Dedicated to the Occasion of Professor A. N. Sharkovskii 65th Birthday

ON GRAPH-REALIZABLE SETS OF PERIODS

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Abstract. We characterize sets of periods of cycles which arbitrary continuous graph maps may have. In this investigation we need the spectral decomposition for graph maps [B3] briefly described in Section 1.

Introduction

Let us call compact one-dimensional branched manifolds *graphs*; we consider this notion in a wide sense, thus we allow non-connected graphs as well as finitely many isolated points in graphs. We study properties of the set $P(f)$ of periods of cycles of a graph map f. A major result on this topic is the Sharkovskii theorem on co-existence of periods of cycles for maps of the real line [S2]. To formulate it let us introduce the *Sharkovskii ordering* $(*)$ for positive integers:

$$(*) \qquad 3 \prec 5 \prec 7 \prec \dots \prec 2\cdot 3 \prec 2\cdot 5 \prec 2\cdot 7 \prec \dots \prec 8 \prec 4 \prec 2 \prec 1$$

Denote by $S(k)$ the set of all such integers m that $k \prec m$ or $k = m$ and by $S(2^\infty)$ the set $\{1, 2, 4, 8, \dots\}$.

Theorem [S2]. *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous map. Then either $P(g) = \emptyset$ or there exists such $k \in \mathbb{N} \cup 2^\infty$ that $P(g) = S(k)$. Moreover for any such k there exists a map $g : [0, 1] \to [0, 1]$ with $P(g) = S(k)$ and there exists a map $g_0 : \mathbb{R} \to \mathbb{R}$ with $P(g_0) = \emptyset$.*

Other information about sets of periods of cycles for one-dimensional maps is contained in papers [AL, M2] for maps of the circle, [ALM] for maps of the letter Y and [Ba] for maps of n-od. Also, a new tool ("rotational theory") was introduced and developed in [B6, BM]; it deals with the idea of a rotation number for interval maps and allows for more delicate study of co-existence of various types of cycles in this context.

The Sharkovskii theorem implies that if a map $f : \mathbb{R} \to \mathbb{R}$ has a cycle of period 3 then it has cycles of all possible periods. The following conjecture, which was formulated by M.Misiurewicz at the Problem Session at Czecho-Slovak Summer Mathematical School near Bratislava in 1990, seems to be closely related.

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Misiurewicz Conjecture. *For a graph X there exists an integer $L = L(X)$ such that for a continuous map $f : X \longrightarrow X$ inclusion $P(f) \supset \{1, 2, \dots, n\}$ implies $P(f) = \mathbb{N}$.*

Misiurewicz conjecture is verified in [B4]. Clearly it implies that sets of periods of cycles of graph maps have some general properties no matter what graph is considered. Moving in this direction we describe in Section 2 sets $A \subset \mathbb{N}$, for which there exists a graph Y and a continuous map $g : Y \longrightarrow Y$ with $P(g) = A$. More precisely, a set $A \subset \mathbb{N}$ is called a *graph-realizable set of periods* iff there exist a graph X and a continuous map $f : X \longrightarrow X$ such that $P(f) = A$. A set B is called a *a zero graph-realizable set of periods* iff there exist a graph X and a continuous map $g : X \longrightarrow X$ such that $h(f) = 0$, $P(f) = B$. Set $lZ \equiv \{li : i \geq 1\}$, $Q(n) \equiv \{2^i n : i \geq 0\}$. Also, say that a set A *almost coincides* with a set B if the symmetric difference $(A \setminus B) \cup (B \setminus A)$ is finite.

The main theorem of this paper is the following Theorem 2.1.

Theorem 2.1. *The following properties take place.*

1. *A set $A \subset \mathbb{N}$ is graph-realizable iff it almost coincides with a finite union of some sets lZ or $Q(n)$.*
2. *A set $A \subset \mathbb{N}$ is zero graph-realizable iff it almost coincides with a finite union of some sets $Q(n)$.*

In what follows we need *the spectral decomposition for graph maps* ([B3]) similar to that for interval maps ([B1, B2, B5]); it is briefly described in Section 1.

Notations

$\operatorname{int} Z$ is the interior of a set Z;
∂Z is the boundary of Z;
$\overline{Z}$ is the closure of Z;
f^n is the n-fold iterate of a map f;
$\operatorname{orb} x \equiv \{f^n x\}_{n=0}^{\infty}$ is the *orbit* of x;
$\omega(x)$ is the limit set of $\operatorname{orb} x$;
$\mathbb{N} \equiv \{1, 2, 3, \dots\}$ is the set of natural numbers;
$\operatorname{Per} f$ is the set of all periodic points of some map f;
$P(f)$ is the set of all periods of periodic points of a map f;
$h(f)$ is a topological entropy of a map f.

1. The Spectral Decomposition

In this section we briefly describe the spectral decomposition for graph maps (for the proofs see [B3]). Let us begin with some historical remarks.

For any map $T : X \to X$ of a compact metric space into itself set $\omega(T) = \bigcup_{x \in X} \omega(x)$. A. N. Sharkovskii constructed the decomposition of the set $\omega(f)$ for continuous interval maps $f : I \to I$ in [S1] and proved that $\omega(f)$ is closed. Then in [JR] Jonker and Rand constructed for unimodal maps the decomposition which is in fact close to that of Sharkovskii; however they used completely different methods based on symbolic dynamics. In [H] the decomposition for piecewise-monotone maps with discontinuities was constructed by Hofbauer and then Nitecki in [N] considered the decomposition for piecewise-monotone continuous maps from more geometrical point of view.

The papers [B1, B2, B5] are devoted to the case of arbitrary continuous interval maps and contain a different approach to the problem. This allows us to obtain new corollaries (e.g. describing generic properties of invariant measures for interval maps). A similar approach is used in [B3] to construct the decomposition for graph maps; now we pass on to the description of the results of [B3].

Let X be a graph, $f : X \to X$ be a continuous map (we do not assume that X is connected). We use terms *edge, vertex, endpoint* in the usual sense; the numbers of edges and endpoints of X are denoted by $\mathrm{Edg}(X)$, $\mathrm{End}(X)$. If necessary, "artificial" vertices are added to make all edges of a graph homeomorphic to an interval. A closed connected set $Y \subset X$ is called a connected *subgraph*. A connected subgraph Y is called *periodic (of period k)* if $Y, fY, \dots, f^{k-1}Y$ are pairwise disjoint and $f^kY = Y$; the union of all iterations of Y is denoted by $\mathrm{orb}\, Y$ and called a *cycle of connected subgraphs*. Let $Y_0 \supset Y_1 \supset \dots$ be periodic connected subgraphs of periods $m_0, m_1, \dots$; then m_{i+1} is divided by m_i $(\forall i)$. If $m_i \to \infty$ then the connected subgraphs Y_i, $i = 1, 2, \dots$ are said to be *generating*. We call any invariant closed set $S \subset Q = \cap(\mathrm{orb}\ Y_i)$ a *solenoidal set* and denote the solenoidal set $Q \cap \omega(f)$ by $S_\omega(Q)$ (note that by [B3] the set $\omega(f)$ is closed).

One can use a transitive shift in an Abelian zero-dimensional infinite group as a model for the map on a solenoidal set. Let $D = \{n_i\}$ be a sequence of integers, n_{i+1} is divided by n_i $(\forall i)$ and $n_i \to \infty$. Consider a subgroup $H(D) \subset \mathbb{Z}_{n_0} \times \mathbb{Z}_{n_1} \times \dots$, defined as follows:

$$H(D) \equiv \{(r_0, r_1, \dots) : r_{i+1} \equiv r_i \pmod{m_i}\, (\forall i)\}.$$

Denote by τ the shift in $H(D)$ by the element $(1, 1, \dots)$; clearly, τ is minimal.

Theorem 1.1 ([B3]). *Suppose that $\{Y_i\}$ are generating connected subgraphs and that they have periods $\{m_i\}$. Let $Q = \bigcap_{i\geq 0} \mathrm{orb}\, Y_i$. Then there exists a continuous surjective map $\varphi : Q \to H(D)$ with the following properties:*

1. *$\tau \circ \varphi = \varphi \circ f$ (i.e. φ semiconjugates $f|Q$ to τ);*
2. *there exists the unique set $S \subset Q \cap \overline{Per f}$ such that $\omega(x) = S$ for any $x \in Q$ and if $\omega(z) \cap Q \neq \emptyset$ then $S \subset \omega(z) \subset S_\omega$;*
3. *for any $\bar{r} \in H(D)$ the set $J = \varphi^{-1}(\bar{r})$ is a connected component of Q and $\varphi|S_\omega$ is at most 2-to-1;*
4. *$h(f|Q) = 0$.*

Let us turn to another type of an infinite limit set. Let $\{Y_i\}_{i=1}^l$ be a collection of connected graphs, $K = \bigcup_{i=1}^l Y_i$. A continuous map $\psi : K \to K$ which permutes these graphs cyclically is called *non-strictly periodic* or *non-strictly l-periodic*; for example if Y is a periodic connected subgraph then $f|\mathrm{orb}\, Y$ is non-strictly periodic. In what follows we will consider monotone semiconjugations between non-strictly periodic graph maps (a continuous map $g : X \to Y$ is *monotone* provided $g^{-1}(Y)$ is connected for any $y \in Y$). We need the following lemma whose proof is left to the reader.

Lemma 1.2. *Let X be a graph. Then there exists a number $r(X)$ such that if $M \subset X$ is a connected subgraph then $\mathrm{card}\{\partial(M)\} \leq r(X)$ where we consider the boundary of M as a subset of X. Moreover, $r(X)$ can be chose in such a way that if L is a connected subgraph of X and $\varphi : L \to N$ is a monotone map onto a graph N then for any connected* *subgraph P of N we have card $\{\partial(P)\} \le r(X)$ where we consider the boundary of P as a subset of N.*

Lemma 1.2 makes the definition below natural. If $\varphi : K \to M$ is continuous, monotone, semiconjugates a non-strictly periodic map $f : K \to K$ to a non-strictly periodic map $g : M \to M$ and $F \subset K$ is a closed f-invariant set such that $\varphi(F) = M$ and $\varphi^{-1}(y) \cap F \subset \partial(\varphi^{-1}(y))\ (\forall y \in M)$ then φ is said to *almost conjugate* $f|F$ to g.

Let Y be an n-periodic connected subgraph, $\operatorname{orb} Y = M$. Denote by $E(M, f)$ the following set:

$$E(M, f) = \{x \in M : \text{for any open } U \ni x, U \subset M \text{ we have } \overline{\operatorname{orb} U} = M\}$$

provided it is infinite. Clearly, $E(M, f)$ is closed; it is called *basic* and denoted by $B(M, f)$ if $\operatorname{Per}(f|M) \neq \emptyset$; otherwise $E(M, f)$ is denoted by $C(M, f)$ and called a *circle-like set.*

Theorem 1.3 ([B3]). *Let Y be an n-periodic connected subgraph, $M = \operatorname{orb} Y$ and $E(M, f) \neq \emptyset$. Then there exists a transitive non-strictly n-periodic map $g : K \to K$ and a monotone continuous surjection $\varphi : M \to K$ which almost conjugates $f|E(M, f)$ to g. Furthermore:*

1. *$E(M, f)$ is a perfect set;*
2. *$f|E(M, f)$ is transitive;*
3. *if $\omega(z) \supset E(M, f)$ then $\omega(z) = E(M, f)$;*
4. *if $E(M, f) = C(M, f)$ is a circle-like set then K is the union of n circles permuted by g, g^n on each of them is an irrational rotation and $h(g) = h(f|E(M, f)) = 0$;*
5. *if $E(M, f) = B(M, f)$ is a basic set then $h(f) > 0$, $B(M, f) \subset \overline{\operatorname{Per} f}$ and there exist a number k and a closed subset $D \subset B(M, f)$ such that $\varphi(D)$ is connected, $f^i(D) \cap f^j(D), 0 \le i < j < kn$, are finite, $g^i(\varphi(D)) \cap g^j(\varphi(D)), 0 \le i < j < kn$, are finite, $f^{kn} D = D$, $\bigcup_{i=0}^{kn-1} f^i D = B(M, f)$ and $f^{kn}|D$, $g^{kn}|\varphi D$ are topologically mixing.*

A number kn from Theorem 1.3(5) is called the *mixing period* of $B(M, f)$.

In Section 2 we will need some results which can be easily deduced from Theorem 1.3. These results establish the connection between the mixing period of $B(M, f)$ and periods of cycles contained in M. Let us state one of them here; in the statement we use the terminology from Theorem 1.3.

Proposition 1.4. *Let M be a cycle of connected subgraphs, $y \in M$ be a periodic point of period l, $E(M, f) = B(M, f) = B$ be a basic set of mixing period m, $D \subset B(M, f)$ and φ be the same as in* Theorem 1.3. *Then the following statements are true:*

1. *$m \le 2l \cdot Edg(X)$;*
2. *if l is not divisible by m then $\varphi(f^i(y)) \notin \operatorname{int}(g^i(\varphi(D)))$ for any i;*
3. *there exists a number $\xi(X)$ such that if A is the union of the boundaries of all sets $g^i(\varphi(D))$ then the cardinality of the union of the boundaries of all sets $\varphi^{-1}(z), z \in A$ (and the cardinality of the set $\varphi^{-1}(A) \cap B$) is at most $m\xi(X)$;*
4. *if $y \in B$ and l is not a multiple of m then $l \le m\xi(X)$.*

Proof. (1) A point y belongs to a connected set $A = \varphi^{-1}(\varphi(y))$. Then by Theorem 1.3 $f^l(A) = A$ and $B \cap A \subset \partial A$. Consider a point $d \in B \cap A$. Clearly there are finitely many small pairwise disjoint open intervals I with an endpoint at d such that the intersection $B \cap \bar{I}$ has d as a non-isolated point and that all other intervals like that are non-disjoint from the chosen ones. The number of such intervals is no more than $2 \cdot Edg(X)$ because every edge can contain no more than 2 such intervals. Repeating this construction for all sets $f(A), \dots, f^{l-1}(A)$ we will define no more than $N = 2l \cdot Edg(X)$ small open intervals non-disjoint from B.

Since the intervals can be chosen arbitrarily small we may assume that for each interval I the intersection $B \cap I$ is contained in only one iteration of D. Denote the collection of all these intervals by $\mathcal{I}$. For each interval $I \in \mathcal{I}$ consider the interval $\varphi(B \cap \bar{I})$ and denote the collection of such intervals by $\mathcal{J}$. Then for each $J \in \mathcal{J}$ there exists a well-defined iteration of D containing J.

Now, for each $J \in \mathcal{J}$ there is a finite collection of other elements of $\mathcal{J}$ with which $g(J)$ intersects over a non-degenerate set. Since there are no more than N elements in $\mathcal{J}$, this implies that there exists a loop of intervals $J_i \in \mathcal{J}, 0 \le i \le s-1$ such that $g(J_i)$ and J_{i+1} have a non-degenerate intersection for each $i \le s-2$ and also $g(J_{s-1}) \cap J_0) \neq \emptyset$. Clearly, $s \le N$. Since distinct iterations of D have degenerate intersections we see that in fact $s = m$ is the mixing period of B. So, $m \le N$ as desired.

(2) If $\varphi(f^i(y)) \in \operatorname{int}(g^i(\varphi(D)))$ for some i then the only powers of g which can map $\varphi(f^i(y))$ back into itself are multiples of the mixing period m of B, and hence so must be the period l of y.

(3) Since by Lemma 1.2 each set $g^i(\varphi(D))$ has at most $r(X)$ points in its boundary and there are m such sets, we conclude that the cardinality of the set A is at most $mr(X)$. Since φ is monotone then for every point $z \in A$ the set $\varphi^{-1}(z)$ is a connected subgraph whose boundary consists of at most $r(X)$ points by Lemma 1.2. Hence the union of the boundaries of sets $\varphi^{-1}(z), z \in A$ consist of at most $mr^2(X)$ points. Setting $\xi(X) = r^2(X)$ we get the desired. Since by Theorem 1.3 φ almost conjugates $f|B$ to g, the entire set $\varphi^{-1}(A) \cap B$ is contained in the union of the boundaries of the sets $\varphi^{-1}(z), z \in A$, so the cardinality of the set $\varphi^{-1}(A) \cap B$ is at most $m\xi(X)$ which completes the proof of the claim (3).

(4) Suppose that $y \in B$ and l is not a multiple of m. Then by claim (2) of this proposition the entire g-orbit of $\varphi(y)$ is contained in the union of boundaries $\partial g^i(\varphi(D))), 0 \le i \le m-1$ and since by Theorem 1.3 the map φ almost conjugates $f|B$ and g we see that the orbit of y is contained in the set $\varphi^{-1}(A) \cap B$. Since by claim (3) of this proposition the cardinality of the set $\varphi^{-1}(A) \cap B$ is at most $m\xi(X)$ this completes the proof. $\square$

To formulate the decomposition theorem denote by Z_f the set of all cycles maximal by inclusion among all limit sets of f.

Theorem 1.5 ([B3]). *Let $f : X \to X$ be a continuous graph map. Then the following statements are true.*

(1) *There exist a finite number of circle-like sets $\{C(K_i, f)\}_{i=1}^k$, an at most countable family of basic sets $\{B(L_j, f)\}$, and a family of solenoidal sets $\{S_\omega(Q_\alpha)\}$ such that*

$$\omega(f) = Z_f \bigcup (\bigcup_{i=1}^k C(K_i)) \bigcup (\bigcup_j B(L_j)) \bigcup (\bigcup_\alpha (S_\omega(Q_\alpha)).$$

(2) *There exist numbers $\gamma(X)$ and $\nu(X)$ such that $k \le \gamma(X)$, the only possible intersections in the decomposition are between basic sets and at most $\nu(X)$ basic sets can intersect at a point.*
(3) *For any ω-limit set $\omega(x)$ there exists a set A from the decomposition containing $\omega(x)$, and A is unique except for the case when $\omega(x)$ is a cycle in which case $\omega(x)$ may be contained in at most $\nu(X)$ basic sets.*

Theorem 1.3 shows that one can consider mixing graph maps as models for graph maps on basic sets. The following theorem seems to be important in this connection; to state it we need the definition of maps with the specification property (see, for example, [DGS]): a map $T : (X, d) \to (X, d)$ of a metric compact set into itself is said to have the *specification property* if for every ε there exists an integer M such that for any $k > 0$, any k points $x_1, \dots, x_k$, and any integers $a_1 \le b_1 < a_2 \le b_2 < \dots < a_k \le b_k < p$ with $a_i - b_{i-1} \ge M (2 \le i \le k)$ and $p \ge M + b_k - a_1$ there exists a point x such that $T^p(x) = x$ and $d(T^n(x), T^n(x_i)) \le \varepsilon$ for $a_i \le n \le b_i, 1 \le i \le k$.

Theorem 1.6 ([B3]). *Let $f : X \to X$ be a continuous mixing graph map. Then f has the specification property.*

It is well-known [DGS] that maps with the specification have nice properties concerning the set of invariant measures. Using them and the above results one can describe generic properties of invariant measures for graph maps. First we need some definitions. Let $T : X \to X$ be a map of a compact metric space into itself. The set of all T-invariant Borel normalized measures is denoted by D_T. A measure $\mu \in D_T$ with *supp* μ containing in one cycle is said to be a *CO-measure*. The set of all CO-measures concentrated on cycles with minimal period p is denoted by $P_T(p)$. Let $V(x)$ be the set of accumulation points of time-averages of iterations of the δ-measure δ_x concentrated at x. A point $x \in X$ is said to have *maximal oscillation* if $V_T(x) = D_T$.

Theorem 1.7 ([B3]). *Let B be a basic set. Then:*
(1) *for any l the set $\bigcup_{p \ge l} P_{f|B}(p)$ is dense in $D_{f|B}$;*
(2) *the set of all ergodic non-atomic invariant measures μ with supp $\mu = B$ is a residual subset of $D_{f|B}$;*
(3) *if $V \subset D_{f|B}$ is a non-empty closed connected set then the set of all such points x that $V(x) = V$ is dense in X (in particular every measure $\mu \in D_{f|B}$ has a generic point);*
(4) *points with maximal oscillation are residual in B.*

Theorem 1.8 ([B3]). *Let μ be an invariant measure. Then the following properties of μ are equivalent:*
(1) *there exists such a point x that supp $\mu \subset \omega(x)$;*
(2) *μ has generic points;*
(3) *μ is concentrated on a circle-like set or can be approximated by CO-measures.*

In particular, CO-measures are dense in all ergodic measures which are not concentrated on circle-like sets.

Let us recall that if $n \ge 1$ then $nZ \equiv \{in : i \ge 1\}$, $Q(n) \equiv \{2^i n : i \ge 0\}$ and that a set A *almost coincides* with a set B if the symmetric difference $(A \setminus B) \cup (B \setminus A)$ is finite. Furthermore, if $B \setminus A$ is finite then say that A *almost contains* B and denote it by $A \underset{a}{\supset} B$. Recall also that $P(t)$ is a set of all periods of cycles of a map T.

In Lemma 1.10 below we need the following easy property of maps with specification.

Property 1.9. *If $T : X \to X$ is a map with specification and X is infinite then $P(T)$ almost coincides with $\mathbb{N}$.*

Proof. Clearly, there exist a point x and a periodic orbit A such that $x \notin A$. Let ε be the distance between x and A, $M = M(\varepsilon/4)$ and $N \geq 3M$ be an integer. Choose a point $z \in A$ and set $a_1 = 0, b_1 = 0, a_2 = M, b_2 = N - M > a_2$ and $p = N$. Set $x_1 = x$ and $x_2 = z$ and apply the specification property to the points x_1, x_2, to the numbers a_1, b_1, a_2, b_2 and to the period p. We conclude that there exists a point y such that $T^p(y) = y$, $d(x_1, y) \leq \varepsilon/4$, and $d(T^i(x_2), T^i(y) \leq \varepsilon/4$ for $a_2 \leq i \leq b_2$.

We need to show that the minimal period k of y is actually N. Indeed, if $k < N$ then $k \leq N/2$. Let us show that then there exists a number of the form jk such that $a_2 \leq jk \leq b_2$. It is obvious if $k \leq M$ because then $b_2 - a_2 \geq M \geq k$ and along any k consecutive integers there must be at least one multiple of k so that there exists a multiple of k among integers $a_2, a_2 = 1, \dots, b_2$. Now, if $k > M$ then since $a_2 - b_1 = M$ and $N - b_2 = M$ we see that $a_2 < k \leq N - k < b_2$ as desired.

Since $T^{jk}(y) = y$ then by the specification property $d(T^{jk}, x_1) \leq \varepsilon/4$ and on the other hand $d(T^{jk}(y), T^{jk}(x_2)) \leq \varepsilon/4$. Since $T^{jk}(x_2) \in A$ we get $d(x_1, A) \leq \varepsilon/2$, a contradiction. $\square$

Our results easily imply the following lemma.

Lemma 1.10. *Let $f : X \to X$ be a graph map, B be a basic set of f, m be a mixing period of B. Then $P(f)$ almost contains mZ while $P(f|B)$ is almost contained in mZ.*

Proof. Let us use notation from Theorem 1.3. In this notation all except for finitely many sets $\varphi^{-1}(y), y \in K$, are intervals. hence only finitely many periodic points of g have non-interval φ-preimages. Now, if y is a periodic point of $g : K \to K$ such that $\varphi^{-1}(y)$ is an interval then f has a periodic point of the same period as y inside $\varphi^{-1}(y)$. Therefore, $P(f)$ almost contains $P(g)$.

By Theorem 1.3 $K = \bigcup_{i=0}^{m-1} g^i(\varphi(D))$ where $g^m(\varphi(D)) = \varphi(D)$, $g^m|\varphi(D)$ has specification and pairwise intersections between iterations of $\varphi(D)$ are finite. Hence by Property 1.9 $P(g)$ almost coincides with mZ, and so $P(f)$ almost contains mZ.

Assume now that M is a cycle of connected subgraphs such that $B = B(M, f)$. To show that $P(f|B)$ is almost contained in mZ observe that by Proposition 1.4(3) the period of a periodic point of $f|B$ which does not divide m must satisfy $l \leq m\xi(X)$. Hence only finitely many period of periodic points of $f|B$ are not multiples of m and so $P(f|B)$ is almost contained in mZ as desired. $\square$

Lemma 1.10 shows why sets mZ appear in graph-realizble sets of periods. The role of sets $Q(n)$ will become clear later; loosely speaking these sets of periods correspond to those invariant connected subgraphs on which a map has zero entropy. To conclude this section let us formulate the following

Lemma 1.11 ([B3]). *Suppose that $y_n \to y$, $y_n \in Per\, f$ and there exists an interval I with an endpoint y such that $y_n \in I\, (\forall n)$. Let*

$$F = F(\{y_i\}) \equiv \{z : orb\, y_n \cap U \neq \emptyset \text{ for any open } U \ni z \text{ and infinitely many } n\}.$$

Then $y \in F$, F is closed, $f(F) = F$, F is a cycle or an infinite set and there exists such x that $\omega(x) \supset F$ and $\omega(x)$ is not a circle-like set.

2. Graph-Realizable Sets of Periods

Now we are ready to prove Theorem 2.1. We will need the following corollary of Sharkovskii theorem.

Corollary S. *Suppose that $f : M \to M$ is a continuous graph map, $I \subset J \subset M$ are closed intervals, p is a number such that $J, f(I), \dots, f^{p-1}(I)$ are disjoint, f^p maps I onto J and either $I = J$ or there is a periodic point y such that $orb\, y \subset I$ and endpoints of I belong to $orb\, y$. Consider the set $Per_I\, f$ of all periodic points of f with orbits contained in $\bigcup_{i=0}^{p-1}$. Then the set $P_I(f)$ of their periods is $pS(k)$ for some $k \in \mathbb{N} \cup 2^\infty$. Moreover, if $h(f) = 0$ then the number k above comes from the set $\{2^i\}_{i=0}^\infty$.*

Proof. By the Sharkovskii Theorem it is sufficient to consider the case when the following holds: $I = [a, b] \subset [c, d] = J$, $a, b \in \mathrm{orb}_f\, y$. Define $g : J \to J$ as follows: $g|\,[a, b] = f$, $g|\,[c, a] = f(a)$, $g|[b, d] = f(b)$. Then $P(g) = S(k)$ for some $k \in \mathbb{N} \cup 2^\infty$. Let us show that g-periodic orbits are exactly those f-periodic orbits which are contained in I. It is enough to show that if z is a g-periodic point then it is f-periodic and its orbit is contained in I. Indeed, otherwise there is a point z' of the orbit of z which does not belong to I. Then $z' \in [c, a] \cup [b, d]$ which implies that $g(z') = f(a)$ or $g(z') = f(b)$ and hence z' must belong to the orbit of y and therefore must belong to I, a contradiction. Hence, g-periodic orbits are exactly those f-periodic orbits which are contained in I, and so $P_I(f) = P(g) = S(k)$.

Now, observe that the family of all limit sets of g is a subfamily of the family of all limit sets of f. Together with well-known properties of the topological entropy it implies that $h(g) \le h(f)$. Hence if $h(f) = 0$ then $h(g) = 0$. Therefore the second claim of the proposition follows from the fact that an interval map has zero entropy if and only if its set of periods is of form $S(k)$ with $k \in \{2^i\}_{i=0}^\infty$ (see [M1]). □

A key role in the proof of Theorem 2.1 is played by Lemma 2.2. In its proof as well as in what follows in Section 2 we will freely use interval notation for arcs inside graphs because it is not causing any confusion and is quite convenient.

Lemma 2.2. *Let $f : X \to X$ be a continuous graph map, y_i be f-periodic points of periods n_i. Then taking a subsequence we may assume that one of the following possibilities holds.*

- (A) *All points y_i have the same period.*
- (B) *For some sequence of cycles of connected subgraphs $M_i \supset \mathrm{orb}\, y_i$ and some number p there exists a basic set $B(M_i, f)$ of mixing period p such that n_i is divided by p $(\forall i)$.*

(C) *There are a sequence of pairs of intervals $J_i \supset I_i$ and a number p such that $f^p I_i = J_i$, intervals $fI_i, \dots, f^{p-1}I_i, J_i$ are pairwise disjoint,* $\operatorname{orb} y_i \subset \bigcup_{j=0}^{p-1} f^j I_i \quad (\forall i)$ *and either $f^p I_i = J_i$ or endpoints of I_i belong to* $\operatorname{orb} y_i$.

Proof. Excluding the case (A) we may assume that $n_i \nearrow \infty, y_i \to y$ and there is an interval $[a, y]$ such that $[a, y)$ contains no vertices of X and $y_i \in [a, y)\, (\forall i)$. Consider the set $F = F(\{y_i\})$ (see Lemma 1.11).

Case 1. *The point y belongs to a circle-like set.*

This possibility is excluded by Lemma 1.11 and Theorem 1.5.

Case 2. *The set F is not a cycle (and therefore is infinite by Lemma 1.11).*

According to Theorem 1.5 and Lemma 1.11 we need to consider two subcases.

Subcase 2a. *The set F is contained in a solenoidal set.*

To investigate this case we rely upon the definition of a solenoidal set and Theorem 1.1. They imply that there exists a generating connected subgraph $Y \ni y$ of arbitrary high period. Let us show that then for some point $b \in [a, y)$ we have that $[b, y] \subset Y \cap [a, y]$. Indeed, otherwise for all integers i such that $y_i \in [a, y]$ is sufficiently close to y we have that $\operatorname{orb} y_i$ is disjoint from $\operatorname{orb} Y$. Since F is infinite, there are points of F not belonging to the union of the boundaries of components of $\operatorname{orb} Y$ and therefore belonging to the union of the interiors of these components. However, then for any such point z we can choose the interior of the containing z component of $\operatorname{orb} Y$ as the neighborhood which will intersect only finitely many orbits of points y_i, a contradiction. We conclude that for any generating connected subgraph Y_i we have that $\operatorname{orb} Y$ contains all but finitely many orbits of y_j.

Now, choose a generating connected subgraph $Y \ni y$ so that its period p is very large; this would guarantee that among the components of the orbit of Y there exists at least one interval denoted by I and containing no vertices of X. By the definition of a solenoidal set the orbit of I is the same as the orbit of Y, hence $\operatorname{orb} y_i \subset \operatorname{orb} I$ for all sufficiently large i. Clearly it is enough to set $I_i = J_i = I$; then, perhaps after further refining of the sequence $\{y_i\}$, the possibility (C) from Lemma 2.2 holds.

Subcase 2b. *The set F is contained in a basic set $B = B(M, f)$.*

Let p be the mixing period of B. Furthermore, let $g : K \to K$ be a transitive non-strictly periodic graph map and $\varphi : M \to K$ be a monotone continuous surjection which almost conjugates $f|B$ to g (such a map φ exists by Theorem 1.3). Finally, let the set D be the same as in Theorem 1.3(5).

Let us show that there are infinitely many points y_i such that points $\varphi(y_i)$ have orbits disjoint from the union A of the boundaries of sets $g^i(\varphi(D))$. Indeed, otherwise for all sufficiently large i the point $\varphi(y_i)$ has the g-orbit non-disjoint from A. Since the points $\varphi(y_i)$ are g-periodic they all come from the union A' of the periodic g-orbits of points of A which is finite. Combining this and the fact that $F \subset B$ we see that $F \subset B \cap \varphi^{-1}(A')$. However, $B \cap \varphi^{-1}(A')$ is contained in the union of the boundaries of finitely many sets $\varphi^{-1}(z), z \in A'$ and by Lemma 1.2 each boundary consists of at most $r(X)$ points. This would imply that F is finite, a contradiction to the standing assumption of Case 2. So, there are infinitely many points y_i such that points $\varphi(y_i)$ have orbits disjoint from A. By Proposition 1.4(2) their periods n_i are multiples of p as desired, i.e. the possibility (B) holds.

Case 3. *The set F is a cycle (i.e. $y \in Per\, f, \operatorname{orb} y_i \to \operatorname{orb} y = F$).*

Let the period of y be k. Consider a map $\psi = f^k$. We may assume that there are small intervals $[y, z_1] = T_1 \subset [y, \zeta_1] = R_1, \dots, [y, z_l] = T_l \subset [y, \zeta_l] = R_l$ such that $n_j > l\, (\forall j)$ and the following holds

(i) $(y, z_s) \cap (y, z_t) = \emptyset\, (s \neq t)$;
(ii) the set $U = \bigcup_{r=1}^{l} [y, z_r)$ is a neighborhood of y;
(iii) $y_i \in T_1$ and $\operatorname{orb}_\psi y_i \subset U\, (\forall i)$;
(iv) $V = \bigcup_{i=1}^{l} R_i$ is a neighborhood of the point y, $V \setminus y$ contains no vertices of X, $\psi^j(U) \subset V$ for $0 \le j \le l$ and also $f^e(V) \cap f^d(V) = \emptyset\, (0 \le e < d < k)$.

Denote by $Y_r^{(i)}$ the smallest subinterval of T_r containing $\{orb_\psi\, y_i \cap T_r\}$; if $Y_r^{(i)} \neq \emptyset$ then set $Y_r^{(i)} = [\alpha_r^{(i)}, \beta_r^{(i)}]$ where $\beta_r^{(i)}$ is closer to the point y than $\alpha_r^{(i)}$. Consider some subcases.

Subcase 3a. *There is an infinite set C of such integers i that for any $j \le l$, $r \le l$ we have $y \notin \psi^j(Y_r^{(i)})$.*

In this case for every $i \in C$ the branch R_j such that $\psi(Y_r^{(i)}) \subset R_j$ for every r with $Y_r \neq \emptyset$ is well-defined. Then clearly $Y_j^{(i)} \subset \psi(Y_r^{(i)})$. Since $\operatorname{orb} y_i \subset V$ and y_i is periodic, we conclude that for every $i \in C$ there exists a number $0 < s_i \le l$ such that $\psi^{s_i}(Y_1^{(i)}) \subset R_1$ and moreover $f^d(Y_1^{(i)}) \cap f^e(Y_1^{(i)}) = \emptyset$ for $0 < d < e \le s_i k$. Taking a subsequence $E \subset C$ we may assume that $s_i = s \le l\, (\forall i \in E)$, so the number $p = ks$, the intervals $Y_1^{(i)} \equiv I_i$ and $\psi^s(Y_1^{(i)}) = f^{ks}(Y_1^{(i)}) \equiv J_i$ are those required in possibility (C).

Subcase 3b. *For any sufficiently large i there exist such $j(i) \le l$ and $r(i) \le l$ that $y \in \psi^{j(i)}(Y_{r(i)}^{(i)})$.*

To consider Subcase 3b we need the following Proposition 3.2. Let us point out here that even though statements similar to Proposition 2.3 are now considered standard in one-dimensional dynamics, major arguments employed here were introduced by A. N. Sharkovskii in his pioneer work. We will only sketch the arguments and leave the detailed proof to the reader.

Proposition 2.3. *In the situation of* Subcase 3b *there exist intervals*

$$L_i \subset Y_{r(i)}^{(i)} = [\alpha_{r(i)}^{(i)}, \beta_{r(i)}^{(i)}], \quad N_i \subset [\beta_{r(i)}^{(i)}, y]$$

and a number t_i such that $\psi^{t_i} N_i = \psi^{t_i} L_i = [\alpha_{r(i)}^{(i)}, y]$. Moreover, there exists a ψ^{t_i}-invariant set $\Sigma_i \subset L_i \cup N_i$ with the following properties:

(1) *$\psi^{t_i}|\Sigma_i$ is at most 2-to-1 semiconjugated to the full 2-shift;*
(2) *for every $\zeta \in \Sigma_i$, every small open interval W such that $\zeta \in W$ and every integer d there exist an interval $U', \zeta \in U' \subset W$ and such integer s that $\psi^{st_i d}(U') = [y, \alpha_{r(i)}^{(i)}]$ and $\psi^{t_i m}(U') \subset N_i \cup L_i\, (0 \le m < sd)$;*
(3) *there exists a point x such that $\Sigma_i = \omega_\psi(x) \subset \omega_f(x)$.*

Proof of Proposition 2.3. We will need the following technical claim which is stated here without a proof.

Claim. *Suppose that I and J are intervals contained in a graph X, J contains no vertices of X in its interior, $T \subset X$ is a tree, $J \subset f(I) \subset T$. Then there exists an interval $K \subset I$ such that $f(K) = J$.*

Let us now sketch the proof of Proposition 2.3. Since by the assumption (iv) $\psi(U) \subset V$ and by the assumption (iii) $\mathrm{orb}_\psi\, y_i \subset U$, we see that for any point $z \in \mathrm{orb}_\psi\, y_i$, $V \supset \psi[y, z] \supset [y, \psi(z)]$ and that by Claim there exists an interval $K \subset [y, z]$ such that $\psi(K) = [y, \psi(z)]$. This construction can be repeated, therefore in fact for any m we can find $K \subset [y, z]$ such that $\psi^m(K) = [y, \psi^m(z)]$. Since both $\alpha^{(i)}_{r(i)}, \beta^{(i)}_{r(i)}$ belong to the periodic orbit of y_i we can find the minimal number s such that $\psi^s(\beta^{(i)}_{r(i)}) = \alpha^{(i)}_{r(i)}$ and then by the above we can find an interval $N' \subset [y, \beta^{(i)}_{r(i)}]$ such that $\psi^s(N') = [y, \alpha^{(i)}_{r(i)}]$.

On the other hand, by the assumptions of Subcase 3b there exists a point $z \in Y^{(i)}_{r(i)}$ such that $\psi^{j(i)}(z) = y$; then similarly to the previous paragraph we can find an interval $L'' \subset [z, \alpha^{(i)}_{r(i)}]$ such that $\psi^{j(i)}(L'') = [y, \psi^{j(i)}(\alpha^{(i)}_{r(i)})]$. Again repeating the arguments from the previous paragraph we can find a number $T \geq 0$ and an interval $L' \subset L''$ such that $\psi^{T+j(i)}(L') = [y, \alpha^{(i)}_{r(i)})]$. Clearly, choosing $t_i = s(T + j(i))$ and applying Claim a few times we can find the intervals $L_i \subset L'$ and $N_i \subset N'$ as desired. The rest of the Proposition 2.3 can be prove by standard one-dimensional methods (see, e.g., [BGMY]). □

Let us continue the proof of Lemma 2.2 in Subcase 3b relying upon the conclusions and the notation of Proposition 2.3. By Theorem 1.5 there exists the unique basic set $B_i = B(M_i, f)$ such that $B_i \supset \omega_f(x) \supset \Sigma_i$. Then by the definition of a basic set and by Proposition 2.3(2) we have $M_i \supset [\alpha^{(i)}_{r(i)}, y]$. By Proposition 1.4(2) we may assume that all B_i have the same mixing period, say, p. Moreover, we may assume that there is a number $r \leq l$ such that $r(i) = r\ (\forall i)$.

Let g, φ, $D_i \subset B_i$ have the same meaning as in Theorem 1.3. Choose them in such a way that $(\Sigma_i \cap D_i)$ is infinite (it is always possible because the finite union of iterates of D_i contains $B_i \supset \Sigma_i$). We will prove that $\varphi[\alpha^{(i)}_r, y] \subset \varphi(D_i)$. Indeed, take such a point $z \in \Sigma_i \cap D_i$ that $\varphi(z) \in \mathrm{int}(\varphi(D_i))$, then by Proposition 2.3 take a small neighborhood W of z and a number s such that $\varphi(W) \subset \mathrm{int}(\varphi(D_i))$, $\psi^{stp}(W) = [\alpha^{(i)}_r, y]$. By the properties of φ we have $\varphi\psi^{st_ip}(W) \subset \varphi(D_i)$, so $\varphi[\alpha^{(i)}_r, y] \subset \varphi(D_i)$. Since $\beta^{(i)}_r$ lies in $[y, \alpha^{(i)}_r]$ between sets $N_i \cap \Sigma_i$ and $L_i \cap \Sigma_i$ belonging to B_i we see that together with the properties of φ it implies that $\varphi(\beta^{(i)}_r) \in \mathrm{int}(\varphi(D_i))$.

Now by Proposition 1.4(2) the fact that $\varphi(\beta^{(i)}_r) \in \mathrm{int}(\varphi(D_i))$ implies that f-period n_i of $\beta^{(i)}_r$ (which is equal to that of y_i) is divided by p. So we get the possibility (B) from Lemma 2.2 which concludes the proof. □

We are ready to prove Theorem 2.1; for the sake of convenience we restate it.

Theorem 2.1. *The following properties take place.*

(1) *A set $A \subset \mathbb{N}$ is graph-realizable iff it almost coincides with a finite union of some sets lZ or $Q(n)$.*
(2) *A set $A \subset \mathbb{N}$ is zero graph-realizable iff it almost coincides with a finite union of some sets $Q(n)$.*

Proof. (1)(i) Let us prove first that if $f : X \to X$ is a continuous graph map then $P(f)$ has the required form. To this end let us consider the family $\mathcal{A}$ of all sets $T(d,n) \equiv \{di : i \geq n\}$ contained in $P(f)$. These sets are *tails* of the previously introduced sets dZ.

Suppose that for some number d there exists such number n that $T(d,n) \in \mathcal{A}$; then d is called a *difference* (for the sake of brevity we omit the references to the map f in our terminology and notation). In other words, d is a difference if and only if dZ is almost contained in $P(f)$. For any difference d, let $n(d)$ be the minimal integer such that $T(d,n(d)) \in \mathcal{A}$; also, denote the family of all sets $T(d,n(d)) \in \mathcal{A}$ by $\mathcal{R}$. Clearly, '$\underset{a}{\supset}$' is a partial ordering in $\mathcal{R}$ and $T(d,n(d))$ is a maximal element of $\mathcal{R}$ if and only if d is not divided by any other difference.

Denote the family of all $\underset{a}{\supset}$-maximal elements of $\mathcal{R}$ by $\mathcal{R}_{max}$ and call d a *basic difference* if $T(d,n(d)) \in \mathcal{R}_{max}$. For any basic difference d choose a prime number $m(d) > dn(d)$ so that if $d_1 \neq d_2$ then $m(d_1) \neq m(d_2)$; the numbers $dm(d)$ are called *starting periods*. By the definitions for any $T(d,n) \in \mathcal{A}$ there exists $T(d',n') \in \mathcal{R}_{max}$ such that $T(d',n') \underset{a}{\supset} T(d,n)$.

Consider the family $\mathcal{B}$ of all sets $Q(m) = \{2^i m : i \geq 0\} \subset P(f)$ for which there is no set $T(d,n) \underset{a}{\supset} Q(m)$. Sets from $\mathcal{B}$ are partially ordered by inclusion; denote by $\mathcal{B}_{max}$ the family of all maximal elements of $\mathcal{B}$. A number m such that $Q(m) \in \mathcal{B}_{max}$ is called a *root*. Finally let us call a number $l \in P(f)$ a *period of finite type* if it does not belong to sets from either $\mathcal{R}_{max}$ or $\mathcal{B}_{max}$; the set of all periods of finite type is denoted by $\mathcal{F}$.

Clearly, to prove Theorem 2.1 it is enough to show that the set of all basic differences, roots and periods of finite type is finite. Indeed, suppose otherwise. Then the set of all starting periods, roots and periods of finite type is infinite too because one starting period may correspond to no more than finite number of basic differences.

So from now on we assume that the set of all starting periods, roots and periods of finite type is infinite, and prove that this is impossible by way of contradiction. Take the correspondent periodic point for every starting period, root and period of finite type (the definitions imply that this is possible). This way we get an infinite sequence $\{y_i\}$ of periodic points of periods N_i and we may assume that $N_i \nearrow \infty$. Let us apply Lemma 2.2 and consider some cases showing that in each case we get a contradiction.

Case A. *There exists d and a set $T(d,n) \in \mathcal{A}$ containing infinitely many numbers N_i.*

There exists $T(d',n') \in \mathcal{R}_{max}$ such that $T(d',n') \underset{a}{\supset} T(d,n)$. Therefore $T(d',n')$ contains infinitely many numbers N_i. By the definitions, if $N_i \in T(d',n')$ then N_i cannot be a root or a period of finite type. So, the only case to be considered is when all infinitely many N_i belonging to $T(d',n')$ are starting periods. In this case $N_i = e_i m(e_i)$ where $m(e_i) = m_i > e_i n(e_i)$ is a prime number. If $d' = 1$ we are done because then $P(f)$ almost coincides with $\mathbb{N}$. On the other hand if $d' > 1$ and $em(e)$ is a starting period divisible by d' then e is divided by d' or $m(e) = d'$ because $m(e)$ is a prime integer. By the choice of prime numbers $m(e)$ we conclude then that $m(e)$ can be equal to d' for no more than one difference e, hence we may assume that for all i the difference e_i is divisible by d', a contradiction to the fact that differences e_i are basic.

Case A$'$. *There is a sequence of cycles of connected subgraphs M_i and a number p such that for every i we have $y_i \in M_i$, a basic set $B_i = B(M_i, f)$ exists and is of mixing period p, and infinitely many N_i are divisible by p.*

By Lemma 1.10 $P(f)$ almost contains the set pZ. Therefore p is a difference and $T(p, n(p)) \in \mathcal{A}$. By Case A this is impossible.

Case B. *Choosing a subsequence we may assume that there is a sequence of pairs of intervals $J_i \supset I_i \ni y_i$ and a number p such that for any i we have $f^p(I_i) = J_i$, intervals $f(I_i), \dots, f^p(I_i) = J_i$ are pairwise disjoint,* $\operatorname{orb} y_i \subset \bigcup_{j=0}^{p-1} I_i$ *and either $f^p(I_i) = I_i = J_i$ or endpoints of I_i belong to* $\operatorname{orb} y_i$.

Let us apply Corollary S to I_i. Consider the set P_{I_i} of periods of all periodic points ζ such that $\operatorname{orb} \zeta \subset \bigcup_{j=0}^{p-1} f^j I_i$. Then by Corollary S there exists $k_i \in \mathbb{N} \cup 2^\infty$ such that $P_{I_i} = pS(k_i)$. Since sets $S(k)$ for all $k \in \mathbb{N} \cup 2^\infty$ are linearly ordered we conclude that there exists $k \in \mathbb{N} \cup 2^\infty$ such that $R = \bigcup_{i=0}^\infty P_{I_i} = pS(k)$. Observe that $N_i \in R$ for any i and consider two subcases.

Subcase B1. $k \in \mathbb{N}$.

Clearly the property $N_i \to \infty$ implies that $k = 2^l(2m+1), m \geq 1$ (in other words, k is not a power of 2). Then $T(2^l p, 2^l p(2m+1)) \in \mathcal{A}$ and at the same time $T(2^l p, 2^l p(2m+1)) \underset{a}{\supset} R \supset \{N_i\}$; so we are done by what has been proven in Case A.

Subcase B2. $k = 2^\infty$.

If there is a set $T(d, n) \in \mathcal{A}$ such that $T(d, n) \underset{a}{\supset} R$ then $T(d, n) \underset{a}{\supset} \{N_i\}$ and by Case A we get a contradiction.

Suppose there is no set $T(d, n)$ such that $T(d, n) \underset{a}{\supset} R$. Then $R \in \mathcal{B}$ and there is a set $Q(l) \in \mathcal{B}_{max}$ such that $R \subset Q(l)$. Hence $\{N_i\} \subset Q(l)$. Now, if a starting period $em(e)$ belongs to $Q(l)$ then the corresponding set $T(e, n(e))$ almost contains $Q(l)$, a contradiction. On the other hand, by the definition the only root which belongs to $Q(l)$ is l. Finally, no period of finite type can belong to $Q(l)$. Therefore, $Q(l)$ cannot contain infinitely many numbers N_i. This contradiction implies that Subcase B2 is also impossible.

It remains now to apply Lemma 2.2 to the sequence of periodic points y_i. According to this lemma, either Case A$'$ or Case B must take place. However, we have just shown that either case is impossible. This proves that indeed if $A = P(f)$ is the set of all periods of periodic points of a graph map $f : X \to X$ then A almost coincides with a finite union of some sets lZ or $Q(n)$ and concludes the proof of the first part of statement (1) of Theorem 2.1.

(1)(ii) Now suppose there is a set A which almost coincides with the finite union of some sets lZ and $Q(m)$. To construct a graph map $f : X \to X$ such that $P(f) = A$ let us first note that we do not suppose X to be connected. So it is enough to show that the following two statements are true.

Proposition 2.4. *For any $m \geq 0$ there exists a graph map $g : Y \to Y$ such that we have $P(g) = \{i : i \geq m\} = T(1, m)$*

Proof. This result follows easily from the results of [AL, M2] which imply that there exists a map $g_m : S^1 \to S^1$ with $P(g_m) = T(1, m)$. □

Proposition 2.5. *There is a map $\psi : [0, 1] \to [0, 1]$ such that $P(\psi) = \{1, 2, 4, 8, \dots\} = Q(1)$.*

Proof. This claim is a part of the Sharkovskii Theorem. □

Taking into account the existence of graph maps g with $Per\, g = \emptyset$ (e.g. irrational rotation) one can easily construct the required graph map by combining the following sets into one graph: 1) finite collections of circles permuted by the map on which the sets of periods of form lZ are realized (here we rely upon Proposition 2.4); 2) finite collections of intervals permuted by the map on which the sets of form $Q(l)$ are realized; 3) finite collections of periodic orbits. (the details of the construction are left to the reader). This completes the proof of the first statement of Theorem 2.1.

(2)(i) We need to prove that every graph map g with zero entropy has a set of periods $P(f)$ which almost coincides with a finite union of some sets $Q(n)$. Let us begin by proving that if $h(f) = 0$ then there exists no number l such that $lZ \subset P(f)$. Indeed, otherwise choose periodic points y_i of periods lp_i where p_i is a sequence of prime numbers and apply to them Lemma 2.2. Since by Theorem 1.3 the entropy of a map on its basic sets is positive we conclude that f has no basic sets. Therefore, of the cases listed in Lemma 2.2 the only possible one in our situation is Case C.

So, we may assume that refining our sequence of periodic points $\{y_i\}$ we get another sequence $\{y'_i\}$ for which the following holds: there is a sequence of pairs of intervals $J_i \supset I_i$ and there is a number r such that $f^r(I_i) = J_i$, intervals $f(I_i), \dots, f^{r-1}(I_i), J_i$ are pairwise disjoint, $\operatorname{orb} y_i \subset \bigcup_{j=0}^{r-1} f^j(I_i)$ $(\forall i)$ and either $f^r(I_i) = J_i$ or endpoints of I_i belong to $\operatorname{orb} y_i$. Here the period of y_i is lp'_i where p'_i is a sequence of prime numbers converging to infinity.

Consider the set P_{I_i} of periods of all periodic points ζ such that $\operatorname{orb} \zeta \subset \bigcup_{j=0}^{p-1} f^j I_i$. Then by Corollary S there exists $k_i \in \{2^j\}_{j=0}^{\infty}$ such that $P_{I_i} = rS(k_i)$, and so in fact there is $k \in \{2^j\}_{j=0}^{\infty}$ such that all periods lp'_i of the points y'_i belong to $rS(k)$. Since p'_i form a growing sequence of prime numbers this is clearly impossible. We conclude that if $h(f) = 0$ then there exists no number l such that $lZ \subset P(f)$ (an alternative proof of this fact follows from [LM]).

Together with claim (1) of Theorem 2.1 this implies that every graph map g with zero entropy has a set of periods $P(f)$ which almost coincides with a finite union of some sets $Q(n)$.

(2)(ii) The construction is similar to that in the proof of the first statement of Theorem 2.1 and is left to the reader. □

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References

[**AL**] L. Alseda,J. Llibre, *A Note on the Set of Periods for Continuous Maps of the Circle which Have Degree One*, Proc. Amer. Math. Soc. **93** (1985), 133–138.

[**Ba**] S. Baldwin, *An extension of Šarkovskiĭ's theorem to the n-od*, Ergodic Theory and Dynamical Systems **11** (1991), 249–271.

[**BGMY**] L. Block, J. Guckenheimer, M. Misiurewicz, L. S. Young, *Periodic Points and Topological Entropy of One Dimensional Maps*, Lect. Notes in Math., vol. 819, Springer, Berlin, 1980, pp. 18–34.

[**B1**] A. M. Blokh, *On the limit behaviour of one-dimensional dynamical systems.1,2* (Russian) , Preprints VINITI #1156-82,#2704-82,Moscow (1982).

[**B2**] A. M. Blokh, *Decomposition of Dynamical Systems on an Interval*, Russ. Math. Surv., #5, **38** (1983), 133–134.

[**B3**] ———, *On Dynamical Systems on One-Dimensional Branched Manifolds.1* (Russian), Theory of Functions, Functional Analysis and Applications, Kharkov, **46** (1986), 8–18; *2*, Theory of Functions, Functional Analysis and Applications, Kharkov, **47** (1986), 67–77; *3*, Theory of Functions, Functional Analysis and Applications, Kharkov, **48** (1987), 32–46.

[**B4**] ———, *The Spectral Decomposition, Periods of Cycles and Misiurewicz Conjecture for Graph Maps*, Lecture Notes in Math., Springer **1514** (1991), 24–31.

[**B5**] ———, *The Spectral Decomposition for One-Dimensional Maps*, Dynamics Reported **4** (1995), 1–59.

[**B6**] ———, *Rotation Numbers, Twists and a Sharkovskii-Misiurewicz-type Ordering for Patterns on the Interval*, Ergodic Theory and Dynamical Systems **15** (1995), 1–14.

[**BM**] A. Blokh, M. Misiurewicz, *New order for periodic orbits of interval maps*, Ergodic Theory and Dynamical Systems **17** (1997), 565–574.

[**DGS**] M. Denker, C. Grillenberger, K. Sigmund, *Ergodic Theory on Compact Spaces*, vol. 527, Springer, Lect. Notes in Math., Berlin, 1976.

[**H**] F. Hofbauer, *The Structure of Piecewise-Monotonic Transformations*, Ergodic Theory and Dynamical Systems **1** (1981), 159–178.

[**JR**] L. Jonker, D. Rand, *Bifurcations in One-Dimensional Systems.1:The non-wandering Set*, Inv. Math. **62** (1981), 347–365.

[**LM**] J. Llibre, M. Misiurewicz, *Horseshoes, Entropy and Periods for Graph Maps*, Topology **32** (1993), 649–664.

[**M1**] M. Misiurewicz, *Horseshoes for Continuous Mappings of an Interval*, Bull. Acad. Pol. Sci., Sér. Sci. Math. **27** (19879), 167–169.

[**M2**] M. Misiurewicz, *Periodic Points of Maps of Degree One of a Circle*, Ergodic Theory and Dynamical Systems **2** (1982), 221–227.

[**N**] Z. Nitecki, *Topological Dynamics on the Interval*, Ergodic Theory and Dynamical Systems, 2., Progress in Math., vol.21, Birkhäuser, Boston, 1982, pp. 1–73.

[**S1**] A. N. Sharkovskii, *Partially Ordered System of Attracting Sets* (in Russian) , DAN SSSR **170** (1966), 1276–1278.

[**S2**] A. N. Sharkovskii, *Coexistence of Cycles of a Continuous Map of a Line into itself*, Ukr. Math. Journal **16** (1964), 61–71.

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