THE JULIA SETS OF QUADRATIC CREMER POLYNOMIALS

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Abstract. We study the topology of the Julia set of a quadratic Cremer polynomial P. Our main tool is the following topological result. Let $f : U \to U$ be a homeomorphism of a plane domain U and let $T \subset U$ be a non-degenerate invariant non-separating continuum. If T contains a topologically repelling fixed point x with an invariant external ray landing at x, then T contains a non-repelling fixed point. Given P, two angles θ, γ are *K-equivalent* if for some angles $x_0 = \theta, \dots, x_n = \gamma$ the impressions of x_{i-1} and x_i are non-disjoint, $1 \le i \le n$; a class of K-equivalence is called a *K-class*. We prove that the following facts are equivalent: 1) there is an impression not containing the Cremer point; 2) there is a degenerate impression; 3) there is a full Lebesgue measure dense G_δ-set of angles each of which is a K-class and has a degenerate impression; 4) there exists a point at which the Julia set is connected im kleinen; 5) not all angles are K-equivalent.

1. Introduction

Polynomial dynamics studies trajectories of points under a polynomial map $P : \mathbb{C} \to \mathbb{C}$ of the complex plane $\mathbb{C}$ into itself. The most interesting dynamics takes place on the Julia set J of P which can be defined as the closure of the set of all repelling periodic points of P. The set J can be either connected or disconnected, and in this paper we concentrate upon the case when J is connected.

Let $\widehat{\mathbb{C}}$ be the complex sphere, $P : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a degree d polynomial with a connected Julia set J_P. Denote by K_P the corresponding filled-in Julia set. Let $\theta = z^d : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ ($\mathbb{D} \subset \mathbb{C}$ is the open unit disk). There exists a conformal isomorphism $\Psi : \mathbb{D} \to \widehat{\mathbb{C}} \setminus K_P$ with $\Psi \circ \theta = P \circ \Psi$ [DH85]. The Ψ-images of radii of $\mathbb{D}$ are called *external rays* (to the

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Julia set J) and are denoted R_α where α is the corresponding angle. If the Julia set is locally connected, the topology and dynamics of J_P are well described. Indeed, if J_P is locally connected, then Ψ extends to a continuous function $\overline{\Psi}: \overline{\mathbb{D}} \to \overline{\widehat{\mathbb{C}} \setminus K_P}$ and $\overline{\Psi} \circ \theta = P \circ \overline{\Psi}$. Let $S^1 = \partial\mathbb{D}$, $\sigma_d = \theta|_{S^1}$, $\psi = \overline{\Psi}|_{S^1}$. Define an equivalence relation $\sim_P$ on S^1 by $x \sim_P y$ if and only if $\psi(x) = \psi(y)$. The equivalence $\sim_P$ is called the *(d-invariant) lamination (generated by P)*. The quotient space $S^1/\sim_P = J_{\sim_P}$ is homeomorphic to J_P and the map $f_{\sim_P}: J_{\sim_P} \to J_{\sim_P}$ induced by σ_d is topologically conjugate to P. The set $J_{\sim_P}$ (with the map $f_{\sim_P}$) is a topological (combinatorial) model of $P|_{J_P}$ and is often called the *topological (combinatorial) Julia set*.

Let us call irrational neutral periodic points *CS-points*. In his fundamental paper [K04] Kiwi extended the above construction to *all* polynomials P with connected Julia set and no CS-points. For such polynomials he obtained a d-invariant lamination $\sim_P$ on S^1. Then $J_{\sim_P} = S^1/\sim_P$ is a locally connected continuum and $P|_{J_P}$ is semi-conjugate to the induced map $f_{\sim_P}: J_{\sim_P} \to J_{\sim_P}$ by a monotone map $m: J_P \to J_{\sim_P}$ (by *monotone* we mean a map whose point preimages are connected). The lamination $\sim_P$ generated by P provides a combinatorial description of the dynamics of $P|_{J_P}$. In addition Kiwi proved that at all periodic points p of P in J_P the set J_P is locally connected at p and $m^{-1} \circ m(p) = \{p\}$.

In what follows we consider a quadratic polynomial P with a *Cremer fixed point* (i.e. with a neutral non-linearizable fixed point $p \in J$ such that $P'(p) = e^{2\pi i\alpha}$ with α irrational). Such a polynomial is said to be a *Cremer polynomial*, and its Julia set is called a *Cremer Julia set.* Let us have an overview of known results concerning the dynamics of Cremer polynomials and the topology of their Julia sets. By Sullivan [Sul83], a Cremer Julia set J is not locally connected (at every point). Still, there are points in J at which rays are landing (e.g., repelling periodic points [DH85]), so it makes sense to study in more detail the pattern in which such landing can occur. In this respect the following important question is due to C. McMullen [McM94]: can a Cremer Julia set contain any points at which at least two rays are landing (so-called *biaccessible points*)? This question was partially answered by Schleicher and Zakeri in [SZ99, Theorem 3] (see also [Zak00, Theorem 3]) where they show that *if* a Cremer Julia set contains a biaccessible point then this point eventually maps to a Cremer point; however it is still unknown if there exist Cremer Julia sets with biaccessible points (we relate our results to the problem of biaccessibility of points in Cremer Julia set later on). Another paper studying the topology of Cremer Julia sets is that of

Sørensen [Sor98]. In this paper the author constructs Cremer polynomials with rays which accumulate on both the Cremer point and its preimage and thus gives examples of Cremer polynomials whose Julia sets have very interesting topological properties.

To some extent our interest in the topology of Cremer Julia sets can be explained by the fact that the results of Kiwi [K04] do not apply to the case when P has CS-points. Thus, one needs other tools to study Cremer Julia sets. The motivation for this paper is to develop such tools; this is facilitated by studying topological properties (such as being *connected im kleinen* at points) of J_P. Our main topological result is Theorem 2.3 below (in its statement we use standard terminology from continuum theory discussed in detail in Section 2). An easy corollary of Theorem 2.3 applies to maps of continua with periodic (rather than fixed) points. These results may be of independent interest.

Given a non-separating continuum K, a k-cross cut C of K is an open arc in $\mathbb{C} \setminus K$ whose closure meets K in two distinct points. The *shadow of* C, denoted by $\mathrm{Sh}(C)$, is the bounded component of $\mathbb{C} \setminus C \cup K$.

Theorem 2.3. *Suppose that K is a non-separating plane continuum or a point, and $f : K \to K$ is a homeomorphism in a neighborhood V of K such that the following holds:*

1. *$f(K) = K$;*
2. *the containment $f(Sh(A)) \subset Sh(A)$ is impossible for any k-crosscut A with $Sh(A) \subset V$;*
3. *any fixed point $y \in K$ is topologically repelling;*
4. *(a) there exists a k-crosscut $B \subset V$ with $f(Sh(B)) \supset Sh(B)$, or (b) there exists a fixed point in K and a fixed external ray which lands at this point.*

Then K is a point.

Dynamical results of the paper are obtained in Section 3 and deal with quadratic polynomials. Recall that by $\Psi : \mathbb{D} \to \widehat{\mathbb{C}} \setminus K_P$ we denote a conformal (Riemann) isomorphism $\Psi : \mathbb{D} \to \widehat{\mathbb{C}} \setminus K_P$ with $\Psi \circ \theta = P \circ \Psi$ [DH85], and that the Ψ-images of radii of $\mathbb{D}$ are called *external rays* (to the Julia set J) and are denoted R_α where α is the corresponding angle. We need to define the *impression* of an external ray R_θ (slightly abusing the language, we will sometimes talk about the *impression of an angle*). It is well-known (see, e.g., [Pom92], Theorem 2.16 on page 34) that this can be done as follows: the impression $\mathrm{Imp}(R_\theta)$ is the set of all limit points of sequences $\Psi(z_n)$ where $z_n \to e^{i\theta}$, $z_n \in \mathbb{D}$.

Now, in the quadratic case, the Julia set J can be either a continuum (if the critical point c does not converge to infinity under iterations of the map) or a Cantor set (if the critical point c converges to infinity under iterations of the map). Fix a Cremer polynomial P with the Julia set J. Then J is connected and the Cremer point p belongs to $\omega(c)$ (this follows from [Mn93] and [Per97], see also [C05, Theorem 1.3]). Following Kiwi [K04] we say that two angles θ, γ are *K-equivalent* if there are angles $x_0 = \theta, \dots, x_n = \gamma$ such that the impressions of x_{i-1} and x_i are non-disjoint, $1 \le i \le n$; a class of K-equivalence is called a *K-class*, and an angle which forms a degenerate K-class is said to be *K-separate*. Our main dynamical result is Theorem 3.11. To state it we need the following definition: a continuum X is *connected im kleinen at a point* x provided for each open set U containing x there exists a connected set $C \subset U$ with x in the interior of C.

Theorem 3.11. *For a quadratic Cremer polynomial P the following facts are equivalent:*

(1) *there is an impression not containing the Cremer point;*
(2) *there is a degenerate impression;*
(3) *the set Y of all K-separate angles with degenerate impressions contains all angles with dense orbits (Y contains a full Lebesgue measure G_δ-set dense in S^1) and a dense in S^1 set of periodic angles, and the Julia set J is connected im kleinen at the landing points of these rays;*
(4) *there is a point at which the Julia set is connected im kleinen;*
(5) *not all angles are K-equivalent.*

The following corollary complements Theorem 3.11.

Corollary 3.12. *Suppose that J is a quadratic Cremer Julia set. Then the following facts are equivalent:*

(1) *All impressions are non-degenerate.*
(2) *The intersection of all impressions is a non-degenerate subcontinuum of J containing the Cremer point and the limit set of the critical point.*
(3) *J is nowhere connected im kleinen.*

Finally we discuss how the conditions from Theorem 3.11 relate to the biaccessibility of points of Cremer Julia sets. Namely, we show that if J satisfies the conditions of Corollary 3.12 then it cannot have biaccessible points. Hence if there are biaccessible points in J then J satisfies the conditions of Theorem 3.11 and thus, in particular, there exists a full measure dense G_δ-set of angles such that the impressions of their rays are degenerate and, moreover, J is connected im kleinen at the landing points of these rays.

In conclusion we would like to thank the referee for thoughtful remarks and suggestions.

2. Fixed point type theorems

By a *continuum* we mean a *non-degenerate* compact, connected and metric space. A continuum $K \subset \mathbb{C}$ is *non-separating* if $\mathbb{C} \setminus K$ is connected. In Section 2 we assume that we are given a non-separating continuum K with a map f defined on its neighborhood (other properties of f will be detailed in the statements). The main applications deal with polynomials, so the reader can think of f as being the restriction of a polynomial; however, the results apply to a larger class of maps.

The *topological hull* $T(X)$ of a plane continuum X is the complement to the unbounded component of $\mathbb{C} \setminus X$ (i.e., the union of X and all of the bounded components of $\mathbb{C} \setminus X$). Observe that $T(X)$ is *always* a non-separating continuum. A simply connected open set U whose boundary is a Jordan curve is said to be a *Jordan disk*. Given a point z we denote by $B_r(z)$ an open ball of radius r centered at z and by $S_r(z)$ the boundary of $B_r(z)$. An f-periodic point x of (minimal) period n is said to be *topologically repelling* if there exists a Jordan disk U containing x and a homeomorphism $\psi : \mathbb{D} \to U$ such that for any $r < 1$ the curve $\psi(S_r(0)) = Q_r$ is mapped outside $\overline{\psi(B_r(0))} = T(Q_r)$ by f^n (in other words, the simple closed curves Q_r are *repelled* from x by the map f^n). For brevity in what follows we say *t-repelling* instead of "topologically repelling". Slightly abusing the language we call the disk U, and curves Q_r, *t-repelling (at x)*.

Consider a *uniformization* $\varphi : \mathbb{C} \setminus \overline{\mathbb{D}} = \mathbb{D}_\infty \to \mathbb{C} \setminus K$ taking infinity to infinity (the notation φ for this map is fixed throughout). The uniformization plane is denoted by $\mathbb{C}_u$. As a rule, for objects in $\mathbb{C}_u$ we use the $'$-notation (e.g., points on S^1 are denoted $a', b', \dots$ etc). The *radial ray* in $\mathbb{D}_\infty$, corresponding to an angle α, is denoted R'_α. The images of radial rays in $\mathbb{C}_u$ are called *external (conformal)* rays (to K). Given an external ray R we define the *(induced) order* on it so that $x <_R y$ $(x, y \in R)$ if and only if $|\varphi^{-1}(x)| < |\varphi^{-1}(y)|$ (in other words, $x <_R y$ in the induced order on R means that the point x is "closer" to K **on the ray** R than y). The impression of the ray R_α is denoted by $\mathrm{Imp}(\alpha)$. For more information see, for example, [Mil00] or [Pom92].

Let X be a non-separating plane continuum. A *crosscut* (of $W = \mathbb{C} \setminus X$) is an open arc $A \subset W$ such that $\overline{A}$ is an arc with exactly two distinct endpoints in ∂W. A crosscut separates W into two disjoint domains, exactly one of which is unbounded. The *(open) shadow* $\mathrm{Sh}(C)$ of a crosscut C is the bounded component of $W \setminus C$. For crosscuts in

$\mathbb{C}_u$ (called *u-crosscuts*) we take $X = \overline{\mathbb{D}}$; for crosscuts in $\mathbb{C} \supset K$ (called *k-crosscuts*) we take $X = K$. If from the context it is clear which type of crosscuts we consider we call them simply *crosscuts.* Given a ray (radial, external) R to X, a crosscut is *R-transversal* if it crosses a ray R transversally and only once. A crosscut is *R-essential* if it is homotopic (with endpoints fixed) outside X to an R-transversal crosscut. If C' is a u-crosscut then the positively oriented arc $[a', b'] \subset S^1 \subset \mathbb{C}_u$ with $[a', b'] = \overline{\mathrm{Sh}(C')} \cap S^1$ is called the *basic arc* (of C'). If $R' = R'_{\theta'}$ is a radial ray in $\mathbb{D}_\infty$ then a u-crosscut is R'-essential if and only if its basic arc contains θ' in its interior. A sequence of crosscuts $\{C_i\}$ is said to be a *fundamental chain* if $\mathrm{Sh}(C_1) \supset \mathrm{Sh}(C_2) \supset \dots$, the closures $\overline{C_i}$ are pairwise disjoint, and $\mathrm{diam}(C_i) \to 0$. Fundamental chains can be introduced for both k-crosscuts and u-crosscuts.

We begin with the following geometric lemma (see, e.g., [Mil00, Main Lemma 17.9, p. 164]).

Lemma 2.1. *Given an angle α there is a fundamental chain of R'_α-transversal u-crosscuts C'_i converging to the point $e^{i\alpha} \in S^1 \subset \mathbb{C}_u$ such that the corresponding R_α-transversal k-crosscuts $C_i = \varphi(C'_i)$ also form a fundamental chain.*

We will say that the u-crosscuts C'_i form a fundamental chain *at* $e^{i\alpha}$. Lemma 2.2 will be useful in what follows.

Lemma 2.2. *Let R be an external ray, $\{C_i\}$ be a fundamental chain of R-transversal k-crosscuts which converges to a point x, and Q be a simple closed curve containing x in its topological hull and such that $K \cap Q$ is non-degenerate. Then there is an arc C complementary to K in Q, non-disjoint from R and such that C is an R-essential k-crosscut.*

Proof. Clearly each component of $Q \setminus K$ is a k-crosscut. It remains to show that one of these arcs is the required R-essential k-crosscut. To this end consider all corresponding u-crosscuts. They are pairwise disjoint, and their shadows are either disjoint or contain one another. We can order the crosscuts by inclusion of their shadows. Denote by $E'_1, E'_2, \dots$ maximal crosscuts in the sense of this order, and by $\mathrm{Sh}'_1, \mathrm{Sh}'_2, \dots$ their respective shadows. The φ-images of these u-crosscuts and their shadows are k-crosscuts $E_1, E_2, \dots$ and their shadows $\mathrm{Sh}_1, \mathrm{Sh}_2, \dots$.

If the collection $\{E_i\}$ of k-crosscuts is infinite, the diameters of E_1, $E_2, \dots$ converge to 0, then so do the diameters of u-crosscuts $E'_1, E'_2, \dots$ (see Corollary 17.11 from [Mil00]). Hence the set $F' = S^1 \cup (\cup \overline{\mathrm{Sh}'_i})$ is compact. Since $x \in \mathrm{int}T(Q)$ then there exists $\varepsilon > 0$

such that the 9ε-ball centered at x is contained in $\operatorname{int} T(Q)$ and so $d(x, Q) > 9\varepsilon$.

Let the radial ray R' correspond to the external ray R. By the choice of x we can choose a point $z' \in R'$ such that $d(\varphi(z'), x) < \varepsilon$, and there is an R-transversal k-crosscut C at $z = \varphi(z')$ of diameter at most ε. By the choice of ε and the previous paragraph $z' \in F'$ and there exists a unique j such that $z' \in \mathrm{Sh}'_j$. We may assume that E'_j is not R'-essential. Then there are points $s', t' \in R' \cap E'_j$ so that the subarc L' of R' with endpoints s', t' is contained inside Sh'_j and contains z'. The points $s', t' \in E'_j$ define a subarc M' of E'_j connecting them. Then the union $N' = L' \cup M' \cup \{s', t'\}$ is a simple closed curve whose topological hull we denote W'. Similar picture holds in the K-plane after the map φ is applied; as always, denote the corresponding objects - φ-images of just introduced ones - by s, t, L, M, N and W. Then $W \cap K = \emptyset$.

The crosscut C is transversal to R, hence one "half" of C, denoted H, enters W. On the other hand, since $d(z, x) < \varepsilon$ then $d(z, Q) > 8\varepsilon$, hence $\operatorname{diam}(C) < \varepsilon$ implies that C is disjoint from Q. This implies that $H \subset W$ (H cannot get out of W through R because C is transversal to R, H cannot get out of W through Q because C is disjoint from Q). Thus the corresponding endpoint of H belongs to both K and W, a contradiction. $\square$

The following theorem relies upon Lemma 2.2.

Theorem 2.3. *Suppose that K is a non-separating plane continuum or a point and $f : K \to K$ is a homeomorphism in a neighborhood V of K such that the following holds:*

1. *$f(K) = K$;*
2. *for any k-crosscut A such that $Sh(A) \subset V$ the containment $f(Sh(A)) \subset Sh(A)$ is impossible;*
3. *any fixed point $y \in K$ is topologically repelling;*
4. *(a) there exists a k-crosscut $B \subset V$ with $f(Sh(B)) \supset Sh(B)$, or (b) there exists a fixed point in K and a fixed external ray which lands at this point.*

Then K is a point.

Proof. We may assume that V is simply connected and its boundary is a Jordan curve. By way of contradiction assume that K is not a point. There exists an open annulus U around $S^1 = \partial\mathbb{D}_\infty$ such that $U = \varphi^{-1}(V \setminus K)$ and φ maps U onto $V \setminus K$ homeomorphically. Hence there exists an induced map $g : U \to \mathbb{D}_\infty$. By the arguments similar to the proof of [FMOT02, Theorem 2.1], g continuously extends over S^1 as a homeomorphism (g is a homeomorphism on the dense set of angles corresponding to the accessible points of K). Denote this extension onto the circle by G while keeping the notation g for the map $g : \overline{U} \to \overline{\mathbb{D}_\infty}$; observe that by (1) $G(S^1) = S^1$ and g is a homeomorphism on a small closed annulus around S^1 since f is a homeomorphism on V.

Let us prove that there are fixed points of G. Indeed, it is obvious if (4)(b) holds. Otherwise by (4)(a) $f(\mathrm{Sh}(B)) \supset \mathrm{Sh}(B)$. Let $B' = \varphi^{-1}(B)$ and let $[a', b']$ be the basic arc of B'. Then $g(\mathrm{Sh}(B')) \supset \mathrm{Sh}(B')$ which implies that $G([a', b']) \supset [a', b']$. Since G is a homeomorphism this implies that there exist fixed points of G.

Let us prove that there are no arcs in S^1 consisting of fixed points. Suppose that an arc $I' \subset S^1$ is such that $G(y') = y'$ for any $y' \in I'$. Choose a point $y' \in \mathrm{int}(I')$ such that $\varphi(y')$ is an accessible point in K. Then clearly $\varphi(y')$ is a fixed point of f. By (3) we can choose a nested sequence of simple closed curves $\{Q_i\}$ such that $\varphi(y') = \cap_i T(Q_i)$ and $f(Q_i) \cap T(Q_i) = \emptyset$. Choose a radial ray $R'_{y'} \subset \mathbb{D}_\infty$; then its φ-image R lands at $\varphi(y')$. Let C_i be a component of $Q_i \setminus K$ which contains a point of R. Then $\overline{C_i}$ has two endpoints each of which is an accessible point of K, and $\mathrm{diam}(C_i) \to 0$. Consider $\varphi^{-1}(C_i) = C'_i$. Then by Corollary 17.11 from [Mil00] $\mathrm{diam}(C'_i) \to 0$. Since each C'_i intersects $R'_{y'}$ this implies that the limit of C'_i is y'. Hence eventually the endpoints of C'_i belong to I' and therefore are fixed under G. Then the corresponding endpoints of C_i are also fixed under f, a contradiction with the fact that Q_i maps off itself. Hence the set of all fixed points of G is a non-empty zero-dimensional compact subset of S^1.

Since G is a homeomorphism of S^1, it follows that there exists a G-fixed point $t' \in S^1$ which is attracting from at least one side. Then by the assumptions for *any* small u-crosscut C' with basic arc $[a', b'], a' < t' < b'$ whose image is a k-crosscut we have $g(C') \cap C' \neq \emptyset$ because otherwise the shadow $\mathrm{Sh}(\varphi(C'))$ maps into itself, a contradiction. By Lemma 2.1 we can choose a fundamental chain of $R'_{t'}$-transversal u-crosscuts C'_i at t' such that $C_i = \varphi(C'_i)$ form a fundamental chain of $R_{t'}$-traversal k-crosscuts. By taking a subsequence we may assume that C_i converge to a point $k \in K$. Since $f(C_i) \cap C_i \neq \emptyset$, we have $f(k) = k$. As before, let $R' = R'_{t'}$ and $R = \varphi(R')$. Observe that by the choice of the point k Lemma 2.2 applies to the external ray R and the point k.

Since k is fixed then it is t-repelling. Take a nested sequence of t-repelling pairwise disjoint simple closed curves Q_i such that $k = \cap_i T(Q_i)$. By Lemma 2.2 for any i there is a complementary to K arc H_i in Q_i such that H_i is an R-essential k-crosscut. Then the u-crosscuts $\varphi^{-1}(H_i) = H'_i$ are R'-essential. Since the diameter of $Q_i \to 0$, the diameter of $H_i' \to 0$ (see by Corollary 17.11 from [Mil00]). Since the crosscuts H_i' are R'-essential we may assume, by taking a subsequence if necessary, that $H_{i+1}' \subset \mathrm{Sh}(H_i')$ and the sequence H_i' is a fundamental chain. Thus, $H_i' \to t'$ as $i \to \infty$. Since t' is attracting from at least one side this implies that for i sufficiently big at least one endpoint of the basic arc of H_i' is mapped by G *inside* this basic arc. Since all curves Q_i are disjoint from their images then so are all u-crosscuts H_i'. Thus we can conclude that for any sufficiently big i the u-crosscut H_i' maps inside $\mathrm{Sh}(H_i')$ by g which implies that g maps the entire shadow $\mathrm{Sh}(H_i')$ into itself. Applying the map φ we observe that then the shadow in the k-plane $\mathrm{Sh}(\varphi(H_i'))$ maps by f into itself, a contradiction to (2). Thus our assumption that K is not a point fails and the theorem is proven. □

Theorem 2.3 can be applied to powers of the map as well. Thus, Corollary 2.4 follows (in this corollary whenever speaking of powers of f we consider them on sufficiently small neighborhoods of K on which they are well-defined).

Corollary 2.4. *Suppose that K is a non-separating plane continuum or a point, and $f : K \to K$ is a homeomorphism in a neighborhood V of K such that the following holds:*

1. *$f(K) = K$;*
2. *for any k-crosscut A with $Sh(A) \subset V$ and any power m the containment $f^m(Sh(A)) \subset Sh(A)$ is impossible;*
3. *any periodic point $y \in K$ is topologically repelling;*
4. *(a) there exists a k-crosscut $B \subset V$ and a positve integer l such that $f^l(Sh(B)) \supset Sh(B)$, or*
   (b) there exists a periodic point in K and a periodic external ray which lands at this point.

Then K is a point.

Observe that without condition (4) of Theorem 2.3 (or condition (4) of Corollary 2.4) the conclusion that K is a point fails. A simple example is a homeomorphism of the unit disk which is an irrational rotation on the angles such that all points not at the center of the disk are attracted to its boundary. A tree-like continuum with analogous properties can be constructed similarly (recall that a planar continuum is *tree-like* if it is non-separating and has empty interior).

3. Quadratic polynomials

Fix a quadratic Cremer polynomial $P = z^2 + v$ with the Julia set J and the Cremer fixed point p. In this case J is a tree-like continuum and, hence, each subcontinuum of J is a non-separating, one-dimensional plane continuum. It follows that the intersection of any two subcontinua of J is connected. Denote the critical point 0 of P by c, and the restriction of z^2 onto S^1 by σ. Denote the external ray (to J) with argument γ by R_γ (while this notation was used above for external rays to a non-separating continuum K, now it is used for J), and denote the impression $\mathrm{Imp}(R_\theta)$ by $\mathrm{Imp}(\theta)$. In what follows speaking of *points* we mean points in the *dynamic plane* while *angles* mean arguments of external rays. We investigate the topology of J; even though it is well known that a Cremer Julia set is not locally connected as a set [Sul83], still there are other topological notions (such as being *connected im kleinen* at points), studied below in the dynamical context.

In a recent paper [CMR05, Theorem 1.1] the following theorem is proven for *all* polynomials.

Theorem 3.1. *The Julia set of a polynomial P is indecomposable if and only if there exists an angle γ such that Imp$(\gamma) = J(P)$; in this case the impressions of all angles coincide with $J(P)$.*

From the topological standpoint, if the Julia set is indecomposable then one cannot use impressions to further study its structure: the latter assumes representing $J = J(P)$ as the union of smaller more primitive continua which is impossible in the case when J is indecomposable. Thus, for the most part the main results of this section will be informative only when the Julia set is decomposable.

Theorem 3.2. *Suppose that γ is a σ-periodic angle. If Imp(γ) does not contain p then Imp(γ) is a singleton.*

Proof. We will use the well-known fact that $p \in \omega(c)$ (see [Mn93, Per97] and also [C05, Theorem 1.3]). Let n be the period of γ, $T = \mathrm{Imp}(\gamma)$. Then $P^n(T) = T$. Suppose that $p \notin T$. Then $P^n|_T$ is a homeomorphism. Indeed, otherwise some iterate of T contains c because if all iterates of T does not contain c then $P^n|_T$ will be a homeomorphism, a contradiction. This in turn will imply that $\cup_{i=0}^{n-1} P^i(T)$ contains $\omega(c)$ and hence p. In this case the Cremer point p will belong to T, a contradiction. So, indeed $P^n|_T$ is a homeomorphism. Also, all periodic points in T are repelling. Now by Theorem 2.3 applied to $P^n|_T$ we see that T is a point. $\square$

Lemma 3.3 studies how impressions of rays intersect.

Lemma 3.3. *Let u, v be two angles such that $X = Imp(u) \cap Imp(v) \neq \emptyset$ and denote the two components of $\mathbb{C} \setminus [R_u \cup Imp(u) \cup R_v \cup Imp(v)]$ by M and H. Suppose that $T \subset J$ is a continuum which satisfies one of the following conditions:*

1. *T is non-disjoint from M and from H;*
2. *$T = Imp(\theta) \cup Imp(\gamma)$ where $\theta \in (u, v), \gamma \in (v, u)$.*

Then $X \subset T$.

Proof. If T satisfies (1) then there exists an open arc A_m whose closure is a closed arc connecting a point $y \in T \cap M$ with infinity inside M; clearly, there exists an analogous arc $A_h \subset H$. If T satisfies (2) then from now on we assume for definiteness that the rays of angles from (u, v) fill out M while the rays of angles from (v, u) fill out H. Set $A_m = R_\theta$, $A_h = R_\gamma$. Then in either case from above $Z = A_m \cup A_h \cup T$ is a connected set whose complement consists of two connected open sets $N \supset R_u$ and $S \supset R_v$ (recall that all continua involved are non-separating).

Let us prove that $\mathrm{Imp}(v) \subset \overline{S}$. Consider cases (1) and (2).

(1) Observe that by the choice of A_m and A_h we have that $\mathrm{Imp}(v)$ is disjoint from A_m and A_h. Thus we can choose an R_v-essential crosscut C of J such that the closure $\overline{\mathrm{Sh}(C)}$ of its shadow $\mathrm{Sh}(C)$ is disjoint from $A_m \cup A_h$ (note also, that C is disjoint from T since $T \subset J$). Then $\mathrm{Sh}(C) \subset S$ and hence $\mathrm{Imp}(v) \subset \overline{\mathrm{Sh}(C)} \subset \overline{S}$.

(2) Since in this case A_m and A_h are external rays then the claim that $\mathrm{Imp}(v) \subset \overline{S}$ follows from the fact that $R_v \subset S$.

Similarly, $\mathrm{Imp}(u) \subset \overline{N}$. Hence $X \subset (\overline{N} \cap \overline{S}) \setminus (A_m \cup A_h) \subset T$. □

Let us now state a few more facts about Cremer polynomials (see, e.g., [GMO99]). The notation introduced here will be used from now on. It is well-known that given a Cremer polynomial P with $P'(p) = e^{2\pi\rho i}$, there exists a special *rotational* Cantor set $F \subset S^1$ such that σ restricted on F is semiconjugate to the irrational rotation by the angle $2\pi\rho$ [BS94, Theorem 1]; the semiconjugacy ψ is not one-to-one only on the endpoints of countably many intervals complementary to F (ψ maps the endpoints of each such interval into one point). Of those complementary intervals to F the most important one is the *critical leaf (diameter)* with the endpoints denoted below by α and $\beta = \alpha + 1/2$ (for definiteness we assume that $0 < \alpha < 1/2$). The set $F = \omega(\alpha)$ is exactly the set of points whose entire orbits are contained in $[\alpha, \beta]$.

Important properties of F are that $p \in \mathrm{Imp}(\gamma)$ for every $\gamma \in F$, and $\{p, c, -p\} \subset \mathrm{Imp}(\alpha) \cap \mathrm{Imp}(\beta) = K \subset B$ (Theorem 4.3 of [GMO99]).

Let $B \subset J$ be the union of all impressions of angles of F. Then B is a continuum because impressions are *upper-semicontinuous* (i.e. if θ_i is a sequence of angles, $\theta_i \to \theta$, and $x_i \in \mathrm{Imp}(\theta_i)$ is a sequence of points converging to a point x, then $x \in \mathrm{Imp}(\theta)$). Moreover, B is a non-separating continuum because so is J. Finally, $P(B) = B$ because $\sigma(F) = F$ and impressions map onto impressions. Hence, and since $c \in B$, we see that the orbit of c (and thus the entire ω-limit set of c) is contained in B. Set $Q = \cap_{\gamma \in F} \mathrm{Imp}(\gamma)$ (the set Q is non-empty since p belongs to all impressions of angles $\gamma \in F$).

A continuum X is *connected im kleinen at x* if for each open set U containing x there exists a connected set $C \subset U$ with x in the interior of C. A continuum X is *locally connected* at $x \in X$ provided for each neighborhood U of x there exists a connected and open set V such that $x \in V \subset U$. Observe that sometimes different terminology is used. For example, in Milnor's book [Mil00, p. 168] the property of local connectivity is called "open local connectivity" while to the property of being connected im kleinen at a point Milnor refers to as the property of being "locally connected at a point". On the other hand, in the textbook by Munkres [Mun00, p. 162] connected im kleinen is called "weakly locally connected". Using our terminology, if a space is locally connected at x, then it is connected im kleinen at x. It is well known that if a continuum is connected im kleinen at each point, then it is locally connected (see, e.g., [Mun00, p. 162, Ex. 6]). However, a continuum can be connected im kleinen at a point without being locally connected at this point.

As an example consider the so-called *infinite broom* (see [Mun00, p. 162, Ex. 7]). Consider an interval $I \subset \mathbb{C}$ with the endpoints $x_0 = a = (0,0)$ and $b = (1,0)$. Also, set $x_i = (2^{-i}, 0)$ and $y^i_j = (2^{-i}, 2^{-i-j}), i = 1, \dots, j = 0, \dots$. In other words, consider "vertical" sequences of points so that each i-th sequence hovers right above x_i, converges to x_i and begins at the height which equals the distance between x_i and x_{i-1}. Now, connect each x_i to all the points from the i_1-st "vertical" sequence, i.e. to the points $y^{i+1}_j, j = 0, \dots$, with segments of straight lines, and consider the union Y of all these segments and the original interval I. It is easy to check that Y is connected im kleinen at a. To see that it is not locally connected at a, suppose that there is an open connected set W containing a of diameter less than 1 . We may assume that $k > 0$ is the smallest number such that $x_i \in W$ if $i \geq k$. Then together with x_k the set W must contain points y^k_j for big enough j. Hence, being connected, the set W must contain x_{k-1} as well, a contradiction.

We show that J is connected im kleinen at certain points. To do so we prove a series of lemmas in which the following terminology is used. A point x is said to be *remote* if $f^n(x) \notin B$ for all n. A recurrent (e.g., periodic) point $x \notin B$ is remote because B is invariant and closed. Also, all remote periodic points are repelling.

Lemma 3.4. *The following claims hold.*

(1) *If γ is a periodic angle and x is the landing point of R_γ then x is remote if and only if $x \notin Q = \cap_{\theta \in F} Imp(\theta)$.*
(2) *If $x \in Imp(\gamma)$ is remote then $x \notin Imp(\theta)$ for any angle $\theta \neq \gamma$.*

Proof. (1) It is enough to show that if $x \notin Q$ then $x \notin B$. By way of contradiction let $x \in \mathrm{Imp}(\theta)$ for some $\theta \in F$. Suppose that $P^n(x) = x$. Choose $\xi \in F$. Then by the well-known properties of the restriction of σ onto F there exists a sequence i_j such that $\sigma^{ni_j}(\theta) \to \xi$. Since impressions are upper-semicontinuous and $x \in \mathrm{Imp}(\sigma^{ni_j}(\theta))$ for every n this implies that $x \in \mathrm{Imp}(\xi)$. Since $\xi \in F$ is chosen arbitrarily, then x belongs to impressions of all angles from F, thus it belongs to Q.

(2) Suppose otherwise. Since x never maps into B then γ and θ never map into F. Since $\theta \neq \gamma$ then we may assume that $\sigma^m(\theta) \in (\alpha, \beta)$ and $\sigma^m(\gamma) \in (\beta, \alpha)$ for some m. By Lemma 3.3, $P^m(x) \in \mathrm{Imp}(\sigma^m(\theta)) \cap \mathrm{Imp}(\sigma^m(\gamma)) \subset K \subset B$, a contradiction with x being remote. Observe that by (1) the above holds for a periodic γ such that the landing point x of R_γ does not belong to Q. $\square$

Now we prove that there are "short connections" in the Julia set which cut impressions consisting of remote points off the rest of the Julia set (in particular this applies to periodic remote points) and apply it to prove that J is connected im kleinen at some points.

Theorem 3.5. *Suppose that θ is an angle such that $Imp(\theta)$ consists of remote points. Then the following holds.*

(1) *$Imp(\theta)$ is disjoint from the impressions of all other angles.*
(2) *Arbitrarily close to θ there are angles $s < \theta < t$ such that $Imp(s) \cap Imp(t) \neq \emptyset$.*
(3) *If $Imp(\theta) = \{x\}$ is a point then J is connected im kleinen at x.*

Proof. (1) Follows immediately from Lemma 3.4

(2) Suppose otherwise. Consider all angles s, t such that $\mathrm{Imp}(s) \cap \mathrm{Imp}(t) \neq \emptyset$. Then there are angles $l_1 \le l_2 < \theta < r_1 \le r_2$ such that the following holds:

(1) $\mathrm{Imp}(l_1) \cap \mathrm{Imp}(r_1) \neq \emptyset$;
(2) $\mathrm{Imp}(l_2) \cap \mathrm{Imp}(r_2) \neq \emptyset$;
(3) if $l \in (l_2, \theta)$ and $r \in (\theta, r_2)$ then $\mathrm{Imp}(l) \cap \mathrm{Imp}(r) = \emptyset$;

(4) if $l \in (l_1, \theta)$ and $r \in (\theta, r_1)$ then $\mathrm{Imp}(l) \cap \mathrm{Imp}(r) = \emptyset$.

Set $N = \mathrm{Imp}(l_1) \cup \mathrm{Imp}(l_2) \cup \mathrm{Imp}(r_1) \cup \mathrm{Imp}(r_2)$. By Lemma 3.3 N is a continuum. Set $A = N \cup (\cup_{z \in [l_2, \theta]} \mathrm{Imp}(z)), B = N \cup (\cup_{z \in [\theta, r_1]} \mathrm{Imp}(z))$. Then A and B are subcontinua of J, while $A \cap B = N \cup \mathrm{Imp}(\theta)$ is not connected because $\mathrm{Imp}(\theta) \cap N = \emptyset$ by (1), a contradiction.

(2) Suppose that U is an open set in J containing x. Since impressions are upper semicontinuous, by (1) and (2) there exist $s < \theta < t$ such that for all $\gamma \in [s, t]$, $\mathrm{Imp}(\gamma) \subset U$, $x \notin \mathrm{Imp}(s) \cup \mathrm{Imp}(t)$ and $\mathrm{Imp}(s) \cap \mathrm{Imp}(t) \neq \emptyset$. Let $C = \cup_{a \in [s,t]} \mathrm{Imp}(a)$. Then $C \subset U$ is connected. We claim that x is in the interior of C.

Indeed, set $A = R_s \cup \mathrm{Imp}(s) \cup R_t \cup \mathrm{Imp}(t)$. Then $\mathbb{C} \setminus A$ consists of two components. Denote the one containing x by W and the other one V. Let $d(x, A \cup \partial U) = \varepsilon$. Consider the $\varepsilon/2$-disk D centered at x. Then $D \subset W$ is disjoint from $A \cup V$, and $D \cap J \subset C$ since points of $D \cap J$ cannot belong to impressions of angles not from (s, t). Thus, x belongs to the interior of C as desired. □

As an example consider a periodic angle θ such that the landing point x of R_θ does not belong to Q and $p \notin \mathrm{Imp}(\theta)$; then by Theorem 3.2 and Lemma 3.4 we see that Lemma 3.5 applies to θ and its impression.

It is well known that quadratic Cremer Julia sets are not locally connected [Sul83]. However it is not known at which points the Julia set is connected im kleinen. Theorem 3.5 contains conditions under which the Julia set is connected im kleinen at a point. In Theorem 3.11 we study when these conditions are satisfied. In fact, Theorem 3.11 establishes a strong connection between local and global properties of a Cremer Julia set. We prove that there exist points at which the Julia set is connected im kleinen (local property) if and only if the polynomial has one of a number of global properties (such as the existence of impressions not containing the Cremer point). First we establish that a quadratic Cremer Julia sets J is never connected im kleinen at the Cremer point p (and hence never locally connected at p).

Theorem 3.6. *Let J be a quadratic Cremer Julia set, then J is not connected im kleinen at the Cremer fixed point p.*

Proof. If there exists θ with $\mathrm{Imp}(\theta) = J$ then by Theorem 3.1 J is an indecomposable continuum. Then J is nowhere connected im kleinen. Hence we may assume that each impression is proper in J. By Theorem 4.3 from [GMO99] (see also [Per94]) $\{p, c, -p\} \subset \mathrm{Imp}(\alpha) \cap \mathrm{Imp}(\beta)$. Hence there are external rays which approach p and have arguments converging to β, and there are external rays which approach $-p$ and have arguments converging to β. To prove our theorem we specify this and show that there are rays which approach *both* p and $-p$ and have arguments converging to β.

To do so we need to the following well-known facts from [BS94, Theorem 1]. Suppose that $P'(p) = e^{2\pi\rho i}$ and $F \subset S^1$ is the Cantor set with rotation number ρ. Then 1) for a rational number $p/q < 1$, given in lowest terms, there exists a *rotational* periodic orbit Q of σ of rotation number p/q (there is an order preserving conjugacy between $\sigma|_Q$ and a rigid rotation by p/q restricted on its periodic orbit); 2) if θ is the least element of Q and γ is the greatest element of Q then $Q \subset [\theta, \theta + 1/2] \cap [\gamma - 1/2, \gamma]$; 3) if $p/q < \rho$ then $\theta < \alpha$, and if $p/q > \rho$ then $\theta > \alpha$; 4) if $p/q \to \rho$ then $\theta \to \alpha$.

By Yoccoz's Theorem [Yoc95] (see also [Per94, Proposition II.12] and [BS94]), if p_n/q_n are the convergents to α in its continued fraction expansion, then there exists a subsequence p_{n_k}/q_{n_k} such that P has a periodic orbit C_k of period q_{n_k}, and without loss of generality we may assume that C_k have the following properties: 1) $C_k \to p$; 2) $p_{n_k}/q_{n_k} < \rho$; 3) for each k the arguments of rays landing at points of C_k form a rotational periodic orbit $Q_k \subset S^1$ with the least element of Q_k denoted $\alpha_k < \alpha$; 4) $\alpha_k \to \alpha$; 5) there are points of F between $\alpha_k + 1/2$ and $\alpha_{k+1} + 1/2$ for every k. Then the external ray R_{α_k} lands on a point $x_k \in C_k$ with $\lim x_k = p$. Set $\beta_k = \alpha_k + 1/2$. Then $\beta_k < \beta, \beta_k \to \beta$, the rays R_{β_k} land on points $-x_k$ with $\lim -x_k = -p$, and there are points of F in (β_k, β_{k+1}). Choose a crosscut I_k from $-x_k$ to $-x_{k+1}$ so that it is contained in the open "strip" E'_k between the rays $R_{\beta_k}, R_{\beta_{k+1}}$, and the Julia set J (E'_k consists of external rays of angles from the arc (β_k, β_{k+1})). Denote by E_k the closure of the unbounded component of $E'_k \setminus I_k$, and let the distance between E_k and p be ε_k (slightly abusing the definitions which apply only to the continua, i.e. bounded sets, but invoking one's geometric intuition we can call E_k the topological hull of the set $R_{\beta_k} \cup I_k \cup R_{\beta_{k+1}}$). Do this construction for all k. Observe that by the construction for any angle $\gamma \in (\beta_k, \beta_{k+1})$ the ray R_γ is disjoint from all constructed crosscuts except for I_k.

Choose $0 < \varepsilon < d(p, -p)/9$ and **fix** k such that $d(-x_i, -p) < \varepsilon$ for $i \geq k$. Let $\eta = 1/2 \min\{\varepsilon, \varepsilon_k, \varepsilon_{k+1}, \varepsilon_{k+2}\}$. Let H be the round circle of radius η centered at p. Since $(\beta_i, \beta_{i+1}) \cap F \neq \emptyset$ and because $p \in \mathrm{Imp}(\theta)$ for all $\theta \in F$ then there exists $\gamma_i \in (\beta_i, \beta_{i+1})$ such that the ray R_{γ_i} comes much closer to p than η. Hence we can find two points $b_i <_{R_{\gamma_i}} a_i$ on R_{γ_i} (according to the order on external rays this means that b_i is farther from infinity on R_{γ_i} than a_i) such that $a_i \in R_{\gamma_i} \cap I_i, b_i \in H$, and the open segment S_i of the ray R_{γ_i} between a_i and b_i is disjoint from either I_i or H.

Observe that the ray R_{γ_i} is disjoint from the crosscut I_j when $i \neq j$; also, all rays are pairwise disjoint. Hence the appropriate pieces of the crosscuts (from a_i to x_{i+1} in I_i and from x_{i+1} to a_{i+1} in I_{i+1}), ray-segments S_j, $j = i, i+1$, and the appropriate arc H_i from b_i to b_{i+1} in the circle H enclose a simply connected domain W_i. Note that $W_k \cap W_{k+1} = \emptyset$. For $i = k, k+1$, let $X_i \subset J \cap \overline{W_i}$ be a subcontinuum which is irreducible with respect to containing x_{i+1} and a point $y_{i+1} \in H$. Then the continua X_k and X_{k+1} are disjoint since they are contained in the closures of W_k and W_{k+1} respectively, and these closures only intersect over a part of the ray $R_{\beta_{k+1}}$ which is disjoint from J. Since $X_k \cap X_{k+1} \subset H$, either X_k or X_{k+1} does not contain c. Assume, without loss of generality, that $c \notin X_k$.

Since $\max\{d(X_k, p), d(X_k, -p)\} < \varepsilon < d(p, -p)/9$ then we have $\operatorname{diam}(X_k) > d(p, -p)/2$. Now the map P is a homeomorphism on X_k, hence the continuum $P(X_k)$ is irreducible between $P(-x_{k+1}) = P(x_{k+1})$ and $P(y_{k+1})$. On the other hand, since $\operatorname{diam}(X_k) > d(p, -p)/2$, then there exists $\delta > 0$ such that $\operatorname{diam}(P(X_k)) > \delta$. Thus, $P(X_k)$ is an irreducible continuum in J of diameter larger than δ, connecting points $P(x_{k+1}), P(y_{k+1})$ each of which can be arbitrarily close to p. It is easy to see that then J is not connected im kleinen at p as desired. Indeed, suppose otherwise. Choose a neighborhood U of p of diameter smaller than δ, and then choose a continuum $C \subset U$ such that $p \in \operatorname{int} C$. Now, by the established properties of J around p there are two points $q, q' \in \operatorname{int} C$ such that the irreducible continuum between them in J is of diameter greater than δ. However C contains them both and is of diameter less than δ, a contradiction. □

If the Julia set is locally connected (see Section 1), all impressions of external rays are points, and the combinatorial properties of the Julia sets can be described with the help of *laminations* and the *pinched disk model*. Moreover, Kiwi showed that similar notions can be used to study polynomials without CS-points [K04]. These tools do not apply to Cremer polynomials in their entirety, still we can mimic them using the obtained results. Namely, given two angles $s, t \in S^1$, connect them with a chord $\operatorname{Ch}(s, t)$ inside the unit disk if and only if $\operatorname{Imp}(s) \cap \operatorname{Imp}(t) \neq \emptyset$, and call them *connected*; call chords like $\operatorname{Ch}(s, t)$ *legitimate*. Suppose that $x, y \in S^1$ and that there are finitely many angles $x_0 = x, x_1, \dots, x_n = y$ such that x_j and x_{j+1} are connected for each $j = 0, \dots, n-1$. Then we say that x and y are *K-equivalent*; also, K-equivalence classes are called simply *K-classes*. Below we study the properties of K-equivalence for Cremer polynomials; observe that a similar equivalence was introduced in [K04] for polynomials without

CS-points. A broken line consisting of a finite union of legitimate chords is called a *K-path*. The fact that points x, y are K-equivalent can be given the following geometric interpretation: if angles x, y are K-equivalent then there is a K-path between them.

Lemma 3.7. *Let x, y be K-equivalent. Then there exists a finite collection of points $x_0 = x, x_1, \dots, x_n = y$ which are ordered, either all belong to the arc $[x, y]$ or all belong to the arc $[y, x]$, and are such that x_j and x_{j+1} are connected for each $j = 0, \dots, n-1$. In other words, there exists a non-self-intersecting K-path from x to y.*

Proof. Suppose that $u, v \in S^1$ are connected, θ, γ are connected, and the chords $\mathrm{Ch}(u, v)$ and $\mathrm{Ch}(\theta, \gamma)$ intersect. We show that at most two of the angles u, v, θ, γ are not connected. Indeed, by Lemma 3.3 $\mathrm{Imp}(u) \cap \mathrm{Imp}(v) \subset \mathrm{Imp}(\theta) \cup \mathrm{Imp}(\gamma)$. Assume for definiteness that there exist common points of $\mathrm{Imp}(u) \cap \mathrm{Imp}(v)$ and $\mathrm{Imp}(\theta)$. Then θ is connected with both u and v. Then the only way there are two pairs of non connected angles among u, v, θ, γ is when γ is connected to neither u, nor v. However by Lemma 3.3 $\mathrm{Imp}(\theta) \cap \mathrm{Imp}(\gamma) \subset \mathrm{Imp}(u) \cup \mathrm{Imp}(v)$, a contradiction.

Consider now all K-paths from x to y, and choose one containing the least number of vertices. If in this K-path there are two chords intersecting not at their endpoints then by the previous paragraph this collection of points can be reduced to a new collection with the same properties, a contradiction. Thus, any K-path with the least number of vertices is non-self-intersecting as desired. □

In Lemma 3.8 we describe the K-class of α (we call it the *critical K-class*). To do so denote by E the set of all angles θ such that $\mathrm{Imp}(\theta) \cap B \neq \emptyset$. Clearly, $E \supset F$ is a closed invariant set of angles which are all K-equivalent to each other; moreover, for each angle $\gamma \in E$ there is an angle $\theta \in F$ such that $\mathrm{Imp}(\theta) \cap \mathrm{Imp}(\gamma) \neq \emptyset$ and $p \in \mathrm{Imp}(\theta)$. Let $E_\infty = \cup_{n \geq 0} \sigma^{-n}(E)$. Also, set $T = \cup_{\theta \in E} \mathrm{Imp}(\theta)$. Then $T \subset \mathbb{C}$ is connected and invariant. Let $T_\infty = \cup_{n \geq 0} P^{-n}(T)$. These sets are used in the proof of the following lemma; to state it we introduce the notion of a *K-separate* angle, i.e. an angle which forms a degenerate K-class (i.e., whose impression is disjoint from all other impressions).

Lemma 3.8. *The critical K-class coincides with E_∞. All other angles are K-separate. Both E_∞ and its complement are fully σ-invariant.*

Proof. Let us prove the first claim. To begin with we show that all angles in E_∞ are K-equivalent. The entire finite set $P^{-i}(p)$ is contained in $P^{-i}(T)$ (clearly, $P^{-i}(p)$ consists of 2^i points); we use this to show that all angles in $\sigma^{-i}(E)$ are K-equivalent. Indeed, let $\gamma \in \sigma^{-i}(E)$. Then $\sigma^i(\gamma) \in E$ and there exists an angle $\theta \in F$ such that $\mathrm{Imp}(\theta) \cap \mathrm{Imp}(\sigma^i(\gamma)) \neq \emptyset$ and $p \in \mathrm{Imp}(\theta)$. The set $A = \mathrm{Imp}(\theta) \cup \mathrm{Imp}(\sigma^i(\gamma))$ is a continuum, so we can find a continuum $H = \mathrm{Imp}(\gamma) \cup \mathrm{Imp}(\theta')$ with $\theta' \in \sigma^{-i}(\theta)$ which maps onto A by P^i (this follows by induction over i). Then there exists a P^i-preimage p' of p which belongs to H, and so γ belongs to the set $N(p')$ of all angles for which there exists $\theta' \in \sigma^{-i}(F)$ with $\mathrm{Imp}(\gamma) \cap \mathrm{Imp}(\theta') \neq \emptyset$ and $p' \in \mathrm{Imp}(\theta')$. By upper-semicontinuity of impressions the set $N(p')$ and the set $Z(p') = \cup_{\gamma \in N(p')} \mathrm{Imp}(\gamma)$ are closed. Since $\gamma \in \sigma^{-i}(E)$ is chosen arbitrarily we see, that the entire continuum $P^{-i}(T)$ is the union of at most 2^i continua $Z(p''), p'' \in P^{-i}(p)$. This easily implies that all angles in $\sigma^{-i}(E)$ are K-equivalent as claimed. Hence, all angles in E_∞ are K-equivalent.

Let us now show that if an angle γ is K-equivalent to α then $\gamma \in E_\infty$. Indeed, for the angle γ there exists an angle θ such that γ and θ are connected. Choose a point $x \in \mathrm{Imp}(\gamma) \cap \mathrm{Imp}(\theta)$. Then by Lemma 3.4 x is not remote, i.e. $P^n(x) \in B$ for some n. Hence $\gamma \in E_\infty$ as desired. The same argument shows that if impressions of two angles intersect then these angles belong to E_∞. Hence besides E_∞ there are only K-classes consisting of single angles. □

By Lemma 3.8 the Julia set is partitioned into the union of impressions of angles from the critical K-class on the one hand, and pairwise disjoint impressions of K-separate angles on the other. An example of a K-separate angle is a periodic angle whose impression does not contain the Cremer point and whose landing point (which by Theorem 3.2 coincides with its impression) is not in Q (and hence, by Lemma 3.4, does not belong to any other impression). Our main theorem studies the case when there are K-separate angles. We will first prove the following lemmas.

Lemma 3.9. *Suppose that X is a non-separating one-dimensional plane continuum for which there exists $\delta > 0$ such that for each $\theta \in S^1$, diam(Imp(θ)) $> \delta$. Then X is nowhere connected im kleinen.*

Proof. Suppose that X is connected im kleinen at x. Choose a continuum K with $\mathrm{diam}(K) < \delta/3$ and $0 < \varepsilon < \delta/3$ such that $B(x, \varepsilon) \cap X \subset K$. Choose a crosscut $C \subset B(x, \varepsilon/2)$, then $\overline{C} \cap X \subset K$. Hence, $\mathrm{diam}(\mathrm{Sh}(C)) \leq \mathrm{diam}(K \cup C) \leq 2\delta/3$, a contradiction. □

Lemma 3.9 yields Lemma 3.10; for a Cremer polynomial P denote by Z_P the intersection of all impressions.

Lemma 3.10. *Suppose that $Z_P \neq \emptyset$ for a Cremer polynomial P. Then $\omega(c) \subset Z_P$ and J is nowhere connected im kleinen.*

Proof. Take an angle θ and consider the set $A = \mathrm{Imp}(\theta) \cap \mathrm{Imp}(\theta + 1/2)$. The set A is a non-empty non-separating continuum or a point. Since $-A = \mathrm{Imp}(\theta + 1/2) \cap \mathrm{Imp}(\theta) = A$ then the map $z \to -z$ maps A onto itself, and so A must contain the fixed point of this map. Since the fixed point $0 = c$ of this map is unique then $c \in A$. Applying this argument to all angles, we conclude that $c \in Z_P$. Since Z_P is an invariant compact set then $\omega(c) \subset Z_P$. Hence there is $\delta > 0$ such that $\mathrm{diam}(\mathrm{Imp}(\theta)) > \delta$ for any angle θ. By Lemma 3.9, J is nowhere connected im kleinen. $\square$

We are ready to prove our main theorem.

Theorem 3.11. *For a quadratic Cremer polynomial P the following facts are equivalent:*

1. *there is an impression not containing the Cremer point;*
2. *there is a degenerate impression;*
3. *the set Y of all K-separate angles with degenerate impressions contains all angles with dense orbits (Y contains a full Lebesgue measure G_δ-set dense in S^1) and a dense in S^1 set of periodic angles, and the Julia set J is connected im kleinen at landing points of these rays;*
4. *there is a point at which the Julia set is connected im kleinen;*
5. *not all angles are K-equivalent.*

In this case a unique infinite K-class and K-classes of K-separate angles form the entire family of K-classes.

Proof. Let us prove that (1) implies (2). Indeed, suppose that there is an angle not containing p in its impression. Then by upper semicontinuity of impressions we can find a periodic angle γ not containing p in its impression. By Theorem 3.2 then $\mathrm{Imp}(\gamma)$ is degenerate as desired.

Let us show that (2) implies (3). Suppose that $\mathrm{Imp}(\gamma) = \{x\}$ is a point. Considering $\mathrm{Imp}(\gamma)$ and $\mathrm{Imp}(\gamma + 1/2) = \{-x\}$ we may assume that $x \neq p$ and hence for all angles from a small arc U containing γ their impressions do not contain p either. So, there are no angles of F in U. Let us show that we may shrink U to a smaller open arc $U' = (\tau', \tau'')$ with $\mathrm{Imp}(\tau') \cap \mathrm{Imp}(\tau'') \neq \emptyset$. Indeed, since $F_\infty = \cup \sigma^{-i}(F)$ is dense in S^1, we can find a number n such that $\sigma^{-n}(F) \cap U$ is infinite. Now, there are only finitely many P^{-n}-preimages of p, and each impression in the union $\cup_{\gamma \in \sigma^{-n}(F)} \mathrm{Imp}(\gamma) = P^{-n}(B)$ contains at least one P^n-preimage of p (observe that $P^{-n}(B)$ is a continuum). Thus, there are two angles $\tau' < \tau''$ from $\sigma^{-n}(F) \cap U$ (the order is considered within U) such that impressions of both angles contain the same P^{-n}-preimage of p and so $\mathrm{Imp}(\tau') \cap \mathrm{Imp}(\tau'') \neq \emptyset$. Set $U' = (\tau', \tau'')$.

Consider the two connected open components of $\mathbb{C} \setminus R_{\tau'} \cup \mathrm{Imp}(\tau') \cup \mathrm{Imp}(\tau'') \cup R_{\tau''}$; let V be the component containing rays of angles from U'. Then there are points of J in V. Indeed, otherwise $\mathrm{Imp}(\tau') \cup \mathrm{Imp}(\tau'')$ contains the impressions of all angles from U' which yields that a forward σ-image of τ' or τ'' will coincide with J implying by Theorem 3.1 that J is indecomposable and all impressions coincide with J, a contradiction. Since $F \subset S^1 \setminus U'$ then $B = \cup_{\gamma \in F} \mathrm{Imp}(\gamma)$ is contained in the union of impressions of angles from $S^1 \setminus U'$ which implies that B is disjoint from V. Take a periodic point $z \in V$ and an angle $\gamma \in U'$ such that R_γ lands at z. Then $\mathrm{Imp}(\gamma) = \{z\}$ by Theorem 3.2, and $z \notin B$ by the above. Hence z never maps into B (recall that B is invariant) which by Lemma 3.8 implies that γ is K-separate. So (2) implies the existence of (periodic) K-separate angles with degenerate impressions in any open set U which contains an angle with degenerate impression not coinciding with $\{p\}$.

Let us prove that if γ is a K-separate angle and θ is an angle with $\gamma \in \omega_\sigma(\theta)$ then θ is K-separate and its impression is a point. Indeed, by Theorem 3.5 and since $\mathrm{Imp}(\gamma)$ and B are disjoint we can find angles $\tau' < \theta < \tau''$ such that $\mathrm{Imp}(\tau') \cap \mathrm{Imp}(\tau'') \neq \emptyset$ and the connected open component V of $\mathbb{C} \setminus R_{\tau'} \cup \mathrm{Imp}(\tau') \cup \mathrm{Imp}(\tau'') \cup R_{\tau''}$ which contains the rays of angles of (τ', τ'') is positively distant from B. Observe that since V is in fact simply connected then we can find two Jordan disks $W' \supset \overline{W''} \supset J \cap V$ which are both positively distant from B. On the other hand, B is a forward invariant closed set containing c, hence the closure of the entire orbit of c is contained in B. Therefore all pull-backs of W' and W'' are univalent. By well-known results of Mañé [Mn93] this implies that the diameter of the pull-backs of W'' converge to 0 as the power of the map approaches infinity. Observe that as σ-images of θ approach γ, the corresponding P-images of $\mathrm{Imp}(\theta)$ get closer and closer to $\mathrm{Imp}(\gamma)$ (because of upper semicontinuity of impressions) and thus we may assume that infinitely many P-images of $\mathrm{Imp}(\theta)$ are contained in W''. Pulling W'' back along the orbit of $\mathrm{Imp}(\theta)$ for more and more time we see that the diameter of $\mathrm{Imp}(\theta)$ cannot be positive, and hence $\mathrm{Imp}(\theta) = \{y\}$ is a point as claimed. Moreover, if θ is not K-separate then by Lemma 3.8 $P^n(y) \in B$ from some time on. However, since by the assumption images of θ approach γ then this would imply that there are points of B in $\mathrm{Imp}(\gamma)$, a contradiction.

Since by the paragraph preceding the previous one there exist K-separate angles then by the previous paragraph all angles whose σ-orbits are dense in S^1, are K-separate and have degenerate impressions. Since the set of all angles whose σ-orbits are dense in S^1, is a full Lebesgue measure dense G_δ-set, the appropriate part of claim (3) follows. Above we showed that in any neighborhood of a K-separate angle with degenerate impression there is a periodic K-separate angle with degenerate impression, hence the second part of (3) follows. Now, for any K-separate angle γ with degenerate impression $\mathrm{Imp}(\gamma) = \{x\}$ the point x is remote by Lemma 3.8 which allows us to apply Theorem 3.5 and conclude that the Julia set is connected im kleinen at such points x. This finally completes the proof of the fact that (2) implies (3). Clearly, (3) implies (4).

Let us show that (4) implies (5). Suppose otherwise. Then the critical class is the entire S^1 which implies that (1) fails. Indeed, if (1) holds then by the already proven so does (3), a contradiction. So, all impressions contain p. By Lemma 3.10 then J is nowhere connected im kleinen, a contradiction.

Finally, (5) implies (1) because if (1) fails then all impressions are trivially K-equivalent.

The rest of the theorem follows from Lemma 3.8. □

Theorem 3.11 addresses the case when there are impressions which do not contain the Cremer point p. The following corollary addresses the case when all impressions contain the Cremer point.

Corollary 3.12. *Suppose that J is a quadratic Cremer Julia set. Then the following facts are equivalent:*

(1) *All impressions are non-degenerate.*
(2) *The intersection of all impressions is a non-degenerate subcontinuum of J containing the Cremer point and the limit set of the critical point.*
(3) *J is nowhere connected im kleinen.*

Proof. By Theorem 3.11, (1) implies that all impressions contain the Cremer point. Hence by Lemma 3.10 (2) holds. By Lemma 3.9, (2) implies (3). By Theorem 3.11, (3) implies (1). □

Finally, in Corollary 3.13 we discuss how the conditions from Theorem 3.11 relate to the biaccessibility of points of Cremer Julia sets.

Corollary 3.13. *Suppose that there is a biaccessible point $z \in J$. Then J satisfies the conditions of Theorem 3.11; in particular, there exists a full measure dense G_δ-set of angles such that the impressions of their rays are degenerate and J is connected im kleinen at the landing points of these rays. Moreover, if θ is an angle such that R_α lands at p then $\theta \in F$.*

Proof. First of all let us remind the reader of the result of Schleicher and Zakeri proven in [SZ99, Zak00] according to which if P is a Cremer polynomial and J is its Julia set and $z \in J$ is a biaccesible point then there exists $n \geq 0$such that $P^n(z) = p$. Thus if J has biaccessible points then the Cremer point p is biaccessible. Let us show now that if θ is an angle such that p is the landing point of R_θ then $\theta \in F$. Indeed, by the known results of Douady and Hubbard [DH85] the ray R_0 lands at a repelling fixed point of P, hence $\theta \neq 0$. Now, if $\theta \neq 0$ and $\theta \notin F$, then it is known (see, e.g., [BS94]) that there will be two iterates θ', θ'' of θ contained in (α, β) and (β, α) respectively. Then the rays $R_{\theta'} = R'$ and $R_{\theta''} = R''$ together with the point p cut the plane into two connected pieces, say, U and V; for the case of definiteness let $R_\alpha \subset U, R_\beta \subset V$. Suppose that $-p \in U$. Then $\mathrm{Imp}(R_\beta)$ cannot contain $-p$, a contradiction with Theorem 4.3 from [GMO99] according to which $p, c, -p \subset \mathrm{Imp}(\alpha) \cap \mathrm{Imp}(\beta)$. Hence $\theta \in F$.

Now, suppose that J satisfies the conditions of Corollary 3.12 and R_1, R_2 are two rays which land at p. Let us pull p back to its preimage $-p$ and denote by $\hat{R}_1$ and $\hat{R}_2$ the pull-backs of the rays R_1, R_2. Then the union of $\hat{R}_1$, $\hat{R}_2$ and $-p$ cuts the plane into two pieces one of which, denoted by W, does not contain p. Clearly, this implies that any angle which originates in W gives rise to an impression not containing p. However this contradicts condition (2) from Corollary 3.12. We conclude that J cannot satisfy these conditions; hence it must satisfy the conditions of Theorem 3.11 as desired. ☐

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