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POINTWISE-RECURRENT MAPS ON UNIQUELY ARCWISE CONNECTED LOCALLY ARCWISE CONNECTED SPACES

ALEXANDER M. BLOKH

ABSTRACT. We prove that self-mappings of uniquely arcwise connected locally arcwise connected spaces are pointwise-recurrent if and only if all their cut-points are periodic while all endpoints are either periodic or belong to what we call "topological weak adding machines". We also introduce the notion of a *ray complete* uniquely arcwise connected locally arcwise connected space and show that for them the above "topological weak adding machines" coincide with classical adding machines (e.g., this holds if the entire space is compact).

1. INTRODUCTION AND THE MAIN RESULTS

There are two main types of results in interval dynamics. First, these are facts about periodic points (for a map f, a point x is called *(f-)periodic (of period $n > 0$)* if $f^n(x) = x$ and $f^t(x) \neq x$ for all $0 < t < n$). The first step here was the celebrated *Sharkovsky Theorem* [Sha64] on the coexistence among periods of periodic points of an interval map. The Sharkovsky Theorem started *combinatorial one-dimensional dynamics* (see a nice book [ALM00] with an extensive list of references). One direction in which the field has developed is the study of the coexistence among periods of periodic points for self-mappings of "graphs", i.e. *one-dimensional compact branched manifolds.*

Results of the second type deal with all limit sets rather than only periodic orbits. This direction has also been initiated by Sharkovsky, who studied maps of the interval from this perspective in a number of papers (see, e.g., [Sha64a, Sha66, Sha66a, Sha67, Sha68]); the scope of our work does not allow us to go into a detailed description of this series of articles which, in our view, laid the foundation of the *one-dimensional topological dynamics.*

It is natural to see if bounds of one-dimensional topological dynamics can be pushed further to cover other (one-dimensional) spaces. As was mentioned, in some works one-dimensional topological dynamics is studied for "graphs"(see, e.g., [ALM00, Blo80s]). In this paper we consider a specific dynamical problem for one-dimensional spaces which can be viewed as more complicated than "graphs".

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All topological spaces considered in this paper are assumed to be Hausdorff. By an *arc* we mean a homeomorphic image of $[0,1]$; by a *Peano subset* we mean a continuous image of $[0,1]$. A very good reference here is Chapter 3 of [HY88].

Definition 1.1 (Uniquely arcwise connected spaces)**.** If for any points $x, y \in X$ there exists an arc $I \subset X$ with endpoints x, y, then X is called *arcwise connected*; if I is *unique*, then X is called *uniquely arcwise connected.*

To give examples we need the following definition.

Definition 1.2 (Endpoints, cutpoints, branchpoints)**.** A point $x \in X$ is said to be of *order* $\mathrm{ord}_X(x)$ *in* X if there are $\mathrm{ord}_X(x)$ components of $X \setminus \{x\}$. A point $x \in X$ is called an *endpoint* of X if $\mathrm{ord}_X(x) = 1$, a *cutpoint* of X if $\mathrm{ord}_X(x) > 1$, and a *branchpoint* of X if $\mathrm{ord}_X(x) > 2$.

Dendrites and trees are known uniquely arcwise connected spaces.

Definition 1.3 (Dendrites and trees)**.** A *dendrite* is a non-degenerated locally connected continuum containing no Jordan curves. A *tree* is a dendrite with finitely many branchpoints.

A lot of arcwise-connected spaces are neither trees nor dendrites.

Definition 1.4 (Locally arcwise connected spaces)**.** A topological space X is *locally arcwise connected* if any point has a basis of arcwise connected open sets.

We study uniquely arcwise connected locally arcwise connected topological Hausdorff spaces. At the suggestion of J. Mayer and L. Oversteegen we call such spaces *generalized dendrites* and denote the family of all such spaces by $\mathcal{GD}$ (it is easy to see that dendrites belong to $\mathcal{GD}$). We rely upon various properties of uniquely arcwise connected spaces and generalized dendrites which we now list together with useful notation. Despite their sometimes complicated structure, uniquely arcwise connected spaces allow, by their nature, for nice notation of their subarcs.

Definition 1.5 (Arcs and notation for them)**.** Let X be uniquely arcwise connected. Then for any points $a \neq b \in X$ a unique closed arc in X with endpoints a and b is denoted $[a, b]$; the notation $(a, b), (a, b]$ and $[a, b)$ is analogous to similar notation in the interval case. Moreover, a homeomorphism $\alpha : [0, 1] \to [a, b]$ with $\alpha(0) = a, \alpha(1) = b$ induces the order on $[a, b]$.

If $K \subset X$ is a Peano subset and $a, b \in K$ then $[a, b] \subset K$ (e.g., if $c, d \in [a, b]$ then $[c, d] \subset [a, b]$). Thus, any Peano subset of X is uniquely arcwise connected.

Definition 1.6 (Path component)**.** Let X be a topological space. Then a maximal by inclusion arcwise connected subset of X is called an *path component* of X. Thus, for a point $x \in X$, the path component of X containing x is the union of all arcs in X containing x. Also, if $x, a, b \in K \subset X$ then we say that x *separates* a *and* b *in* K if a and b belong to distinct path components of $K \setminus \{x\}$.

If X is uniquely arcwise connected and $a, b \in X$, then u separates a and b in X if and only if $u \in (a, b)$. Hence $[a, x] \cup [x, b] = [a, b]$ if and only if $x \in [a, b]$, and if $y \neq x$ then either x separates a and y, or x separates b and y. Later in the paper we will use the following simple fact: if $a, b, c, d \in X$ with $b \in (a, c)$ and $c \in (b, d)$ then $b, c \in [a, d]$.

Definition 1.7 (Arc cutpoints)**.** For a point $x \in X$, denote by $\mathrm{ord}_{X,arc}(x)$ the number of path components of $X \setminus \{x\}$. Depending on $\mathrm{ord}_{X,arc}(x)$, we call x an *arc endpoint of* X (if $\mathrm{ord}_{X,arc}(x) = 1$), an *arc cutpoint of* X (if $\mathrm{ord}_{X,arc}(x) > 1$), and an *arc branchpoint of* X (if $\mathrm{ord}_{X,arc}(x) > 2$).

Clearly, x is an arc cutpoint if and only if there exist a, b such that $x \in (a, b)$. Cutpoints are arc cutpoints, but the opposite is not always true.

Example 1.8 ("Warsaw Circle")**.** The "Warsaw Circle" W_c is defined as follows: take the graph of the function $\sin(\frac{1}{x}), 0 < x \le \frac{1}{2\pi}$, add to it a vertical segment from $(0, -1)$ to $(0, 1)$, and then complete the thus constructed continuum C with an arc I connecting $(0, -1)$ with $(\frac{1}{2\pi}, 0)$ and avoiding C. Then no point of W_c is a cutpoint of W_c, but any point of $W_c \setminus \{(0, 1)\}$ is its arc cutpoint. Observe that W_c is uniquely arcwise connected, but not locally arcwise connected. For generalized dendrites the situation is different.

Lemma 1.9. *If X is a generalized dendrite then all components of an open set $U \subset X$ are open and are generalized dendrites; thus, path components of $X \setminus \{x\}$ are components of $X \setminus \{x\}$, are open and locally arcwise connected (so that* $\mathrm{ord}_{X,arc}(x) = \mathrm{ord}_X(x)$*). In particular, the sets of arc endpoints, arc cutpoints, and arc branchpoints of X coincide with the sets of endpoints, cutpoints and branchpoints of X, respectively. Moreover, if A is a component of $X \setminus \{x\}$ then $\overline{A} = A \cup \{x\}$.*

Proof. We claim that, for $x \in X$ and A, a path component A of $X \setminus \{x\}$ is open. Take a point $a \in A$. Since X is locally arcwise connected, we can find a neighborhood U of a in X such that U is arcwise connected and $x \notin U$. This implies that $U \subset A$ and shows that A is open and locally arcwise connected. The set $X \setminus \{x\}$ decomposes into pairwise disjoint path components of $X \setminus \{x\}$ each of which is open, connected and locally arcwise connected. Hence path components of $X \setminus \{x\}$ are components of $X \setminus \{x\}$ with desired properties. Let $U \subset X$ be open and let $C \subset U$ be a component of U. Every point $x \in C$ comes into C with a small arcwise connected neighborhood. Thus, C is locally arcwise connected and open. If $x, y \in C$ but there exists $z \in [x, y] \setminus C$ then x, y belong to distinct path components of $X \setminus \{z\}$ and C is not connected, a contradiction. Finally, if A is a component of $X \setminus \{x\}$ then the complement of $A \cup \{x\}$ is open as the union of all other path components of $X \setminus \{x\}$. Hence $A \cup \{x\}$ is closed; since $x \in \overline{A}$ we get the desired. □

Few dynamical results were obtained for continuous maps on dendrites (see, e.g. [MT89, AEO07]). So-called $\mathbb{R}$-*trees* give another example of generalized dendrites; however results on $\mathbb{R}$-trees are either not dynamical (see, e.g., [Nik89], [MO90] or [MNO92]) or arise in the study of groups of isometries of hyperbolic space [Thu88, MS84, MS88, Bes88] and do not deal with the dynamics on $\mathbb{R}$-trees. The author is not aware of any dynamical results for generalized dendrites. However their one-dimensional nature allows one to consider for them some classical problems of topological dynamics. To describe a particular problem which we tackle in this paper, we need more definitions. Recall that a point z is called a *limit* point of a sequence $x_0, x_1, \dots$ if in any neighborhood of z there exists a point $x_i \neq z$.

Definition 1.10 (Recurrent points; pointwise-recurrent maps)**.** Let $f : X \to X$. Given a point $x \in X$, the sequence $(x, f(x), \dots) = \mathrm{O}_f(x) = \mathrm{O}(x)$ is called the *(f-)orbit* of x. The set $\omega_f(x) = \omega(x)$ of all limit points of $\mathrm{O}(x)$ is said to be the *(ω-)limit set of* x. A point which belongs to its own limit set is said to be *recurrent* (in other words, a point x such that $f(x)$ visits any neighborhood of x is said to be *recurrent*). A map such that all points are recurrent is called *pointwise-recurrent.*

An important and nice property of recurrent points is the following theorem due to Gottschalk, Erdös and Stone.

Theorem 1.11 ([ES45, Got44])**.** *If g is a continuous map of a Hausdorff topological space then, for any positive integer n, the set of recurrent points of g and the set of recurrent points of g^n coincide.*

The most obvious example of a recurrent point is a periodic point; in this case the recurrence manifests itself in the most transparent way. Accordingly, an easy example of a pointwise-recurrent map is a one-to-one map of a finite set as in this case all points are periodic. A more complicated case is that of a *minimal* map, i.e. such a map $g : X \to X$ that all points of X have dense orbit in X. This shows that in general pointwise-recurrent maps can have a complicated nature.

However with some additional restrictions on the space (often assumed a *manifold* or a *continuum*) and the map (often assumed a *homeomorphism*) one can establish a close connection between pointwise-recurrent maps and maps whose all points (or vast majority of points) are periodic. In some cases it is even possible to show that their periods are uniformly bounded; a lot of classic results are obtained for pointwise-recurrent homeomorphisms along these lines (see, e.g., [KP98, Mon37, OT90, Wea72]). The aim of this paper is to show that if we replace the restriction on the map (normally required to be a homeomorphism) by that on the space (required by us to be from $\mathcal{GD}$) we can still obtain similar results. This reconfirms a heuristic observation according to which in a lot of cases results valid for homeomorphisms of higher dimensional spaces have analogs for continuous maps of one-dimensional spaces.

Given a map $f : X \to X$, define the *grand orbit* $\mathrm{GO}_f(x) = \mathrm{GO}(x)$ as the set of all points which eventually map to $\mathrm{O}(x)$. A set $A \subset X$ is *invariant* if $x \in A$ implies $\mathrm{O}(x) \subset A$ (equivalently, $f(A) \subset A$). A set $B \subset X$ is *fully invariant* if $y \in B$ implies $\mathrm{GO}(y) \subset X$ (equivalently, $f(B) \cup f^{-1}(B) \subset B$). Our arguments will be based, in particular, on the fact that pointwise-recurrent maps have some restrictive properties which can be used in their description. Indeed, suppose that $f : X \to X$ is pointwise-recurrent. Let us show that then for any point $x \in X$ we have $\mathrm{GO}(x) \subset \omega(x)$. Indeed, let $y \in \mathrm{GO}(x)$. Then $\omega(y) = \omega(x)$ while, on the other hand, the fact that y is recurrent implies that $y \in \omega(y)$. Hence $y \in \omega(x)$. Also, it follows that $f(X)$ is dense as otherwise a point from $X \setminus \overline{f(X)}$ is not recurrent. In particular, if $f(X)$ is closed then $f(X) = X$. This yields the following property.

Property A. *Let $f : X \to X$ be a pointwise-recurrent self-mapping of X. Then $\overline{f(X)} = X$ and $\mathrm{GO}(x) \subset \omega(x)$ for any $x \in X$ so that any periodic orbit is fully invariant under f. In particular, if it is known that $f(X)$ is closed then $f(X) = X$.*

Property A can be used to characterize pointwise-recurrent self-mappings of X.

Lemma 1.12. *A continuous map $f : [0, 1] \to [0, 1]$ is pointwise-recurrent if and only if f^2 is the identity map.*

Proof. Suppose that f is not the identity map. We may assume that one of the following two cases holds: (1) there exists an interval (a, b) which is fixed point free and such that at least one of the points a, b is fixed and $f(a, b) \cap (a, b) \neq \varnothing$, or (2) such interval does not exist, the set of all fixed points is a point d, and the map f "flips" $[0, d]$ and $[d, 1]$.

Indeed, $[0, 1] = Y \cup Z$ where $Y \neq \varnothing$ is the set of all fixed points of f and $Z \neq 0$ is an at most countable union of open intervals whose endpoints are fixed except possibly for the intervals with endpoints 0 or 1. If Y is not connected, we can find a component (a, b) of Z whose both endpoints are fixed, and (1) holds. Otherwise suppose that $Y = [u, v]$. We may assume that $0 < u$. If $f(0, u) \cap (0, u) \neq \varnothing$ then again (1) holds. Otherwise $f[0, u) \subset [u, 1]$. If $u < v$ this implies by continuity that there are points in $(0, u)$ which are not fixed but map to fixed points in $[u, v]$, a contradiction with Property A. Hence in this case $u = v$ and $f(0, u) \subset (u, 1]$. Therefore $u < 1$ and similar arguments show that $f(u, 1] \subset [0, u)$, i.e. in the end case (2) holds. Consider these cases separately.

(1) Let a be fixed; assume that (a, b) is maximal by inclusion interval with listed properties. By Property A and since $f(a, b) \cap (a, b) \neq \varnothing$ we see that $f((a, b)) > a$. Since (a, b) is fixed point free, all its points map in the same direction by f. If they map towards a then they are attracted by a and clearly there are non-recurrent points. If $f(x) > x$ for any $x \in (a, b)$ then, since (a, b) is maximal, either $b < 1$ is fixed or $b = 1$ which also forces b to be fixed. Similar to the above this implies that, by Property A, $f((a, b)) < b$ and hence all points of (a, b) are attracted by b and there are non-recurrent points.

(2) By Property A we see that $f^2([0, d]) \subset [0, d]$ and $f^2([d, 1]) \subset [d, 1]$. Now (1) leads to a contradiction unless f^2 is the identity map. $\square$

I. Naghmouchi [Nag12] recently obtained far more general results. Namely, let D be a dendrite whose set of endpoints $\mathrm{End}(D)$ is countable; the first result of [Nag12] is that $f : D \to D$ is pointwise-recurrent if and only if f is a pointwise-periodic homeomorphism. Suppose now that the set $B(D)$ of branchpoints of D is discrete. Then it is proven in [Nag12] that $f : D \to D$ is pointwise-recurrent if and only if all cutpoints of D (i.e., all points of $D \setminus \mathrm{End}(D)$) are periodic.

The aim of this paper is to consider pointwise-recurrent maps on generalized dendrites. We need a few definitions. Observe that in Definitions 1.13 and 1.14 we include no topological requirements on either a set or a map.

Definition 1.13 (Periodic sets)**.** A set A is said to be *(f-)periodic* if A, $f(A)$, $\dots$, $f^{n-1}(A)$ are pairwise disjoint while $f^n(A) \subset A$. More generally, the union of n pairwise disjoint sets $A_0, \dots, A_{n-1}$ is said to be an *(f-)cycle of sets (of period n)* if $f(A_i) \subset A_{i+1}, i = 0, \dots, n-2$ and $f(A_{n-1}) \subset A_0$. Each set A_i is then said to be a *set from a cycle of sets.*

Periodic singletons (orbits) are the simplest periodic sets (cycles of sets).

Definition 1.14 (Adding machines)**.** Let $\mathcal{C} = \{C_0 \supset C_1 \supset \dots\}$ be a nested sequence of f-cycles of sets of periods $m_n \nearrow \infty$ (clearly, m_{i+1} is a multiple of m_i for any i). We say that $\mathcal{C}$ *generates a weak adding machine* $C_\infty = \bigcap_{n=0}^{\infty} C_n$ *(of type* $(m_0, m_1, \dots)$. If the intersection of each *nested* sequence of sets from the cycles of sets C_n is non-empty, then we call C_∞ a *full weak adding machine.*

A weak adding machine is an f-invariant set. For a nested sequence $\mathcal{C} = C_0 \supset \dots$ of cycles of sets of periods m_i, choose a nested sequence of sets from these cycles $\mathcal{R} = \{T^0 \supset \dots\}$ and call it the *root* of C_∞; there are infinitely many ways to choose the root. Once it chosen each set X in each cycle C_n of sets from $\mathbb{C}$ acquires a natural index from 0 to $n-1$ depending on the least power of f mapping T_n into X. We denote sets from the cycle C_i by setting $T^0_n = T^0$ and then $T^i_n, 0 \le i \le n-1$ so that $f^i(T^0_0) \subset T^i_n$. Clearly, a sequence $\{T^{j_i}_i,\ i = 0, 1, \dots\}$ is nested if and only if $j_{i+1} \equiv j_i \pmod{m_{i+1}}$ (if, for some i, $j_{i+1} \not\equiv j_i \pmod{m_{i+1}}$, then $T^{j_{i+1}}_{i+1} \cap T^{j_i}_i = \varnothing$). Some *nested* sequences $\{T^{j_i}_i, i = 0, 1, \dots\}$ may have empty intersections.

Set $Z_\infty = \mathbb{Z}_{m_0} \times \mathbb{Z}_{m_1} \times \dots$ and define $H(m_0, m_1, \dots) = H \subset Z_\infty$ as the set of all sequences $(j_0, j_1, \dots) \in Z_\infty$ with $j_{i+1} \equiv j_i \pmod{m_{i+1}}$. Let $\tau : H \to H$ be such that $\tau(j_0, j_1, \dots) = (j_0 + 1 \pmod{m_0}, j_1 + 1 \pmod{m_1}, \dots)$. The map τ models $f|_{C_\infty}$ for an adding machine C_∞ of type $(m_0, m_1, \dots)$ generated by f-periodic sets $C_0 \supset C_1 \supset \dots$; to each non-empty intersection $\bigcap T^{j_i}_i$ we associate the sequence $\bar{j} = (j_0, j_1, \dots)$. By the above $\bar{j} \in H(m_0, \dots)$. This gives a map $\psi : C_\infty \to H(m_0, \dots)$ of a weak adding machine to an invariant subset $\psi(C_\infty)$ of $H(m_0, \dots)$. Clearly, ψ semiconjugates $f|_{C_\infty}$ with $\tau|_{\psi(C_\infty)}$.

Definition 1.15 (Models of adding machines)**.** Suppose that cycles of sets $C_0 \supset \dots$ generate a weak adding machine C_∞ of type $(m_0, \dots)$. Then C_∞ is said to be a *topological weak adding machine* if ψ is continuous and one-to-one onto image, and *topological adding machine* if ψ is a homeomorphism onto $H(m_0, \dots)$.

Lemma 1.16 uses terminology and notation from Definitions 1.13 - 1.15.

Lemma 1.16. *Suppose that $f : X \to X$ is a map of a topological space X and that cycles of sets $C_0 \supset C_1 \supset \dots$ generate a weak adding machine C_∞. Then:*

(1) *if for each i and for all $J > i$ the sets of C_i are open in the relative topology of of the corresponding sets of C_i then the map ψ is a continuous map of C_∞ onto $\psi(C_\infty) \subset H(m_0, \dots)$;*

(2) *if for each j all sets in C_j are compact then ψ is a continuous map and $\psi(C_\infty) = H(m_0, \dots)$.*

Proof. (1) As the basis in $H(m_0, \dots)$ we can choose *cylinders* (sets consisting of sequences in $H(m_0, \dots)$ for which a few initial parameters are fixed). Then for each such cylinder K its ψ-preimage is the appropriate set B from C_i intersected with C_∞. By the assumption there is an open set $U \subset X$ such that $U \cap C_i = B$. It follows that $U \cap C_\infty = B \cap C_\infty = \psi^{-1}(K)$. Hence $\psi^{-1}(K)$ is open in C_∞.

(2) Follows from (1), from the fact that nested sequences of compact sets have non-empty intersections, and from the assumptions of the lemma. □

Observe that $H(m_0, \dots)$ - and therefore any topological weak adding machine associated with $H(m_0, \dots)$ - is uncountable.

Definition 1.17. A *ray* R is the image of $\mathbb{R}_+ \cup \{0\}$ under an embedding F into a topological space X. If $F(t)$ converges as $t \to \infty$, we say that R converges at infinity. If X is a uniquely arcwise connected locally arcwise connected topological space, then we say that X is *ray complete* if every ray in X converges at infinity.

For a map f, set $F_n(f) = F_n = \{x : f^n(x) = x\}$ and $D_n(f) = D_n = \bigcup_{i=1}^n F_i$.

Theorem 1.18. *Let X be a uniquely arcwise connected locally arcwise connected topological space. Then a continuous map $f : X \to X$ is pointwise-recurrent if and only if all its cutpoints are periodic. Moreover, in this case the following holds.*

(1) *The map f is one-to-one; the set of all cutpoints of X is fully invariant.*

(2) *The sets $F_n(f)$ and $D_n(f)$ are arcwise connected and closed for any n.*

(3) *An endpoint x of X is periodic or belongs to a topological weak adding machine (then x is a limit point of a sequence of branchpoints of X).*
(4) *If X is ray complete (e.g., if X is compact) then an endpoint of X is periodic or belongs to a topological adding machine. If X is a tree then there exists N such that f^N is the identity map.*

Let us describe a map f from Theorem 1.18. For each n with $D_n \underset{\neq}{\subsetneq} D_{n+1}$, cycles of connected sets of periods $n+1$ are added to D_n. Let A be one of sets from such cycle of sets C. Then A is attached to D_n at a point x of period $m \le n$ with $n+1 = km$. There are $k > 1$ sets from C attached to x; they "rotate" around x under iterations of f^m and have no points mapped to D_n (in particular, x is a branchpoint of X as there are at least two sets from C and the set D_n which meet at x). As n increases, the growth of D_n can stop at some place to never resume; then the corresponding part of X consists of periodic points only (with bounded from above periods). Otherwise the periods of sets like A grow to infinity which results in creation of recurrent points from topological weak adding machines.

Theorem 1.18 does not hold for uniquely arcwise connected spaces which are not locally arcwise connected. Indeed, consider a compact topological space X formed by a set of radii of the unit circle whose arguments form a Cantor set $C \subset \mathbb{S}^1$. Define a minimal map f on C and then extend it onto X so that each radius R_x defined by a point $x \in C$ maps to the radius $R_{f(x)}$ defined by the point $f(x)$, and the map is an isometry on R_x. Then all points of X are recurrent.

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2. Uniquely arcwise connected topological spaces

From now on we ***always*** consider a uniquely arcwise connected space X. Call a map $f : X \to X$ *continuous on arcs* if, for any arc $I \subset X$, the restriction $f|_I$ is continuous. From now on we ***always*** consider a map $f : X \to X$ continuous on arcs. Continuity on arcs does not imply continuity.

Example 2.1. Set $X \subset \mathbb{C}$ to be the union of a closed interval I connecting $(0, -1)$ and $(0, 0)$ and a countable collection of closed intervals J_k of lengths $\frac{1}{k}$ emanating from $(0, 0)$ and forming the angles $\frac{2\pi}{k}, k \ge 2$ with the positive direction on x-axis. Clearly X is a dendrite. Now, define a map $f : X \to X$ as follows. First set $f(I) = (0, -1)$; in other words, we assume that I collapses to the point $(0, -1)$. To define f on each J_k, denote by x_k the midpoint of J_k for each k. Denote by A_k and B_k the two closed subintervals into which x_k divides (except for the common endpoint x_k) the interval J_k so that $(0, 0) \in B_k$. Set $f|_{A_k}$ to be the identity map and $f|_{B_k}$ to be a linear (with respect to the plane metric restricted on X) map which stretches B_k onto $I \cup B_k$). Then not only is our map f not continuous, but also even the set of all f-fixed points is $(-1, 0) \cup (\bigcup A_k)$ which is not closed while f is clearly continuous on arcs. This shows the limitations of conclusions which we can make by only assuming that f is continuous on arcs.

Another unpleasant property of maps continuous on arcs is that they need not have continuous on arcs iterates. The author is grateful to the referee for the following example.

Example 2.2. Take the dendrite X from Example 2.1. Take a sequence $\{y_k = (0, -1 + 2^{-k}) : k = 0, 1, \dots\}$ and set $I_k = [y_k, y_{k+1}]$. Construct now our map f as follows. Let f take the points $\{y_k, k = 1, 2, \dots\}$ to $(0, 0)$ while taking $y_0 = (0, 0)$ to $(0, -1)$. Map I_0 linearly onto I and each I_k onto J_{k+1} so that the midpoint of I_k maps to the endpoint of J_{k+1} and linearly otherwise. Thus, as t moves down along I_k, the point $f(t)$ moves along J_{k+1} first from $(0, 0)$ out to the other endpoint of J_{k+1} and then back down to $(0, 0)$. Finally, map $(0, -1)$ to $(0, 0)$, and leave f as it was defined in Example 2.1 on $\bigcup_{k=1}^{\infty} J_{k+1}$.

Clearly, f is continuous on arcs. However $f^2|_I$ is not continuous. To see that, observe that for any $s \in I$ there is a point $s_k \in I_k$ with $f^2(s_k) = s$. Notice also that $f|_I$ is continuous and maps I onto the entire X (so that X is a Peano subset).

This shows that results on maps continuous on arcs require special tools. As we see below, these tools are of one-dimensional nature. They and based upon the fact that some other standard facts still hold for maps continuous on arcs. E.g., let $A \subset X$ be arcwise connected. Then $f(A)$ is also arcwise connected. Indeed, take two points $f(x) \in f(A), f(y) \in f(A)$ and consider $f|_{[x,y]}$. Since f is continuous on arcs, the set $f([x, y])$, as a continuous image of an arc, is arcwise connected as desired. Observe also, that if f is continuous on arcs then it is continuous on *trees* (finite unions on arcs in X).

Definition 2.3. A set Z is said to be *closed on arcs* if for any arc $[a, b]$ the intersection $[a, b] \cap Z$ is closed in $[a, b]$.

As an example of how this notion is used, let us prove Lemma 2.4.

Lemma 2.4. *Suppose that X is uniquely arcwise connected, $f : X \to X$ is continuous on arcs, and $Y \subset X$ is an arcwise connected set such that $f^k(Y) \subset Y$. If the set $F_n = \{x : x \in Y, f^{kn}(x) = x\}$ is arcwise connected, then for any closed arc $I = [a, b] \subset Y$ the intersection $F_n \cap I$ is a closed arc.*

In other words, F_n is closed on arcs if it is considered as a subset of Y.

Proof. Clearly, $F_n \cap I$ is an interval with endpoints, say, u and v, where u (v) either belongs to $F_n \cap I$ or not. Observe that $f|_I$ is continuous and one-to-one on (u, v). Hence $f(I)$ is an arc with endpoints $f(u), f(v)$ such that $(f(u), f(v)) \subset P_f$. Moreover, $f|_{[f(u),f(v)]}$ is continuous and hence $f^2|_{[u,v]}$ is continuous. Repeating this argument n times, we see that $f^n|_{[u,v]}$ is continuous and identity on (u, v). Hence $f^n(u) = u, f^n(v) = v$ and $u, v \in F_n$ as desired. □

Lemma 2.5 allows us to "project" points in X to its subsets closed on arcs.

Lemma 2.5. *Let $Y \subset X$ be an arcwise connected set closed on arcs. Let $z \notin Y$ be a point of X. Then there exists a unique point $w \in Y$ such that $(w, z] \cap Y = \varnothing$. Moreover, if z and z' belong to the same path component of $X \setminus Y$, the corresponding point w serves both z and z'.*

In the proof we repeatedly use the fact that X is uniquely arcwise connected.

Proof. Choose a point $x \in Y$ and consider $[x, z]$. Then for some point w we have that $[x, w] \subset Y$ while $(w, z] \cap Y = \varnothing$. Let us show that w with these properties is unique. Suppose that $w' \in Y, w' \neq w$ is such that $(w, z] \cap Y = \varnothing$. Then for some point $u \in [z, w] \cap [z, w']$ we must have that $(u, w] \cap (u, w'] = \varnothing$. Connecting w and w' with an arc inside Y we will get a contradiction with the fact that X is uniquely arcwise connected as $[u, w] \cup [w, w']$ and $[w', u]$ are two distinct arcs connecting w' and u and . Thus, w is well-defined. The remaining claim is left to the reader. □

Lemma 2.5 leads to the following definition.

Definition 2.6. Denote the point w from Lemma 2.5 by $p_Y(z)$. Moreover, for any point $z \in Y$ we set $p_Y(z) = z$.

In general the map $p_Y(z)$ is not continuous. E.g., take the "Warsaw circle" (see Example 1.8) and choose $Y = [a, b]$ to be a closed arc inside I. Then choose $z \in [(0, 0), (0, 1)] = K$. It follows that if for points of K the "projection" to Y is, say, a, then for all other close by points of W_c the "projection" to Y is b.

However it is easy to see that if the set $X' \subset X$ is locally connected then $p_Y|_{X'}$ is continuous. Thus, if $Y \subset X'$ then $p_Y|_{X'}$ is a *retraction*.

Definition 2.7. Let $E = \{e_1, \dots, e_k\}$ be a collection of points of X. Then the smallest connected set $\mathrm{Ch}(E)$ containing E is called the *connected hall* of E.

Before we prove the next lemma observe that if $Y \subset X$ is a tree then $f(Y)$ is a dendrite (i.e., a locally connected uniquely arcwise connected compactum). In particular this implies that for any dendrite $D \subset f(Y)$ all components (equivalently, path components) of $f(Y) \setminus D$ are open in $f(Y)$.

Lemma 2.8. *Let $E = \{e_1, \dots, e_n\} \subset X$ and set $Y = \mathrm{Ch}(E)$. Let $Z = \mathrm{Ch}(f(E))$ and let $T = Y \cap f^{-1}(Z)$. Then $f(T) = Z$ and $f|_T$ can be extended over the entire Y as a continuous map F so that on any component of $Y \setminus T$ the map F is a constant.*

Notice that $f|_T$ is continuous. Also, if $y \in Y$ is such that F is not a constant on a neighborhood of y then $y \in T$ and so in fact $F(y) = f(y)$.

Proof. The dendrite $f(Y)$ contains Z. Hence the map p_Z on $f(Y)$ is a retraction. Define the map F as $(p_Z \circ f)|_Y$. Then F coincides with f on T. Moreover, continuity of f on Y, the fact that $f(Y)$ is a dendrite, and the above listed properties of "projections" imply the rest of the lemma. □

The construction of the map F from Lemma 2.8 can be iterated. This immediately yields Corollary 2.9.

Corollary 2.9. *Let $E = \{e_1, \dots, e_n\} \subset X$ and set $Y = \mathrm{Ch}(E)$. Let $Z_i = \mathrm{Ch}(f^i(E)), i = 0, 1, \dots$. Let $T_n \subset Y$ be a set of all points $y \in Y$ such that $f^i(y) \in Z_i, i = 0, 1, \dots, n$. Then $f^n(T_n) = Z_n$ and $f^n|_{T_n}$ can be extended over the entire Y as a continuous map F_n so that on any component of $Y \setminus T_n$ the map F_n is a constant.*

This leads to Lemma 2.10.

Lemma 2.10. *Let $E = \{e_1, \dots, e_n\} \subset X$ and set $Y = \mathrm{Ch}(E)$. Suppose that $\mathrm{Ch}(f^n(E)) \supset Y$. Then there are periodic points of f in Y whose entire f^n-orbit is contained in Y.*

Proof. Consider a map $F_n : Y \to \mathrm{Ch}(f^n(E))$ constructed in Corollary 2.9. Then compose it with p_Y to construct a continuous map $g = p_Y \circ F_n : Y \to Y$. Take a fixed point x of g (it is well known [Nad92] that such point exists). If g is not a constant on a neighborhood of x in Y then it follows from the construction that $f^n(x) = g(x)$ as desired. Otherwise choose the open set W of points attracted to x (since g is a constant on a neighborhood of x, the set W is open), and then the component U of W containing x. It is well-known that the (finite) boundary of U maps to itself. This implies that there are g-periodic points in $\mathrm{Bd}(U)$. If one such point belongs to an open set on which g is a constant, then close by points of U will not be attracted to x, a contradiction. Hence g and f^n coincide on all g-periodic points in $\mathrm{Bd}(U)$ which completes the proof. □

Lemma 2.10 allows one to make conclusions about the f^n-orbits of points under certain circumstances. To make such conclusions we need the following definition.

Definition 2.11. Given a map $g : X \to X$, and a point $y \in X$ with $g(y) \neq y$, let $A_g(y)$ *be the path component of* $X \setminus \{y\}$ *containing* $g(y)$. It follows that $z \in A_g(y)$ if and only if $y \notin [z, g(y)]$.

In Corollary 2.12 we study maps without periodic arc cutpoints.

Corollary 2.12. *Let* $f : X \to X$ *be a map continuous on arcs without periodic arc cutpoints and* $x \in X$ *be a point with* $f^n(x) \neq x$. *Then the entire* f^n*-orbit of* x *is contained in* $A_{f^n}(x) \cup \{x\}$, *so that if* x *is not periodic then* $\mathrm{O}_{f^n}(f^n(x)) \subset A_{f^n}(x)$.

Proof. First observe that if x is an arc endpoint then the claim holds because then $A_{f^n}(x) = X \setminus \{x\}$. Assume now that x is not an arc cutpoint. Suppose by way of contradiction that there exists the minimal m such that $f^{mn}(x) \notin A_{f^n}(x)$. By the assumption $f^{mn}(x) \neq x$. Set $E = \{x, f^n(x), f^{n(m-1)}(x)\}$ and $Y = \mathrm{Ch}(E)$. Then $\mathrm{Ch}(f^n(E)) \supset Y$. By Lemma 2.10 there is a periodic point $y \in Y$ with $\mathrm{O}_{f^n}(y) \subset Y$. Since $f^{mn}(x) \notin A_{f^n}(x)$ then $y \notin E$ which implies that y is an arc cutpoint, a contradiction. □

In the interval case Corollary 2.12 deteriorates to an obvious statement according to which if there are no interior periodic points of $f : [0, 1] \to [0, 1]$ then all points of $(0, 1)$ map in the same direction under f.

3. Proofs of main results

We need the following definition inspired by that of a recurrent point.

Definition 3.1. Consider a point x of a uniquely arcwise connected space X. Suppose that a map $g : X \to X$ is given. If for any $y \in X, y \neq x$ there exists $n > 0$ such that x and $g^n(x)$ belong to the same path component of $X \setminus \{y\}$ then x is said to *return (to path components, under g)* (or to be a *returning (to path components, under g)* point). If x returns to components under any power of g then we say that x *totally returns (to path components, under g)* (or is a *totally returning (to path components, under g)* point).

By a *preperiodic* point we mean a non-periodic point which eventually maps to a periodic point. We need the following simple observation.

Lemma 3.2. *If* $g : X \to X$ *is given and a point* $x \in X$ *is such that* $x \neq g(x) = g^2(x)$ *then* x *is not returning to path components under* g. *If* y *is totally returning then* y *is not preperiodic.*

Proof. Choose z separating x from $g(x)$; it follows that $g(x)$ does not return to the path component of $X \setminus \{z\}$ containing x and proves the claim. Applying this claim to y and g^N with sufficiently large N completes the proof of the lemma. □

To prove lemmas leading to the proof of Theorem 3.6 which implies Theorem 1.18 we make the following Standing Assumption about the map we are working with.

Standing Assumption. *We assume that $f : X \to X$ is a continuous on arcs map such that all points totally return to path components.*

Suppose that $Y \subset X$ is arcwise connected and such that $f^N(Y) \subset Y$. Then $f^N|_Y$ is such that all points totally return to path components. However as example 2.2 shows we cannot guarantee that $f^N|_Y$ is continuous on arcs. Still, Lemma 2.8 and Corollary 2.9 allow us to work with powers of f.

The next key lemma is an important technical result.

Lemma 3.3. *Suppose that $x' \in X$ is such that $f^n(x') = x'$. Then it is impossible that for some $t \neq x'$ we have $[x', t] \subset [x', f^n(t))$.*

Proof. Suppose otherwise. Then by Corollary 2.9 there exists a point $z = z_0 \in (x', t)$ such that $f^n(z) = t$, a point $z_1 \in (x, z)$ such that $f^n(z_1) = z$, etc. The sequence z_i is ordered on $[x', t]$ in the sense of induced order so that z_{i+1} separates $z_i = f^n(z_{i+1})$ from x' on $[x', t]$, and $z_i \to x$ for some point $x \in [x', t]$. By Corollary 2.9 $f^n([z_{i+1}, z_i]) \supset [z_i, z_{i-1}]$. Thus for any m we have $f^{nm}(x, z_m) \supset (x, z)$.

Consider the union $Y = \bigcup_i f^{ni}(x, z]$. Since $(x, z] \subset f^n(x, z]$, it follows that $f^n(Y) = Y$. Clearly, $Y \subset X$ is uniquely arcwise connected. Let us show that Y contains no periodic points. Indeed, suppose that Y contains a periodic point u. Then we can choose N so big that z_N is very close to x and $(x, z_N]$ contains no points of $\mathrm{O}_f(u)$. Since $f^{Nn}(x, z_N] \supset (x, z]$, then the periodic point u has eventual preimages which do not belong to $\mathrm{O}_f(u)$. As this contradicts Lemma 3.2, we see that indeed Y contains no periodic points. By Corollary 2.9 this implies that, e.g., z_1 does not totally return to path components, a contradiction. □

We will need the following simple fact.

Lemma 3.4. *A continuous map of a tree to itself has a fixed point. In particular, suppose that $Z \subset X$ is a tree with all its cutpoints periodic such that f^n maps its endpoints map to Z. Then there is an f^n-fixed point in Z.*

Proof. The first claim of the lemma is well-known (see, e.g., [Nad92]). To prove the second observe that f is one-to-one on its cutpoints. Since by the assumption f is continuous on Z, it follows that in fact f maps Z homeomorphically onto its image $f(Z)$. Hence, the f-images of endpoints of Z are the endpoints of $f(Z)$. Repeating this argument n times we see that $f^n|Z$ is a homeomorphism of Z onto $f^n(Z)$ and that the f^n-images of the endpoints of Z are the endpoints of $f^n(Z)$. By the assumption this implies that $f^n(Z) \subset Z$. Hence by the first claim of the lemma there are f^n-fixed points in Z. □

Though assumptions on continuity of f are weak, we prove for f some standard properties; recall, that $F_n(f)$ is the set of all f^n-fixed points of f. Thus, if $Y \subset X$ is such that $f^n(Y) \subset Y$, then $F_k(f^n|_Y)$ is the set of all f^{nk}-fixed points in Y.

Lemma 3.5. *Let $Y \subset X$ be arcwise connected and such that $f^k(Y) \subset Y$. Then the set $F_n(f^k|_Y)$ is arcwise connected and closed on arcs. The set $F_1(f^k|_Y)$ is non-empty (and so all sets $F_n(f^k|_Y)$ are non-empty).*

If $z \notin F_n(f^k|_Y)$ then by Lemma 3.5 we can define the point $p_{F_n(f^k|_Y)}(z) = x_n(z)$ for which the path component of $Y \setminus \{x_n(z)\}$ which contains z contains no points of $F_n(f^k|_Y)$ (in particular, $[z, x_n(z)) \cap F_n(f^k|_Y) = \varnothing$).

Proof. For brevity throughout the proof we set $F_i(f^k|_Y) = F_i, i = 1, 2, \dots$. Let us assume that $F_n \neq \varnothing$. First we show that F_n is arcwise connected. Indeed, otherwise there are two points $x, y \in F_n$ such that $[x, y] \not\subset F_n$. Choose a point $z \in (x, y)$ such that $f^{kn}(z) \neq z$. Clearly, then at least one these two statements holds: (1) $[x, z] \subset [x, f^{kn}(z))$, or (2) $[y, z] \subset [y, f^{kn}(z))$. By Lemma 3.3 this contradicts our Standing Assumption. Thus, F_n is arcwise connected. By Lemma 2.4 this implies that F_n is closed on arcs.

We claim that $F_n \neq \varnothing$ for some n. Assume otherwise and consider $x \in Y$. Then $f^k(x) \neq x$, and by Corollary 2.12 $\mathrm{O}_{f^k}(x) \subset A_{f^k}(x)$. If for a point $y \notin A_{f^k}(x)$ there exists an integer n with $f^{nk}(y) = x$, then $\mathrm{O}_{f^k}(x) \subset A_{f^k}(x)$ implies that y does not return to path components under f^k, a contradiction. Hence $x \notin f^{kn}(Y \setminus A_{f^k}(x))$ for every n. We claim that then $f^{kn}(Y \setminus A_{f^k}(x)) \subset A_{f^k}(x)$ for every n. Indeed, otherwise we can choose a point $y \in Y \setminus A_{f^k}(x)$ with $f^{kn}(y) \notin A_{f^k}(x) \cup \{x\}$. Since $f^{kn}(x) \in A_{f^k}(x)$ then by Lemma 2.8 there exists a point $z \in [y, x]$ with $f^{kn}(z) = x$, a contradiction. Hence $F_n \neq \varnothing$ for some n.

Now, take $x \in F_n$ and consider the set $\mathrm{O}_{f^k}(x) \subset F_n$. Then $Z = \mathrm{Ch}(\mathrm{O}_{f^k}(x)) \subset F_n$ is a tree. By Lemma 3.4 there are f^k-fixed points in Z. Hence $F_1 \neq \varnothing$. □

Recall that $D_n(f) = D_n$ is the union of set $F_n(f) = F_n, n = 1, \dots, n$.

Theorem 3.6. *If X is uniquely arcwise connected and $f : X \to X$ is continuous on arcs then all points of X totally return to path components under f if and only if all arc cutpoints of X are periodic. Moreover, then the following holds.*

1. *The map f is one-to-one; the set of all arc cutpoints of X is fully invariant.*
2. *The sets $F_n(f)$ and $D_n(f)$ are arcwise connected for any n.*
3. *An endpoint x of X is periodic or belongs to a weak adding machine generated by cycles of arcwise connected sets (then x is a limit point of a sequence of branchpoints of X).*
4. *If X is ray complete (e.g., if X is compact) then an endpoint of X is periodic or belongs to a full weak adding machine. If X is a tree then there exists N such that f^N is the identity map.*

Proof. Denote by Ar the set of all arc cutpoints of X. Also, let $P_f = \bigcup D_n$ be the set of all periodic points of f. First we prove that *if all points of X totally return to path components under f then* $\mathrm{Ar} \subset P_f$. By Lemma 3.5 $F_1 \neq \varnothing$ and since $F_1 \subset F_i$, then D_n is arcwise connected for any n. Hence P_f is invariant and uniquely arcwise connected. Let us show that $\mathrm{Ar} \subset P_f$.

Indeed, otherwise there exists an arc cutpoint $x \notin P_f$ and a non-degenerate path component A of $X \setminus \{x\}$ disjoint from P_f. Take a point $y \in A$. Since y returns to path components under f, there exists N such that $f^N(y) \in A$. Connect x and a fixed point $a \in P_f$ to create an interval $[a, x]$ which intersects P_f over an interval $[a, b]$ or over an interval $[a, b)$. Denote by B the path component of $X \setminus \{b\}$ containing x.

We claim that $f^N(b) = b$. Indeed, $[a, b) \subset P_f$, hence $f|_{[a,b)}$ is one-to-one which implies that in fact $f|_{[a,b]}$ is one-to-one. Repeatedly applying this, we see that $f^N[a, b] = [a, f^N(b)]$ and that $[a, f^N(b)) \subset P_f$. If $f^N(b) \in B$ then there are points of P_f close to $f^N(b)$ which belong to B, a contradiction. If $f^N(b) \notin B$ then, since $f^N(y) \in A$, we have $[f^N(b), f^N(y)] \supset [f^N(b), x] \supset [f^N(b), b]$. Since X is uniquely arcwise connected, an interval (z, b) of points of $[a, b)$ is contained in $(f^N(b), b)$ and hence in $f^N[b, y]$. Since by Lemma 3.2 the map f has no preperiodic points, we arrive at a contradiction.

This implies that no point of B ever maps to b. We claim that $f^N(B) \subset B$. Indeed, $f^N(y) \in A \subset B$. If now there is a point $d \in B$ with $f^N(d) \notin B$ then by Lemma 2.8 there is a point $u \in B$ with $f^N(u) = b$ again contradicting Lemma 3.2. Thus, $f^N(B) \subset B$. Since A contains no periodic points of f^N, by Lemma 3.5 it contains some points which do not totally return to path components under $f^N|_A$, and hence do not totally return to path components under f, a contradiction. This completes the proof of the fact that if all points of X return to path components under f then P_f contains all arc cutpoints of X.

We denote by Ar the set of all arc cutpoints of X. Assume now that all points $x \in \mathrm{Ar}$ are periodic. Then f is one-to-one on Ar. Let $x \neq y$ but $f(x) = f(y) = z$. If no point $t \in (x, y)$ maps to z we can choose $t \in (x, y)$ and observe that points z and $f(t)]$ can be connected with two arcs, $f[x, t]$ and $f[t, y]$. If there exists $t \in (x, y)$ with $f(t) = z$ we can apply the same argument to $[t, y]$. Thus, f is one-to-one, and hence all powers of f are one-to-one (in particular, for any closed arc $I = [a, b] \subset X$ and any N, the map $f^N|_I$ is a homeomorphism onto image).

This immediately implies that Ar is forward invariant. On the other hand if an arc endpoint x maps to an arc cutpoint $f(x)$ then $f(x)$ is periodic of period, say, n, and $f(x)$ has two distinct preimages: x and $f^{n-1}(f(x))$ (by the above $f^{n-1}(f(x))$ is an arc cutpoint of X and hence $f^{n-1}(f(x)) \neq x$), a contradiction. Hence Ar is fully invariant (both its image and its preimage are contained in it).

We claim that the set F_N is arcwise connected for any N. Indeed, if $x, y \in F_N$, then $F^N|_{[x,y]}$ is a homeomorphism onto $[x, y]$ with all points being periodic. By Lemma 1.12 this implies that $f^N|_{[x,y]}$ is the identity map and hence $[x, y] \subset F_n$. Thus, F_N is arcwise connected. By Lemma 2.4 this implies that F_N is closed on arcs. Moreover, $P_f \neq \varnothing$ implies by Lemma 3.4 that $F_1 \neq \varnothing$. Then D_n is arcwise connected for any n. Since each F_i is closed on arcs, then so is D_n for any n.

Let us show that all points of X totaly return under f. We may assume that x is an arc endpoint of X. Suppose that a number n and a point $y \neq x$ are given. Choose points $u, v \in (x, y)$ so that $x < u < v < y$ in the induced order on $[x, y]$. Choose a number k such that $f^k(u) = u, f^k(v) = v$. Then f^{ik} is identity on $[u, v]$. Sincef^{ik} is continuous on arcs and one-to-one then $f^{ik}(x)$ belongs to the arc component of $X \setminus \{y\}$ containing x. Hence, x totally returns to path components under f as desired.

Let us prove claims (1)-(4) assuming that $\mathrm{Ar} \subset P_f$ (and hence, by the above, all points of X totally return to path components under f). We have already proven (1) and (2) for such maps. To prove (3) we first make some observations. Take a path component A_n of $X \setminus D_n$. Choose the smallest N with $f^N(A_n) \cap A_n \neq \varnothing$. Choose $z' \in A_n$ and let $p_{D_n}(z') = t_n^0$. Then A_n is a path component of $X \setminus \{t_n^0\}$ disjoint from D_n. Since f has no preperiodic points, no point of A_n ever maps to D_n, and so $f^N(A_n) \subset A_n$. Clearly, $f^i(A_n)$ is a subset of a path component T_n^i of $X \setminus D_n$ and by the choice of N the sets $T_n^i, i = 0, \dots, N-1$ are all pairwise disjoint. Thus, the sets $A_n = T_n^0, T_n^i, \dots, T_n^{N-1}$ form a cycle of sets. Similar to the above, for each set T_n^i there is a unique point $t_n^i = p_{D_n}(T_n^i) \in D_n$ such that T_n^i is a path component of $X \setminus \{t_n^i\}$.

Let the period of t_n^0 be m. To prove that t_n^0 is an arc branchpoint, choose a point $u \in T_n^0$ and set $I = [t_n^0, u]$. Since u is an arc cutpoint, u is periodic of period km with $k > 1$. It follows that $f^{km}|_I$ is the identity map and the set $Y = \bigcup_{i=0}^{k-1} f^{im}(I)$ is a finite tree on which f^{km} is the identity map. If t_n^0 is not an arc branchpoint of Y, then the fact that $f^m(t_n^0) = t_n^0$ implies that a small subarc of Y with an endpoint t_n^0 consists of f^m-fixed points, a contradiction with the fact that all f^m-fixed points are contained in D_n. Thus, t_n^0 is an arc branchpoint of X. Similarly, all points t_n^i are branchpoints of X.

Now, let z be an arc endpoint of X which is not periodic. By the above for any n we can choose an arc component T_n^0 of $X \setminus D_n$ so that $z \in T_n^0$ and then the cycle of the sets $C_n = T_n^0 \cup \dots \cup T_n^{N-1}$ for some N. As the number n grows, we will find a nested sequence of cycles of sets $C_0 \supset C_1 \supset \dots$ containing z of periods $m_0 < m_1 < \dots$. To show that $C_\infty = \bigcap C_i$ is a weak adding machine, it suffices to show that a nested sequence of sets $T_0^{j_0} \supset T_1^{j_1} \supset \dots$ from cycles of sets C_i is such that the intersection $Z = \bigcap T_i^{j_i}$ is either empty or a singleton. Indeed, otherwise Z is a non-degenerate arcwise connected subset of X which is *wandering* (i.e., all its images are pairwise disjoint). Clearly, there are arc cutpoints of X in Z. This contradicts the periodicity of all arc cutpoints of X and completes the proof of (3).

To prove (4), take a nested sequence of sets $T_0^{j_0} \supset T_1^{j_1} \supset \dots$ and the points $t_i^{j_i}$ defined above. Then there is a unique ray R connecting the points $t_i^{j_i}$. If X is ray complete then the intersection $Z = \bigcap T_i^{j_i}$ is non-empty because it contains the point to which R converges at infinity. Finally, the claim in (4) about trees immediately follows from (3) because trees have finitely many branchpoints. $\square$

Now let us prove that Theorem 3.6 implies Theorem 1.18. Lemma 3.7 shows how recurrent and totally returning points are related.

Lemma 3.7. *If $f : X \to X$ is a continuous map of $X \in \mathcal{GD}$ and x is a recurrent point of f then x is totally returning.*

Proof. Choose $y \neq x$ and denote by A the component of $X \setminus \{y\}$ containing x. Choose a small neighborhood B of x so that $B \subset A$. Finally, suppose that a positive integer N is given. Since y is recurrent, then by Theorem 1.11 there exists n such that $f^{Nn}(x) \in B \subset A$ as desired. $\square$

Proof of Theorem 1.18. First observe that continuous maps are continuous on arcs. This and Lemma 3.7 imply that Theorem 3.6 holds in our setting. By Lemma 1.9, Theorem 3.6(1) implies Theorem 1.18(1). Clearly, Theorem 3.6(2) and continuity of f imply Theorem 1.18(2). To prove Theorem 1.18(3) we need to show that the weak adding machine C_∞ from Theorem 3.6(3) is topological. Suppose that C_∞ is generated by cycles of sets $C_i, i = 0, 1, \dots$ of periods $N_i, i = 0, 1, \dots$. By Lemma 1.16 it suffices to show that sets from cycles of sets C_i are open in C_i in relative topology for any i. However this follows from Lemma 1.9. Finally, Theorem 1.18(4) immediately follows from Theorem 3.6(4). $\square$

In conclusion observe that a generalized dendrite X admits a canonical *ray completion* $\widehat{X}$. A sketch of the construction follows. Consider all rays in X which do not converge at infinity. Two such rays R_1, R_2 have either coinciding (from some moment on), or disjoint (from some moment on) tails. In the former case we consider them equivalent. To each class of equivalence we associate a point of $\widehat{X}$ called a *point at infinity*. Define $\widehat{X}$ as the union of X and the just defined points at infinity; as neighborhoods of those points we take components C of sets $X \setminus \{b\}$ where b is a point of X united with all points at infinity defined by rays contained in C. It is easy to see that the space $\widehat{X}$ is a ray complete generalized dendrite.

A pointwise-recurrent continuous map $f : X \to X$ can be extended to a pointwise-recurrent continuous map $\hat{f} : \widehat{X} \to \widehat{X}$ of the ray completion $\widehat{X}$ of X. Then $f : X \to X$ can be viewed as a result of removing from $\widehat{X}$ of a few backward orbits of endpoints of X. It is not necessarily so that removed points belong to topological adding machines; some removed points my be periodic. Removing a periodic endpoint creates a ray in X which does not converge at infinity and is such that its tail consists of points of the same period. The space $\widehat{X}$ may be a dendrite or even a tree.

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(Alexander M. Blokh) Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294-1170

E-mail address, Alexander M. Blokh: `ablokh@math.uab.edu`