# ROTATION NUMBERS, TWISTS AND A SHARKOVSKII-MISIUREWICZ-TYPE ORDERING FOR PATTERNS ON THE INTERVAL


A. M. Blokh

Department of Mathematics, University of Alabama at Birmingham,
Birmingham, AL 35294-2060, USA



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Abstract. We introduce *rotation numbers* and *pairs* characterizing cyclic patterns on an interval and a special order among them; then we prove the theorem which specializes the Sharkovskii theorem in this setting.


## 0. Introduction

One of the remarkable results in one-dimensional dynamics is the Sharkovskii theorem. To state it let us first introduce the *Sharkovskii ordering* for positive integers:

$$(*)\qquad 3 \succ_S 5 \succ_S 7 \succ_S \dots \succ_S 2\cdot 3 \succ_S 2\cdot 5 \succ_S 2\cdot 7 \succ_S \dots \succ_S 8 \succ_S 4 \succ_S 2 \succ_S 1$$

Denote by $Sh(k)$ the set of all integers $m$ such that $k \succeq_S m$ and by $Sh(2^\infty)$ the set $\{1, 2, 4, 8, \dots\}$. Also denote by $P(\varphi)$ the set of periods of cycles of a map $\varphi$.

**Theorem S[S].** *If $g : [0,1] \to [0,1]$ is continuous, $m \succ_S n$ and $m \in P(g)$ then $n \in P(g)$ and so there exists $k \in \mathbb{N} \cup 2^\infty$ with $P(g) = Sh(k)$.*

Theorem S characterizes sets of periods of interval maps. Similar result concerning circle maps of degree one is due to Misiurewicz. To state it we need some more definitions. Let $f : S^1 \to S^1$ be a map of degree 1, $\pi : \mathbb{R} \to S^1$ be the natural projection (i.e. the one which maps an interval $[0, 1)$ onto the whole circle); let us fix a lifting $F$ of $f$. If $x \in S^1, X \in \pi^{-1}x$ then the set of all limit points of the sequence $\dfrac{F^n(X)}{n} \equiv I_F(x)$ is an interval which does not depend on $X$. If $I_F(x) = \{\rho_F(x)\}$ is a one-point set then $\rho_F(x)$ is the *$F$-rotation number* of $x$. The following theorem summarizes some results from [I],[NPT].

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**Theorem INPT [I],[NPT].** (1) *The set* $\bigcup_{x\in S^1} I_F(x) \equiv I_F$ *of all limit points of sequences* $\{\frac{1}{n}F^n(x)\}_{n=1}^{\infty}$, $x \in S^1$, *is a closed interval.*

(2) *If* $f$ *has at least one periodic point then the set of all rotation numbers of periodic points is dense in* $I_F$; *otherwise* $f$ *is monotonically semiconjugate to an irrational rotation by an angle* $2\pi\alpha$ *and* $I_F = \{\alpha\}$.

The set $I_F$ is called the *rotation interval* of $F$; fixing $F$ we write $I_f$ instead of $I_F$. If $x$ is an $f$-periodic orbit of period $q$ and $X$ is its lifting then there exists a well-defined integer $p$ such that $F^q(X) = X + p$. Denote a pair $(p,q)$ by $rp(x)$ and call it the *rotation pair* of $x$; then $\rho(x) = p/q$. Denote by $RP(f)$ the set of all rotation pairs of cycles of $f$. For real numbers $a \le b$ let $N(a,b) = \{(p,q) \in \mathbb{Z}^2 : p/q \in (a,b)\}$. For $a \in \mathbb{R}$ and $l \in \mathbb{Z}^+ \cup \{2^\infty\} \cup \{0\}$ let $Q(a,l)$ be empty if $a$ is irrational or $l = 0$; otherwise let it be $\{(ks, ns) : s \in Sh(l)\}$ where $a = k/n$ with $k, n$ coprime (see [M1]).

**Theorem M1[M1].** *For a continuous circle map* $f$ *of degree* 1 *there exist* $a, b \in \mathbb{R}, a \le b$ *and* $l, r \in \mathbb{Z}^+ \cup \{2^\infty\}$ *such that* $I_f = [a,b]$, $RP(f) = N(a,b) \cup Q(a,l) \cup Q(b,r)$.

In fact the aforementioned rotation numbers for circle maps of degree one are a particular case of more general *functional rotation numbers*; we consider them in [B3] using the approach which is somewhat close to that of Misiurewicz-Ziemian (see [MZ]) so we will touch the subject here very briefly. Let $X$ be a one-dimensional branched manifold (a *graph*), $\phi : X \longrightarrow \mathbb{R}^n$ be a Borel measurable bounded function. Then the set $I_{f,\phi}(x)$ of all limit points of the sequence $\frac{1}{n}\sum_{i=0}^{n-1}\phi(f^i x)$ is closed and connected. If $I_{f,\phi}(x) = \{\rho_{f,\phi}(x)\}$ is a one-point set then $\rho_{f,\phi}(x)$ is called the *$\phi$-rotation number* of $x$. The union $I_f(\phi)$ of all sets $I_{f,\phi}(x)$ taken over the appropriate set of points (generally those whose orbits to some extent avoid the set of discontinuities of $\phi$) is called in [B3] the *$\phi$-rotation set* of $f$. The above introduced rotation numbers and sets are called *functional*.

In [B3] we study functional rotation numbers and sets relying on the "spectral decomposition theorem" ([B1-B2]). We prove that $\phi$-rotation numbers of periodic points and of points with $\omega$-limit sets on which the map is somewhat similar to an irrational rotation are dense in the rotation set. We show that if $\gamma \in I_{f,\phi}$ then there is a point $y$ such that $\rho_{f,\phi}(y) = \gamma$. We also study sufficient conditions for the functional rotation set of circle or interval maps to be connected; the fact that classical rotation numbers are a particular case of the above defined functional rotation numbers (see, e.g. [MZ]) allows to obtain Theorem INPT as a corollary of our results. Indeed, let $f : S^1 \longrightarrow S^1$ be a map of degree 1, $\pi : \mathbb{R} \longrightarrow S^1$ be a natural projection and $F$ be a lifting of $f$. Define $\phi_f : S^1 \longrightarrow \mathbb{R}$ so that $\phi_f(x) = F(X) - X$ for any point $X \in \pi^{-1}x$ (it is easy to see that $\phi_f$ is well-defined). Then the classical rotation number of the point $z$ equals $\rho_{f,\phi_f}(z)$ whenever it exists, and the classical rotation set is $I_f(\phi_f)$. The choice of $\phi_f$ here is important; provided there is a connection between $\phi_f$ and $f$ one may be able to derive a lot of information about the map $f$ from the $\phi_f$-rotation interval. The main idea of what follows is that by choosing an appropriate function $\phi$ we may get results for interval maps similar to Theorems INPT and M1.

Let $f : [0,1] \longrightarrow [0,1]$ be continuous, $Per f$ be its set of periodic points, $Fix f$ be its set of fixed points. It is easy to see that if $Per f = Fix f$ then $\omega(x)$ is a fixed point for any $y$. Let $Perf \neq Fixf$ and $A = \{y : \omega(y)$ does not contain a fixed point $\}$. Let $L = \{x : x > f(x)\}$ and $\chi_L = \chi$ be the indicator function of the set $L$. In this paper we mainly study $\chi$-rotation numbers of periodic points of $f$. More precisely, for any non-fixed periodic point $y$ of period $p(y)$ the number $l(y) = card\{orb(y) \cap L\}$ is well-defined; we call the pair $rp(y) = (l(y), p(y))$ the *rotation pair* of $y$ and denote the set of all rotation pairs of periodic non-fixed points of $f$ by $RP(f)$. Also, the $\chi$-rotation number $\rho_\chi(y) = \dfrac{l(y)}{p(y)}$ will be simply called the *rotation number* of $y$. Let us introduce the following ordering among all pairs of positive integers $(k, n)$ such that $k < n$:

1) if $k/l \in (1/2, p/q)$ then $(p, q) \gtrdot (k, l)$;

2) if $p/q \neq 1/2$ then $(p, q) \gtrdot (k, 2k)$ for any $k$;

3) if $p/q = k/l = m/n$ where $m, n$ are coprime then $(p, q) \gtrdot (k, l)$ if and only if $(p/m) \succ_S (k/m)$ (note that both $(p/m)$ and $(k/m)$ are integers).

**Main Theorem.** *If $(p, q) \gtrdot (k, l)$ and $(p, q) \in RP(f)$ then $(k, l) \in RP(f)$.*

It is easy to see that the Main Theorem implies Corollary 3.4; before we state it let us note that for any rational $p/q$, $p, q$ coprime, the set $Q(p/q, 3)$ is in fact the set of all pairs $(ps, qs)$, $s \in \mathbb{Z}^+$.

**Corollary 3.4.** (1) *For a continuous interval map $f$ with non-fixed periodic points there exist $0 \leq a \leq 1/2 \leq b \leq 1$ and $l, r \in \mathbb{Z}^+ \cup \{2^\infty\} \cup \{0\}$ such that $RP(f) = N(a, b) \cup Q(a, l) \cup Q(b, r)$, if $a < b = 1/2$ then $r = 3$, if $a = 1/2 < b$ then $l = 3$, if $a = b = 1/2$ then $r = l \neq 0$, if $a = 0$ then $l = 0$ and if $b = 1$ then $r = 0$.*

(2) *If $a, b, l, r$ are numbers satisfying all the properties from the statement (1) then there is a continuous interval map $f$ such that $RP(f) = N(a, b) \cup Q(a, l) \cup Q(b, r)$.*

In the second statement of Corollary 3.4 we rely upon the following result from a forthcoming paper [B4] where we deal with rotation numbers and pairs for unimodal maps.

**Theorem 0.1 [B4].** (1) *For any $0 < a < 1$, $a \neq 1/2$ and any $l \in \mathbb{Z}^+$ there is a unimodal map $f$ such that $RP(f) = N(a, 1/2) \cup Q(a, l) \cup Q(1/2, 3)$.*

(2) *For any $l \in \mathbb{Z}^+$ there is a unimodal map $f$ such that $RP(f) = Q(1/2, l)$.*

Corollary 3.4 may be considered as an analog of Theorem M1.

The results of the paper were the subject of the author's talks at the Topology and Dynamical Systems seminar at UAB in the September, 1992, semi-annual conference on dynamical systems at Maryland University in March, 1993 and 879th AMS meeting in Knoxville in March, 1993 (see Abstracts of AMS, # 2, **14** (1993), p. 298).

## 1. Preliminaries

We begin this section with some preliminary definitions and results concerning so-called *combinatorial dynamics*; the best sources are [MN] and [ALM] and we refer to them for more extensive list of relevant publications.

A *pattern* is a cyclic permutation of the set $1, 2, \ldots, n$; strictly speaking we should use the term *cyclic pattern* but we deal in the present paper only with cyclic patterns, so we use our terminology for the sake of convenience (in [ALM] they are called *oriented patterns*). If an interval map $f$ on its periodic orbit $P$ is conjugate to a pattern $\pi$ by a strictly monotone increasing map then $P$ is a *representative* of $\pi$ in $f$ and $f$ *exhibits* $\pi$ on $P$ ([MN]). A pattern $\pi$ *forces* a pattern $\theta$ if every continuous interval map $f$ which exhibits $\pi$ also exhibits $\theta$. Baldwin [Ba] showed that forcing is a partial ordering. Let a map $f : [0,1] \longrightarrow [0,1]$ exhibit a pattern $\pi$ on $P$. Two points $x, y \in P$ are called *adjacent* if $(x, y) \cap P = \emptyset$. By $P$-interval (or $\pi$-*interval*) we mean any of the closed intervals bounded by adjacent elements of $P$. The map $f$ is $P$-*monotone* (or $\pi$-*monotone*) if $0, 1 \in P$ and $f$ is monotone on each $P$-interval. By [MN] for any pattern $\pi$ there is a $\pi$-monotone map $L_\pi$ such that $\pi$ forces a pattern $\theta$ if and only if $L_\pi$ exhibits $\theta$; $L_\pi$ is called $P$-*adjusted* (or $\pi$-*adjusted*) map. For a pattern $\pi$ let $RP(\pi) = RP(L_\pi)$. Let $I_0, \dots$ be intervals such that $f(I_j) \supset I_{j+1}$ for $0 \le j$; then we say that $I_0, \dots$ is a *chain* of intervals (note that a chain of intervals need not to be finite). If a finite chain of intervals $I_0, \dots, I_{k-1}$ is such that $f(I_{k-1}) \supset I_0$ then we call $I_0, \dots, I_{k-1}$ a *loop* of intervals. Let us now state the following folklore

**Lemma 1.1.** (1) *Let $I_0, \dots$ be an infinite chain of intervals. Then there is an interval $M$ such that $f^j(M) \subset I_j$ for all $j$.*

(2) *Let $I_0, \dots, I_k$ be a finite chain of intervals. Then there is an interval $M$ such that $f^j(M) \subset I_j$ for $0 \le j \le k-1$ and $f^k(M) = I_k$.*

(3) *Let $I_0, \dots, I_k$ be a loop of intervals. Then there is a periodic point $x$ such that $f^j(x) \in I_j$ for $0 \le j \le k-1$ and $f^k(x) = x$.*

Let us say that a map $f$ has the *right horseshoe* if there are points $a, b, c$ such that $f(c) \le a = f(a) < b < c \le f(b)$ and the *left horseshoe* if there are points $a, b, c$ such that $f(c) \ge a = f(a) > b > c \ge f(b)$.

**Lemma 1.2.** *If a map $f$ has the right (resp. the left) horseshoe $a, b, c$ then $(s, t) \in RP(f)$ for any $(s, t)$ such that $0 < s/t \le 1/2$ (resp. $1 > s/t \ge 1/2$) and the corresponding periodic orbit $Q$ with $rp(Q) = (s, t)$ lies completely to the right (resp. left) of $a$; in particular if a map $f$ has both the right and the left horseshoe then $RP(f) = \{(s, t) : 0 < s < t\}$.*

*Proof.* Let $f$ have the right horseshoe. We may assume that $f(b) = c, f(c) = a$ and $x < f(x)$ for any $a < x \le b$. If $d$ is the closest from the left to $c$ fixed point then also $f(y) < y$ for any $d < y \le c$. Let $J = [a, b], K = [b, d], L = [d, c]$; obviously $f(J) \supset J \cup L \cup K, f(K) \supset L, f(L) \supset J \cup K$. Then for a given $(s, t)$ such that $0 < s/t \le 1/2$ let $\{I_r\}_{l=0}^{t-1}$ be a sequence of intervals such that $I_0 = \dots = I_{t-2s-1} = J, I_{t-2s+2r} = J, I_{t-2s+2r+1} = L (0 \le r \le s-2), I_{t-2} = K, I_{t-1} = L$. By Lemma 1.1 there is a periodic point $x$ such that $f^j(x) \in I_j$ for $0 \le j \le t-1$ and $f^t(x) = x$; the choice of the intervals $I_j$ easily implies that $rp(x) = (s, t)$ and by the construction the whole orbit of $x$ lies to the right of $a$ which completes the proof. □

**Lemma 1.3.** *Let $\pi$ be a pattern which forces more than one fixed point. Then:*

(1) *for any $(s, t)$, $0 < s < t$ there is a forced by $\pi$ pattern $\theta$ such that $rp(\theta) = (s, t)$;*

(2) *$RP(\pi) = \{(s, t) : 0 < s < t\}$.*

*Proof.* Let $f = L_\pi$, $P$ be a representative of $\pi$ in $f$. A $P$-interval contains an $f$-fixed point if and only if its endpoints move in different directions. Since $f$ moves 0 to the right and 1 to the left it is easy to see that if there is more than one fixed point forced by $\pi$ then there exists a $P$-interval $[x, y]$ and a fixed point $a \in (x, y)$ such that $f(x) < x < y < f(y)$. Let $i$ be the smallest such that $f^i(y) < a$;

then $i > 1$. Let $j$ be the smallest such that $f^j(y) \geq f^{i-1}(y)$. Then we have $f^i(y) \leq a = f(a) < f^{j-1}(y) < f^{i-1}(y) \leq f^j(y)$ which means that $f$ has the right horseshoe; similarly $f$ has the left horseshoe and so Lemma 1.2 implies the required. $\square$

We need more definitions. If a pattern $\pi$ is semiconjugate to a pattern $\pi'$ by a monotone (not necessarily strictly) increasing map then $\pi'$ is said to be a *reduction*) of $\pi$ and $\pi$ is said to have a *block structure* over $\pi'$ [MN]. A fixed point pattern is a reduction (called *trivial*) of any pattern. The following lemma is stated without proof.

**Lemma 1.4.** *A pattern forces all its reductions; furthermore, all its reductions have periods smaller or equal then the period of the pattern itself.*

A pattern which has a block structure over the two-periodic pattern is said to have *division* [LMPY]. If we speak *mixing* maps we always mean *topologically mixing*; a pattern $\pi$ is said to be *mixing* if its $\pi$-adjusted map is mixing.

**Theorem MN1 [MN].** *If $\pi$ is a pattern then it has either division or a block structure over a mixing pattern.*

The following lemma is obvious, so we state it without proof.

**Lemma 1.5.** (1) *If $\pi'$ is a non-trivial reduction of $\pi$ then $\rho(\pi') = \rho(\pi)$.*

(2) *Any pattern $\pi$ forces a pattern $\theta$ such that $\rho(\theta) = \rho(\pi)$, period of $\theta$ is smaller or equal than that of $\pi$ and $\theta$ is either mixing or of period* 2.

Let $\mathcal{U}$ be the family of all self-mappings of the interval $[0, 1]$ with a unique fixed point, say, $a$; then all the points to the left of $a$ are mapped to the right and all the points to the right of $a$ are mapped to the left. If an adjusted map corresponding to a pattern has more than one fixed point then by Lemma 1.3 its family of rotation pairs is $\{(s, t) : 0 < s < t\}$; thus the Main Theorem in case when a pattern $\pi$ with $rp(\pi) = (p, q)$ forces more than one fixed point follows immediately from Lemma 1.3 and so *from now on we assume that the patterns we consider force a unique fixed point and maps we consider belong to $\mathcal{U}$.* We study a kind of symbolic dynamics for these maps. Let us call a non-degenerate interval $I$ *admissible* if one of its endpoints is $a$. We call a chain (a loop) of admissible intervals $I_0, I_1, \dots$ *admissible.* Note that if $I_0, \dots, I_{k-1}$ is an admissible loop then $k > 1$ since the image of an admissible interval cannot contain this interval.

Let $\phi$ be a function defined on the family of all admissible intervals such that $\phi([b, a]) = 0$ if $b < a$ and $\phi([a, c]) = 1$ if $a < c$. For any admissible loop $\bar{\alpha} = \{I_0, \dots, I_{k-1}\}$ let us call the pair of numbers $(p, k)$ the *rotation pair* of $\bar{\alpha}$ where $p$ is the number of indices $0 \leq s \leq k-1$ such that $\phi(I_s) = 1$; also let us call the number $\rho(\bar{\alpha}) = k^{-1} \cdot \sum_{j=0}^{k-1} \phi(I_j)$ the *rotation number* of $\bar{\alpha}$. We finish this series of definitions with the following one: a sequence $\{y_1, \dots, y_l\}$ is called *non-repetitive* if it cannot be represented as several repetitions of a smaller sequence.

**Lemma 1.6.** *Let $f \in \mathcal{U}$ and $\bar{\alpha} = \{I_0, \dots, I_{k-1}\}$ be an admissible loop. Then there are the following possibilities.*

(1) *Let $k$ be even, $\phi(I_j) = 0$ if $j$ is even and $\phi(I_j) = 1$ if $j$ is odd. Then $f$ has a point $x$ of period* 2.

(2) *Let the first possibility fail. Then there is a periodic point* $x \in I_0$ *such that* $x \neq a, f^j(x) \in I_j (0 \leq j \leq k-1), f^k(x) = x$ *and so* $\rho(x) = \rho(\bar{\alpha})$. *Moreover, if the sequence of numbers* $\{\phi(I_0), \dots, \phi(I_{k-1})\}$ *is non-repetitive then* $rp(x) = rp(\bar{\alpha})$.

*Proof.* By Lemma 1.1 there is an interval $M = M(\bar{\alpha})$ such that $f^j(M) \subset I_j (0 \leq j \leq k-1)$ and $f^k(M) = I_0$. Clearly, $f^j(M)$ is non-degenerate for all $0 \leq j \leq k$. Let us show that neither $f^j(M) \supset f^{j+1}(M)$ nor $f^j(M) \subset f^{j+1}(M)$ for some $0 \leq j \leq k-1$ is possible. Indeed, then $f^l(M) \supset f^{l+1}(M)$ for $l \geq j$ and in particular $I_0 = f^k(M) \supset f^{k+1}(M) = f(I_0) \supset I_1$. Since all the intervals $I_0, I_1, \dots$ have $a$ as one of the endpoints we conclude that $I_0 \supset I_1 \supset \dots \supset I_0$ and so $I_0 = f(I_0) = I_1$; clearly it is impossible for $f \in \mathcal{U}$. On the other hand $f^j(M) \subset f^{j+1}(M)$ is impossible either due to obvious properties of maps from $\mathcal{U}$ and the fact that $a \notin int\, f^j(M)$.

Let us consider the two possibilities from Lemma 1.6. Suppose the first one holds and $f$ has no 2-periodic points. Then the only $f^2$-fixed point is $a$, $f(0) > 0$ and so $f^2(x) > x$ for any $x \in [0, a)$; thus $M \subset I_0$ and $f^k(M) = I_0$ is impossible which contradicts Lemma 1.1 and shows that there must be 2-periodic points. Suppose now that the second possibility holds. Then there is $j$ such that $I_j$ and $I_{j+1}$ lie both to the same side of $a$. Let us show that $a \notin f^j(M)$. Indeed, otherwise intervals $f^j(M)$ and $f^{j+1}(M)$ lie to the same side of $a$ vand contain $a$, so one of them contains the other which is impossible. Hence $a \notin f^j(M)$ and there is a periodic point $x$ with the required properties. The rest of the lemma easily follows. □

Any point $x$ with the properties from Lemma 1.6 is said to be *generated* by $\bar{\alpha}$.

## 2. Interval twists and their properties

In this section we study properties of patterns of special kind called *twists*; they are used in the proof of the Main Theorem but seem to be of interest by themselves. We call a pattern $\pi$ a *twist pattern* or simply *twist* if it does not force other patterns of smaller or equal period which have the rotation number $\gamma$ such that $[\gamma, 1/2] \supset [\rho(\pi), 1/2]$. Let $\pi$ be a pattern of period $m$ and the only representative of $\pi$ in $f = L_\pi$ be $Q$. Then due to the properties of adjusted maps $\pi$ is a twist if and only if $Q$ is the only periodic orbit of $f$ of period less than or equal to $m$ such that $[\rho(Q), 1/2] \supset [\rho(\pi), 1/2]$. The following corollary follows from Lemma 1.3.

**Corollary 2.1.** *A twist forces a unique fixed point.*

Corollary 2.2 follows from Lemma 1.5.

**Corollary 2.2.** *A twist is either of period* 2 *or mixing.*

Let us introduce notations we use till the end of this section. From now on let $\pi$ be a pattern with $rp(\pi) = (r, n)$ such that $f = L_\pi \in \mathcal{U}$ (i.e. $f$ has a unique fixed point); by Corollary 2.1 this is true if $\pi$ is a twist. Clearly the cases when $2r \leq n$ and $2r \geq n$ are similar, so assume that $2r \leq n$ (and so $\rho(\pi) = r/n \leq 1/2$). Let $P$ be the representative of $\pi$ in $f$, $A_0 = \{x \in P : x < a, f(x) < a\}$, $A_1 = \{x \in P : x < a, f(x) > a\}$, $A_2 = \{x \in P, a < x\}$. Also, for any periodic orbit $X = \{x_0, x_1 = f(x_0), \dots, x_{k-1} = f^{k-1}(x_0)\}$ of $f$ let us denote the admissible loop $\{[x_0, a], [x_1, a], \dots, [x_{k-1}, a]\}$ by $\bar{\alpha}(X)$.

**Lemma 2.3.** *Let* $\pi$ *be a twist. Then the following holds.*

(1) $f(A_2)$ *lies to the left of* $a$.

(2) *f is increasing on* $A$ *and decreasing on* $A \cup A$

(3) *$f^2(x) < x$ for $x \in A_1$.*

*Proof.* (1) Let $x \in A_2$, $f(x) > a$; consider a new admissible chain $\bar\beta$ obtained from $\bar\alpha(P)$ by omitting the interval $[a, f(x)]$. Then $\bar\beta$ is an admissible loop and $0 < \rho(\bar\beta) = \dfrac{r-1}{n-1} < \dfrac{r}{n}$. By Lemma 1.6 a periodic point $x$ generated by $\bar\beta$ has the period smaller than that of $P$; also $\rho(x) = \rho(\bar\beta) < \rho(P)$ contradicting the fact that $\pi$ is a twist.

(2) If the statement of the lemma fails then there exist $z', z'' \in P$, $I' = [a, z'], I'' = [a, z'']$ such that $I' \subset I'', [a, f(z')] \supset [a, f(z'')]$. Let us consider the sequence of intervals $\bar\beta$ which is obtained from $\bar\alpha(P)$ by replacing $I''$ with $I'$. Then $\bar\beta$ is admissible. If $x \in I'$ is a periodic point generated by $\bar\beta$ then its period is not bigger than $n$ and $\rho(x) = \rho(\pi)$. Let us prove that $x \notin P$. Indeed, let $b$ be the closest from the left to $a$ point from $P$. Suppose that $l < n$ is such that $f^l(z'') = b$; then by the construction and the properties of $x$ we have $f^l(x) \in [b, a]$. If $f^l(x) \neq b$ then $x \notin P$ for there are no points from $P$ in $(b, a]$. But if $f^l(x) = b$ then $f^{n-l}(f^l(x)) = x = f^{n-l}(b) = f^{n-l}(f^l(z'')) = z''$ which is impossible since $x \in I'$. This proves that $x \notin P$ and so $\pi$ is not a twist which is a contradiction.

(3) If the statement of the lemma fails then there is $x \in A_1$ such that $[x, a] \supset [f^2(x), a]$ (note that due to (1) $f^2(x) < a$). Omitting the intervals $[x, a], [f(x), a]$ from the admissible loop $\bar\alpha(P)$ we obtain a new admissible loop $\bar\beta$ with $\rho(\bar\beta) = \dfrac{r-1}{n-1} < \rho(P)$ for $\rho(P) \le 1/2$ which implies that $\pi$ is not twist. $\square$

The following technical lemma plays an important role in our studying of properties of twists and in the proof of the Main Theorem.

**Lemma 2.4.** *Suppose that $rp(\pi) = (r, n)$, $\rho(\pi) = r/n \le p/q \le 1/2$, $y \in P$, numbers $m, k$ are such that $f^m(y) < y < f^{m+1}(y) < a$ and among points $y, f(y), \dots, f^m(y)$ there are exactly $k$ those lying to the right of $a$. If $\pi$ does not force a pattern $\theta$ such that $rp(\theta) = (s, t), s \le \max(r, k), \rho(\theta) \le p/q$ then $pm + p > kq > pm$; in particular, the above described dynamics is impossible for $k = p$.*

*Proof.* Suppose that $\pi$ does not force a pattern $\theta$ such that $rp(\theta) = (s, t)$, $s \le \max(r, k)$ and $\rho(\theta) \le p/q$. Consider an admissible loop $\bar\alpha = \{[y, a], [f(y), a], \dots, [f^{m-1}(y), a]\}$; by Lemma 1.6 $\bar\alpha$ generates a periodic point $x$ with the rotation number $\rho(x) = k/m$; also $x \notin P$ since $f^m(x) = x$ and $f^m(y) \neq y$. So by the assumption $k/m > p/q$ which implies that $kq > pm$ proving one of the parts of the required inequality.

Now, let $i$ be such that $(i-1)n < m < m+1 < in$. Let us consider an admissible loop $\bar\beta = \{[f^{m+1}(y), a], [f^{m+2}(y), a], \dots, [f^{in-1}(y), a]\}$. Obviously among points $f^{m+1}(y), \dots, f^{in-1}(y)$ there are $ir - k$ those lying to the right of $a$. Hence by Lemma 1.6 $\bar\beta$ generates a periodic point $z$ with the rotation number $\rho(z) = \dfrac{ir-k}{in-m-1}$. The choice of $i$ implies that $ir - k < \max(r, k)$, so by the assumption $\dfrac{ir-k}{in-m-1} > \dfrac{p}{q}$ which implies that $irq + pm + p > kq + inp$. Since $r/n \le p/q$ we conclude that $pm + p > kq$ which completes the proof. $\square$

Corollary 2.5 follows from Lemma 2.4 if $(p, q) = (r, n)$. It seems that this corollary may be important for describing twists.

**Corollary 2.5.** *Suppose that $\pi$ is a twist, $y \in P$ and numbers $m < n, k$ are such that $f^m(y) < y < f^{m+1}(y) < a$ and among points $y, f(y), \dots, f^m(y)$ there are* *exactly $k$ those lying to the right of $a$. Then $rm + r > kn > rm$.*

## 3. The proof of the Main Theorem

The main step in this section is the following Lemma 3.1; in its proof we rely upon Lemma 2.4 and properties of twists.

**Lemma 3.1.** *If $\pi$ is a mixing pattern, $rp(\pi) = (r, n)$ and $(p, q)$ is such that $[r/n, 1/2] \supset [p/q, 1/2]$ then $\pi$ forces a pattern $\theta$ with $rp(\theta) = (p, q)$ unless $r/n = p/q$ and $r, n$ are coprime.*

Before we prove Lemma 3.1 let us show how it implies Theorem S and the Main Theorem.

*Proof of Theorem S.* Let $m \succ_S n$ and $m \in P(f)$; we need to show that $n \in P(f)$. Let $(p, m)$ be a rotation pair for an $m$-periodic point $y$ whose orbit exhibits a pattern $\pi$. We consider some cases.

(1) $m > 1$ *is odd.* For the sake of definiteness let $p/m < 1/2$. By Theorem MN1 $\pi$ has a mixing reduction $\pi'$. If $rp(\pi') = (p', m')$ then $m' \le m$ is odd, so $m' \succ_S n$. On the other hand by Lemma 1.4 $\pi$ forces $\pi'$ so there is a periodic point $y'$ which exhibits the pattern $\pi'$. Since $m' \succ n$ there is an integer $s$ such that $p'/m' < s/n \le 1/2$. By Lemma 3.1 it implies that $f$ has a point of period $n$.

(2) $m = 2^k, k > 1$. Then $n = 2^i, i < k$. We may assume that $k > 1$; clearly it is enough to show that then $f$ has a periodic point of period $2^{k-1}$. Indeed, first let $k = 2$ and $m = 4$. Then if $\pi$ has division then by Lemma 1.4 it forces 2-periodic pattern; otherwise $\pi$ is mixing and forces 2-periodic pattern by Lemma 3.1. Now if $k > 2$ then observe that $x$ is a 4-periodic point of $g = f^{2^{k-2}}$ and thus $g$ has a 2-periodic orbit which completes the consideration of this case.

(3) $m = 2^k(2l + 1), k, l > 0$. Then $n = 2^j(2i + 1)$ and either $j > k$, or $j = k$ and $i > l$, or $j \le k, i = 0$. To cover the first two possibilities one may apply the case (1) to $f^{2^k}$; in particular $f$ has a point of period $2^{k+1}$ which due to case (2) covers the third possibility and completes the proof. □

*Proof of the Main Theorem.* Let $\pi$ be a pattern, $rp(\pi) = (p, q)$ and $(p, q) \gtrdot (s, t)$; for the sake of definiteness let $\rho(\pi) \le 1/2$. By Theorem MN1 $\pi$ has a mixing or 2-periodic reduction $\pi'$, $rp(\pi') = (p', q')$ which is by Lemma 1.4 forced by $\pi$ and by Lemma 1.5 has the rotation number $\rho(\pi) = \rho(\pi')$. If $\rho(\pi) \ne 1/2$ then $\pi'$ is mixing which by Lemma 3.1 implies that $\pi'$ forces a pattern $\theta$ with $rp(\theta) = (s, t)$. Now let $\rho(\pi) = s/t$ and consider some possibilities.

(1) $p', q'$ *are coprime and* $p \ne p'$. The fact that $\pi'$ is a reduction of $\pi$ means that for the $\pi$-adjusted map $f = L_\pi$ there is an interval $I$ such that $I, f(I), \dots, f^{q'-1}(I)$ are pairwise disjoint, $f^{q'}(I) = I$ and $orb\, I \supset P$ where $P$ is the representative of $\pi$ in $f$. Since $(p, q) \gtrdot (s, t)$ and $\rho(\pi) = s/t$ then $q/q' \succ_S t/q'$, and so by Theorem S the fact that $P \cap I$ is a $q/q'$-periodic orbit for $f^{q'}|I$ implies that $f^{q'}|I$ has a periodic orbit $Q$ of period $t/q'$. Obviously $orb_f Q$ is a $t$-periodic orbit of rotation number $p/q = p'/q' = s/t$ which therefore has the rotation pair $(s, t)$; this completes the consideration of the first case.

(2) $p', q'$ *are coprime and* $p = p'$. Then the definition of the $\gtrdot$-order and the assumption that $p/q = s/t, (p, q) \gtrdot (s, t)$ imply that $s = p, t = q$.

(3) $p', q'$ *are not coprime.* Then Lemma 3.1 implies the required which completes the proof of the Main Theorem. □

To prove Lemma 3.1 we need the following

**Lemma 3.2.** *Let $\pi$ be a mixing pattern which forces another pattern $\pi'$ such that $rp(\pi') = (m, n)$, $m, n$ coprime, $P$ be the representative of $\pi$ in $L_\pi = f$, $Q = \operatorname{orb} z$ be a representative of $\pi'$ in $f$ and $a$ be the unique $f$-fixed point. Then the following holds.*

(1) *There exist intervals $K, J$ with disjoint interiors such that $f^i(K), f^i(J)$ and $f^i(z)$ lie to the same side of $a$ for any $0 \le i \le n-1$ and $f^n(K) \cap f^n(J) \supset K \cup J$.*

(2) *For any $(s,t)$ such that $s/t = m/n$ there is a pattern $\theta$ forced by $\pi$ such that $rp(\theta) = (s,t)$.*

*Proof.* (1) Let $x \in Q, x < a$ and consider the maximal closed interval $I = [b, c] \ni x$ such that $f^j(I)$ and $f^j(x)$ lie to the same side of $a$ for all $0 \le j \le n$. Then the maximality of $I$ implies that $f^n(c) = a$ and either $b = 0$ or $f^n(b) = a$. Let us consider $f^n|I$ and prove that there are two subintervals of $I$ whose $f^n$-images contain their union. Note that $f^n$ has only finitely many fixed points (otherwise there are two of them, say, $\zeta'$ and $\zeta''$, and $f^n|[\zeta'', \zeta]$ is monotone which contradicts the fact that $f$ is mixing). Hence if $A$ is the set of $f^n$-fixed points in $I$ then the number of points in $A$ is finite and positive (for $x \in A$). The set $B = I \setminus A$ consists of points $y$ such that $f^n(y) \ne y$; a component $C$ of $B$ is called *left (right)* if all the points from $C$ are mapped by $f^n$ to the left (right). Clearly, two neighboring components of $B$ are both left (right) if and only if their common endpoint is an extremum for $f^n$; since an extremum of any power of $f$ must be a preimage of a point from $P$ we see that one of the two components of $B$ with the common endpoint $x$ is left and the other one is right. Hence there are components of $B$ of both kinds.

Let $[\alpha, \beta]$ be a left component of $B$; we prove that $\alpha, \beta$ are $f^n$-fixed points. Clearly the endpoints of any component of the set $B$ are either $f^n$-fixed points or $b$ or $c$. If $f^n\beta \ne \beta$ then $f^n(\beta) < \beta \le a$ and so $\beta \ne c$ which is a contradiction. If $f^n(\alpha) \ne \alpha$ then $f^n(\alpha) < \alpha < a$ and so $\alpha \ne 0$, $f^n(\alpha) \ne a$ which is also a contradiction. Hence $f^n(\alpha) = \alpha, f^n(\beta) = \beta$. Let $[\alpha, \beta]$ be the leftmost among the left components of $B$, $\gamma \in [\alpha, \beta]$ be a minimum for $f^n|[\alpha, \beta]$. Since $f$ is mixing then $f^n(\gamma) < \alpha$ and $\alpha < \gamma < \beta$. Let us show that $\beta \in f^n[f^n\gamma, \alpha]$. Indeed, if $f^n(\gamma) < b$ then $b \ne 0$ and so $f^n(b) = a$ which implies the required. If on the other hand $f^n(\gamma) \ge b$ and $\beta \notin f^n[f^n\gamma, \alpha]$ then by the choice of $[\alpha, \beta]$ we have $f^n[f^n(\gamma), \beta] = [f^n(\gamma), \beta]$ which is impossible for a mixing $f$. Thus $K = [f^n(\gamma), \alpha]$ and $J = [\gamma, \beta]$ are two subintervals of $I$ whose $f^n$-images cover their union and the first statement of the lemma is proven.

(2) The second statement of the lemma easily follows from the first one. $\square$

*Remark.* M. Misiurewicz suggested another proof of Lemma 3.2. It relies upon the following corollary of Theorem 9.12 from [MN].

**Lemma MN2 [MN].** *Let $\pi$ and $\theta'$ be two patterns, $\pi$ does not have a block structure over $\theta'$, $rp(\theta') = (m', n')$ and $\rho(\theta') \ne 1/2$. If $\pi$ forces $\theta'$ then for any $k$ there exists a pattern $\theta$ with $rp(\theta) = (km', kn')$ forced by $\pi$.*

Obviously mixing patterns cannot have non-trivial block structure, and so if $\pi'$ in Lemma 3.2 is not 2-periodic then Lemma 3.2 follows from Lemma MN2. It remains to prove that a mixing pattern forces patterns $\pi'$ with rotation pairs $rp(\pi') = (k, 2k)$ which can be done directly.

*Proof of Lemma 3.1.* Let $f = L_\pi$. By Lemma 3.2 we may assume that $p, q$ are coprime. Let $p/q = 1/2$. If $\pi$ has division then by Lemma 1.4 $f$ has a 2-periodic orbit, so we may assume that $\pi$ has no division. Then $a$ has at least one preimage $b \neq a$; let $b < a$. Since $f$ is surjective there is a point $c < a$ such that $f(c) = 1$. Thus we have $f[b, c] \supset [a, 1], f[a, 1] \supset [0, a]$ which implies that $f$ has a 2-periodic point and completes the case $p/q = 1/2$.

Now assume that $r/n < 1/2$. Let us show that for any $y$ there is at least one $0 \leq i \leq n - r$ such that $f^i(y) \geq a$ and so if $\pi'$ is a pattern forced by $\pi$ and $rp(\pi') = (s, t)$ then $s/t \geq \dfrac{1}{n - r + 1}$ and $t \leq s(n - r + 1)$. Indeed, otherwise there is a point $y$ such that $y < f(y) < \cdots < f^{n-r}(y) < a$. Let $P$ be the representative of $\pi$ in $f$, $k_i = card\, P \cap (f^i(y), a]$. Then clearly $k_0 > k_1 > \cdots > k_{n-r} \geq 0$ implying that $k_0 \geq n - r$ which is obviously not possible since $card\, P \cap (0, a] = n - r - 1$.

Let us consider the family $\mathcal{A}$ of all patterns $\theta'$ forced by $\pi$ such that $rp(\theta') = (s, t)$ with $s \leq \max(r, p) = k$ and $s/t \leq p/q$. The family $\mathcal{A}$ is non-empty since $\pi \in \mathcal{A}$ and by the previous paragraph it is finite. Let $\theta \in \mathcal{A}$, $rp(\theta) = (s, t)$ be forcing-minimal in $\mathcal{A}$ and show that it is a twist. Indeed, otherwise it forces a pattern $\pi'$ with $rp(\pi') = (s', t')$ such that $s'/t' \leq s/t$ and $t' \leq t$. Clearly it implies that $s' \leq s \leq k$ and thus contradicts the choice of $\theta$. We may also assume that $(s, t) \neq (p, q)$ since otherwise there is nothing to prove.

Let $g = L_\theta$, $Q$ be a representative of $\theta$ in $g$, $a$ is a unique $g$-fixed point. Let us show that $p$ is not a multiple of $s$. Indeed, otherwise let us consider an admissible loop $\bar\beta$ obtained by the repetition of the loop $\bar\alpha(Q)$ $p/s$ times. Since $p, q$ are coprime and $s/t \leq p/q$ there are $v \geq q + 1$ intervals in $\bar\beta$. Also, since $\theta$ has no division then there is a point $z \in Q$ such that $z < gz < a$. Erasing now the interval $[z, a]$ from $\bar\beta$ we will get a new admissible loop which by Lemma 1.6 generates a periodic orbit $Q'$ with $rp(Q') = (s', t')$ such that $\rho(Q') = s'/t' = \dfrac{p}{v - 1} \leq p/q$ which contradicts the choice of $\theta$ (notice that $s' \leq p \leq \max(r, p)$).

Let us show that there is a point $y \in Q$ and a number $m$ such that $g^m(y) < y < g^{m+1}(y) < a$ and the $y$-orbit from $y$ to $g^m(y)$ enters $[a, 1]$ exactly $p$ times (this will allow to apply Lemma 2.4). Consider all the points from $Q \cap [0, a]$. They are divided into time segments (if $z = g^w(y)$ and $y, g(y), \dots, g^{w-1}(y) \neq z$ then the *time segment from $y$ to $z$* is the finite sequence $y, g(y), \dots, g^w(y) = z$); by Lemma 2.3 each time-segment begins at its *entry point* (say, $c$) and ends up at its *exit point* (say, $d$), then in the orbit follows a point $g(d) > a$ and then the point $g^2(d) < d$, the next entry point. Let $EN$ be the set of entry points and $EX$ be that of exit points. Let $\mathbf{x}$ be the sequence of entry and exit points as they appear in the orbit while the number of iterates grows (there may be points in $Q$ which are both entry and exit points, they will appear in $\mathbf{x}$ twice in succession). Assuming that $x_0$ is an entry point we have that $\{x_0, x_2, \dots, x_{2s-2}\} = EN$ and $\{x_1, x_3, \dots, x_{2s-1}\} = EX$. Lemma 2.3 implies the following properties.

**Property 1.** $x_{2k+2} < x_{2k+1}$.

**Property 2.** *If* $x_{2i+1} < x_{2j+1}$ *then* $x_{2i+2} < x_{2j+2}$.

**Property 3.** $x_{2k} \leq x_{2k+1}$.

Let $\phi(u) = u + 2p \pmod{2s}$. We claim that there is a point $x_{2i+1} \in EX$ such that $x_{\phi(2i)} < x_{2i+1} < x_{\phi(2i+1)}$. Indeed, since $p$ is not a multiple of $s$ there is more than one number in the set $1, \phi(1), \dots$ which implies that there are points $x$

such that $x_{\phi(2i+1)} > x_{2i+1}$ and there are points $x_{2j+1}$ such that $x_{\phi(2j+1)} < x_{2j+1}$. Hence there are two exit points $x_{2i-1}$ and $x_{2i+1}$ such that $x_{2i+1} < x_{\phi(2i+1)}$ and $x_{2i-1} > x_{\phi(2i-1)}$. Applying Property 2 to two exit points $x_{2i-1} > x_{\phi(2i-1)}$ we have $x_{2i} > x_{\phi(2i)}$. On the other hand by Property 3 $x_{2i} \leq x_{2i+1}$ and by the choice of $i$ we have $x_{2i+1} < x_{\phi(2i+1)}$. Thus $x_{\phi(2i)} < x_{2i+1} < x_{\phi(2i+1)}$. Obviously the $g$-orbit of $x_{2i+1} = y$ enters $[a, 1]$ exactly $p$ times before it gets mapped into $x_{\phi(2i)}$ (this follows from the definition of $\phi$). On the other hand $x_{\phi(2i)}$ and $x_{\phi(2i+1)}$ are consecutive entry and exit points, so for the corresponding iterate $g^m$ of $g$ we will have that $g^m(y) < y < g^{m+1}(y) < a$ and the time segment from $y$ to $g^m(y)$ enters $[a, 1]$ exactly $p$ times.

Now, by the choice of $\theta$ it does not force a pattern $\pi'$ with $rp(\pi') = (s', t')$ such that $s' \leq \max(s, p)$ and $s'/t' \leq p/q$ (note that $s \leq \max(r, p) = k$ and so $\max(s, p) \leq k$). Therefore by Lemma 2.4 the above found dynamics is impossible; this contradiction proves the lemma. $\square$

The following corollary may now be added to the list of properties of twists.

**Corollary 3.3.** *If $\pi$ is a twist and $rp(\pi) = (p, q)$ then $p, q$ are coprime.*

As was stated in the Introduction the Main Theorem leads to the description of all possible sets $RP(f)$ given in Corollary 3.4

*Proof of Corollary 3.4.* (1) Let $f$ be a continuous interval map with non-fixed periodic points. Let us consider the family $\mathcal{A}$ of all its rotation pairs $(p, q)$ such that $p/q \leq 1/2$. Clearly there are two possibilities. It may be that there is a rotation pair $(p, q) \in \mathcal{A}$ such that for any $(s, t) \in \mathcal{A}$ we have $(p, q) \gtrdot (s, t)$. Then by the Main Theorem $\mathcal{A} = \{(s, t) : (p, q) \gtrdot (s, t)\}$. Let $p/q = a = m/n$ and $m, n$ are coprime. If $p/q < 1/2$ then clearly $\mathcal{A} = Q(a, p/m) \cup N(a, 1/2) \cup Q(1/2, 3)$; if $p/q = 1/2$ then $\mathcal{A} = Q(1/2, p)$. The other possibility is that there is a sequence of rotation pairs $(p_i, q_i)$ such that $p_i/q_i = a_i \searrow a, a_0 > a_1 > \dots$ and for any $(s, t) \in \mathcal{A}$ we have $s/t > a$. Then by the Main Theorem $(s, t) \in \mathcal{A}$ for any $(s, t)$ such that $a < s/t \leq 1/2$ and so $\mathcal{A} = Q(a, 0) \cup N(a, 1/2) \cup Q(1/2, 3)$. Similarly considering the set $\mathcal{B} = \{(s, t) \in RP(f) : s/t \leq 1/2\}$ and using the definition of $\gtrdot$-ordering we see that indeed there exist $0 \leq a \leq 1/2 \leq b \leq 1$ and $l, r \in \mathbb{Z}^+ \cup \{2^\infty\} \cup \{0\}$ such that $RP(f) = N(a, b) \cup Q(a, l) \cup Q(b, r)$; moreover by the Main Theorem if $a < b = 1/2$ then $r = 3$, if $a = 1/2 < b$ then $l = 3$, if $a = b = 1/2$ then $r = l \neq 0$, if $a = 0$ then $l = 0$ and if $b = 1$ then $r = 0$ which completes the proof of the first statement.

(2) Let $\mathcal{F} = \{f_i\}_{i=0}^{\infty}$ be a sequence of maps of the interval $[0, 1]$ into itself, $0 = b_0 < a_1 < b_1 < \dots$ be a sequence of points such that $a_i \to 1$, $[b_j, a_{j+1}] = I_{j+1}, [a_{j+1}, b_{j+1}] = M_{j+1} (0 \leq j)$. Define a map $g_{\mathcal{F}} = g : [0, 1] \longrightarrow [0, 1]$ so that all intervals $I_j$ are $g$-invariant, $g|I_j$ is conjugate to $f_j$ by an increasing map and $g|M_j$ is linear for all $j$. Then $g$ is continuous.

Let numbers $0 \leq a \leq 1/2 \leq b \leq 1$, $l, r \in \mathbb{Z}^+ \cup \{2^\infty\} \cup \{0\}$ be such that if $a < b = 1/2$ then $r = 3$, if $a = 1/2 < b$ then $l = 3$, if $a = b = 1/2$ then $r = l \neq 0$, if $a = 0$ then $l = 0$ and if $b = 1$ then $r = 0$. We show that the construction from the previous paragraph allows to obtain a map $g$ such that $RP(g) = N(a, b) \cup Q(a, l) \cup Q(b, r)$. First note that by Theorem 0.1 we may assume that $a = b = 1/2$ is not the case. If $0 < a < 1/2$ is irrational or if $a$ is rational and $l \neq 0$ then there is a unimodal map $f_1$ such that $RP(f_1) = N(a, 1/2) \cup Q(a, l) \cup Q(1/2, 3)$; let $f_i = f_1$ for all $i$. If $a = 1/2$ let $f_i$ be the identity map for all $i$. Finally, let $a < 1/2$ be rational and $l = 0$. Then there is a sequence $p_i/q_i = a_i \searrow a$ of rational numbers smaller than $1/2$ ($p_i, q_i$ are coprime for all $i$) and thus by Theorem 0.1 there is a sequence of unimodal maps $f_i$ such that $RP(f_i) = N(a_i, 1/2) \cup Q(a_i, 3) \cup Q(1/2, 3)$. Similarly we can construct the sequence of maps $h_i : [0,1] \to [0,1]$ by numbers $b$ and $r$. Let $\mathcal{C} = \{f_i, h_i\}_{i=0}^{\infty}$; then it is easy to see that for the map $g = g_{\mathcal{C}}$ we have $RP(g) = N(a, b) \cup Q(a, l) \cup Q(b, r)$ which completes the proof. $\square$

Finally we would like to prove the following Corollary 3.5 which was first proved in [LMPY].

**Corollary 3.5 [LMPY].** *Let $\pi$ be a pattern of even period $n$ with no division. Then $\pi$ forces a pattern of period $n/2$ if $n/2$ is odd and $n/2+1$ if $n/2$ is even.*

*Proof.* Let $f = L_\pi$, $P$ be a representative of $\pi$ in $f$; by Lemma 1.3 we may assume that $f$ has a unique fixed point $a$. Since $\pi$ has no division we may assume without loss of generality that there is a point $y \in P$ such that $y < f(y) < a$. Let us consider now the admissible loop $\bar{\beta}$ which is obtained from $\bar{\alpha}(P)$ as follows: all intervals $[z, a]$, $z < a$ remain, but all the maximal subchains of intervals in $\bar{\alpha}(P)$ of the form $[a, x], [a, f(x)], \dots, [a, f^k(x)]$, $x > f(x) > \dots > f^k(x) > a$ are replaced by $[a, x]$. Clearly $\bar{\beta}$ is admissible, so by Lemma 1.6 it generates a periodic orbit $Q$ of rotation number $\rho(\bar{\beta})$. At the same time $\rho(\bar{\beta}) < 1/2$ (since there are no consecutive admissible intervals in $\bar{\beta}$ which would lie to the right of $a$ and on the other hand the intervals $[a, y], [a, f(y)]$ both lie to the left of $a$); also the period of $\bar{\beta}$ is smaller or equal than $2n$. So $\rho(\bar{\beta}) = \rho(Q) \le \dfrac{m-1}{2m}$ where $n = 2m$. Consider the two possibilities for $m$.

(1) $m = 2k$ is even. Then $\rho(Q) = \dfrac{2k-1}{4k} < \dfrac{k}{2k+1} < 1/2$ which by the Main Theorem implies that $f$ has a periodic orbit of period $2k+1 = m+1 = n/2+1$.

(2) $m = 2k+1$ is even. Then $\rho(Q) = \dfrac{2k}{4k+2} = \dfrac{k}{2k+1} < 1/2$; since $k$ and $2k+1$ are coprime it implies by the Main Theorem that $f$ has a periodic orbit of period $2k+1 = m = n/2$ and completes the proof. $\square$

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Department of Mathematics, University of Alabama at Birmingham, UAB Station, Birmingham, AL 35294-2060, USA

*E-mail address*: `ablokh@math.uab.edu`