BO08] A. Blokh, L. Oversteegen, *Wandering gaps for weakly hyperbolic cubic polynomials*, in: "Complex dynamics: Families and Friends", ed. by D. Schleicher, A K Peters (2008), 121–150.

[DH84] A. Douady, J. H. Hubbard, *Étude dynamique des polynômes complexes I, II* Publications Mathématiques d'Orsay **84-02** (1984), **85-04** (1985).

[Fis89] Y. Fisher, *The classification of critically preperiodic polynomials.* Ph.D. Thesis, Cornell University, 1989.

[GM93] L. Goldberg, J. Milnor, *Fixed points of polynomial maps II: Fixed point portraits*, Ann. Scient. École Norm. Sup., 4^e série, **26** (1993), 51–98

[Kiw02] J. Kiwi, *Wandering orbit portraits*, Trans. Amer. Math. Soc. **354** (2002), 1473–1485.

[Kiw04] J. Kiwi, *$\mathbb{R}$eal laminations and the topological dynamics of complex polynomials*, Advances in Math. **184** (2004), 207–267.

[Kiw05] J. Kiwi, *Combinatorial continuity*, Proc. London Math. Soc. **91** (2005), 215–248.

[KvS09] O. Kozlovski, S. van Strien, *Local connectivity and quasi-conformal rigidity of non-renormalizable polynomials*, Proc. Lond. Math. Soc. (3) **99** (2009), 275-296.

[Lev98] G. Levin, *On backward stability of holomorphic dynamical systems*, Fundamenta Mathematicae **158** (1998), 97–107.

[Mil06] J. Milnor, *Dynamics in One Complex Variable, 3rd ed*, Princeton University Press, Princeton, 2006.

[Nad92] S. B. Nadler, Jr., *Continuum theory*, Marcel Dekker Inc., New York, 1992.

[Poi93] A. Poirier, *On postcritically finite polynomials, parts I, II*, Stony Brook IMS Preprint 1993/5 and 1993/7 (1993).

[PM97] R. Perez-Marco, *Fixed points and circle maps*, Acta Math. **179** (1997), 243–294.

[Rog92] J. Rogers, *Singularities in the boundaries of local Siegel disks*, Erg. Th. and Dyn. Syst., **12** (1992), 803–821.

[RY08] P. Roesch, Y. Yin, *The boundary of bounded polynomial Fatou components*, C. R. Math. Acad. Sci. Paris, **346** (2008), 877-880.

[Thu08] W. Thurston. *The combinatorics of iterated rational maps* (1985), published in: "Complex dynamics: Families and Friends", ed. by D. Schleicher, A K Peters (2008), 1–108.

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