

SLICES OF THE PARAMETER SPACE OF CUBIC POLYNOMIALS

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ABSTRACT. In this paper, we study slices of the parameter space of cubic polynomials, up to affine conjugacy, given by a fixed value of the multiplier at a non-repelling fixed point. In particular, we study the location of the *main cubioid* in this parameter space. The *main cubioid* is the set of affine conjugacy classes of complex cubic polynomials that have certain dynamical properties generalizing those of polynomials $z^2 + c$ for c in the filled main cardioid.

1. INTRODUCTION

By *classes* of polynomials, we mean affine conjugacy classes. For a polynomial f , let $[f]$ be its class. For any polynomial f , we write $K(f)$ for the filled Julia set of f , and $J(f)$ for the Julia set of f . The *connectedness locus* \mathcal{M}_d of degree d is the set of classes of degree d polynomials whose critical points *do not escape* (i.e., have bounded orbits). Equivalently, \mathcal{M}_d is the set of classes of degree d polynomials f whose Julia set $J(f)$ is connected. The connectedness locus \mathcal{M}_2 of degree 2 is otherwise called the *Mandelbrot set*; the connectedness locus \mathcal{M}_3 of degree 3 is also called the *cubic connectedness locus*. The *principal hyperbolic domain* \mathcal{PHD}_3 of \mathcal{M}_3 can be defined as the set of classes of hyperbolic cubic polynomials with Jordan curve Julia sets. Equivalently, we have $[f] \in \mathcal{PHD}_3$ if both critical points of f are in the immediate attracting basin of the same attracting (or super-attracting) fixed point. Recall that a polynomial is *hyperbolic* if the orbits of all its critical points converge to attracting or super-attracting cycles.

We define the *main cubioid* \mathcal{CU} as the set of classes $[f] \in \mathcal{M}_3$ with the following properties: f has a non-repelling fixed point, f has no repelling periodic cutpoints in $J(f)$, and all non-repelling periodic points of f , except at most one fixed point, have multiplier 1. The main cubioid is a cubic analogue of the main cardioid (by the *main cardioid*

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we mean the closure of the family of all polynomials $z^2 + c$ that have an attracting fixed point). In this paper, we discuss properties of \mathcal{CU} -polynomials, i.e., cubic polynomials f such that $[f] \in \mathcal{CU}$.

Theorem 1.1 ([BOPT14]). *We have that $\overline{\mathcal{PHD}}_3 \subset \mathcal{CU}$.*

It is conjectured in [BOPT14] that $\overline{\mathcal{PHD}}_3 = \mathcal{CU}$. Some properties of the complement $\mathcal{CU} \setminus \overline{\mathcal{PHD}}_3$ are discussed in [BOPT14b].

Let \mathcal{F} be the space of polynomials

$$f_{\lambda,b}(z) = \lambda z + bz^2 + z^3, \quad \lambda \in \mathbb{C}, \quad b \in \mathbb{C}.$$

An affine change of variables reduces any cubic polynomial f to the form $f_{\lambda,b}$. Clearly, 0 is a fixed point for every polynomial in \mathcal{F} . Define the λ -slice \mathcal{F}_λ of \mathcal{F} as the space of all polynomials $g \in \mathcal{F}$ with $g'(0) = \lambda$. The λ -slice \mathcal{F}_λ maps onto the space of classes of all cubic polynomials with a fixed point of multiplier λ as a finite branched covering. Indeed, it is easy to see that polynomials $f_{\lambda,b}$ and $f_{\lambda,-b}$ are affinely conjugate and belong to the same class consisting of *exactly* these polynomials. Hence, this branched covering is equivalent to the map $b \mapsto a = b^2$, i.e., classes of polynomials $f_{\lambda,b} \in \mathcal{F}_\lambda$ are in one-to-one correspondence with the values of a . Thus, if we talk about, say, points $[f]$ of \mathcal{M}_3 , then it suffices to take $f \in \mathcal{F}_\lambda$ for some λ . There is no loss of generality in that we consider only perturbations of f in \mathcal{F} . The family \mathcal{F} has been studied by Zakeri [Z99], Buff and Henriksen [BuHe01]. The main result of this paper is a description of \mathcal{CU} through λ -slices where $|\lambda| \leq 1$.

We use calligraphic (script) font for parameter space objects like \mathcal{F} , \mathcal{M}_3 , etc., to distinguish them from the dynamical plane objects. We mostly use German Gothic fonts for various objects in the closed disk related to *laminations* and used in the combinatorial models of polynomials (laminations will be introduced in Subsection 5.2). We mostly use Greek letters for *angles*, i.e. elements of \mathbb{R}/\mathbb{Z} .

We need a few combinatorial concepts. Given an angle $\alpha \in \mathbb{R}/\mathbb{Z}$, we write $\bar{\alpha}$ for the corresponding point $e^{2\pi i \alpha}$ of the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. The angle tripling map $\sigma_3 : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ identifies with the self-map of \mathbb{S}^1 taking z to z^3 . We write $(\bar{\alpha}, \bar{\beta})$ for an open arc of the unit circle with endpoints $\bar{\alpha}$ and $\bar{\beta}$ if the direction from $\bar{\alpha}$ to $\bar{\beta}$ within the arc is positive. A closed chord of the closed unit disk $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ with endpoints $\bar{\alpha}, \bar{\beta} \in \mathbb{S}^1$ is denoted by $\overline{\alpha\beta}$. Given a closed set $X \subset \mathbb{S}^1$, define *holes* of X as components of $\mathbb{S}^1 \setminus X$.

Let $\mathfrak{U} \subset \overline{\mathbb{D}}$ be the convex hull of $\mathfrak{U}' = \mathfrak{U} \cap \mathbb{S}^1$. By definition, *holes* of \mathfrak{U} are holes of \mathfrak{U}' ; *edges* of \mathfrak{U} are chords on the boundary of \mathfrak{U} . The set \mathfrak{U} is said to be a (*stand alone*) *invariant gap* if $\sigma_3(\mathfrak{U}') = \mathfrak{U}'$, and, for every hole $(\bar{\alpha}, \bar{\beta})$ of \mathfrak{U} , we either have $\sigma_3(\bar{\alpha}) = \sigma_3(\bar{\beta})$ (then

the chord $\overline{\alpha\beta}$ is called *critical*), or the circular arc $(\sigma_3(\overline{\alpha}), \sigma_3(\overline{\beta}))$ is also a hole of \mathfrak{U} . Extend the map $\sigma_3 : \mathfrak{U}' \rightarrow \mathfrak{U}'$ to every edge of \mathfrak{U} linearly so that a critical edge maps to a point, and a non-critical edge $\overline{\alpha\beta}$ maps to $\overline{(3\alpha)(3\beta)} = \sigma_3(\alpha)\sigma_3(\beta)$ and keep the notation σ_3 for this extension. The *degree* of \mathfrak{U} is the number of its edges mapping *onto* a non-degenerate edge of \mathfrak{U} ; it is well-defined. Degree two gaps are also called *quadratic gaps*.

We measure arc length in \mathbb{S}^1 so that the total length of the entire circle is 1. The *length of a chord* ℓ of \mathbb{D} is the length of the shorter circle arc in \mathbb{S}^1 connecting the endpoints of ℓ . A hole of \mathfrak{U} is called a *major hole* if its length is greater than or equal to $\frac{1}{3}$. The edge of \mathfrak{U} connecting the endpoints of a major hole is called a *major edge*, or simply a *major*, of \mathfrak{U} . Any *quadratic* invariant gap \mathfrak{U} has exactly one major that is either critical or periodic [BOPT13]. By [BOPT13], there exists a Cantor set $Q \subset \mathbb{S}^1$ with the following property. If we collapse every hole of Q to a point, we obtain a topological circle whose points are in one-to-one correspondence with *all* quadratic invariant gaps \mathfrak{U} such that $\mathfrak{U} \cap \mathbb{S}^1$ is a Cantor set. Moreover, the following holds:

- (1) for each point $\overline{\theta} \in \mathbb{S}^1$ of Q that is not an endpoint of a hole of Q , the critical chord $\overline{(\theta + \frac{1}{3})(\theta + \frac{2}{3})}$ is the major of a quadratic invariant gap \mathfrak{U} such that $\mathfrak{U} \cap \mathbb{S}^1$ is a Cantor set;
- (2) for each hole $(\overline{\theta}_1, \overline{\theta}_2)$ of Q the chord $\overline{(\theta_1 + \frac{1}{3})(\theta_2 + \frac{2}{3})}$ is the periodic major of a quadratic invariant gap \mathfrak{U} such that $\mathfrak{U} \cap \mathbb{S}^1$ is a Cantor set.

The convex hull \mathfrak{Q} of Q in the plane is called the *Principal Quadratic Parameter Gap* (see Figure 1); it is similar to the Main Cardioid of \mathcal{M}_2 .

Let us now introduce some analytic notions. Fix λ with $|\lambda| \leq 1$. The λ -*connectedness locus* \mathcal{C}_λ is defined as the set of all $f \in \mathcal{F}_\lambda$ such that $K(f)$ is connected, equivalently, such that $[f] \in \mathcal{M}_3$. This is a full continuum [BrHu88, Z99] (a compact set $X \subset \mathbb{C}$ is *full* if $\mathbb{C} \setminus X$ is connected). For every polynomial $f \in \mathcal{F}_\lambda$ and every angle α , we define the *dynamic ray* $R_f(\alpha)$. Also, for every angle θ , in the parameter plane of \mathcal{F}_λ we define the *parameter ray* $\mathcal{R}_\lambda(\theta)$. We use rays to show that the picture in \mathcal{F}_λ resembles the picture in the parameter plane of quadratic polynomials. Let \mathcal{CU}_λ be the set of all polynomials $f \in \mathcal{F}_\lambda$ with $[f] \in \mathcal{CU}$.

Main Theorem. *The set \mathcal{CU}_λ is a full continuum. The set \mathcal{C}_λ is the union of \mathcal{CU}_λ and a countable family of limbs \mathcal{LI}_H of \mathcal{C}_λ parameterized by holes H of \mathfrak{Q} . The union is disjoint (except when $\lambda = 1$). For a hole $H = (\theta_1, \theta_2)$ of \mathfrak{Q} , the following holds.*

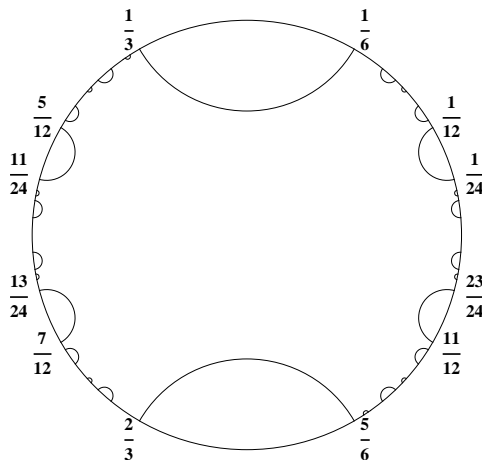


FIGURE 1. The Principal Quadratic Parameter Gap

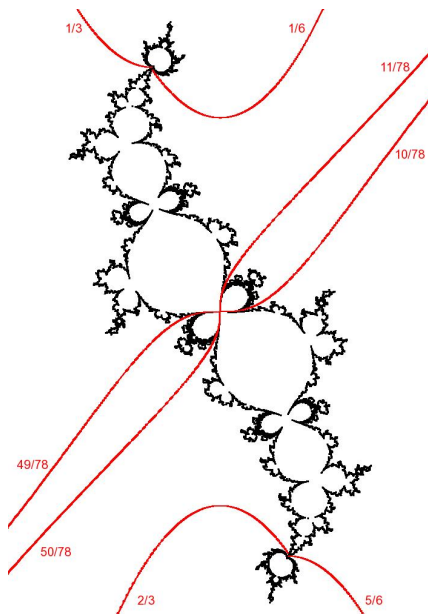
- (1) The parameter rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ land at the same point $f_{\text{root}(H)}$.
- (2) Let $\mathcal{W}_\lambda(H)$ be the component of $\mathbb{C} \setminus \overline{\mathcal{R}_\lambda(\theta_1) \cup \mathcal{R}_\lambda(\theta_2)}$ containing the parameter rays with arguments from H . Then, for every $f \in \mathcal{W}_\lambda(H)$, the dynamic rays $R_f(\theta_1 + \frac{1}{3})$, $R_f(\theta_2 + \frac{2}{3})$ land at the same point, either periodic and repelling for all $f \in \mathcal{W}_\lambda(H)$, or 0 for all $f \in \mathcal{W}_\lambda(H)$. Moreover, $\mathcal{LI}_H = \mathcal{W}_\lambda(H) \cap \mathcal{C}_\lambda$.
- (3) The dynamic rays $R_{f_{\text{root}(H)}}(\theta_1 + \frac{1}{3})$, $R_{f_{\text{root}(H)}}(\theta_2 + \frac{2}{3})$ land at the same parabolic periodic point, and $f_{\text{root}(H)}$ belongs to \mathcal{CU}_λ .

Figure 2 shows the parameter slice $\mathcal{F}_{e^{2\pi i/3}}$ in which several parameter rays and several wakes are shown.

2. DETAILED STATEMENT OF THE MAIN RESULTS

In this section, we break our Main Theorem into steps called Theorem A, Theorem B and Theorem C.

2.1. The structure of \mathcal{F}_λ . A (pre)critical point of a polynomial f is defined as a point mapped to a critical point of f by some iterate $f^{\circ r}$ of the polynomial f (here $r \geq 0$). Let $G_{K(f)}$ be the Green function for $K(f)$. Call unbounded trajectories of the gradient flow for $G_{K(f)}$ *dynamic rays* of f . Dynamic rays of f can be of two types. All but countably many of them accumulate in $K(f)$ and are called *smooth rays* (of f). Otherwise a dynamic ray extends from infinity until it *crashes* at an escaping (pre)critical point of f ; such rays can exist only if $K(f)$ is disconnected, and escaping critical points exist. The remaining part of $\mathbb{C} \setminus K(f)$ consists of bounded trajectories of the gradient flow for $G_{K(f)}$

FIGURE 2. Parameter slice $\mathcal{F}_{e^{2\pi i/3}}$ with some rays

called *ideal rays* (of f). An ideal ray can extend from one escaping (pre)critical point to another escaping (pre)critical point or from an escaping (pre)critical point to the Julia set. Thus, if c is an escaping (pre)critical point, then several dynamic rays crash at c , and some ideal rays accumulate at c . We conclude that $\mathbb{C} \setminus K(f)$ is the union of dynamic rays and ideal rays.

Let V_f be the union of all dynamic rays of f . Then V_f is homeomorphic to $\mathbb{C} \setminus \overline{\mathbb{D}}$ and coincides with the basin of attraction of infinity with countably many ideal rays removed. The *Böttcher coordinate* is an analytic map $\phi_f : V_f \rightarrow \mathbb{C}$ such that $\phi_f(V_f)$ is $\mathbb{C} \setminus \overline{\mathbb{D}}$ united with countably many finite radial segments that originate at the boundary of $\overline{\mathbb{D}}$ and form a null sequence. Moreover, we can choose ϕ_f so that the derivative $\phi'_f(z)$ tends to a positive real number as $z \rightarrow \infty$ and $\phi_f \circ f = \phi_f^3$. The existence of Böttcher coordinates was established by Douady and Hubbard in [DH8485]. Theorem 2.1 stated below is a consequence of the analytic dependence of the Böttcher coordinate on parameters [DH8485, BrHu88].

Theorem 2.1 ([BuHe01], Proposition 2). *Fix λ , and let \mathcal{V} be the union of $\{b\} \times V_{f_{\lambda,b}}$ over all $b \in \mathbb{C}$. This set is open in \mathbb{C}^2 . The map $\Psi : \mathcal{V} \rightarrow \mathbb{C}^2$ given by the formula $\Psi(b, z) = (b, \phi_{f_{\lambda,b}}(z))$ is an analytic embedding of \mathcal{V} into \mathbb{C}^2 .*

Note that dynamic rays of f are preimages under ϕ_f of unbounded radial segments. Hence dynamic rays can be parameterized by *angles*, i.e., elements of \mathbb{R}/\mathbb{Z} : the dynamic ray $R = R_f(\theta)$ of argument $\theta \in \mathbb{R}/\mathbb{Z}$ is such that for every $z \in R$, we have $\phi_f(z) = re^{2\pi i\theta}$, where $r > 1$. A dynamic ray R *lands* at a point $z \in \mathbb{C}$ if $z \in K(f)$, and z is the only accumulation point of R in \mathbb{C} . The complement of V_f in $\mathbb{C} \setminus K(f)$ consists of ideal rays. Every accessible point of $K(f)$ is the landing point of some dynamic ray or some ideal ray.

Recall that $|\lambda| \leq 1$. Since 0 is a non-repelling fixed point of $f \in \mathcal{F}_\lambda$, it follows from the Fatou–Shishikura inequality [Fat20, Shi87] that at least one critical point of f is non-escaping. Thus, each f in $\mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ must have two *distinct* critical points, one of which escapes to infinity while the other is non-escaping. In this case, the non-escaping critical point of f will be denoted by $\omega_1(f)$. Let $\omega_2(f) \neq \omega_1(f)$ denote the other critical point of f . Let $\omega_2^*(f)$ be the corresponding *co-critical point* (i.e., $\omega_2^*(f)$ is the unique point with $\omega_2^*(f) \neq \omega_2(f)$ and $f(\omega_2^*(f)) = f(\omega_2(f))$).

We write \mathcal{P}_λ for the set of polynomials $f \in \mathcal{F}_\lambda$ such that $[f] \in \overline{\mathcal{PHD}_3}$. By [BOPT14], if $f \in \mathcal{P}_\lambda$, then $[f] \in \mathcal{CU}$. By [BOPT14a], the points $\omega_1(f)$ and $\omega_2(f)$ can be consistently defined for all $f \in \mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$. Moreover, for every component \mathcal{W} of $\mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$, the points $\omega_1(f)$ and $\omega_2(f)$ depend holomorphically on f as f moves through \mathcal{W} . Observe that if $f \in \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$, then $\omega_2^*(f) \in V_f$. Then the map $\Phi_\lambda(f) = \phi_f(\omega_2^*(f))$ is a conformal isomorphism between the complement of \mathcal{C}_λ in \mathcal{F}_λ and the complement of the closed unit disk [BuHe01]. This isomorphism can be used to define *parameter rays*. Namely, the ray $\mathcal{R}_\lambda(\theta)$ is defined as the preimage of the straight ray (unbounded radial segment) $\{re^{2\pi i\theta} \mid r > 1\}$ under Φ_λ . We emphasize here that for *dynamic rays* we use the notation $R(\cdot)$ (with possible subscripts and superscripts) while for *parameter rays* we use the notation $\mathcal{R}(\cdot)$ (with possible subscripts). This is consistent with our convention to use the calligraphic font for parameter space objects. We have $f \in \mathcal{R}_\lambda(\theta)$ if and only if $\omega_2^*(f)$ belongs to the ray $R_f(\theta)$, i.e., if and only if both $R_f(\theta + 1/3)$ and $R_f(\theta + 2/3)$ crash into the critical point $\omega_2(f)$.

2.2. Immediate renormalization. In [Lyu83, MSS83], the notion of *J-stability* was introduced for any holomorphic family \mathcal{T} of rational functions: a map from \mathcal{T} is *J-stable* with respect to \mathcal{T} if its Julia set admits an equivariant holomorphic motion over some neighborhood of the map in the family. Say that $f \in \mathcal{F}_\lambda$ is *stable* if it is *J-stable* with respect to \mathcal{F}_λ , otherwise we call f *unstable*. The set \mathcal{F}_λ^{st} of all stable polynomials $f \in \mathcal{F}_\lambda$ is an open subset of \mathcal{F}_λ . A component of \mathcal{F}_λ^{st} is called a (λ) -*stable component*, or a *domain of (λ) -stability*. It is easy

to see that, given λ , the polynomial $f_{\lambda,b}$ has a disconnected Julia set if $|b|$ is sufficiently big. Hence, if $f = f_{\lambda,b}$ is stable and $J(f)$ is connected, then its domain of stability is bounded. Let \mathcal{CU}_λ be the set of all polynomials $f \in \mathcal{F}_\lambda$ with $[f] \in \mathcal{CU}$.

Recall some topological concepts. Observe that, given a compact set $A \subset \mathbb{C}$, there is a unique unbounded complementary domain U of A . The set A is said to be *unshielded* if A coincides with the boundary of U . For example, a polynomial Julia set is unshielded whether it is connected or not. For an unshielded compact set A , the *topological hull* $\text{Th}(A)$ of A is by definition the union of A with all bounded complementary components of A . Equivalently, $\text{Th}(A)$ is the complement of U .

Theorem 2.2 ([BOPT14, BOPT14a]). *All bounded components of $\mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$ consist of stable \mathcal{CU} -polynomials. Moreover, $\mathcal{P}_\lambda \subset \text{Th}(\mathcal{P}_\lambda) \subset \mathcal{CU}_\lambda$.*

The following classic definition and major result are due to Douady and Hubbard [DH85].

Definition 2.3. A *polynomial-like* map is a proper holomorphic map $f : U \rightarrow f(U)$ of degree $k > 1$, where $U, f(U) \subset \mathbb{C}$ are open subsets isomorphic to a disk, and $\bar{U} \subset f(U)$. The *filled Julia set* $K(f)$ of f is the set of points in U that never leave U under iteration. The *Julia set* $J(f)$ of f is defined as the boundary of $K(f)$. Two polynomial-like maps $f : U \rightarrow f(U)$ and $g : V \rightarrow g(V)$ are said to be *hybrid equivalent* if there is a quasi-conformal map φ from a neighborhood of $K(f)$ to a neighborhood of $K(g)$ conjugating f to g in the sense that $g \circ \varphi = \varphi \circ f$ wherever both sides are defined and such that $\bar{\partial}\varphi = 0$ almost everywhere on $K(f)$. If $k = 2$, then the corresponding polynomial-like maps are said to be *quadratic-like*. Note that U can always be chosen as a Jordan domain.

Straightening Theorem ([DH85]). *Let $f : U \rightarrow f(U)$ be a polynomial-like map. Then f is hybrid equivalent to a polynomial P of the same degree. Moreover, if $K(f)$ is connected, then P is unique up to (global) conjugation by an affine map.*

We will need the following definition.

Definition 2.4. Let f be a polynomial, $f^* = f|_U$ be a polynomial-like map, and g be the polynomial hybrid equivalent to f^* . The curves in $\mathbb{C} \setminus K(f^*)$ corresponding to dynamic rays of g are called *polynomial-like rays* of f . If the degree of f^* is two, then we will talk about quadratic-like rays. We will denote polynomial-like rays $R^*(\beta)$, where β is the argument of the external ray of g corresponding to $R^*(\beta)$.

Note that polynomial-like rays of f are only defined in a bounded neighborhood of $K(f^*)$. Observe also that the polynomial-like (quadratic-like) map will be always specified when we talk about polynomial-like (quadratic-like) rays, which is why we omit it from our notation.

Say that a cubic polynomial $f \in \mathcal{F}$ is *immediately renormalizable* if there are Jordan domains U^* and V^* such that $0 \in U^*$, and $f^* = f : U^* \rightarrow V^*$ is a quadratic-like map (we will use the notation f^* at several occasions in the future when we talk about immediately renormalizable maps). If $f \in \mathcal{F}_\lambda$ with $|\lambda| \leq 1$ is immediately renormalizable, then the quadratic-like Julia set $J(f^*) = J^*$ is connected. Indeed, f^* is hybrid equivalent to a quadratic polynomial g . Since $0 \in K(f^*)$ is a non-repelling f -fixed point, it corresponds to a non-repelling fixed point of g . Hence, $J(g)$ and $J(f^*)$ are connected, and $g(z) = z^2 + c$ with c in the filled main cardioid. The filled quadratic-like Julia set of f^* is denoted by $K(f^*)$. In [BOPT14a], some sufficient conditions on polynomials for being immediately renormalizable are obtained.

Theorem 2.5 ([BOPT14a]). *All polynomials in the unbounded component of $\mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$ are immediately renormalizable.*

Theorem 2.5 and Theorem 2.2 imply the following corollary.

Corollary 2.6. *All polynomials in $\mathcal{F}_\lambda \setminus \mathcal{CU}_\lambda$ are immediately renormalizable.*

Proof. By Theorem 2.2, $\text{Th}(\mathcal{P}_\lambda) \subset \mathcal{CU}_\lambda$. Therefore, any polynomial $f \in \mathcal{F}_\lambda \setminus \mathcal{CU}_\lambda$ in fact belongs to $\mathcal{F}_\lambda \setminus \text{Th}(\mathcal{P}_\lambda)$ and therefore, by Theorem 2.5, is immediately renormalizable as desired. \square

The discussion below, following [BCLOS10], aims at relating quadratic-like rays to dynamic and ideal rays (observe that [BCLOS10] deals with polynomials of arbitrary degree). Define an *external ray* R^e as a smooth dynamic ray, or a one-sided limit of smooth dynamic rays. Slightly abusing the language, we will call such rays either *smooth rays* or *one-sided rays*. The analysis of the structure of $\mathbb{C} \setminus K(f)$ given above shows that every external ray R^e has its initial (unbounded) part coinciding with the dynamic ray of a well-defined argument θ . If R^e is itself a smooth ray, this completely defines its argument as θ . The ray R^e is then approximated by smooth external rays from either side. If, however, R^e is not smooth, then the same analysis shows that R^e is the one-sided limit of smooth rays from exactly one side, in which case we associate to R^e the appropriate *one-sided external argument*, θ^+ or θ^- , and denote R^e by $R_f^e(\theta^-)$ or $R_f^e(\theta^+)$, respectively. Observe that parts of $R_f^e(\theta^-)$ and $R_f^e(\theta^+)$ from infinity to a certain precritical

point in the plane coincide. If R^e is a smooth ray of argument θ , then we set $R^e(\theta^+) = R^e(\theta^-) = R^e$.

An external ray R^e of a polynomial f accumulates in a unique component of $J(f)$. If R^e accumulates in a component E of $J(f)$, then R^e is called *an external ray to E* . For a non-closed oriented curve l from infinity or a finite point in \mathbb{C} to a bounded region in \mathbb{C} , define its *principal* (or *limit*) set $\text{Pr}(l)$, analogously to how it is done for conformal external rays, as the set of all accumulation points of l in the forward direction, i.e., as the ω -limit set of l . Thus, the *principal set* $\text{Pr}(R^e)$ of R^e is the set of all accumulation points of R^e in $J(f)$. Possible intersections of external rays are described in [BCLOS10].

Lemma 2.7 (Lemma 6.1 [BCLOS10]). *If two distinct external rays R_0^e, R_1^e have a common point, then R_0^e, R_1^e are both non-smooth. The intersection $L = R_0^e \cap R_1^e$ is connected and can contain a (pre)critical point only as an endpoint. Furthermore, one and only one of the following cases holds:*

- (1) L is a smooth curve joining infinity and a (pre)critical point;
- (2) L is a single (pre)critical point;
- (3) L is a smooth closed arc between two (pre)critical points;
- (4) L is a smooth curve from a (pre)critical point to $J(f)$ and, moreover, the rays R_0^e, R_1^e are not periodic.

Except for the last case, the rays R_0^e, R_1^e have their principal sets in different components of $J(f)$.

In our setting, external rays are closely related to quadratic-like rays. To describe this relation, we need Lemma 2.8, which follows immediately from [BrHu88]. Recall that any $f \in \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ with $|\lambda| \leq 1$ is immediately renormalizable.

Lemma 2.8. *If $f \in \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ with $|\lambda| \leq 1$, then $J(f^*)$ is an invariant component of $J(f)$, and $\text{Th}(J(f^*)) = K(f^*)$ is a component of $K(f)$.*

Hence, if $J(f)$ is disconnected, then we can choose neighborhoods $V^* \supset U^* \supset J(f^*)$ enclosed by components of equipotentials. In addition to quadratic-like rays accumulating in $J(f^*)$, it is natural to consider external rays that penetrate U^* ; by definition and by the choice of U^* , any such ray has its entire tail inside U^* so that its principal set is located inside U^* .

Definition 2.9. A quadratic-like ray R^* and an external ray R^e to $J(f^*)$ correspond to each other if $\text{Pr}(R^*) = \text{Pr}(R^e)$, and R^*, R^e are homotopic in $\mathbb{C} \setminus K(f^*)$ among curves with the same limit set.

Let $\psi : \mathbb{C} \setminus K(f^*) \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$ be the Riemann map with $\psi(z) \sim kz$ as $z \rightarrow \infty$ with some $k > 0$. It follows that, in the situation of Definition 2.9, the rays $\psi(R^*) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ and $\psi(R^e) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ land at the same point in $\mathbb{S}^1 = \text{Bd}(\mathbb{D})$. Observe also that no two distinct quadratic-like rays can correspond to each other in the sense of Definition 2.9. Theorem 2.10 is a special case of a more general result obtained in Theorem 6.9 of [BCLOS10].

Theorem 2.10 (cf [BCLOS10], Theorem 6.9). *Consider a polynomial $f \in \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ with $|\lambda| \leq 1$. Then every external ray R^e to $J(f^*)$ corresponds to exactly one quadratic-like ray $l = \xi(R^e)$. The mapping $\xi : R^e \mapsto l$ from external rays to $J(f^*)$ to quadratic-like rays is onto and $\xi^{-1}(l)$ consists of finitely many rays. Moreover, if $\xi^{-1}(l) = \{R_1^e, \dots, R_k^e\}$ with $k > 1$, then one of the following holds:*

- (i) *we have $k = 2$, both rays R_1^e, R_2^e are non-smooth, meet at a precritical point x , and share a common arc from x to E ;*
- (ii) *the rays R_1^e, \dots, R_k^e land at the same preperiodic point, and include at least one pair of disjoint rays.*

2.3. Combinatorics of the angle tripling map: an overview. In this subsection, we briefly describe some results of [BOPT13].

Let $\mathfrak{U} \subset \overline{\mathbb{D}}$ be the convex hull of $\mathfrak{U}' = \mathfrak{U} \cap \mathbb{S}^1$. It is said to map under σ_3 in a *quasi-covering fashion* if,

- (1) we have $\sigma_3(\mathfrak{U}') = \mathfrak{V}'$ for some set \mathfrak{V}' and $\mathfrak{V} = \text{CH}(\mathfrak{V}')$, and
- (2) for every hole $(\bar{\alpha}, \bar{\beta})$ of \mathfrak{U} , we either have $\sigma_3(\bar{\alpha}) = \sigma_3(\bar{\beta})$, or the circular arc $(\sigma_3(\bar{\alpha}), \sigma_3(\bar{\beta}))$ is also a hole of \mathfrak{V} .

So, the set \mathfrak{U} is a (*stand alone*) *invariant gap* if $\sigma_3(\mathfrak{U}') = \mathfrak{U}'$, and σ_3 maps \mathfrak{U} in a quasi-covering fashion. The set \mathfrak{U} is a (*stand alone*) *periodic gap/leaf* if there exists $n > 0$ such that $\sigma_3^{\circ n}(\mathfrak{U}') = \mathfrak{U}'$, convex hulls of sets $\sigma_3^{\circ i}(\mathfrak{U}')$, $i = 0, \dots, n-1$ intersect **at most** by common edges, and each such convex hull maps under σ_3 in a quasi-covering fashion.

An invariant gap \mathfrak{g} can have one or two majors [BOPT13]. Any *quadratic* invariant gap \mathfrak{U} has exactly one major. It can be shown that any major edge is either critical or periodic. If the major of \mathfrak{U} is critical and has no periodic endpoints, then \mathfrak{U} is said to be of *regular critical type*. If the major of \mathfrak{U} is critical and has a periodic endpoint, then \mathfrak{U} is said to be of *caterpillar type*. Finally, if the major of \mathfrak{U} is periodic, then \mathfrak{U} is said to be of *periodic type*. Since any major edge is either critical or periodic, then any quadratic invariant gap is of one of the three types just introduced.

Quadratic invariant gaps can be generated as follows. Let \mathfrak{c} be any critical chord. Set $L(\mathfrak{c})$ to be the longer closed arc of \mathbb{S}^1 connecting the

endpoints of \mathbf{c} , and $X(\mathbf{c})$ be the set of all points in $L(\mathbf{c})$, whose forward orbits stay in $L(\mathbf{c})$. Let $\mathfrak{U}(\mathbf{c})$ be the convex hull of $X(\mathbf{c})$ in the plane.

Theorem 2.11 ([BOPT13]). *If \mathbf{c} is any critical chord, then $\mathfrak{U}(\mathbf{c})$ is a quadratic invariant gap. If one of the endpoints of \mathbf{c} is periodic, then $\mathfrak{U}(\mathbf{c})$ is of caterpillar type. Otherwise, $\mathfrak{U}(\mathbf{c})$ is of regular critical type or of periodic type depending on whether or not the forward orbits of the endpoints of \mathbf{c} are contained in $L(\mathbf{c})$. Any quadratic invariant gap is of the form $\mathfrak{U}(\mathbf{c})$ for some \mathbf{c} .*

For any critical chord \mathbf{c} , we let $\mathfrak{U}_c(\mathbf{c})$ denote the convex hull of all non-isolated points in $\mathfrak{U}'(\mathbf{c})$ (here the subscript c stands for “clean”). Then $\mathfrak{U}_c(\mathbf{c}) = \mathfrak{U}(\mathbf{c})$ unless $\mathfrak{U}(\mathbf{c})$ is of caterpillar type. If the gap $\mathfrak{U}(\mathbf{c})$ is of caterpillar type, then $\mathfrak{U}_c(\mathbf{c})$ is a quadratic invariant gap of periodic type while the set $\mathfrak{U}'(\mathbf{c})$ has isolated points and is obtained from $\mathfrak{U}'_c(\mathbf{c})$ by adding the non-periodic endpoint of \mathbf{c} and countably many iterated preimages of it. The boundary of the caterpillar gap $\mathfrak{U}(\mathbf{c})$ consists of $\mathfrak{U}'_c(\mathbf{c})$ and countable concatenations of edges inserted into holes of $\mathfrak{U}_c(\mathbf{c})$.

We also have a description of the parameter space of all quadratic invariant gaps. First of all, we need to parameterize critical chords. This is done by the map taking a point $\bar{\theta} \in \mathbb{S}^1$ to the critical chord $\overline{(\theta + \frac{1}{3})(\theta + \frac{2}{3})}$, which will be denoted by \mathbf{c}_θ . Consider the map π from \mathbb{S}^1 to the set of all quadratic invariant gaps taking $\bar{\theta}$ to $\mathfrak{U}(\mathbf{c}_\theta)$. The parameter picture of quadratic invariant gaps is somewhat similar to that of the rotation number in an analytic family of circle diffeomorphisms.

Theorem 2.12 ([BOPT13]). *The mapping π is surjective but not injective. Non-trivial fibers of π are exactly those of quadratic invariant gaps of periodic type. The fiber of the invariant quadratic gap of periodic type with major $\overline{(\theta_1 + \frac{1}{3})(\theta_2 + \frac{2}{3})}$ is the open arc $(\bar{\theta}_1, \bar{\theta}_2)$.*

Nontrivial fibers of π are the holes of a certain compact subset $Q \subset \mathbb{S}^1$. The convex hull \mathfrak{Q} of Q in the plane is called the *Principle Quadratic Parameter Gap*. Holes of \mathfrak{Q} will play an important role in our description of \mathcal{CU}_λ . Sometimes, we will also need the mapping π_c taking $\bar{\theta}$ to the quadratic invariant gap $\mathfrak{U}_c(\mathbf{c}_\theta)$.

Now we classify finite (i.e., having finitely many edges) invariant gaps. Such gaps \mathbf{g} have also finitely many *vertices*, i.e., points of \mathbf{g}' . Our classification of gaps \mathbf{g} mimics Milnor’s classification of hyperbolic components in slices of cubic polynomials and quadratic rational functions [Mil06, Mil93]. Say that \mathbf{g} is of *type A* (for “adjacent”) if \mathbf{g} has only one major, and \mathbf{g} is of *type B* (for “bi-transitive”) if \mathbf{g} has two majors that belong to the same σ_3 -orbit of edges. Finally, we say that \mathbf{g} is of *type D* (for “disjoint”) if \mathbf{g} has two majors, whose σ_3 -orbits

are disjoint (except for common endpoints). Every finite invariant gap must be of one of these three types.

Let $A \subset \mathbb{S}^1$ be a finite invariant set. Then we associate with it a unique *rotation number* p/q if there is an orientation preserving homeomorphism of the circle that conjugates $\sigma_3|_A$ with the restriction of the rotation by the angle p/q onto an invariant subset of \mathbb{S}^1 . We also talk about rotation numbers of periodic angles and finite invariant gaps. Clearly, not every periodic point/orbit has a rotation number; also, to have a rotation number p/q , a periodic point/orbit must be of period q . On the other hand, every finite invariant gap has a rotation number by definition. We will use the following simple count.

Lemma 2.13. *There are q type D finite invariant gaps of rotation number p/q .*

Indeed, a type D finite invariant gap \mathfrak{g} of a fixed rotation number is determined by the number of edges that separate the two majors of \mathfrak{g} in a counterclockwise motion from the major whose hole contains $\bar{0}$ to the major whose hole contains $\bar{\frac{1}{2}}$. The next lemma also easily follows from [BOPT13]. Observe that if an endpoint of a hole $(\bar{\theta}_1, \bar{\theta}_2)$ of \mathfrak{Q} is such that $\theta_1 + \frac{1}{3}$ (resp., $\theta_2 + \frac{2}{3}$) has a well-defined rotation number p/q , then the convex hull of the entire orbit of $\overline{\theta_1 + \frac{1}{3}\theta_2 + \frac{2}{3}}$ is a finite invariant gap of rotation number p/q .

Lemma 2.14. *Suppose that $(\bar{\theta}_1, \bar{\theta}_2)$ is a hole of \mathfrak{Q} such that $\theta_1 + \frac{1}{3}$ and $\theta_2 + \frac{2}{3}$ are of rotation number p/q . Then $\overline{\theta_1 + \frac{1}{3}\theta_2 + \frac{2}{3}}$ is a major of a finite invariant gap of type D.*

Theorem 2.15 follows from [BOPT13]. It relates type D finite invariant gaps and quadratic invariant gaps of periodic type.

Theorem 2.15. *If \mathfrak{M} is a major of a type D finite invariant gap, then \mathfrak{M} is the major of some quadratic invariant gap of periodic type.*

Let us provide more detail concerning the relation between finite invariant gaps of type D and quadratic invariant gaps; our description here is based upon [BOPT13]. Let \mathfrak{g} be a finite invariant gap of type D. Let \mathfrak{M}_1 be a major of \mathfrak{g} . Then \mathfrak{M}_1 defines an invariant quadratic gap \mathfrak{U}_1 of periodic type such that $\mathfrak{U}_1 \cap \mathbb{S}^1$ is the set of all points of the circle whose orbits stay at the same side of \mathfrak{M}_1 as \mathfrak{g} . We have $\mathfrak{g} \subset \mathfrak{U}_1$. Similarly, the other major \mathfrak{M}_2 of \mathfrak{g} determines another invariant quadratic gap \mathfrak{U}_2 .

2.4. Main results. The proof of Theorem A uses methods close to those used by J. Milnor in [Mil00].

Theorem A (Wakes in λ -slices). *Fix a complex number λ such that $|\lambda| \leq 1$. For every hole (θ_1, θ_2) of \mathfrak{Q} , the parameter rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ land at the same point.*

Thus the parameter rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$, together with their common landing point, divide the plane into two (open) parts.

Definition 2.16. Let (θ_1, θ_2) be a hole of \mathfrak{Q} and $\mathcal{W}_\lambda(\theta_1, \theta_2)$ be the component of $\mathbb{C} \setminus \overline{\mathcal{R}_\lambda(\theta_1) \cup \mathcal{R}_\lambda(\theta_2)}$ containing the parameter rays with arguments from (θ_1, θ_2) . The set $\mathcal{W}_\lambda(\theta_1, \theta_2)$ is called the (parameter) *wake* (of \mathcal{F}_λ). The joint landing point of the rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ is called the *root point* of the parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$. Let the *period* of the parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ be the period of $\theta_1 + \frac{1}{3}$ under the angle tripling map.

Theorem B gives a dynamical description of parameter wakes. Observe that by definition for almost all holes (θ_1, θ_2) of \mathfrak{Q} the corresponding major $\overline{\theta_1 + \frac{1}{3}\theta_2 + \frac{2}{3}}$ is in one-to-one correspondence with (θ_1, θ_2) . The only exception is the major $\overline{0\frac{1}{2}}$, which corresponds to two holes of \mathfrak{Q} , namely to holes $(\frac{1}{6}, \frac{1}{6})$ and $(\frac{2}{3}, \frac{5}{6})$. This in turn is related to the fact that $\overline{0\frac{1}{2}}$ as the major of an invariant quadratic gap \mathfrak{U} does not uniquely define \mathfrak{U} . A unique quadratic invariant gap with major $\overline{0\frac{1}{2}}$ contained in the upper half of the unit disk (above $\overline{0\frac{1}{2}}$) is denoted by FG_a while a unique quadratic invariant gap with major $\overline{0\frac{1}{2}}$ contained in the lower half of the unit disk (below $\overline{0\frac{1}{2}}$) is denoted by FG_b . Both FG_a and FG_b have the same major $\overline{0\frac{1}{2}}$. Define the set $\mathcal{W}'_\lambda(\theta_1, \theta_2)$ as the wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ except for the holes $(\frac{1}{6}, \frac{1}{3})$ and $(\frac{2}{3}, \frac{5}{6})$ for which we set

$$\mathcal{W}'_\lambda\left(\frac{1}{6}, \frac{1}{3}\right) = \mathcal{W}'_\lambda\left(\frac{2}{3}, \frac{5}{6}\right) = \mathcal{W}_\lambda\left(\frac{1}{6}, \frac{1}{3}\right) \cup \mathcal{W}_\lambda\left(\frac{2}{3}, \frac{5}{6}\right).$$

Theorem B. *Fix a hole $I = (\theta_1, \theta_2)$ of \mathfrak{Q} . Then the set $\mathcal{W}'_\lambda(I)$ coincides with the set of polynomials f for which the dynamic rays $R_f(\theta_1 + \frac{1}{3}), R_f(\theta_2 + \frac{2}{3})$ land at the same periodic point that is either repelling for all $f \in \mathcal{W}'_\lambda(I)$ or equals 0 for all $f \in \mathcal{W}'_\lambda(I)$. If f_{root} is the root point of $\mathcal{W}'_\lambda(I)$, then the dynamic rays $R_{f_{\text{root}}}(\theta_1 + \frac{1}{3}), R_{f_{\text{root}}}(\theta_2 + \frac{2}{3})$ land at the same parabolic periodic point, and f_{root} belongs to \mathcal{CU}_λ .*

Define a *limb* of the λ -connectedness locus \mathcal{C}_λ in the λ -slice \mathcal{F}_λ as the intersection of \mathcal{C}_λ with a parameter wake. A compact set $X \subset \mathbb{C}$ is said to be *full* if the complement $\mathbb{C} \setminus X$ is connected. Theorem C describes some topological properties of \mathcal{C}_λ and \mathcal{CU}_λ . Recall that \mathcal{CU}_λ is the set of all polynomials $f \in \mathcal{F}_\lambda$ with $[f] \in \mathcal{CU}$.

Theorem C. *The set \mathcal{CU}_λ is disjoint from all parameter wakes, unless $\lambda = 1$. The λ -connectedness locus \mathcal{C}_λ is the union of \mathcal{CU}_λ and all limbs of \mathcal{C}_λ . The set \mathcal{CU}_λ is a full continuum.*

The case $\lambda = 1$ is addressed in Theorem 7.14. In this case, the two parameter wakes of period one intersect the set \mathcal{CU}_1 ; the intersection is described explicitly. Other parameter wakes are disjoint from \mathcal{CU}_1 .

3. PROOF OF THEOREM A

In this section, we discuss the geometry of parameter rays in λ -slices. In particular, we prove Theorem A. We first recall Lemma B.1 from [GM93] that goes back to Douady and Hubbard [DH8485].

Lemma 3.1. *Let g be a polynomial, and z be a repelling periodic point of g . If a smooth ray with rational argument θ in the dynamical plane of g lands at z , then, for every polynomial \tilde{g} sufficiently close to g , the ray \tilde{R} with argument θ in the dynamical plane of \tilde{g} is smooth and lands at a repelling periodic point \tilde{z} close to z . Moreover, \tilde{z} depends holomorphically on \tilde{g} .*

Lemma 3.1 easily implies the following lemma that we will need.

Lemma 3.2. *Let g_t , $t \in [0, 1]$ be a continuous family of polynomials of the same degree, θ be a rational angle, and let R_t be a smooth ray with argument θ in the dynamical plane of g_t . Denote the point, at which the ray R_t lands, by z_t . Suppose that the points z_t are repelling for $t \in [0, 1)$ but the landing point z_1 is not the limit of landing points z_t as $t \rightarrow 1$. Then the point z_1 is parabolic.*

Lemmas 3.1 and 3.2 deal with continuity of rays landing at repelling periodic points. The situation with parabolic periodic points is studied below. The main objects we consider are repelling petals and rays landing at parabolic periodic points.

3.1. Polynomials with parabolic points and their petals. Let g be a polynomial of arbitrary degree such that 0 is a fixed parabolic point of g of multiplier 1. Suppose that $g(z) = z + az^{q+1} + o(z^{q+1})$, where q is a positive integer and $a \neq 0$. Recall from [Mil06] that an *attracting vector* for g is defined as a vector (=complex number) v such that av^q is a negative real number, i.e., v and av^{q+1} have opposite directions. Clearly, there are q straight rays consisting of attracting vectors that divide the plane of complex numbers into q *repelling sectors*.

Consider a repelling sector S . Note that the set $S^{-q} = \{z \in \mathbb{C} \mid z^{-q} \in S\}$ is the complement of the ray $\{-ta \mid t > 0\}$ in \mathbb{C} . Let U be a sufficiently small disk around 0. We will write F for the composition

of the function $w \mapsto w^{-1/q}$ mapping $(S \cap U)^{-q}$ onto $S \cap U$, the function g mapping $S \cap U$ onto $g(S \cap U)$, and the function $z \mapsto z^{-q}$ mapping $g(S \cap U)$ to \mathbb{C} . We have $F(w) = w - qa + \alpha(w)$, where $\alpha(w)$ denotes a power series in $w^{-1/q}$ that converges in a neighborhood of infinity, and whose free term is zero (note that this function is single valued and holomorphic on S^{-q}). It follows that there exists a positive real number r with the property $|\alpha(w)| < \frac{|a|}{2}$ whenever $|w| > r|a|$. Consider the half-plane Π given by the inequality $\Re(w/a) > r$. Since this inequality implies that $|w| > r|a|$, we have $F(\Pi) \supset \Pi$, and also that the shortest distance from a point on the boundary of Π to a point the boundary of $F(\Pi)$ is at least $(q - \frac{1}{2})|a|$. The preimage of the half-plane Π under the map $S \rightarrow S^{-q}$, $z \mapsto z^{-q}$ is called a *repelling petal* of g .

Every repelling sector includes a repelling petal; thus, our polynomial $g(z) = z + az^{q+1} + o(z^{q+1})$ has q repelling petals. Hence there are at least q external rays landing at 0. A repelling petal P of g has the property $g(P) \supset P$. The dependence of the repelling petals on parameters is described in Lemma 3.3 proved in [BOPT14] (the proof follows the same lines as the proof of Lemma 5 in [BuHe01]).

Lemma 3.3. *Let $g_t(z) = z + a_t z^{q+1} + o(z^{q+1})$ be a continuous family of polynomials, in which $a_t \neq 0$, and t belongs to a locally compact metric space that is a countable union of compact spaces. Then all q repelling petals of g_t can be chosen to vary continuously with respect to t .*

3.2. Stability of rays and their perturbations. In this subsection, we fix $\lambda = \exp(2\pi ip/q)$ for some relatively prime p and q (i.e., λ is a root of unity). The ratio p/q is called the *rotation number* (of the fixed point 0). We discuss conditions that imply that a dynamic ray $R_f(\theta)$ in the dynamic plane of a polynomial $f \in \mathcal{F}_\lambda$ landing at 0 is stable (i.e., for $\tilde{f} \in \mathcal{F}_\lambda$ close to f , the ray $R_{\tilde{f}}(\theta)$ also lands at 0).

Proposition 3.4 (cf Proposition 3.3, [BOPT14]). *We have $f_{\lambda,b}^{\circ q}(z) = z + T_{p/q}(b)z^{q+1} + o(z^{q+1})$, where $T_{p/q}(b)$ is a non-zero polynomial in b . Moreover, the degree of $T_{p/q}$ is at most q and if, for some b , we have $T_{p/q}(b) = 0$, then $f_{\lambda,b}^{\circ q}(z)$ has $2q$ parabolic Fatou domains at 0 forming two cycles under f as well as $2q$ external rays landing at 0.*

The representation for $f_{\lambda,b}^{\circ q}(z)$ and the fact that $T_{p/q}$ is a non-zero polynomial are proved in [BOPT14, Proposition 3.3]. The claim about the degree follows from Lemma 3.5 below. The last claim in Proposition 3.4 follows from [Bea91] (see the Petal Theorem 6.5.4 and Theorem 6.5.8) and the fact that our maps are cubic.

Lemma 3.5. *Let \mathbb{V} be the vector space of all polynomials in b and z given by $p(b, z) = \sum_{n=1}^N a_n(b)z^n$, where a_n is a polynomial of degree $\leq n - 1$. Then $f_{\lambda, b}^{\text{or}}(p) \in \mathbb{V}$ for each $p \in \mathbb{V}$ and each integer $r > 0$.*

Proof. Let $p(b, z) = a_1(b)z + \cdots + a_N(b)z^N$ be a polynomial such that the degree of $a_j(b)$ is at most $j - 1$, for every j . It follows that $p(b, z)^k = c_k(b)z^k + \cdots + c_{Nk}(b)z^{Nk}$, where the degree of each $c_i(b)$ is at most $i - k$ for each $i \geq k$. Thus, $f_{\lambda, b}(p(b, z)) = \lambda p(b, z) + bp^2(b, z) + p^3(b, z)$ can be written as $d_1(b)z + \cdots + d_{3N}(b)z^{3N}$, where the degree of each $d_i(b)$ is at most $i - 1$. The result follows. \square

Proposition 3.6 deals with rays landing at parabolic points.

Proposition 3.6 ([BOPT14] Proposition 3.4). *Suppose that an external ray $R_{f_{\lambda, b_*}}(\theta)$ with periodic argument $\theta \in \mathbb{R}/\mathbb{Z}$ lands at 0, and $T_{p/q}(b_*) \neq 0$. Then, for all b sufficiently close to b_* , the ray $R_{f_{\lambda, b}}(\theta)$ lands at 0.*

3.3. External and quadratic-like rays of polynomials in $\mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$.

In this subsection, λ is a complex number of modulus at most 1. We study the parameter plane \mathcal{F}_λ . The set $\mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ is foliated by parameter rays. Let us describe the dynamics of a polynomial $f \in \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ choosing f from the parameter ray $\mathcal{R}_\lambda(\varkappa)$. We will establish a correspondence between the dynamical properties of f and the properties of the invariant quadratic gap $\mathfrak{U}_c(\varkappa)$. Let us emphasize that in what follows we deal with both *external* rays, *quadratic-like* rays (defined in Subsection 2.2), and *dynamic* rays (defined in Subsection 2.1), so it is important to distinguish between these types of rays.

A polynomial $f \in \mathcal{F}_\lambda$ belongs to a *parameter ray* $\mathcal{R}_\lambda(\varkappa)$ if and only if the dynamic rays $R_f(\varkappa + \frac{1}{3})$ and $R_f(\varkappa + \frac{2}{3})$ crash into $\omega_2(f)$; by Theorem 2.1 then no other dynamic ray crashes into $\omega_2(f)$. Set $R_f(\varkappa + \frac{1}{3}) \cup R_f(\varkappa + \frac{2}{3}) \cup \omega_2(f) = \Gamma_f$; clearly, Γ_f is a curve dividing \mathbb{C} , the dynamic plane of f , in two parts. Let $\Sigma_f = W_f(\varkappa + \frac{1}{3}, \varkappa + \frac{2}{3})$ be the part of the plane bounded by Γ_f and containing the rays $R_f(\theta)$ for all $\theta \in (\varkappa + \frac{1}{3}, \varkappa + \frac{2}{3})$; we call such sets Σ_f (enclosed by two dynamic rays of f landing or crashing at the same point) *dynamic wedges*. In general, a *dynamic wedge* $W_f(\alpha, \beta)$ is a part of the plane bounded by $\overline{R_f(\alpha) \cup R_f(\beta)}$ and containing the rays $R_f(\theta)$ for all $\theta \in (\alpha, \beta)$, where the dynamic rays $R_f(\alpha)$ and $R_f(\beta)$ crash or land at the same point.

Then Σ_f maps one-to-one onto the complement of $f(\Gamma_f)$ and contains the dynamic rays with arguments in $\mathbb{S}^1 \setminus L(\varkappa)$, where $L(\varkappa)$ is the longer closed arc with endpoints $\varkappa + \frac{1}{3}$ and $\varkappa + \frac{2}{3}$. By Theorem 2.5, the polynomial f is immediately renormalizable. The quadratic-like

filled Julia set $K(f^*)$ is disjoint from Γ_f , since every point of Γ_f is escaping. Clearly, $K(f^*) \not\subset \Sigma_f$ because $f|_{\Sigma_f}$ is one-to-one while $f|_{K(f^*)}$ is generically two-to-one. Hence, $K(f^*) \cap \overline{\Sigma}_f = \emptyset$. In this subsection, we fix a polynomial $f \in \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ that belongs to the parameter ray $\mathcal{R}_\lambda(\varkappa)$.

We need more information about the Jordan domains U^* and V^* , for which the map $f^* : U^* \rightarrow V^*$, the restriction of f to U^* , is a quadratic-like map, cf. [BrHu88]. The domain V^* can be defined as the connected component of the set of all point $z \in \mathbb{C}$ such that $G_{K(f)}(z) < G_{K(f)}(\omega_2(f))$ containing $K(f^*)$, and the domain U^* is defined as the f -pullback of V^* in V^* . In particular, V^* is disjoint from Γ_f .

Lemma 3.7. *A dynamic ray $R_f(\alpha)$ does not accumulate in $K(f^*)$ if and only if there is an integer $k \geq 0$ such that $R_f(3^k\alpha) \subset \overline{\Sigma}_f$. In particular, a smooth ray $R_f(\alpha)$ does not accumulate in $K(f^*)$ if and only if there is an integer $k \geq 0$ such that $R_f(3^k\alpha) \subset \Sigma_f$.*

Proof. By definition, a dynamic ray $R_f(\alpha)$ accumulates inside V^* if and only if it penetrates V^* . On the other hand, $R_f(\alpha)$ accumulates in $K(f^*)$ if and only if all its images accumulate in U^* . We claim that this is equivalent to all its images accumulating in V^* . Let us show that if all images of $R_f(\alpha)$ accumulate in V^* , then they all accumulate in U^* . Indeed, otherwise there must exist an image of the ray accumulating in V^* but not in U^* . This would imply that the next image of the ray would be contained in $\overline{\Sigma}_f$, a contradiction. Thus, $R_f(\alpha)$ accumulates in $K(f^*)$ if and only if all its images accumulate in V^* . Hence $R_f(\alpha)$ does not accumulate in $K(f^*)$ if and only if there is an integer $k \geq 0$ such that $R_f(3^k\alpha)$ is disjoint from V^* . Since rays with arguments from $(\varkappa + \frac{2}{3}, \varkappa + \frac{1}{3})$ penetrate V^* , then $3^k\alpha \in [\varkappa + \frac{1}{3}, \varkappa + \frac{2}{3}]$, as desired. The case of a smooth ray now follows because smooth rays cannot pass through ω_2 and hence cannot intersect the boundary of Σ_f . \square

We want to describe *external* rays that accumulate in $K(f^*)$. Recall that external rays R^e have one-sided arguments. If R^e is a smooth ray with argument θ , then it is associated with both one-sided external arguments θ^+ and θ^- . If R^e is not smooth, then R^e is the one-sided limit of smooth rays from exactly one side, and we associate to R^e the appropriate *one-sided* external argument, θ^+ or θ^- . In what follows, we write $R_f^e(\theta^\tau)$, where $\tau = +$ or $\tau = -$. For any set $A \subset \mathbb{S}^1$ and an angle $\alpha \in \mathbb{R}/\mathbb{Z}$, say that α^+ (respectively, α^-) is a *one-sided argument (of α)* in A if $\alpha \in A$, and α is not isolated in A from the positive side (resp., from the negative side). Recall that $(\varkappa + \frac{1}{3})(\varkappa + \frac{2}{3})$ is denoted by \mathbf{c}_\varkappa .

Proposition 3.8. *Consider a polynomial $f \in \mathcal{R}_\lambda(\varkappa)$ and its immediate renormalization $f^* : U^* \rightarrow V^*$. Then the set of arguments of external rays to $K(f^*)$ coincides with the union of the set of arguments α^\pm where $\alpha \in \mathfrak{U}'_c(\mathbf{c}_\varkappa)$ never maps to an endpoint of \mathbf{c}_\varkappa (these external rays are smooth) and the set of one-sided arguments β^τ , where β eventually maps to an endpoint of \mathbf{c}_\varkappa , and β^τ is the one-sided argument of β in $\mathfrak{U}'_c(\mathbf{c}_\varkappa)$ (these external rays are not smooth).*

In particular, the (one-sided) argument of an external ray accumulating in $K(f^*)$ always belongs to $\mathfrak{U}'_c(\mathbf{c}_\varkappa)$.

Proof. Smooth rays are exactly the rays with arguments that never map to the endpoints of \mathbf{c}_\varkappa . Moreover, smooth rays are both external and dynamic. Thus, by Lemma 3.7, a smooth ray $R_f^e(\alpha)$ accumulates in $K(f^*)$ if and only if, for any $k \geq 0$, the ray $R_f(3^k\alpha)$ is disjoint from $\overline{\Sigma}_f$. This is equivalent to the fact that $3^k\alpha$ belongs to the interior of $L(\mathbf{c}_\varkappa)$ for any integer $k \geq 0$, which, in turn, is equivalent to the fact that $\alpha \in \mathfrak{U}'_c(\mathbf{c}_\varkappa)$ and never maps to the endpoints of \mathbf{c}_\varkappa .

Let $R^e = R^e(\alpha^\tau)$ be a non-smooth external ray. Since R^e is non-smooth if and only if $\sigma_3^{o k}(\bar{\alpha})$ is an endpoint of \mathbf{c}_\varkappa for the least integer $k \geq 0$, then, if all points $3^m\bar{\alpha}$ are in the interior of $L(\varkappa)$ for $m < k$, and $R^e((3^k\alpha)^\tau) = f^{o k}(R^e)$ accumulates in $K(f^*)$, the orbit of the principal set of R^e is disjoint from $\overline{\Sigma}_f$, and R^e accumulates in $K(f^*)$. So, it suffices to consider rays with one-sided arguments $\varkappa + \frac{1}{3}^\pm$, $\varkappa + \frac{2}{3}^\pm$ and to choose those of them, that accumulate in $K(f^*)$. A simple analysis shows that $R^e(\alpha^\tau)$ (where $\bar{\alpha}$ is an endpoint of \mathbf{c}_\varkappa) accumulates in $K(f^*)$ if and only if α^τ is a one-sided argument in $\mathfrak{U}'_c(\mathbf{c}_\varkappa)$. This completes the proof. Observe that in the regular critical case both endpoints of \mathbf{c}_\varkappa have one-sided argument in $\mathfrak{U}'_c(\mathbf{c}_\varkappa)$, in the periodic case neither endpoint of \mathbf{c}_\varkappa has one-sided arguments in $\mathfrak{U}'_c(\mathbf{c}_\varkappa)$, and, in the caterpillar case only the periodic endpoint of \mathbf{c}_\varkappa has one-sided argument in $\mathfrak{U}'_c(\mathbf{c}_\varkappa)$. \square

We defined the *correspondence* between external rays and quadratic-like rays in Definition 2.9.

Proposition 3.9. *For any hole $(\bar{\alpha}, \bar{\beta})$ of $\mathfrak{U}_c(\mathbf{c}_\varkappa)$, the external rays $R_f^e(\alpha^-)$ and $R_f^e(\beta^+)$ correspond to the same quadratic-like ray to $K(f^*)$. More precisely, there are two cases:*

- (1) *the gap $\mathfrak{U}_c(\mathbf{c}_\varkappa)$ is of regular critical type, the external rays $R_f^e(\alpha^-)$ and $R_f^e(\beta^+)$ meet at an eventual preimage x of $\omega_2(f)$, and then continue along a joint arc from x to $K(f^*)$;*

- (2) *the gap $\mathfrak{U}_c(\mathfrak{c}_\varkappa)$ is of periodic type with major hole $(\overline{\alpha_*}, \overline{\beta_*})$, the external rays $R_f^e(\alpha^-)$ and $R_f^e(\beta^+)$ both land at an eventual preimage of the common periodic landing point for the rays $R_f^e(\alpha_*^-)$ and $R_f^e(\beta_*^+)$.*

Proof. Suppose that the external rays $R_f^e(\alpha^-)$ and $R_f^e(\beta^+)$ do not correspond to the same quadratic-like ray to $K(f^*)$. By Theorem 2.10, these rays correspond to distinct quadratic-like rays, say, $T_\alpha \neq T_\beta$, to $K(f^*)$. Then there are quadratic-like rays T in either of the two components U_1^*, U_2^* of $U^* \setminus (T_\alpha \cup T_\beta \cup K(f^*))$. Clearly, both components U_1^* and U_2^* contain segments of quadratic-like rays to $K(f^*)$. Therefore, by Theorem 2.10, both components contain segments of external rays to $K(f^*)$. However, by Proposition 3.8, there are no external rays to $K(f^*)$ with arguments in (α, β) , a contradiction. Hence, $R_f^e(\alpha^-)$ and $R_f^e(\beta^+)$ correspond to the same quadratic-like ray T to $K(f^*)$.

Suppose that $\mathfrak{U}_c(\mathfrak{c}_\varkappa)$ has major $\mathfrak{M} = \overline{\alpha_*\beta_*}$. First assume that it is of regular critical type. Then the external rays $R_f^e(\alpha_*^-)$, $R_f^e(\beta_*^+)$ meet at $\omega_2(f)$ and then, as follows from [BCLOS10, Lemma 6.1], extend together towards $K(f^*)$. If $\overline{\alpha\beta}$ is an edge of $\mathfrak{U}_c(\mathfrak{c}_\varkappa)$, then $\overline{\alpha\beta}$ is an iterated σ_3 -pullback of \mathfrak{M} , so that α and β are appropriate preimages of α_* and β_* , and the external rays $R_f^e(\alpha^-)$, $R_f^e(\beta^+)$ meet at a preimage x of $\omega_2(f)$ and then extend together towards $K(f^*)$. In fact, their union is a pullback of the union of rays $R_f^e(\alpha_*^-)$, $R_f^e(\beta_*^+)$. This covers case (1) of the proposition and corresponds to case (i) of Theorem 2.10. Assume now that $\mathfrak{U}_c(\mathfrak{c}_\varkappa)$ is of periodic type. Then, by [BCLOS10, Lemma 6.1], the rays $R_f^e(\alpha_*^-)$, $R_f^e(\beta_*^+)$ cannot intersect, which implies that they have to land at the same periodic point of $K(f^*)$. Since all edges of $\mathfrak{U}_c(\mathfrak{c}_\varkappa)$ are preimages of $\overline{\alpha_*\beta_*}$, claim (2) follows.

There are two distinct cases within claim (2). If $\mathfrak{U}(\varkappa) = \mathfrak{U}_c(\varkappa)$ is of periodic type, the rays $R_f^e(\alpha_*)$ and $R_f^e(\beta_*)$ are smooth, disjoint, and land at the same periodic point. The situation is a little more complicated if $\mathfrak{U}(\varkappa)$ is of caterpillar type. In that case, we may assume that $\alpha_* = \varkappa + \frac{1}{3}$. Then the dynamic ray $R_f(\alpha_*)$ crashes (together with the dynamic ray $R_f(\alpha_* + \frac{1}{3})$) at the escaping critical point $\omega_2(f)$. However, in this case, the external non-smooth ray $R_f^e(\alpha_*^-)$ still lands at the same periodic point as the smooth external ray $R_f^e(\beta_*^+)$, and both rays correspond to the same quadratic-like ray to $K(f^*)$. \square

Let $f \in \mathcal{F}_\lambda$ with $|\lambda| \leq 1$ be immediately renormalizable. We say that a chord $\overline{\alpha\beta}$ *generates a cut* if there are signs ρ and τ , each of which equals $+$ or $-$, such that the external rays $R^e(\alpha^\rho)$ and $R^e(\beta^\tau)$ have intersecting principal sets in $K(f^*)$. The *cut* (generated by the

chord $\overline{\alpha\beta}$) is then defined as the smallest connected closed subset of $R_f^e(\alpha^\rho) \cup R_f^e(\beta^\tau)$ containing the union of *dynamic rays* $R_f(\alpha) \cup R_f(\beta)$. We claim that if $\overline{\alpha\beta}$ generates a cut, then this cut is unique. In other words, if the cut Γ generated by $\overline{\alpha\beta}$ exists for some specific choice of the signs ρ and τ , then, for every other choice of the signs, the corresponding cut either does not exist or coincides with Γ . This is a straightforward consequence of the following observation: if $R^e(\alpha^+) \neq R^e(\alpha^-)$, then at most one of these two external rays has the principal set in $K(f^*)$; similarly for $R^e(\beta^+)$ and $R^e(\beta^-)$.

Denote the cut generated by $\overline{\alpha\beta}$ by $\Gamma(\overline{\alpha\beta})$. If the (one-sided) rays defining a cut Γ land at the same point z , then z is called the *vertex* (of Γ).

Suppose now that f lies in the parameter ray $\mathcal{R}_\lambda(\varkappa)$. By Proposition 3.9, there are cuts generated by edges of $\mathfrak{U}_c(\mathbf{c}_\varkappa)$. In the regular critical case, $\Gamma(\overline{\alpha\beta})$ is formed by dynamic rays $R_f(\alpha)$ and $R_f(\beta)$ crashing into the same (pre-)critical point. Clearly, the cut $\Gamma(\overline{\alpha\beta})$ is disjoint from $K(f^*)$. This corresponds to case (1) of Proposition 3.9. In the periodic and caterpillar cases the external rays $R_f^e(\alpha^-)$, $R_f^e(\beta^+)$ are disjoint but land at the same (pre)periodic point. This corresponds to case (2) of Proposition 3.9. Observe that, in the caterpillar case, exactly one of the rays forming a cut is non-smooth. Clearly, $\Gamma(\overline{\alpha\beta})$ separates all dynamic rays, whose arguments belong to (α, β) , from all dynamic rays, whose arguments belong to (β, α) .

By Proposition 3.8, the (one-sided) argument of an external ray accumulating in $K(f^*)$ belongs to $\mathfrak{U}'_c(\mathbf{c}_\varkappa)$. No chord connecting two points of $\mathfrak{U}'_c(\mathbf{c}_\varkappa)$ can cross an edge of $\mathfrak{U}_c(\mathbf{c}_\varkappa)$ (say that two chords of \mathbb{D} cross if they intersect in \mathbb{D} and do not coincide). This yields Lemma 3.10.

Lemma 3.10. *Suppose that a polynomial f lies in $\mathcal{R}_\lambda(\varkappa)$ with $|\lambda| \leq 1$. Each edge of $\mathfrak{U}_c(\mathbf{c}_\varkappa)$ generates a cut in the dynamical plane of f consisting of two rays that accumulate either on a (pre)critical point (in the regular critical case) or on a (pre)periodic point (in the periodic and caterpillar cases). In either case, the corresponding external rays correspond to the same quadratic-like ray. If a chord of \mathbb{D} generates a cut and crosses an edge of $\mathfrak{U}_c(\mathbf{c}_\varkappa)$ in \mathbb{D} , then it coincides with this edge.*

The following is a partial converse of Proposition 3.9.

Proposition 3.11. *Suppose that external rays $R_f^e(\alpha^\epsilon)$ and $R_f^e(\beta^\delta)$ correspond to the same quadratic-like ray to $K(f^*)$ (here $\epsilon, \delta \in \{-, +\}$). Then $\overline{\alpha\beta}$ is an edge of $\mathfrak{U}_c(\mathbf{c}_\varkappa)$.*

Proof. Suppose not. Let $\psi : \mathbb{C} \setminus K(f^*) \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$ be the Riemann map with $\psi(z) \sim kz$ as $z \rightarrow \infty$, for some $k > 0$. Then, by definition, the rays $\psi(R_f^e(\alpha^\epsilon))$ and $\psi(R_f^e(\beta^\delta))$ land at the same point $z \in \mathbb{S}^1$. We may assume that the open wedge in the positive direction from $\psi(R_f^e(\alpha^\epsilon))$ to $\psi(R_f^e(\beta^\delta))$ contains no points of \mathbb{S}^1 ; otherwise we simply interchange α^ϵ and β^δ . Since $\overline{\alpha\beta}$ is not an edge of $\mathfrak{U}_c(\mathbf{c}_\varkappa)$, there are infinitely many external rays to $K(f^*)$ with arguments in (α, β) . It follows that, for any such ray R^e , the ray $\psi(R^e)$ can only land at z . By Theorem 2.10, this implies that there are infinitely many quadratic-like rays R^* such that $\psi(R^*)$ lands at z . Since in fact there exist only finitely many such quadratic-like rays, we obtain a contradiction. \square

3.4. Landing properties. In this subsection, we fix a hole $(\bar{\theta}_1, \bar{\theta}_2)$ of Ω , consider a polynomial $f \in \mathcal{R}_\lambda(\varkappa) \subset \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ with $|\lambda| \leq 1$, and study the mutual location of the point $\bar{\varkappa}$ and the hole $(\bar{\theta}_1, \bar{\theta}_2)$, under which the *dynamic* rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ can have a common landing point in $K(f^*)$. Note that, by Theorem 2.12, the hole $(\bar{\theta}_1, \bar{\theta}_2)$ defines an invariant quadratic gap \mathfrak{U} with periodic major $\mathfrak{M} = (\theta_1 + \frac{1}{3})(\theta_2 + \frac{2}{3})$. We refer the reader to Subsection 2.3 for the notation and the main notions and concepts we deal with in this subsection.

Lemma 3.12. *Suppose that the dynamic rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ are contained in external rays R_1^e, R_2^e , respectively, with the same landing point z . Then $z \in K(f^*)$. Moreover, if neither $(\theta_1, \theta_2) = (\frac{2}{3}, \frac{5}{6})$ nor $(\theta_1, \theta_2) = (\frac{1}{6}, \frac{1}{3})$, then the following are equivalent:*

- (1) $\varkappa \notin [\theta_1, \theta_2]$;
- (2) the plane cut $R_1^e \cup \{z\} \cup R_2^e$ separates $K(f^*)$.

Proof. Since $f \in \mathcal{R}_\lambda(\varkappa) \subset \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$, the dynamic rays $R_f(\varkappa + \frac{1}{3})$ and $R_f(\varkappa + \frac{2}{3})$ crash at the critical point $\omega_2(f)$ and cut the plane in two wedges. One of them (W_1) contains dynamic rays with arguments from $(\varkappa + \frac{1}{3}, \varkappa + \frac{2}{3})$ and the other one (W_2) contains dynamic rays with arguments from $(\varkappa + \frac{2}{3}, \varkappa + \frac{1}{3})$. It follows that $K(f^*) \subset W_2$. Observe also that both arcs $(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$ and $(\theta_1 + \frac{2}{3}, \theta_2 + \frac{1}{3})$ are longer than $\frac{1}{3}$. Hence both angles $\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3}$ are in the arc $[\varkappa + \frac{2}{3}, \varkappa + \frac{1}{3}]$, the rays R_1^e and R_2^e are contained in $\overline{W_2}$, one of these rays lies in W_2 , and the point z belongs to W_2 .

Let $W_{(1)}, W_{(2)}$ be the two wedges defined by R_1^e, R_2^e . If $K(f^*)$ meets both these wedges, then $z \in K(f^*)$ as desired. Moreover, then $\varkappa \notin [\theta_1, \theta_2]$ because otherwise $\mathfrak{U}'_c(\mathbf{c}_\varkappa) \subset [\theta_2 + \frac{2}{3}, \theta_1 + \frac{1}{3}]$, a contradiction with the fact that $K(f^*)$ meets both wedges defined by R_1^e, R_2^e .

Now assume that $K(f^*) \subset W_{(1)} \cup \{z\}$. We claim that then $\varkappa \in [\theta_1, \theta_2]$. Indeed, suppose otherwise. Then $K(f^*)$ is contained in W_2 but is disjoint from $W_{(2)}$; hence σ_3 is one-to-one on arguments of external rays landing in $K(f^*)$ except possibly for $\varkappa + \frac{1}{3}$, $\varkappa + \frac{2}{3}$, a contradiction. Thus, the endpoints of \mathbf{c}_\varkappa belong to $[\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3}]$, and at least one of the endpoints of \mathbf{c}_\varkappa belongs to $(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$. Hence, at least one of the points $\theta_1 + \frac{1}{3}$, $\theta_2 + \frac{2}{3}$ (for definiteness suppose that this is $\theta_2 + \frac{2}{3}$) belongs to $(\varkappa + \frac{2}{3}, \varkappa + \frac{1}{3})$. By Theorem 2.12, the chord $\overline{(\theta_1 + \frac{1}{3})(\theta_2 + \frac{2}{3})}$ is the major of an invariant quadratic gap of periodic type whose intersection with \mathbb{S}^1 is contained in $[\theta_2 + \frac{2}{3}, \theta_1 + \frac{1}{3}]$. Hence the orbit of $\theta_2 + \frac{2}{3}$ is completely contained in the arc $[\varkappa + \frac{2}{3}, \varkappa + \frac{1}{3}]$, which implies that the landing point z of R_2^e belongs to $K(f^*)$, as desired. \square

Observe that if $(\theta_1, \theta_2) = (\frac{2}{3}, \frac{5}{6})$ or $(\theta_1, \theta_2) = (\frac{1}{6}, \frac{1}{3})$, then the plane cut $R_1^e \cup \{z\} \cup R_2^e$ defined in Lemma 3.12 cannot separate $K(f^*)$.

The next lemma specifies Lemma 3.12.

Lemma 3.13. *The following claims are equivalent.*

- (1) *The dynamic rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ are contained in a pair of external rays with the same periodic landing point.*
- (2) *One of the following holds.*
 - (a) *The angle \varkappa is in $[\theta_1, \theta_2]$.*
 - (b) *The chord $\overline{(\theta_1 + \frac{1}{3})(\theta_2 + \frac{2}{3})}$ is a major edge of a type D invariant gap or leaf \mathbf{g} while $[\varkappa + \frac{1}{3}, \varkappa + \frac{2}{3}]$ is contained in the closure of the hole behind the other major edge of \mathbf{g} . Moreover, for every $\overline{\alpha\beta}$ in the σ_3 -orbit of $\overline{(\theta_1 + \frac{1}{3})(\theta_2 + \frac{2}{3})}$, the rays $R_f^e(\alpha^+)$ and $R_f^e(\beta^-)$ land at a fixed point z ; we have $z = 0$ unless $(\theta_1, \theta_2) = (\frac{2}{3}, \frac{5}{6})$ or $(\frac{1}{6}, \frac{2}{3})$.*

Except for the case when $\mathbf{g} = \overline{0\frac{1}{2}}$, the set \mathbf{g} is a gap.

Proof. We first prove that (2) implies (1). Assume that $\varkappa \in [\theta_1, \theta_2]$. Then, by Proposition 3.9, the rays $R_f^e(\theta_1 + \frac{1}{3})^-$ and $R_f^e(\theta_2 + \frac{2}{3})^+$ land at a common point, as desired. If (b) holds, the claim follows immediately.

Assume now that (1) holds. Denote the external rays containing $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ and landing at a common point z by R_1^e , R_2^e ; then, by Lemma 3.12, we have $z \in K(f^*)$. Suppose that $\varkappa \notin [\theta_1, \theta_2]$, and prove that (b) holds.

To begin with, consider the case when $(\theta_1, \theta_2) = (\frac{2}{3}, \frac{5}{6})$ so that $(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3}) = (0, \frac{1}{2})$. In this case $\mathbf{g} = \overline{0\frac{1}{2}}$. This is a leaf but it can be informally viewed as a gap with two distinct edges $\overline{0\frac{1}{2}}$ and $\overline{\frac{1}{2}0}$. Our

assumption that $\varkappa \notin [\theta_1, \theta_2]$ means that $\varkappa \notin [\frac{2}{3}, \frac{5}{6}]$. Since the critical cut formed by the rays $R_f(\varkappa + \frac{1}{3})$ and $R_f(\varkappa + \frac{2}{3})$ cannot cross the cut formed by the rays R_1^e and R_2^e , then, as in the proof of Lemma 3.12, it follows that $[\varkappa + \frac{1}{3}, \varkappa + \frac{2}{3}] \subset [\frac{1}{2}, 0]$. In other words, $[\varkappa + \frac{1}{3}, \varkappa + \frac{2}{3}]$ is contained in the closure of the hole behind the other major edge $\frac{1}{2}0$ of \mathbf{g} as desired. Moreover, by Proposition 3.9, then $R_f^e(\frac{1}{2})^-$ and $R_f^e(0)^+$ land at the same fixed point as desired. The same can be proven in the case when $(\theta_1, \theta_2) = (\frac{1}{6}, \frac{1}{3})$.

Thus, from now on we may assume that $(\theta_1, \theta_2) \neq (\frac{2}{3}, \frac{5}{6})$ and $(\theta_1, \theta_2) \neq (\frac{1}{6}, \frac{1}{3})$. Recall that we suppose that $\varkappa \notin [\theta_1, \theta_2]$, and want to prove that then (b) holds. By Theorem 2.10, external rays R_1^e and R_2^e correspond to quadratic-like rays. If they correspond to the same quadratic-like ray, then the cut $R_1^e \cup \{z\} \cup R_2^e$ does not separate $K(f^*)$. By Lemma 3.12, then $\varkappa \in [\theta_1, \theta_2]$, a contradiction. Hence the external rays R_1^e and R_2^e correspond to different quadratic-like rays to $K(f^*)$.

By Lemma 3.10, the chord $\mathfrak{M} = \overline{(\theta_1 + \frac{1}{3})(\theta_2 + \frac{2}{3})}$ lies inside $\mathfrak{U}_c(\mathfrak{c}_\varkappa)$, except for the endpoints. Since the arguments of both rays are periodic, z is periodic. Moreover, since the external rays R_1^e and R_2^e correspond to different quadratic-like rays, y is a cutpoint of $K(f^*)$. Since $|\lambda| \leq 1$, there are no periodic cutpoints of $K(f^*)$ that are not fixed, and the fixed cutpoint 0 must be parabolic. Therefore, $z = 0$ is parabolic. Since \mathfrak{M} is a major of some quadratic invariant gap \mathfrak{U} , the orbit of \mathfrak{M} consists of pairwise disjoint chords; since z is parabolic, σ_3 restricted to them preserves their circular order. Completing the orbit of \mathfrak{M} to its convex hull \mathbf{g} , we see that \mathbf{g} is an invariant finite gap of type D. Hence $[\varkappa + \frac{1}{3}, \varkappa + \frac{2}{3}]$ is contained in the closure of the hole behind a major edge of \mathbf{g} ; by definition, \mathfrak{M} must be the other major edge of \mathbf{g} . The remaining claim that for every $\overline{\alpha\beta}$ in the orbit of $\overline{(\theta_1 + \frac{1}{3})(\theta_2 + \frac{2}{3})}$ the rays $R_f^e(\alpha^+)$ and $R_f^e(\beta^-)$ land at the same point follows from Proposition 3.9. \square

3.5. Parameter wakes. In Section 3.5, we suppose that (θ_1, θ_2) is a hole of \mathfrak{Q} . Fix $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$. Define the set $\widehat{\mathcal{W}}_\lambda(\theta_1, \theta_2)$ as the set of all polynomials such that the *dynamic* rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ land at the same *repelling* periodic point of f (observe that then both rays are smooth). By Lemma 3.1, the set $\widehat{\mathcal{W}}_\lambda(\theta_1, \theta_2)$ is open.

Recall that, by Theorem A, for every hole (θ_1, θ_2) of \mathfrak{Q} , the parameter rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ land at the same point. Thus the parameter rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$, together with their common landing point, divide the plane into two (open) parts. In Definition 2.16 we consider a hole (θ_1, θ_2) of \mathfrak{Q} and denote by $\mathcal{W}_\lambda(\theta_1, \theta_2)$ the component of $\mathbb{C} \setminus$

$\overline{\mathcal{R}_\lambda(\theta_1) \cup \mathcal{R}_\lambda(\theta_2)}$ containing the parameter rays with arguments from (θ_1, θ_2) . The set $\mathcal{W}_\lambda(\theta_1, \theta_2)$ is called the (parameter) *wake* (of \mathcal{F}_λ). The joint landing point f_{root} of the rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ is called the *root point* of the parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$. Let the *period* of the parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ be the period of $\theta_1 + \frac{1}{3}$ under σ_3 .

A parabolic periodic point z of f_{root} is not necessarily equal to 0; also, if $z \neq 0$, then the multiplier at z is 1. Recall that, by Proposition 3.4, if $\lambda = e^{2\pi ip/q}$ and $T_{p/q}(b) = 0$, then $f_{\lambda,b}^{oq}$ has $2q$ parabolic Fatou domains at 0 forming two cycles under $f_{\lambda,b}$.

We also need an observation concerning external and dynamic rays. Namely, if an argument α is periodic, then either $R_f(\alpha) = R_f^e(\alpha)$ is a smooth ray, or the dynamic ray $R_f(\alpha)$ with argument α crashes at a *critical* point. In the latter case, the two one-sided external rays $R_f^e(\alpha^-)$ and $R_f^e(\alpha^+)$ extend the ray $R_f(\alpha)$ and contain infinitely many precritical points. We will need this observation when talking about the boundary of the set $\widehat{\mathcal{W}}_\lambda(\theta_1, \theta_2)$.

Recall that, for a hole (θ_1, θ_2) of \mathfrak{Q} , the set $\mathcal{W}'_\lambda(\theta_1, \theta_2)$ is defined as the wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ except for the holes $(\frac{1}{6}, \frac{1}{3})$ and $(\frac{2}{3}, \frac{5}{6})$ for which

$$\mathcal{W}'_\lambda\left(\frac{1}{6}, \frac{1}{3}\right) = \mathcal{W}'_\lambda\left(\frac{2}{3}, \frac{5}{6}\right) = \mathcal{W}_\lambda\left(\frac{1}{6}, \frac{1}{3}\right) \cup \mathcal{W}_\lambda\left(\frac{2}{3}, \frac{5}{6}\right).$$

Proposition 3.14. *If $\widehat{\mathcal{W}}_\lambda(\theta_1, \theta_2) \neq \emptyset$, then the parameter rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ land at a point f_{root} , where f_{root} is a polynomial with a parabolic periodic point. We have $\widehat{\mathcal{W}}_\lambda(\theta_1, \theta_2) = \mathcal{W}'_\lambda(\theta_1, \theta_2)$; moreover, sets $\mathcal{W}_\lambda(\frac{1}{6}, \frac{1}{3})$ and $\mathcal{W}_\lambda(\frac{2}{3}, \frac{5}{6})$ are the two components of $\widehat{\mathcal{W}}_\lambda(\frac{1}{6}, \frac{1}{3}) = \widehat{\mathcal{W}}_\lambda(\frac{2}{3}, \frac{5}{6})$.*

Proof. By Lemma 3.10, if $\varkappa \in (\theta_1, \theta_2)$ and $f \in \mathcal{R}_\lambda(\varkappa)$, then the corresponding dynamic rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ land at the same periodic point z and are smooth. Assume that $\widehat{\mathcal{W}} = \widehat{\mathcal{W}}_\lambda(\theta_1, \theta_2) \neq \emptyset$; then, for every $f \in \widehat{\mathcal{W}}$, the point z is repelling.

For a polynomial $f = f_{\lambda,b}$ in the boundary of $\widehat{\mathcal{W}}$, either one of the rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ crashes into a critical point, or both rays are smooth but one of the landing points fails to be repelling (this follows from Lemma 3.1). Consider these cases separately.

(1) Suppose that at least one of the rays $R_f(\theta_1 + \frac{1}{3})$, $R_f(\theta_2 + \frac{2}{3})$ is not smooth. We claim that if $(\theta_1, \theta_2) \neq (\frac{2}{3}, \frac{5}{6})$ and $(\theta_1, \theta_2) \neq (\frac{1}{6}, \frac{2}{3})$, then our assumption implies that $f \in \mathcal{R}_\lambda(\theta_1) \cup \mathcal{R}_\lambda(\theta_2)$; on the other hand, if $(\theta_1, \theta_2) = (\frac{2}{3}, \frac{5}{6})$ or $(\theta_1, \theta_2) = (\frac{1}{6}, \frac{2}{3})$, then the non-smoothness

of either $R_f(0)$ or $R_f(\frac{1}{2})$ implies that

$$f \in \mathcal{R}_\lambda\left(\frac{2}{3}\right) \cup \mathcal{R}_\lambda\left(\frac{5}{6}\right) \cup \mathcal{R}_\lambda\left(\frac{1}{6}\right) \cup \mathcal{R}_\lambda\left(\frac{1}{3}\right).$$

Indeed, suppose that $R_f(\theta_1 + \frac{1}{3})$ is not smooth. We claim that then $f \in \mathcal{R}_\lambda(\theta_1)$. By the assumption, either $R_f(\theta_1)$ and $R_f(\theta_1 + \frac{1}{3})$ crash at the escaping critical point (and then $f \in \mathcal{R}_\lambda(\theta_1 - \frac{1}{3})$), or $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_1 + \frac{2}{3})$ crash at the escaping critical point (and then $f \in \mathcal{R}_\lambda(\theta_1)$). We claim that the latter option takes place.

Suppose otherwise: $f \in \mathcal{R}_\lambda(\theta_1 - \frac{1}{3})$, and the rays $R_f(\theta_1)$ and $R_f(\theta_1 + \frac{1}{3})$ crash at the escaping critical point. Arbitrarily close to f , there are polynomials $g \in \widehat{\mathcal{W}}$. These polynomials g must satisfy one of the conditions (2)(a) and (2)(b) from Lemma 3.13. However, in case $(\theta_1, \theta_2) \neq (\frac{2}{3}, \frac{5}{6})$ and $(\theta_1, \theta_2) \neq (\frac{1}{6}, \frac{2}{3})$, we see that (2)(b) is impossible as it would imply that the rays $R_g(\theta_1 + \frac{1}{3})$, $R_g(\theta_2 + \frac{2}{3})$ land at 0, which is parabolic. Thus, in this case polynomials g belong to external parameter rays $\mathcal{R}_\lambda(\varkappa)$ with $\varkappa \in (\theta_1, \theta_2)$, a contradiction with $f \in \mathcal{R}_\lambda(\theta_1 - \frac{1}{3})$.

Suppose now that $(\theta_1, \theta_2) = (\frac{2}{3}, \frac{5}{6})$ and choose f from the boundary of $\widehat{\mathcal{W}}$ so that $R_f(\theta_1 + \frac{1}{3}) = R_f(0)$ is not smooth. Choose $g \in \widehat{\mathcal{W}}$ very close to f . Then the smooth dynamic rays $R_g(0)$, $R_g(\frac{1}{2})$ land at the same repelling fixed point of g . By Lemma 3.13, this implies that $g \in \mathcal{R}_\lambda(\varkappa)$ with $\varkappa \in (\frac{2}{3}, \frac{5}{6})$ or $\varkappa \in (\frac{1}{6}, \frac{1}{3})$, and both cases are possible.

(2) Suppose that the point y , at which one of the rays $R_f(\theta_1 + \frac{1}{3})$ or $R_f(\theta_2 + \frac{2}{3})$ lands, is a parabolic periodic point of f . If $y = 0$, then $\lambda = e^{2\pi ip/q}$ for some integers p and q . We claim that then $T_{p/q}(b) = 0$. Indeed, suppose that $T_{p/q}(b) \neq 0$. Then, by Proposition 3.6, a small neighborhood of f_{root} is disjoint from $\widehat{\mathcal{W}}(\theta_1, \theta_2)$. This contradicts the fact that f_{root} is on the boundary of $\widehat{\mathcal{W}}$. If $y \neq 0$, and m is the period of θ_1 (and θ_2), then, by the Yoccoz inequality, f^{om} has multiplier 1 at the point y (cf. the proof of Theorem 1.1 in [BOPT14]). In both cases, $b \in \mathcal{Z}$, where the set \mathcal{Z} consists of all parameter values b such that either $T_{p/q}(b) = 0$, or f^{om} has a parabolic fixed point of multiplier 1 different from 0. Note that \mathcal{Z} is a finite set.

Hence, if $(\theta_1, \theta_2) \neq (\frac{2}{3}, \frac{5}{6})$ and $(\theta_1, \theta_2) \neq (\frac{1}{6}, \frac{2}{3})$, then the boundary of $\widehat{\mathcal{W}}$ lies in the union of $\mathcal{R}_\lambda(\theta_1) \cup \mathcal{R}_\lambda(\theta_2)$ and a finite set of points. Since $\widehat{\mathcal{W}}$ is open, it follows that the rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ land at the same point f_{root} . By the previous paragraph, the point f_{root} is a polynomial with a parabolic periodic point. Since polynomials with disconnected Julia sets located in the wedge between $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ belong

to $\widehat{\mathcal{W}}$ (see Proposition 3.9; here we mean the wedge in the positive direction from $\mathcal{R}_\lambda(\theta_1)$ to $\mathcal{R}_\lambda(\theta_2)$), it follows that $\widehat{\mathcal{W}}$ coincides with this wedge, except, perhaps, for finitely many points of \mathcal{Z} removed from this wedge. However, the existence of such removed points would contradict the maximum modulus principle applied to the multiplier of the landing point of $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$. The arguments in the case of $(\theta_1, \theta_2) = (\frac{2}{3}, \frac{5}{6})$ or $(\theta_1, \theta_2) = (\frac{1}{6}, \frac{2}{3})$ are almost literally the same except that now instead of one wedge we need to talk about the union of two wedges: one in the positive direction from $\mathcal{R}_\lambda(\frac{2}{3})$ to $\mathcal{R}_\lambda(\frac{5}{6})$ and the other one in the positive direction from $\mathcal{R}_\lambda(\frac{1}{6})$ to $\mathcal{R}_\lambda(\frac{2}{3})$. \square

The set $\widehat{\mathcal{W}}_\lambda(\theta_1, \theta_2)$ is called a *non-special (parameter) wake*. By Proposition 3.14, the parameter rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ land at the same point *under the assumption that there is at least one polynomial $f \in \mathcal{F}_\lambda$ with $R_f(\theta_1 + \frac{1}{3})$, $R_f(\theta_2 + \frac{2}{3})$ landing at the same repelling periodic point*. Recall that the common landing point of the rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ is called the *root point* of the parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$.

It remains to consider the case, where the dynamic rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ *never* land at the same *repelling* periodic point, no matter what polynomial $f \in \mathcal{F}_\lambda$ we choose. We claim that, in this case, λ must be a root of unity. Indeed, take an angle $\varkappa \in (\theta_1, \theta_2)$. Then, by Theorem 2.12, the chord $(\theta_1 + \frac{1}{3})(\theta_2 + \frac{2}{3})$ is the major of the quadratic invariant gap $\mathfrak{U}(\varkappa)$ of periodic type. It follows from Proposition 3.9 that, for every $f \in \mathcal{R}_\lambda(\varkappa)$, the dynamic rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ land at the same periodic point x . This point is either repelling or parabolic. By our assumption, x is not repelling, hence x is parabolic. Since $f \notin \mathcal{C}_\lambda$ and by the Fatou–Shishikura inequality, there is at most one non-repelling cycle of f . Thus we must have $x = 0$, which means that 0 is a parabolic point, i.e., the multiplier λ is a root of unity.

Fix $\lambda = e^{2\pi ip/q}$ and consider $f \in \mathcal{F}_\lambda$. If the rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ land at a point x , then let $W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$ be the (dynamic) wedge bounded by these rays and the point x that contains all dynamic rays with arguments in $(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$. The boundary of $W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$ is denoted by Θ_f . Recall that a *parabolic domain of f at 0* is a Fatou component of f that contains an attracting petal of 0.

Lemma 3.15. *If there exists $f \in \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ such that the dynamic rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ both land at 0 and the parabolic domains of f at 0 are disjoint from $W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$, then the convex hull \mathfrak{g} of the set of all points $\bar{\theta}$ such that the dynamic ray $R_f(\theta)$ lands at 0 is a finite invariant gap of type D. Moreover, if $f \in \mathcal{R}_\lambda(\varkappa)$, then $\varkappa \in (\theta_1, \theta_2)$.*

Proof. By [Mil06, Lemma 18.12], there are finitely many rays landing at 0, and they are permuted with combinatorial rotation number p/q , i.e., \mathfrak{g} is a finite invariant gap. Moreover, \mathfrak{g} is of type D. Indeed, otherwise the angles $\theta_1 + \frac{1}{3}$ and $\theta_2 + \frac{2}{3}$ would be in the same orbit under the angle tripling map. This contradicts Theorem 2.12, since the endpoints of the major of a quadratic invariant set cannot belong to one orbit.

Let $f \in \mathcal{R}_\lambda(\varkappa)$. Then \varkappa belongs to (θ_1, θ_2) . Indeed, the rays $R_f(\varkappa + \frac{1}{3})$ and $R_f(\varkappa + \frac{2}{3})$ crash at the escaping critical point $\omega_2(f)$. Thus, both $\varkappa + \frac{1}{3}$ and $\varkappa + \frac{2}{3}$ belong to the same major hole of \mathfrak{g} . Since $(\theta_1 + \frac{1}{3})(\theta_2 + \frac{2}{3})$ is the major of an invariant quadratic gap, the σ_3 -orbits of $\theta_1 + \frac{1}{3}$ and $\theta_2 + \frac{2}{3}$ do not enter the arc $(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$. Hence, $(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$ is a major hole of \mathfrak{g} . There are two major holes of \mathfrak{g} , either giving rise to the corresponding dynamic wedge containing a critical point of f . Since, by the assumption, the parabolic domains of f at 0 are disjoint from $W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$ and one of these dynamic wedges contains a non-escaping critical point of f , then the escaping critical point $\omega_2(f)$ is in $W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$, i.e., we have $\varkappa \in (\theta_1, \theta_2)$. \square

Consider the set $\widetilde{\mathcal{W}}_\lambda(\theta_1, \theta_2)$ of all polynomials $f = f_{\lambda, b}$ such that the rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ land at 0, and the attracting petals at 0 are disjoint from $W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$. Since there are at most two cycles of dynamic rays landing at 0, then f can have only one cycle of parabolic domains at 0. By Proposition 3.4, this implies that $T_{p/q}(b) \neq 0$.

Proposition 3.16. *If $\widetilde{\mathcal{W}}_\lambda(\theta_1, \theta_2) \neq \emptyset$, then the parameter rays $\mathcal{R}_\lambda(\theta_1)$, $\mathcal{R}_\lambda(\theta_2)$ land at the same point.*

Proof. The proof is similar to that of Proposition 3.14. If $\widetilde{\mathcal{W}}_\lambda(\theta_1, \theta_2) \neq \emptyset$, then, by Lemma 3.15, the boundary of $\widetilde{\mathcal{W}}_\lambda(\theta_1, \theta_2)$ lies in the union of the rays $\mathcal{R}_\lambda(\theta_1)$, $\mathcal{R}_\lambda(\theta_2)$ and the polynomials $f_{\lambda, b}$ corresponding to roots b of the polynomial $T_{p/q}$. Thus, the rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ land at the same point. \square

Let us now define *special (parameter) wakes*.

Definition 3.17. Let $(\bar{\theta}_1, \bar{\theta}_2)$ be a hole of \mathfrak{Q} . Suppose that there exists $f \in \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ such that the dynamic rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ both land at 0 and the parabolic domains of f at 0 are disjoint from $W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$. Then the set $\mathcal{W}_\lambda(\theta_1, \theta_2)$ bounded by the parameter rays $\mathcal{R}_\lambda(\theta_1)$, $\mathcal{R}_\lambda(\theta_2)$ and their common landing point is called a *special (parameter) wake*. Recall that the landing point of the rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ is called the *root point* of the parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$. Recall also that points in the parameter planes \mathcal{F}_λ are polynomials.

The next claim complements Proposition 3.16.

Proposition 3.18. *If $\widetilde{\mathcal{W}}_\lambda(\theta_1, \theta_2) \neq \emptyset$, and $f_{root} = f_{\lambda, b_0}$ is the common landing point of the parameter rays $\mathcal{R}_\lambda(\theta_1)$, $\mathcal{R}_\lambda(\theta_2)$, then $T_{p/q}(b_0) = 0$. Moreover, $\widetilde{\mathcal{W}}_\lambda(\theta_1, \theta_2)$ coincides with $\mathcal{W}_\lambda(\theta_1, \theta_2)$, possibly with several punctures f_{λ, b_i} , where $1 \leq i \leq N$ and $T_{p/q}(b_i) = 0$. In particular, the map f_{λ, b_i} , for every $0 \leq i \leq N$, has two cycles of parabolic domains at its unique parabolic point 0.*

Proof. The proof is similar to that of Proposition 3.14. The last claim holds by Proposition 3.4. \square

We are now ready to prove Theorem A.

Proof of Theorem A. Consider any hole $(\bar{\theta}_1, \bar{\theta}_2)$ of Ω . Choose $\varkappa \in (\theta_1, \theta_2)$ and a polynomial $f \in \mathcal{R}_\lambda(\varkappa)$. By Lemma 3.13, the dynamic rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ land at the same periodic point z . Since $\omega_2(f)$ is escaping, then Fatou-Shishikura inequality implies that the point z is either repelling or 0. If z is repelling, then, by Proposition 3.14, the polynomial f belongs to the non-special parameter wake bounded by the parameter rays $\mathcal{R}_\lambda(\theta_1)$, $\mathcal{R}_\lambda(\theta_2)$ and their common landing point. In particular, the parameter rays $\mathcal{R}_\lambda(\theta_1)$, $\mathcal{R}_\lambda(\theta_2)$ land at the same point.

Now let $z = 0$. If $W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$ is disjoint from all attracting petals at 0, then, by Proposition 3.18, the polynomial f belongs to the special parameter wake bounded by the parameter rays $\mathcal{R}_\lambda(\theta_1)$, $\mathcal{R}_\lambda(\theta_2)$ and their common landing point. We claim that the remaining case (where $z = 0$ but the wedge $W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$ contains some parabolic domain Ω at 0) is impossible. Indeed, suppose f has the just listed properties. Since $J(f)$ is disconnected, f is immediately renormalizable; we let f^* be the polynomial-like restriction of f . The fact that the orbits of all points in Ω converge to 0 implies that $\bar{\Omega} \subset K(f^*)$. Hence there are dynamic rays in the wedge $W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$ that land at points of $\bar{\Omega} \subset K^*$. However, by Lemma 3.7, no dynamic ray in the wedge $W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$ accumulates in $K(f^*)$, a contradiction. \square

4. DYNAMICAL DESCRIPTION OF PARAMETER WAKES

In this section, we expand the dynamical description of polynomials in a given parameter wake. We will also discuss the dynamics of polynomials corresponding to the root points of parameter wakes.

4.1. Special parameter wakes. In this subsection, we assume that $\lambda = e^{2\pi ip/q}$ is a root of unity. By definition, special parameter wakes can exist only in these λ -slices \mathcal{F}_λ . We will need the following proposition.

Proposition 4.1. *Let $\mathcal{W}_\lambda(\theta_1, \theta_2)$ be a special parameter wake, and f_{root} be its root point. Then, for every $f \in \mathcal{W}_\lambda(\theta_1, \theta_2) \cup \{f_{root}\}$, the rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ land at 0.*

Proof. Consider $f = f_{\lambda,b} \in \mathcal{W}_\lambda(\theta_1, \theta_2)$ or $f = f_{root}$. If $T_{p/q}(b) \neq 0$, then, by Propositions 3.16 and 3.18, the rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ land at 0. Suppose now that $T_{p/q}(b) = 0$ but at least one of the rays $R_f(\theta_1 + \frac{1}{3})$, $R_f(\theta_2 + \frac{2}{3})$ (say, the former) fails to land at 0. The ray $R_f(\theta_1 + \frac{1}{3})$ has to land somewhere, say, at a point $z \neq 0$. The point z is repelling or parabolic. It cannot be repelling because then, by Lemma 3.1, for polynomials g sufficiently close to f the ray $R_g(\theta_1 + \frac{1}{3})$ will have to land at a point close to z while, by Proposition 3.18, there are polynomials g arbitrarily close to f such that $R_g(\theta_1 + \frac{1}{3})$ lands at 0, a contradiction. On the other hand, by the Fatou–Shishikura inequality, the point $z \neq 0$ cannot be parabolic either (observe that both critical points of f are in parabolic domains at 0). \square

A choice of the combinatorial rotation number p/q provides a classification of holes of \mathfrak{Q} into *special* holes and *non-special* holes.

Definition 4.2. A hole $(\bar{\theta}_1, \bar{\theta}_2)$ of \mathfrak{Q} is (p/q) -*special* if the map σ_3 preserves the cyclic order on the periodic orbit of $\theta_1 + 1/3$, and the combinatorial rotation number of this orbit under σ_3 equals p/q . If $(\bar{\theta}_1, \bar{\theta}_2)$ is a special hole, then the orbit of $\theta_2 + 2/3$ has combinatorial rotation number p/q as well. Indeed, since the orbit of $(\theta_1 + \frac{1}{3})(\theta_2 + \frac{2}{3})$ is on the boundary of some invariant quadratic gap, σ_3 permutes this orbit as a combinatorial rotation. All holes of \mathfrak{Q} that are not special are called *non-special holes*.

Lemma 4.3 easily follows from Definition 4.2.

Lemma 4.3. *A hole $(\bar{\theta}_1, \bar{\theta}_2)$ is p/q -special if and only if there exists a finite type D invariant gap \mathfrak{g} with major $\theta_1 + \frac{1}{3}\theta_2 + \frac{2}{3}$. There are exactly $2q$ special holes of \mathfrak{Q} corresponding to the rotation number p/q .*

Proof. The first claim of the lemma is left to the reader. Now, by Lemma 2.13, there are q type D finite invariant gaps of rotation number p/q . Every type D finite invariant gap of rotation number p/q has two major holes giving rise to two p/q -special holes. Overall this yields $2q$ special holes as desired. \square

By Lemma 4.3, there are exactly $2q$ special holes of Ω corresponding to the rotation number p/q . Theorem 4.4 helps to distinguish among various parameter wakes. Given $f \in \mathcal{F}_\lambda$, denote by \mathbf{g}_f the convex hull of the set of points $\bar{\theta}$ such that $R_f(\theta)$ lands at 0.

Theorem 4.4 (Special parameter wakes vs. special holes). *The parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ is a special parameter wake if and only if the hole (θ_1, θ_2) of Ω is p/q -special. In particular, there are $2q$ special parameter wakes in \mathcal{F}_λ .*

Proof. Let $\mathcal{W}_\lambda(\theta_1, \theta_2)$ be a special parameter wake. Consider its root point $f = f_{root}$. Then, by Proposition 3.18, there are two cycles of parabolic domains at 0 and two cycles of external rays of f landing at 0. It follows that \mathbf{g}_f contains two periodic cycles under σ_3 . If $p/q = 0$, these two cycles are $\bar{0}$ and $\bar{\frac{1}{2}}$, and $\mathbf{g}_f = \overline{0\frac{1}{2}}$. If $p/q \neq 0$, then, by [Kiw02], there are no other rays landing at 0. Hence, in this case, the set \mathbf{g}_f is a finite type D invariant gap of rotation number p/q . On the other hand, by Proposition 4.1, the points $\bar{\theta}_1 + \frac{1}{3}$ and $\bar{\theta}_2 + \frac{2}{3}$ belong to \mathbf{g}_f . The chord $\overline{\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3}}$ must be an edge of \mathbf{g}_f as otherwise an eventual image of it will cross it. Thus, the hole $(\bar{\theta}_1, \bar{\theta}_2)$ of Ω is p/q -special.

Now, assume that the hole $(\bar{\theta}_1, \bar{\theta}_2)$ is p/q -special. By Theorem A, the parameter rays $\mathcal{R}(\theta_1), \mathcal{R}(\theta_2)$ land at the same point, and the parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ exists. We claim that, in our case, the parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ is special. Indeed, otherwise, for every $f \in \mathcal{W}_\lambda(\theta_1, \theta_2)$, the dynamic rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ land at the same repelling periodic point z . Suppose first that $p/q = 0$ and $\bar{\theta}_1 + \frac{1}{3}, \bar{\theta}_2 + \frac{2}{3}$ equals either $\overline{0\frac{1}{2}}$ or $\overline{\frac{1}{2}0}$. Then both rays $R_f(0)$ and $R_f(\frac{1}{2})$ land at a repelling fixed point z . However, in this case 0 is a parabolic point with $f'(0) = 1$, and at least one fixed external ray lands at 0, a contradiction. Hence a 0-special hole corresponds to a special parameter wake.

Let us now assume that $p/q \neq 0$. Let A be the component of $\overline{\mathbb{D}} \setminus \overline{\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3}}$ disjoint from the circle arc $(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$, and let B be the other component. We claim that $\mathbf{g}_f \subset A$. Indeed, by [BOPT13], the arc $B \cap \mathbb{S}^1$ contains either $\bar{0}$ or $\overline{1/2}$, and it is easy to see that $B \cap \mathbb{S}^1$ contains no other σ_3 -invariant sets (recall that $p/q \neq 0$), a contradiction. Then \mathbf{g}_f is a finite invariant gap of rotation number p/q located in the component A of $\overline{\mathbb{D}} \setminus \overline{\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3}}$ and disjoint from the circle arc $(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$, while the leaf $\overline{\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3}}$ rotates in A with rotation number p/q . We will show in the next paragraph that this is impossible for combinatorial reasons.

Let \mathfrak{U} be the quadratic invariant gap such that its major hole is $(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$. The existence of such a quadratic gap is a consequence of Theorem 2.12. It is well-known (see, e.g., [BOPT13]) that there is a continuous projection $\psi_{\mathfrak{U}} : \mathfrak{U}' \rightarrow \mathbb{S}^1$ that identifies the endpoints of every arc in $\mathbb{S}^1 \setminus \mathfrak{U}'$ and that semi-conjugates the mapping $\sigma_3 : \mathfrak{U}' \rightarrow \mathfrak{U}'$ with the mapping $\sigma_2 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$. Apply this projection to the vertices of \mathfrak{g}_f (by definition of \mathfrak{U} , all vertices of \mathfrak{g}_f belong to \mathfrak{U}'). We obtain a σ_2 -invariant finite set of points of \mathbb{S}^1 permuted by σ_2 as a combinatorial rotation with rotation number p/q . However, a subset of \mathbb{S}^1 with these properties is unique. Therefore, it must coincide with the $\psi_{\mathfrak{U}}$ -image of the σ_3 -orbit of $\theta_1 + \frac{1}{3}$. It follows that one of the vertices of \mathfrak{g}_f must coincide either with $\theta_1 + \frac{1}{3}$ or with $\theta_2 + \frac{2}{3}$, a contradiction. The last claim of the theorem follows from Lemma 4.3. \square

4.2. Root points of special parameter wakes. In this subsection, we still assume that $\lambda = e^{2\pi ip/q}$ is a fixed root of unity. We will prove that all zeros of $T_{p/q}$ correspond to root points of special parameter wakes. In particular, it will follow that the set $\widetilde{\mathcal{W}}_{\lambda}(\theta_1, \theta_2)$ introduced in Section 3.5 coincides with the special parameter wake $\mathcal{W}_{\lambda}(\theta_1, \theta_2)$, where $(\bar{\theta}_1, \bar{\theta}_2)$ is any p/q -special hole of \mathfrak{Q} . Lemma 4.5 describes the situation, when two special parameter wakes share a root point.

Lemma 4.5. *If two special parameter wakes $\mathcal{W}_{\lambda}(\theta_1, \theta_2)$ and $\mathcal{W}_{\lambda}(\theta'_1, \theta'_2)$ have the same root point, then $(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$ and $(\theta'_1 + \frac{1}{3}, \theta'_2 + \frac{2}{3})$ are the two major holes of the same type D finite invariant gap.*

Proof. Let $f = f_{root}$ be the common root point of the parameter wakes $\mathcal{W}_{\lambda}(\theta_1, \theta_2)$ and $\mathcal{W}_{\lambda}(\theta'_1, \theta'_2)$. By Proposition 4.1, the four dynamic rays

$$R_f\left(\theta_1 + \frac{1}{3}\right), \quad R_f\left(\theta_2 + \frac{2}{3}\right), \quad R_f\left(\theta'_1 + \frac{1}{3}\right), \quad R_f\left(\theta'_2 + \frac{2}{3}\right)$$

land at 0. Therefore, the four arguments of these rays correspond to the vertices of the same type D finite invariant gap \mathfrak{g} . Clearly, these vertices are on the boundaries of major holes of \mathfrak{g} . \square

Two majors \mathfrak{M} and \mathfrak{M}' of the same type D finite invariant gap G , as well as the corresponding major holes of G , will be called *conjugate*.

Proposition 4.6. *Every zero b of the polynomial $T_{p/q}$ corresponds to a common root point $f_{\lambda,b}$ of two special parameter wakes $\mathcal{W}_{\lambda}(\theta_1, \theta_2)$ and $\mathcal{W}_{\lambda}(\theta'_1, \theta'_2)$, where arcs $(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$ and $(\theta'_1 + \frac{1}{3}, \theta'_2 + \frac{2}{3})$ are conjugate major holes depending on b . The degree of the polynomial $T_{p/q}$ is equal to q .*

Proof. By Lemmas 4.4 and 4.5, the $2q$ special parameter wakes in \mathcal{F}_λ have at least q different root points. By Proposition 3.18, each special parameter wake has a zero of the polynomial $T_{p/q}$ as its root point. Since, by Proposition 3.4, the degree of the polynomial $T_{p/q}$ is at most q , the degree of $T_{p/q}$ equals q , and each of the zeros of $T_{p/q}$ corresponds to a common root point of two special parameter wakes. \square

Note that an alternative way of proving that the degree of the polynomial $T_{p/q}$ is equal to q may follow the methods of a paper by Buff, Écalle and Epstein [BEE13], in which the authors prove a similar statement for parameter slices of quadratic rational functions.

Theorem 4.7 (Dynamics of special parameter wakes). *Assume that the wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ is a special parameter wake. A polynomial $f = f_{\lambda, b}$ belongs to $\mathcal{W}_\lambda(\theta_1, \theta_2)$ if and only if the dynamic rays $R_f(\theta_1 + \frac{1}{3})$, $R_f(\theta_2 + \frac{2}{3})$ land at 0, the parabolic domains at 0 are disjoint from the wedge $W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$, and $T_{p/q}(b) \neq 0$. A polynomial f is the root point of the parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ if and only if $T_{p/q}(b) = 0$, and the rays $R_f(\theta_1 + \frac{1}{3})$, $R_f(\theta_2 + \frac{2}{3})$ land at 0. Moreover, then the polynomial f has a parabolic point 0 with two cycles of parabolic domains at 0.*

Proof. Let $f = f_{\lambda, b} \in \mathcal{W}_\lambda(\theta_1, \theta_2)$. Then, by Proposition 4.1, the dynamic rays $R_f(\theta_1 + \frac{1}{3})$, $R_f(\theta_2 + \frac{2}{3})$ land at 0. By Proposition 4.6, we have $T_{p/q}(b) \neq 0$ for all $f = f_{\lambda, b}$ in the parameter wake, and, by Proposition 3.18, the parabolic domains at 0 are disjoint from the wedge $W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$. On the other hand, if the dynamic rays $R_f(\theta_1 + \frac{1}{3})$, $R_f(\theta_2 + \frac{2}{3})$ land at 0, the parabolic domains at 0 are disjoint from the wedge $W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$, and $T_{p/q}(b) \neq 0$, then, by Proposition 3.18 and by Proposition 4.6, we have $f \in \mathcal{W}_\lambda(\theta_1, \theta_2)$. The characterization of the root point of $\mathcal{W}_\lambda(\theta_1, \theta_2)$ follows from Proposition 4.6. \square

4.3. Non-special parameter wakes. In this section, we assume that λ is a complex number with $|\lambda| \leq 1$. We characterize dynamics of non-special parameter wakes in \mathcal{F}_λ . To describe the properties of root points of special wakes, we need a lemma.

Lemma 4.8. *Suppose that $f \mapsto z_f$ is a complex-valued holomorphic function defined on some open subset $\mathcal{U} \subset \mathcal{F}_\lambda$ such that, for every $f \in \mathcal{U}$, the point z_f is a repelling periodic point of f of period m . If f_0 is a boundary point of \mathcal{U} , and z_f tends to 0 as $f \in \mathcal{U}$ tends to f_0 , then $\lambda^m = 1$, and 0 is a degenerate parabolic point of f_0 , i.e., there are at least two cycles of parabolic domains at 0.*

Note that if $f_0 = f_{\lambda, b_0}$ is as in the statement of the lemma, then we must necessarily have $T_{p/q}(b_0) = 0$, where p/q is the rotation number associated with λ .

Proof. Set $f = f_{\lambda, b}$ and $f_0 = f_{\lambda, b_0}$. We have $f^{\circ m}(z) - z = (z - z_f)\varphi_f(z)$, where φ_f is a polynomial function of z depending holomorphically on $f \in \mathcal{U}$ such that $\varphi_f(z_f) \neq 0$ for $f \in \mathcal{U}$. It is easy to see that all coefficients of the polynomial φ_f are algebraic functions of b without any poles in the finite part of the plane. This follows from the Euclidean algorithm for polynomials. In particular, there is a well-defined limit polynomial φ_{f_0} of φ_f as $f \in \mathcal{U}$ tends to f_0 .

On the other hand, since 0 is a fixed point of f , we have $f(0) = 0$. Note that $z_f \neq 0$ for $f \in \mathcal{U}$, since otherwise we must have $|\lambda| > 1$, a contradiction with our assumption. Therefore, we have $\varphi_f(0) = 0$ for all $f \in \mathcal{U}$. It follows that we also have $\varphi_{f_0}(0) = 0$. Therefore, the polynomial $f_0^{\circ m}(z) - z$ is divisible by z^2 . This means that 0 is a parabolic fixed point of $f_0^{\circ m}$ with multiplier 1. In particular, we have $\lambda^m = 1$.

Suppose now that $\lambda = e^{2\pi ip/q}$, where p and q are co-prime. The number m is then necessarily divisible by q so that $m = qr$ for some positive integer r . We want to prove that $f_0^{\circ q}(z) - z$ is $o(z^{q+1})$. To this end, we note that $f^{\circ q}(z) = z + a(b)z^{q+1} + h(b, z)$, where $a(b)$ is a polynomial function of b , and h is a polynomial in two variables divisible by z^{q+2} . This follows from the fact that 0 is a parabolic fixed point of f of multiplier λ . What we need to show is that $a(b_0) = 0$. Assume the contrary: $a(b_0) \neq 0$.

We have $f^{\circ m}(z) = z + ra(b)z^{q+1} + h_r(b, z)$, where h_r is a polynomial of two variables divisible by z^{q+2} . This follows from a simple computation based on the fact that the composition of two polynomials $f_1(z) = z + a_1z^{q+1} + \dots$ and $f_2(z) = z + a_2z^{q+1} + \dots$, where, in both cases, dots denote the terms of order $q+2$ or higher with respect to z , is equal to $z + (a_1 + a_2)z^{q+1} + o(z^{q+1})$. Since $(z - z_f)\varphi_f(z)$ is divisible by z^{q+1} as a polynomial in z , we have $\varphi_f(z) = z^{q+1}\tilde{\varphi}_f(z)$, where $\tilde{\varphi}_f(z)$ is a polynomial function of z , whose coefficients are algebraic functions of b without any poles in the finite part of the plane. We can now substitute f_0 for f to obtain that $f_0^{\circ m}(z) = z + z^{q+2}\tilde{\varphi}_f(z)$. This implies that $a(b_0) = 0$, as desired. \square

We now prove the following proposition, which characterizes root points of non-special parameter wakes.

Proposition 4.9. *Let $\mathcal{W}_\lambda(\theta_1, \theta_2)$ be a non-special parameter wake, and f_{root} its root point. Then the rays $R_{f_{root}}(\theta_1 + \frac{1}{3})$ and $R_{f_{root}}(\theta_2 + \frac{2}{3})$ land at the same parabolic periodic point different from 0.*

Proof. Consider the function $f \in \mathcal{W}_\lambda(\theta_1, \theta_2) \mapsto z_f$, where z_f is the common landing point of $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$. Clearly, z_f is a holomorphic function of f . Being a local branch of a globally defined multivalued analytic function, the function z_f has a well-defined limit z_* at the root point $f_{root} = f_{\lambda, b_0}$. The point z_* cannot be repelling because any neighborhood of f_{root} in \mathcal{F}_λ contains polynomials f , for which the rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ do not land at a common repelling point (see Lemma 3.1). Therefore, the point z_* is neutral.

Suppose that $z_* \neq 0$. Then there are no parabolic cycles different from those of 0 and z_* . This follows from a simple version of the Fatou-Shishikura inequality stating that the number of cycles of parabolic basin plus the number of irrationally neutral cycles of a degree d polynomial does not exceed $d - 1$. The ray $R_{f_{root}}(\theta_1 + \frac{1}{3})$ lands either at 0 or at z_* (in the latter case, z_* must be parabolic). If it lands at 0, then, by Proposition 3.6, we must have $T_{p/q}(b_0) = 0$. It follows that there are two cycles of parabolic domains of 0, a contradiction with the assumption $z_* \neq 0$. The contradiction shows that the ray $R_{f_{root}}(\theta_1 + \frac{1}{3})$ lands at z_* . Similarly, the ray $R_{f_{root}}(\theta_2 + \frac{2}{3})$ also lands at z_* .

Suppose now that $z_* = 0$. In this case, it also follows that $T_{p/q}(b_0) = 0$ since 0 becomes a degenerate parabolic point at the moment when it merges with z_f as $f \rightarrow f_{root}$. This follows from Lemma 4.8. Hence there are no parabolic cycles different from 0. The rays $R_{f_{root}}(\theta_1 + \frac{1}{3})$ and $R_{f_{root}}(\theta_2 + \frac{2}{3})$ must land at parabolic points. Therefore, they land at 0. By the dynamical characterization of special wakes, Theorem 4.7, the polynomial f_{root} is the root point of two special wakes, one of which must coincide with $\mathcal{W}_\lambda(\theta_1, \theta_2)$. A contradiction with our assumption that the wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ is non-special. \square

We can now give a dynamical description of non-special parameter wakes, which follows easily from Propositions 3.14 and 4.9. Recall that, for a hole (θ_1, θ_2) of \mathfrak{Q} , the set $\mathcal{W}'_\lambda(\theta_1, \theta_2)$ is defined as the wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ except for the holes $(\frac{1}{6}, \frac{1}{3})$ and $(\frac{2}{3}, \frac{5}{6})$ for which we set

$$\mathcal{W}'_\lambda\left(\frac{1}{6}, \frac{1}{3}\right) = \mathcal{W}'_\lambda\left(\frac{2}{3}, \frac{5}{6}\right) = \mathcal{W}_\lambda\left(\frac{1}{6}, \frac{1}{3}\right) \cup \mathcal{W}_\lambda\left(\frac{2}{3}, \frac{5}{6}\right).$$

We will call sets $\mathcal{W}'_\lambda(\theta_1, \theta_2)$ *non-special* if they correspond to non-special parameter wakes. Observe that parameter wakes $\mathcal{W}_\lambda(\frac{1}{6}, \frac{1}{3})$ and $\mathcal{W}_\lambda(\frac{2}{3}, \frac{5}{6})$ are either both special or both non-special, so the notion of a non-special set $\mathcal{W}'_\lambda(\theta_1, \theta_2)$ is well-defined.

Theorem 4.10 (Dynamics of non-special parameter wakes). *A polynomial f belongs to a non-special set $\mathcal{W}'_\lambda(\theta_1, \theta_2)$ if and only if the dynamic rays $R_f(\theta_1 + \frac{1}{3})$, $R_f(\theta_2 + \frac{2}{3})$ land at the same repelling periodic point. If a polynomial f_{root} is the root point of $\mathcal{W}_\lambda(\theta_1, \theta_2)$, then the dynamic rays $R_{f_{root}}(\theta_1 + \frac{1}{3})$, $R_{f_{root}}(\theta_2 + \frac{2}{3})$ land at the same parabolic periodic point of multiplier 1 different from 0.*

Theorem B follows from Theorems 4.7 and 4.10, except for its last claim, according to which, for every root polynomial f_{root} , we have $f_{root} \in \mathcal{CU}_\lambda$. We postpone the proof of this claim until Section 7.

5. FIXED POINTS AND (GEO)LAMINATIONS

The purpose of this section is to prove Theorem C. In the first several subsections of it we develop the tools necessary for the proof.

5.1. Fixed points. Recall some topological results of [BFMOT12]. First define *special pieces* of a polynomial f with connected Julia set; Loosely, these are subcontinua X of $K(f)$ carved in it by *exit continua* E_1, \dots, E_n located on the boundary of X so that $f(X)$ grows out of X only “through” the exit continua of X . The definition below is a little more special and less general than that in [BFMOT12].

Definition 5.1 (Central component and exit continua). Let E_1, \dots, E_n be a finite (perhaps empty) collection of (possibly degenerate) continua in $J(f)$, each consisting of principal sets of two or more external rays. Denote the union of E_i with these external rays by \tilde{E}_i . Suppose that there is a component T of $\mathbb{C} \setminus \cup \tilde{E}_i$, whose boundary intersects *all* E_1, \dots, E_n . Then the continua E_i are called *exit continua (of T)* while T is called the *central component (of the exit continua E_1, \dots, E_n)*.

Observe that the collection of exit continua may be empty, in which case the central component T coincides with \mathbb{C} .

Definition 5.2 (Special pieces). Let T be the central component of the exit continua E_1, \dots, E_n . For every i , let the wedge W_i be the component of $\mathbb{C} \setminus \tilde{E}_i$ containing T . Any non-separating continuum X is said to be a *special piece (of f)* if the following holds:

- (1) $X \subset [T \cap K(f)] \cup (\cup E_i)$ is a continuum and X contains $\cup E_i$;
- (2) each E_i is either a fixed point or maps forward in such a way that $f(E_i) \subset W_i$ (loosely, sets E_i are mapped “towards T ”);
- (3) the set $f(X) \setminus X$ is disjoint from T (loosely speaking, X can only grow “through exit continua”).

Observe that the set $[T \cap K(f)] \cup (\cup E_i)$ above is a continuum.

Theorem 5.3 ([BFMOT12]). *Let f be a polynomial with connected Julia set, and X be a special piece of f . Then X contains a fixed Cremer or Siegel point, or an invariant attracting or parabolic Fatou domain, or a fixed repelling or parabolic point at which at least two rays land so that f non-trivially rotates these rays.*

In particular, Theorem 5.3 applies to invariant continua in $K(f)$ that are non-separating in \mathbb{C} (with an empty collection of exit continua).

A major tool for us is the notion of an *impression*.

Definition 5.4. Given a (possibly disconnected) Julia set J , choose an angle α and consider the set L^- of all limit points of sequences $x_i \in R_f(\alpha_i)$ where $R_f(\alpha_i)$ are smooth rays with arguments α_i , and α_i converge to α from the negative side in the usual sense (i.e., $\alpha_i < \alpha$). Then $\text{Imp}(\alpha^-) = L^- \setminus R_f^e(\alpha^-)$ is the *one-sided impression of α from the negative side*. Similarly we define the *one-sided impression $\text{Imp}(\alpha^+)$ of α from the positive side*. In case J is connected, the *impression $\text{Imp}(\alpha)$* is defined as $\text{Imp}(\alpha^-) \cup \text{Imp}(\alpha^+)$.

Corollary 5.5. *Let f be a polynomial with connected Julia set and let T be the central component of exit continua E_1, \dots, E_n , each of which is a repelling or parabolic periodic point. Set $Y = [T \cap K(f)] \cup (\cup E_i)$. Suppose that Y contains no periodic Cremer points, no periodic Siegel points, and no periodic attracting or parabolic Fatou domains. Then there are infinitely many periodic points in Y , at each of which finitely many external rays land so that the minimal iterate of f that fixes the periodic point non-trivially rotates these external rays.*

Proof of Corollary 5.5. Choose the iterate f^{or} of f that fixes all rays landing at E_1, \dots, E_n . It is easy to see that $Y = Y_0$ is a special piece for f^{or} . By Theorem 5.3, there exists an f^{or} -fixed repelling point $y_0 \in Y_0$ with several external rays of f^{or} -period $m_0 > 1$ landing at y_0 . Observe that, by the choice of r , the point y_0 is not equal to any of the points E_0, \dots, E_n . Consider one of the wedges formed by the external rays of f landing at y_0 , say, W_1 , and set $Y_1 = W_1 \cap Y_0$. It follows that Y_1 is a special piece for a suitable iterate of f , whose exit continua are periodic points of f . We can now apply the same argument to Y_1 , etc. \square

The following is a consequence of Corollary 5.5.

Corollary 5.6. *Suppose that Z is a non-separating continuum in \mathbb{C} that is obtained as a finite union of one-sided impressions of periodic external angles and that contains no periodic Cremer points and no periodic Siegel points. Then Z is a singleton.*

Proof. Let us first observe that one-sided impressions cannot contain parabolic or attracting periodic Fatou domains. Notice also that, passing to a suitable iterate of the map, we may assume that all the arguments of the external rays involved are invariant. Then, by the previous paragraph, we can consider some iterate $f^{\circ n}$ of f such that $f^{\circ n}|_Z$ has a fixed point z at which several external ray land, and these external rays rotate under $f^{\circ n}$. Since the arguments of the external rays whose impressions form Z are invariant, we see that these impressions cannot contain z , a contradiction. \square

5.2. (Geo)laminations. Let us recall some known facts about invariant geolaminations. A *geodesic* (also called *geometric*) lamination, which we will abbreviate as a *geolamination*, is a closed collection \mathcal{L} of closed chords in $\overline{\mathbb{D}}$ that do not cross in \mathbb{D} . We will always assume that all points of \mathbb{S}^1 (i.e., all degenerate chords) are included into \mathcal{L} . The collection \mathcal{L} being closed means that the union \mathcal{L}^+ of all chords in \mathcal{L} is a closed subset of $\overline{\mathbb{D}}$. Elements of \mathcal{L} are called *leaves*.

As in the beginning of Subsection 2.3, we identify \mathbb{S}^1 with \mathbb{R}/\mathbb{Z} . Then the d -tupling map $t \mapsto d \cdot t \pmod{1}$ identifies with the map $\sigma_d : z \mapsto z^d$ on the unit circle \mathbb{S}^1 in the complex plane \mathbb{C} . If a chord ℓ has endpoints $\bar{\alpha}$ and $\bar{\beta}$, then we denote this chord by $\overline{\alpha\beta}$. We will always extend σ_d over all leaves of a given geolamination \mathcal{L} by linearly mapping any leaf $\ell = \overline{\alpha\beta}$ onto the chord $\sigma_d(\bar{\alpha})\sigma_d(\bar{\beta}) = \overline{(d\alpha)(d\beta)}$.

Definition 5.7 (Invariant geolaminations, cf [BMOV13]). A geolamination \mathcal{L} is said to be (sibling) σ_d -invariant if:

- (1) for each $\ell \in \mathcal{L}$, we have $\sigma_d(\ell) \in \mathcal{L}$,
- (2) for each $\ell \in \mathcal{L}$ there exists $\ell^* \in \mathcal{L}$ such that $\sigma_d(\ell^*) = \ell$,
- (3) for each $\ell \in \mathcal{L}$ such that $\sigma_d(\ell)$ is a non-degenerate leaf, there exist d **pairwise disjoint** leaves ℓ_1, \dots, ℓ_d in \mathcal{L} such that $\ell_1 = \ell$ and $\sigma_d(\ell_i) = \sigma_d(\ell)$ for all $i = 2, \dots, d$.

The leaves ℓ_1, \dots, ℓ_d are called *siblings* of ℓ .

Gaps of \mathcal{L} are the closures of components of $\overline{\mathbb{D}} \setminus \mathcal{L}^+$. Gaps of \mathcal{L} are *finite* or *infinite* according to whether they have finitely many or infinitely many points in \mathbb{S}^1 . We say that a σ_3 -invariant geolamination \mathcal{L} *co-exists* with a stand-alone invariant quadratic gap \mathfrak{U} if there are no leaves of \mathcal{L} crossing edges of \mathfrak{U} in \mathbb{D} and different from these edges. We say that the geolamination \mathcal{L} *tunes* the gap \mathfrak{U} if all edges of \mathfrak{U} are leaves of \mathcal{L} .

Recall that a continuous map between topological spaces is said to be *monotone* if the full preimage of any connected set is connected. When talking about monotone maps, we will always assume their continuity.

Definition 5.8 ([BCO11]). Given a continuum Q , define a *finest map* of Q onto a locally connected continuum Z as a monotone map $\varphi_Q : Q \rightarrow Z$ such that for any monotone map $t : Q \rightarrow T$ onto a locally connected continuum T there exists a monotone map $h : Z \rightarrow T$ with $t = h \circ \varphi_Q$. It is easy to see that φ_Q and Z are defined up to a homeomorphism, and we can talk about *the* finest map $\varphi_Q = \varphi$. A continuum $X \subset \mathbb{C}$ is *unshielded* if it coincides with the boundary of the unbounded component of the set $\mathbb{C} \setminus X$.

The finest map exists for unshielded continua.

Proposition 5.9 ([BCO11]). *For an unshielded continuum Q , there is an equivalence relation \sim_Q on \mathbb{S}^1 such that the finest monotone map φ_Q maps Q onto \mathbb{S}^1 / \sim_Q . The map φ_Q can be extended onto the whole complex plane \mathbb{C} so that $\varphi_Q : \mathbb{C} \rightarrow \mathbb{C}$ is one-to-one outside of Q ; in what follows, when talking about φ_Q , we mean the extended map $\varphi_Q : \mathbb{C} \rightarrow \mathbb{C}$.*

The following terminology will be used in the rest of the paper.

Definition 5.10 ([BCO11]). The equivalence relation \sim_Q is called the *finest lamination (of Q)*. Let $p_{\sim_Q} : \mathbb{S}^1 \rightarrow \mathbb{S}^1 / \sim_Q$ be the canonical projection. Given $y \in Q$, let \sim_Q -class generated by y be the \sim_Q -class $p_{\sim_Q}^{-1}(\varphi_Q(y))$. Full preimages of points under φ_Q are called *fibers* of φ_Q .

If $Q = J(f)$ is the connected Julia set of a polynomial f of degree d , this construction is compatible with the dynamics of σ_d .

Definition 5.11 ([BCO11]). Set $\sim_{J(f)} = \sim_f$ and call it the *lamination generated by f* . Denote \mathbb{S}^1 / \sim_f by $J_{\sim}(f)$ and call it the *topological Julia set*. If f^* is a polynomial-like map, then, similarly, one defines the *lamination generated by f^** . Define a *geolamination \mathcal{L}_f* as follows: a chord ℓ is a leaf of \mathcal{L}_f if ℓ is a boundary chord of the convex hull of some \sim_f -class. Call \mathcal{L}_f the *geolamination generated by f* .

Theorem 5.12 shows that the finest map onto the connected Julia set preserves the dynamics.

Theorem 5.12 ([Kiw04, BCO11]). *If a polynomial f has connected Julia set, then the map $\varphi_{J(f)} : \mathbb{C} \rightarrow \mathbb{C}$ semiconjugates f and a branched covering map of the plane $f_{\sim} : \mathbb{C} \rightarrow \mathbb{C}$. On $J(f)$, the map $\varphi_{J(f)}$ collapses all impressions of angles to points; it is one-to-one on the boundaries of Fatou domains eventually mapped onto an attracting or parabolic periodic Fatou domain and maps these boundaries onto Jordan curves. Moreover, $\varphi_{J(f)}$ is one-to-one on the set of all (pre)periodic points $x \in J(f)$ such that the \sim_f -class generated by x is finite.*

Theorem 5.12 justifies the following terminology.

Definition 5.13. The map $f_\sim : \mathbb{C} \rightarrow \mathbb{C}$ from Theorem 5.12 is called a *topological polynomial*.

Note the following immediate consequence of Theorem 5.12: if points $\bar{\alpha}$ and $\bar{\beta}$ are periodic or preperiodic under σ_d and belong to the same finite \sim_f -class, then the rays $R_f(\alpha)$, $R_f(\beta)$ land at the same point. By [BCO11], if $\bar{\alpha}$ belongs to the \sim_f -class corresponding to a point $x \in J_\sim(f)$, then the impression of $R_f(\alpha)$ is contained in $\varphi_f^{-1}(x)$; otherwise, the impression of $R_f(\alpha)$ is disjoint from $\varphi_f^{-1}(x)$.

Infinite gaps of the geolamination \mathcal{L}_f may be of two types. Firstly, they may be convex hulls of infinite \sim_f -classes. Secondly, they may correspond to bounded Fatou components of f as follows. Let U be a bounded Fatou component of f such that φ_f does not collapse the boundary of U . Then φ_f maps the boundary of U to some Jordan curve S_U . For each $y \in S_U$ consider the convex hull in the unit disk of $p_{\sim_f}^{-1}(y)$. Then take the closure of the component of the unit disk minus the union of these convex hulls whose boundary intersects all of them; denote this closure by \mathfrak{U} . Then \mathfrak{U} is an infinite gap of \mathcal{L}_f corresponding to U . The correspondence between Fatou components of f and gaps of \mathcal{L}_f just described will be heavily used in the sequel.

Proposition 5.14. *A periodic fiber Q of φ_f that corresponds to an infinite \sim_f -class must contain a Cremer or a Siegel periodic point. Moreover, if Q has a repelling periodic cutpoint, then Q contains at least two irrationally indifferent periodic points.*

Proof. Suppose that Q contains neither Cremer nor Siegel periodic points. Since Q is a fiber of φ_f , by Theorem 5.12, the continuum Q cannot contain a Fatou domain eventually mapped to a parabolic or attracting periodic Fatou domain. Thus, Corollary 5.5 applies and shows that there are infinitely many points in Q , at which the landing rays rotate in a non-trivial fashion. However, by [BCO11, Proposition 40], the fiber Q can only contain a finite collection of periodic cutpoints. Finally, let x_0 be a repelling periodic cutpoint of Q . Then, arguing as above, we see that every component of $Q \setminus \{x_0\}$ contains a Cremer or a Siegel periodic point. This proves the last part of the proposition. \square

6. THE INVARIANT QUADRATIC GAP $\mathfrak{U}(f)$

In this section, we consider an immediately renormalizable polynomial $f \in \mathcal{F}_\lambda$ with $|\lambda| \leq 1$ (in particular, by Theorem 2.5 any polynomial f in the unbounded component of $\mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$ is immediately renormalizable), and define an invariant quadratic gap $\mathfrak{U}(f)$ associated with 0;

by $f^* : U^* \rightarrow V^*$, we denote the corresponding polynomial-like map. When $J(f)$ is disconnected, the gap that is the convex hull of all arguments of rays to $K(f^*)$ was introduced in Subsection 3.3 (though using different approach based upon the fact that $J(f)$ is disconnected in an essential way). Our approach here is necessarily different as we deal with immediately renormalizable polynomials and their quadratic-like Julia sets in both connected and disconnected cases. Once we introduce $\mathfrak{U}(f)$, it will be easy to see that the gap introduced in the disconnected case in Subsection 3.3 does coincide with $\mathfrak{U}(f)$.

Lemma 6.1. *Let f be a complex cubic polynomial with a non-repelling fixed point a . Suppose that there exists a quadratic-like Julia set J^* such that $a \in J^*$. Then J^* is unique.*

Proof. Suppose that U_1 and U_2 are Jordan disks such that $f_1^* = f|_{U_1}$ and $f_2^* = f|_{U_2}$ are quadratic-like maps with filled Julia sets K_1^*, K_2^* , both containing the non-repelling fixed point a . Observe that, by definition, the map $f : U_1 \cap U_2 \rightarrow f(U_1 \cap U_2)$ is proper. Let U be the component of $U_1 \cap U_2$ containing a . Then $f : U \rightarrow f(U)$ is also proper. Moreover, by definition $f(\text{Bd}(U)) \cap \bar{U} = \emptyset$. Since $f(a) = a$, it follows from the Maximum Modulus Principle that $\bar{U} \subset f(U)$. Replacing $f(U)$ with a slightly smaller Jordan disk V and redefining U as the component of $f^{-1}(V)$ containing a , we obtain a polynomial-like map $f^* = f|_U$.

If f^* is univalent, then, by the Schwarz Lemma, a is repelling, a contradiction. Hence the degree of f^* is greater than one, and hence equals two. Then, by Theorem 5.11 from [McM94], it follows that $J_1^* = J_2^*$. \square

To define the quadratic invariant gap $\mathfrak{U}(f)$ associated with $K(f^*)$, we use (pre)periodic points of f . Observe that, by definition, $K(f^*)$ is a component of $f^{-1}(K(f^*))$.

Definition 6.2. Let $\widehat{X}(f) = \widehat{X}$ be the set of all σ_3 -(pre)periodic points $\bar{\alpha} \in \mathbb{S}^1$ such that $R_f(\alpha)$ lands in $K(f^*)$. Let $X(f) = X$ be the closure of \widehat{X} . Let $\mathfrak{U}(f)$ be the convex hull of X . Let $\widetilde{K}(f^*)$ be the component of $f^{-1}(K(f^*))$ different from $K(f^*)$ (such a component of $f^{-1}(K(f^*))$ exists because $f|_{K(f^*)}$ is two-to-one). Let $Y(f) = Y$ be the closure of the set of all preperiodic points $\bar{\alpha} \in \mathbb{S}^1$ with $R_f(\alpha)$ landing in $\widetilde{K}(f^*)$.

The gap $\mathfrak{U}(f)$ is the combinatorial counterpart of the set $K(f^*)$. Clearly, $X \neq \emptyset$ and the map $\sigma_3 : X \rightarrow X$ is such that any point of X is the image of two or three different points of X . Denote the length

of any circle arc T by $|T|$. The lengths of circle arcs are measured in radians/ 2π so that the total length of the unit circle is equal to 1.

Lemma 6.3. *The map $\sigma_3|_{\widehat{X}}$ is exactly two-to-one. Moreover, Y lies in the closure of a hole $I = (\overline{\alpha}, \overline{\beta})$ of X .*

Proof. The set Y lies in the closure $[\overline{\alpha}, \overline{\beta}]$ of a hole $I = (\overline{\alpha}, \overline{\beta})$ of X because otherwise $K(f^*)$ and $\widetilde{K}(f^*)$ are non-disjoint. Let $\overline{\theta} \in \widehat{X}$ and let $z \in K(f^*)$ be the landing point of $R_f(\theta)$. If z is critical, then we may assume that $R_f(\theta + \frac{1}{3})$ lands at z , and hence $\overline{\theta + \frac{1}{3}} \in \widehat{X}$; if z is not critical, then we can find a point $z' \in K(f^*)$ with the same image as z and may again assume that, say, $R_f(\theta + \frac{1}{3})$ lands at z' , and hence $\overline{\theta + \frac{1}{3}} \in \widehat{X}$. Thus the map $\sigma_3|_{\widehat{X}}$ is at least two-to-one. Now, suppose that $\theta, \theta + \frac{1}{3}, \theta + \frac{2}{3} \in \widehat{X}$. Let $z \in K(f^*)$ be the landing point of $R_f(\theta)$. Choose a point $\tilde{z} \in \widetilde{K}(f^*)$ with the same image as z ; then there exists $\overline{\theta} \in Y$ such that $R_f(\overline{\theta})$ lands at \tilde{z} . Thus, $\overline{\theta} \notin \widehat{X}$ whereas $\overline{\theta}$ must coincide with either $\overline{\theta + \frac{1}{3}}$ or $\overline{\theta + \frac{2}{3}}$, a contradiction. \square

From now on, in the setting when f is immediately renormalizable, $I = (\overline{\alpha}, \overline{\beta})$ is the hole of X whose closure $[\overline{\alpha}, \overline{\beta}]$ contains the set Y .

Lemma 6.4. *We have that $\widetilde{K}(f^*)$ is disjoint from U^* . The gap $\mathfrak{U}(f)$ is an invariant quadratic gap of regular critical or periodic type, and I is its major hole.*

Proof. The fact that $\widetilde{K}(f^*)$ is disjoint from U^* follows because otherwise points of $\widetilde{K}(f^*) \cap U^*$ must belong to $K(f^*)$, a contradiction. Suppose that $|I| < \frac{1}{3}$. If $\overline{\alpha} \in \widehat{X}$, then points $\overline{\alpha} + \frac{1}{3}, \overline{\alpha} + \frac{2}{3}$ belong to $(\overline{\beta}, \overline{\alpha})$ and cannot belong to $Y \subset [\overline{\alpha}, \overline{\beta}]$. Hence in this case three points in \widehat{X} have the same σ_3 -image, a contradiction with Lemma 6.3. If $\overline{\alpha} \notin \widehat{X}$, we can find a point $\overline{\theta} \in \widehat{X}$ so close to α that points $\overline{\theta}, \overline{\theta} + \frac{1}{3}, \overline{\theta} + \frac{2}{3}$ belong to $(\overline{\beta}, \overline{\alpha})$, which similarly leads to a contradiction. Hence $|I| \geq \frac{1}{3}$.

We claim that if $|I| = \frac{1}{3}$, then neither $\overline{\alpha}$ nor $\overline{\beta}$ is periodic. Otherwise let $\overline{\alpha}$ be periodic and set $\overline{\alpha\beta} = \mathbf{c}$. Since X is invariant, $X \subset \mathfrak{U}(\mathbf{c})$. By [BOPT13] (see Subsection 2.3), the gap $\mathfrak{U}(\mathbf{c})$ is of caterpillar type, $\overline{\beta}$ is isolated in $\mathfrak{U}(\mathbf{c})$ and thus isolated in X . Thus, $\overline{\beta} \in \widehat{X}$, which implies that $\overline{\alpha} \in \widehat{X}$ by definition and because an eventual σ_3 -image of $\overline{\beta}$ equals $\overline{\alpha}$. Since $\overline{\alpha} + \frac{2}{3} \in (\overline{\beta}, \overline{\alpha})$, we have $\overline{\alpha} + \frac{2}{3} \notin Y$, and hence the landing point of $R_f(\overline{\alpha} + \frac{2}{3})$ must also belong to $K(f^*)$. Therefore, $\overline{\alpha} + \frac{2}{3} \in \widehat{X}$, a contradiction with Lemma 6.3. We conclude that if $|I| = \frac{1}{3}$, then $\mathfrak{U}(f)$ is of regular critical type.

Assume that $|I| > \frac{1}{3}$. Choose a critical chord \mathbf{c} with endpoints in $[\bar{\alpha}, \bar{\beta}]$; clearly, we can always choose \mathbf{c} to be a critical chord with no periodic endpoint. Then $X \subset \mathfrak{U}'(\mathbf{c})$, where $\mathfrak{U}(\mathbf{c})$ is the invariant quadratic gap defined in Subsection 2.3; by [BOPT13], the set $\mathfrak{U}'(\mathbf{c})$ has no isolated points, and, for any point $\bar{\gamma} \in \mathfrak{U}'(\mathbf{c})$, the backward orbit of $\bar{\gamma}$ in $\mathfrak{U}'(\mathbf{c})$ is dense in $\mathfrak{U}'(\mathbf{c})$ (see Subsection 2.3 for details). Hence $X = \mathfrak{U}'(\mathbf{c})$ is a quadratic invariant gap of periodic type as desired. \square

One-sided impressions were introduced in Definition 5.4.

Proposition 6.5. *If f is a cubic polynomial with connected Julia set $J(f)$ then one-sided impressions are subcontinua of $J(f)$.*

Proof. Let us show that $\text{Imp}(\gamma^-)$ is a continuum. Recall that ϕ_f is the Böttcher coordinate on $\mathbb{C} \setminus K(f)$. For $\varepsilon > 0$ and $r > 1$, let $T(\varepsilon, r)$ be the closure of the set $\phi_f^{-1}(S(\varepsilon, r))$, where $S(\varepsilon, r)$ consists of all points $\rho e^{2\pi i\theta}$ with $\rho < r$ and $\theta \in (\gamma - \varepsilon, \gamma)$. Clearly, each set $T(\varepsilon, r)$ is a continuum. By definition, $\text{Imp}(\gamma^-)$ is the intersection of a nested family of sets $T(\varepsilon, r)$ taken for $\varepsilon \rightarrow 0$ and $r \rightarrow 1$. It follows that $\text{Imp}(\gamma^-)$ is a continuum. Moreover, since points of such closures need more and more time to escape to infinity as $r \rightarrow 0$, it follows that $\text{Imp}(\gamma^-) \subset J(f)$, as desired. \square

Proposition 6.6. *Suppose that $f \in \mathcal{F}_\lambda$ with $|\lambda| \leq 1$ has a disconnected Julia set. Let $\bar{\alpha}\bar{\beta}$ be any edge of $\mathfrak{U}(f)$. Then the principal sets of $R^e(\alpha^-)$ and $R^e(\beta^+)$ intersect. Moreover, if α and β are periodic, then the rays $R^e(\alpha^-)$ and $R^e(\beta^+)$ land at the same point of $K(f^*)$.*

Proof. Recall that $f^* : U^* \rightarrow V^*$ is a quadratic-like restriction of f . Let $\psi : \mathbb{C} \setminus K(f^*) \rightarrow \mathbb{C} \setminus \mathbb{D}$ be a conformal isomorphism. By way of contradiction let us assume that the principal sets of $R^e(\alpha^-)$ and $R^e(\beta^+)$ are disjoint.

Since $J(f)$ is disconnected, the set $K(f^*)$ is a component of the filled Julia set $K(f)$. Then $R_f^e(\alpha^-)$, $R_f^e(\beta^+)$ correspond to different quadratic-like rays $R^*(\alpha)$ and $R^*(\beta)$ (recall that, by Theorem 2.10, for every external ray R^e to $J(f^*)$, there exists a unique quadratic-like ray R^* such that $\text{Pr}(R^e) = \text{Pr}(R^*)$, and these rays are homotopic in $\mathbb{C} \setminus K(f^*)$ among all curves with the same limit set). We can define a quasi-conformal isomorphism ψ^* between $U^* \setminus K(f^*)$ and some topological annulus A , whose inner boundary is the unit circle.

The map ψ^* is the composition of the straightening map (taking U^* to some Jordan neighborhood of $K(g)$, where g is the quadratic polynomial hybrid equivalent to f^*) and the Böttcher coordinate for g . The quadratic-like rays $R^*(\alpha)$ and $R^*(\beta)$ are mapped to different

radial rays under ψ^* . These two radial rays disconnect the annulus A . Each complementary component has infinitely many radial rays with periodic arguments. The associated quadratic-like rays correspond to external rays (by Theorem 2.10) that must all land in $K(f^*)$, a contradiction with the definition of $\mathfrak{U}(f)$. Finally, if an argument of a one-sided external ray is periodic, then the one-sided ray in question must land. This implies the last claim of the lemma. \square

In the rest of Section 6, we suppose that $J(f)$ is connected. Let us relate the gap $\mathfrak{U}(f)$ and the lamination \sim_f . For an edge $\overline{\alpha\beta}$ of $\mathfrak{U}(f)$, we let $\Delta(\alpha, \beta)$ stand for the set

$$\text{Imp}(\alpha^-) \cup \text{Imp}(\beta^+) \cup R_f^e(\alpha^-) \cup R_f^e(\beta^+).$$

This set is clearly a closed set.

Let us now quote Theorem 6.7, which is proven in [BOPT16] and is similar to Theorem 2.10 but applicable to the case when the Julia set is connected. Given a set $T \subset \mathbb{S}^1$ and a map $h : T \rightarrow \mathbb{S}^1$ we say that h is *monotone extendable* if it has a monotone extension $m : \mathbb{S}^1 \rightarrow \mathbb{S}^1$.

Theorem 6.7 ([BOPT16], Lemma 4.9). *Let a polynomial f of degree d have a connected Julia set and $K(f^*) \subset K(f)$ be a polynomial-like connected filled Julia set of degree k . Let $\mathcal{B}_f(K(f^*)) = \mathcal{B}$ be the set of all angles $\theta \in \mathbb{S}^1$ such that the external ray $R_f^e(\theta)$ lands in $J(f^*)$. Then there is a monotone extendable continuous map $\psi : \mathcal{B} \rightarrow \mathbb{S}^1$ such that:*

- (1) *the set of all (pre)periodic angles from $\psi(\mathcal{B})$ is dense in \mathbb{S}^1 ;*
- (2) *for every $\theta \in \mathcal{B}$, the polynomial-like ray $R^*(\psi(\theta))$ lands at the same point y_θ as $R_f^e(\theta)$ and is homotopic to $R_f^e(\theta)$ outside of $K(f^*)$ under a homotopy that fixes y_θ ;*
- (3) *we have $\psi \circ \sigma_d(\theta) = \sigma_k \circ \psi(\theta)$ for every $\theta \in \mathcal{B}$.*

By Theorem 6.7, the ψ -images of σ_d -(pre)periodic points of \mathcal{B} are σ_k -(pre)periodic. Since for any connected polynomial Julia set J a dense in \mathbb{S}^1 set of (pre)periodic angles gives rise to a dense in J set of their landing points, it follows that the set of landing points of all (pre)periodic angles from \mathcal{B} is dense in $J(f^*)$.

Proposition 6.8. *If $\overline{\alpha\beta}$ is an edge of $\mathfrak{U}(f)$, then $\text{Imp}(\alpha^-) \cap \text{Imp}(\beta^+) \cap J(f^*)$ is non-empty. Moreover, either $J(f^*) \subset \text{Imp}(\alpha^-) \cup \text{Imp}(\beta^+)$ or, otherwise, $J(f^*) \setminus [\text{Imp}(\alpha^-) \cup \text{Imp}(\beta^+)]$ is contained in the unbounded component of $\mathbb{C} \setminus \Delta(\alpha, \beta)$ containing external rays with arguments from (β, α) . In particular, $\overline{\alpha} \sim_f \overline{\beta}$.*

Recall that, since $J(f)$ is connected, the lamination \sim_f generated by f is defined. Unshielded compacta A and their topological hulls $\text{Th}(A)$

were defined in Subsection 2.2. The topological hull of a compact unshielded subset of the plane is the union of topological hulls of its components.

Proof. Set $T = \text{Imp}(\alpha^-) \cap J(f^*)$. We claim that the topological hull $\text{Th}(T) = T'$ of T coincides with the intersection of the topological hull $\text{Th}(\text{Imp}(\alpha^-))$ of $\text{Imp}(\alpha^-)$, and $K(f^*)$. Indeed, let S be a bounded complementary component of T . Then $S \subset \text{Th}(\text{Imp}(\alpha^-)) \cap K(f^*)$. Hence $T' \subset \text{Th}(\text{Imp}(\alpha^-)) \cap K(f^*)$. On the other hand, if $z \in \text{Th}(\text{Imp}(\alpha^-)) \cap K(f^*)$, then either $z \in T$, or z belongs to a Fatou domain of $J(f)$ that is also a bounded complementary domain of both $\text{Imp}(\alpha^-)$ and $K(f^*)$. In either case $z \in T'$ as desired. Thus, $T' = \text{Th}(\text{Imp}(\alpha^-)) \cap K(f^*)$.

We claim that T is a continuum. Recall that one-sided impressions are continua. We will use Theorem 6.7, including the notation from that theorem. Observe that any bounded complementary component of $J(f^*)$ is a Fatou component of $J(f)$; similarly, any bounded complementary component of $\text{Imp}(\alpha^-)$ is a Fatou component of $J(f)$. Therefore, bounded complementary components of either $J(f^*)$ or $\text{Imp}(\alpha^-)$ are disjoint from $J(f^*) \cup \text{Imp}(\alpha^-)$. Moreover, if a bounded complementary domain of $\text{Imp}(\alpha^-)$ and a bounded complementary domain of $J(f^*)$ are non-disjoint, then they coincide.

If T is disconnected, then T' is disconnected too. Divide the components of T' in two groups whose unions A and B are disjoint non-separating compact sets. Choose open Jordan disks $U \supset A$ and $V \supset B$ such that \bar{U} and \bar{V} are disjoint closed Jordan disks. Then, by Moore's theorem [Dav86], a map Ψ that collapses \bar{U} and \bar{V} maps \mathbb{C} to a space homeomorphic to the plane, in which the continuum $\Psi(\text{Th}(\text{Imp}(\alpha^-)))$ and the continuum $\Psi(K(f^*))$ intersect over a two-point set. By Theorem 61.4 from [Mun00], this implies that $\Psi(\text{Th}(\text{Imp}(\alpha^-))) \cup \Psi(K(f^*))$ is a separating continuum, which implies that $T' = \text{Th}(\text{Imp}(\alpha^-)) \cap K(f^*)$ is a separating continuum. Since both $\text{Th}(\text{Imp}(\alpha^-))$ and $K(f^*)$ are non-separating, there exists a bounded complementary component W of T' and an open (in relative topology of $\text{Bd}(W)$) set $E \subset \text{Bd}(W) \cap K(f^*)$. Then, by Theorem 6.7, we can find a (pre)periodic angle $\gamma \in \mathcal{B}$ such that the (pre)periodic polynomial-like ray $R^*(\psi(\gamma))$ has a terminal interval contained in W , lands in E and is homotopic to the external ray $R_f^e(\gamma)$ by a homotopy outside $K(f^*)$ fixing its landing point, a contradiction. Thus, T is connected. Similarly, $\text{Imp}(\beta^+) \cap J(f^*)$ is a continuum.

By way of contradiction, assume that the continua $\text{Imp}(\alpha^-) \cap J(f^*)$ and $\text{Imp}(\beta^+) \cap J(f^*)$ are disjoint. By choosing U^* sufficiently tight

around $K(f^*)$ we may assume that there are Jordan disks V_α, V_β with $\text{Imp}(\alpha^-) \cap J(f^*) \subset V_\alpha \subset \overline{V_\alpha} \subset U^*$, $\text{Imp}(\beta^+) \cap J(f^*) \subset V_\beta \subset \overline{V_\beta} \subset U^*$

such that $\overline{V_\alpha} \cap \overline{V_\beta} = \emptyset$. By Theorem 6.7, we can find a pair of (pre)periodic angles $\psi(\gamma), \psi(\eta)$ in $\psi(\mathcal{B})$ such that $R^*(\psi(\gamma)) \cup R^*(\psi(\eta)) \cup J(f^*)$ separates $\overline{V_\alpha} \setminus K(f^*)$ from $\overline{V_\beta} \setminus K(f^*)$. Then $R_f(\gamma) \cup R_f(\eta) \cup J(f^*)$ also separates $\overline{V_\alpha} \setminus K(f^*)$ from $\overline{V_\beta} \setminus K(f^*)$. It follows that $\{\gamma, \eta\}$ separates $\{\alpha, \beta\}$ in \mathbb{R}/\mathbb{Z} . Thus γ or η belongs to (α, β) ; assume that it is γ . The ray $R_f(\gamma)$ is periodic and lands in $J(f^*)$, a contradiction with the definition of $\mathfrak{U}(f)$.

Let us prove the remaining claims of the lemma. If $J(f^*)$ has any points outside of $\text{Imp}(\alpha^-) \cup \text{Imp}(\beta^+)$, then these points cannot belong to the unbounded component of $\mathbb{C} \setminus \Delta(\alpha, \beta)$ that contains external rays with arguments from (α, β) . On the other hand, no point of $J(f^*)$ can belong to a bounded component of $\mathbb{C} \setminus \Delta(\alpha, \beta)$ because otherwise some points of $J(f^*)$ do not belong to the closure of $\mathbb{C} \setminus K(f)$. Hence $J(f^*) \setminus (\text{Imp}(\alpha^-) \cup \text{Imp}(\beta^+))$ is contained in the unbounded component of $\mathbb{C} \setminus \Delta(\alpha, \beta)$ containing external rays with arguments from (β, α) .

The last claim of the lemma follows from the fact that $\text{Imp}(\alpha^-) \cap \text{Imp}(\beta^+) \neq \emptyset$ and from [BCO11, Kiw04]. \square

7. PROOF OF THEOREM C

This section is mainly devoted to a proof of Theorem C; along the way we also complete the proof of Theorem B and show that root polynomials belong to \mathcal{CU}_λ . First we show that parameter wakes in \mathcal{F}_λ are disjoint from \mathcal{CU}_λ where $|\lambda| \leq 1$. Recall that if f is immediately renormalizable, then we denote by f^* its quadratic-like restriction upon the appropriate planar domain U^* .

7.1. Parameter wakes are disjoint from \mathcal{CU} . In this subsection, λ is a fixed complex number such that $|\lambda| \leq 1$. Recall, that the *main cuboid* \mathcal{CU} is the set of classes $[f] \in \mathcal{M}_3$ with the following properties: f has a non-repelling fixed point, f has no repelling periodic cutpoints in $J(f)$, and all non-repelling periodic points of f , except at most one fixed point, have multiplier 1. Recall that \mathcal{CU}_λ is the set of all polynomials $f \in \mathcal{F}_\lambda$ with $[f] \in \mathcal{CU}$. We will need Theorem 5.3 when proving that certain polynomials $f \in \mathcal{F}_\lambda$ do not belong to \mathcal{CU}_λ .

We will prove that all parameter wakes in the λ -slice \mathcal{F}_λ are disjoint from \mathcal{CU}_λ , for $\lambda \neq 1$. Recall, that the λ -connectedness locus \mathcal{C}_λ is the set of all $f \in \mathcal{F}_\lambda$ such that $K(f)$ is connected. A *limb* of \mathcal{C}_λ is the intersection of \mathcal{C}_λ with a parameter wake.

Lemma 7.1. *Suppose that f lies in a limb of \mathcal{C}_λ . Then either f has a repelling periodic cutpoint in $K(f)$, or f belongs to a special wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ of some period k , the dynamic wedge $W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$ contains a non-repelling periodic point $x \neq 0$ of period k of multiplier different from 1 such that for every i , $0 < i < k$, the point $f^{\circ i}(x)$ belongs to the dynamic wedge $W_f(\sigma_3^{\circ i}(\theta_1), \sigma_3^{\circ i}(\theta_2))$.*

Recall that $W_f(\alpha, \beta)$ is the wedge bounded by the dynamic rays $R_f(\alpha)$, $R_f(\beta)$ and the point z on which they land or crash; it contains external rays with arguments from (α, β) .

Proof. By our assumptions, the filled Julia set $K(f)$ is connected, and f lies in some parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ of some period k . The rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ land at the point x_1 , which is either a repelling periodic point (if the parameter wake is non-special), or the parabolic point 0 (if the parameter wake is special). If the parameter wake is non-special, we are done. Hence we may assume that the parameter wake is special. Set $X_1 = \overline{W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})} \cap K(f)$. Then X_1 is a special piece for $f^{\circ k}$ with exit continuum $\{0\}$. By Theorem 5.3 applied to X_1 and $f^{\circ k}$, the set X_1 contains either (1) a non-repelling $f^{\circ k}$ -fixed point in X_1 , whose multiplier is different from 1, or (2) an $f^{\circ k}$ -invariant parabolic domain with the $f^{\circ k}$ -fixed point on its boundary and multiplier 1, or (3) a repelling $f^{\circ k}$ -fixed cutpoint of X_1 . Observe that case (3) is impossible because we assume that f does not have repelling periodic cutpoints.

Let us show that case (2) is impossible. Suppose that there is an $f^{\circ k}$ -invariant parabolic domain Ω in X_1 . Let x be the corresponding parabolic $f^{\circ k}$ -fixed point of multiplier 1. Since the parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ is special, then, by Theorem 4.7, we have $x \neq 0$. Since the multiplier at x is 1, there exists an external $f^{\circ k}$ -fixed ray $R_f(\gamma)$ landing at x with $\gamma \in (\theta_1, \theta_2)$. We claim that this is impossible for combinatorial reasons. Indeed, the arc $[\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3}]$ maps onto its image under the angle tripling map so that the arc $[3\theta_1, 3\theta_2]$ is covered twice while the rest of the circle is covered exactly once. Since the map is two-to-one on Ω , it follows that all external rays landing at points from Ω must have arguments that map to $[3\theta_1, 3\theta_2]$. However, a simple analysis shows that the only two fixed angles under the k th iterate of the angle tripling map in $[3\theta_1, 3\theta_2]$ are $3\theta_1$ and $3\theta_2$ themselves, a contradiction with the existence of γ .

Finally, consider case (1). Then since x is non-repelling, then by Theorem 4.3 from [BCLOS10] there exists a critical point d such that no (pre)periodic cut separates d and x . Hence x belongs to the strip bound by the two cuts: the cut formed by the rays $R_f(\theta_1 + \frac{2}{3})$, $R_f(\theta_2 +$

$\frac{2}{3}$) (these rays land on the point 0) and the cut formed by the rays $R_f(\theta_2 + \frac{1}{3}), R_f(\theta_1 + \frac{2}{3})$ (these rays land at the preimage of 0 not equal to 0). It follows that for every $i, 0 < i < k$, the point $f^{oi}(x)$ belongs to the dynamic wedge $W_f(\sigma_3^{oi}(\theta_1), \sigma_3^{oi}(\theta_2))$ as claimed. \square

We are now ready to prove the following theorem.

Theorem 7.2. *Parameter wakes in \mathcal{F}_λ , where $\lambda \neq 1$, are disjoint from \mathcal{CU}_λ .*

Proof. Suppose that $\lambda \neq 1$. Let $f \in \mathcal{W}_\lambda(\theta_1, \theta_2)$. If $K(f)$ is disconnected, then, by definition, $f \notin \mathcal{CU}_\lambda$. Suppose that $K(f)$ is connected. By Lemma 7.1, the map f has a non-repelling periodic point $x \neq 0$, whose multiplier is different from 1, or a periodic repelling cutpoint in $K(f)$. In the latter case, we have (by definition of \mathcal{CU}_λ) that $f \notin \mathcal{CU}_\lambda$. In the former case, since $\lambda \neq 1$, we again see that, by definition of \mathcal{CU}_λ , we have $f \notin \mathcal{CU}_\lambda$. \square

More details on the mutual location of \mathcal{CU}_1 and the wakes in \mathcal{F}_1 are given in Section 7.4.

7.2. Properties of polynomials from $\mathcal{C}_\lambda \setminus \mathcal{CU}_\lambda$. We need the *Separation Lemma* of Jan Kiwi [Kiw00] inspired by [GM93]; a convenient version is given below (in fact, a stronger statement is proved in [Kiw00]).

Lemma 7.3 (Separation Lemma [Kiw00]). *Suppose that f is a polynomial with connected Julia set. Then there is a finite collection of periodic dynamic rays for f (landing at periodic repelling or parabolic points) such that the closure of the union of these rays divides the plane into parts with the following property: every part contains at most one non-repelling periodic point of f or one parabolic basin of f .*

A cut formed by periodic dynamic rays from Lemma 7.3 will be called a *K-cut*. Let Υ be the closure of the union of the periodic rays from Lemma 7.3; every component of $\mathbb{C} \setminus \Upsilon$ can contain at most one non-repelling periodic point of f or parabolic basin of f . Let $x \neq y$ be periodic non-repelling points of f and let Γ be a K-cut. Say that Γ *separates domains at x* from y if Γ separates x and y and, if x is attracting or parabolic, then Γ separates all attracting or parabolic domains at x from y .

We need a few laminational results from [BOPT13].

Lemma 7.4 ([BOPT13]). *If a geolamination \mathcal{L} has an invariant quadratic gap \mathfrak{U} of regular critical type, then, for every chord $\overline{\alpha\beta}$ of the unit disk such that $\sigma_3^{on}(\overline{\alpha\beta})$ does not cross an edge of \mathfrak{U} for all n , we have that $\overline{\alpha\beta}$ is eventually mapped to a subset of \mathfrak{U} .*

We will use the following corollary of Lemma 7.4.

Corollary 7.5. *If $f \in \mathcal{C}_\lambda$ with $|\lambda| \leq 1$ is immediately renormalizable, then there are no cuts $\Gamma(\overline{\alpha\beta})$ with (pre)periodic vertex x such that $\overline{\alpha\beta}$ crosses an edge of $\mathfrak{U}(f)$. Moreover, if there exists a periodic cut with vertex $x \neq 0$, then $\mathfrak{U}(f)$ is of periodic type, and, for any periodic cut $\Gamma(\overline{\alpha\beta})$ with vertex x , the chord $\overline{\alpha\beta}$ does not cross the siblings of the major \mathfrak{M} of $\mathfrak{U}(f)$.*

Proof. Suppose that there exists a cut $\Gamma(\overline{\alpha\beta})$ with (pre)periodic vertex x such that $\overline{\alpha\beta}$ crosses an edge of $\mathfrak{U}(f)$. Then, by definition, there are points of $J(f^*)$ in either component of $\mathbb{C} \setminus \Gamma(\overline{\alpha\beta})$, which implies that $x \in J(f^*)$, and hence $\overline{\alpha}, \overline{\beta} \in \widehat{X} \subset \mathfrak{U}'(f)$, a contradiction (indeed, by the assumption at least one of $\overline{\alpha}, \overline{\beta}$ must not belong to $\mathfrak{U}'(f)$).

Suppose now that $x \neq 0$ and that $\mathfrak{U}(f)$ is of regular critical type. Let \mathfrak{h} be the convex hull of the finite set $\mathfrak{h}' \subset \mathbb{S}^1$ of points $\overline{\gamma}$, where γ are arguments of rays landing at x . By the previous paragraph, images of \mathfrak{h} do not cross edges of $\mathfrak{U}(f)$. Then, by Lemma 7.4, the set \mathfrak{h} is eventually mapped inside $\mathfrak{U}(f)$. However, no image of x can be a cutpoint of $J(f^*)$ since $|\lambda| \leq 1$ and, hence, there are no periodic cutpoints of $J(f^*)$ different from 0. This contradiction proves the second claim.

Now, let $\Gamma(\alpha\beta)$ be a periodic cut with vertex x ; set $\ell = \overline{\alpha\beta}$. By the above, no image of ℓ crosses edges of \mathfrak{U} . Suppose that an image $\tilde{\ell}$ of ℓ crosses the sibling \mathfrak{M}^* of \mathfrak{M} that is disjoint from \mathfrak{U} . If $\tilde{\ell}$ is disjoint from \mathfrak{U} then $\sigma_3(\tilde{\ell})$ crosses an edge of \mathfrak{U} , a contradiction. Otherwise an endpoint of $\tilde{\ell}$ must coincide with an endpoint of \mathfrak{M} . Then either $\sigma_3(\tilde{\ell}) \subset \mathfrak{U}$, and all leaves from the orbit $\tilde{\ell}$ are contained in \mathfrak{U} , a contradiction (ℓ is periodic!), or $\sigma_3(\tilde{\ell})$ crosses an edge of \mathfrak{U} , again a contradiction. Hence no image of ℓ crosses \mathfrak{M}^* as desired. \square

Below, we prove a sequence of lemmas in which we study polynomials $f \in \mathcal{C}_\lambda \setminus \mathcal{CU}_\lambda$. Our aim is to prove an analog of Proposition 6.6. We will do this in a step by step fashion. Theorem 2.10 and other tools applicable in the disconnected case do not apply anymore. Instead we have to use various planar tools combining continuum theory and dynamics of (geo)laminations.

We now investigate the situation, where $\mathfrak{U}(f)$ is of periodic type.

Definition 7.6 (cf. [BOPT13]). Let \mathfrak{U} be a quadratic invariant gap of periodic type. Denote its major by $\overline{\theta_1 + \frac{1}{3}\theta_2 + \frac{2}{3}} = \mathfrak{M}$ and assume that it is of period k with sibling $\overline{\theta_2 + \frac{1}{3}\theta_1 + \frac{2}{3}} = \mathfrak{M}^*$. Let \mathfrak{Y}' be the set of all points $\overline{\alpha}$ of the circle such that, for every n , we have $3^n\alpha \in [3^n(\theta_1 +$

$\frac{1}{3}$), $3^n(\theta_2 + \frac{2}{3})$], and let \mathfrak{V} be the convex hull of \mathfrak{V}' . In this setting, \mathfrak{U} is called the *senior gap* while \mathfrak{V} is called the *vassal gap* (of \mathfrak{U}). If $\mathfrak{U} = \mathfrak{U}(f)$ is defined by an immediately renormalizable polynomial $f \in \mathcal{C}_\lambda$, we also consider the continuum $I(f) = I = \text{Imp}^-(\theta_1 + \frac{1}{3}) \cup \text{Imp}^+(\theta_2 + \frac{2}{3})$ (the fact that I is a continuum follows from Proposition 6.8).

In Theorem 7.7, specific (geo)laminations with a quadratic invariant gap \mathfrak{U} of periodic type are studied. Recall that, given a geolamination \mathcal{L} , the map σ_3 extends to all leaves of \mathcal{L} so that leaves are mapped to leaves or points in \mathbb{S}^1 . We also extend σ_3 to all gaps of \mathcal{L} so that gaps are mapped to gaps, leaves or points and so that the obtained extension is a continuous self-map of the closed unit disk.

Theorem 7.7 ([BOPT13]). *There exists a unique invariant lamination $\sim_{\mathfrak{U}}$ with a given senior gap \mathfrak{U} of periodic type and such that the vassal gap $\text{CH}(\mathfrak{V}') = \mathfrak{V}$ of period k is a gap of $\sim_{\mathfrak{U}}$. The gap \mathfrak{V} maps two-to-one onto itself under $\sigma_3^{3^k}$, all other gaps of $\sim_{\mathfrak{U}}$ are one-to-one pullbacks of \mathfrak{U} or \mathfrak{V} and all leaves of the corresponding geolamination $\mathcal{L}_{\mathfrak{U}}$ are edges of these gaps. Also, there are no points of period k located between the major \mathfrak{M} and its sibling \mathfrak{M}^* except for the endpoints of \mathfrak{M} .*

Theorem 7.7 implies the following corollary.

Corollary 7.8. *Let \mathfrak{U} be a senior gap of periodic type with major \mathfrak{M} and vassal gap \mathfrak{V} . Let ℓ be a chord. Then one of the following holds.*

- (1) *An eventual σ_3 -image of ℓ crosses an edge of \mathfrak{U} .*
- (2) *An eventual σ_3 -image of ℓ is contained in \mathfrak{U} .*
- (3) *An eventual σ_3 -image of ℓ separates \mathfrak{M} from \mathfrak{M}^* in \mathbb{D} .*

Proof. Set $\ell = \overline{\alpha\beta}$. Assume that neither (1) nor (2) holds. Let an eventual image of ℓ intersect \mathfrak{U} . Then, by the assumption, this image is a chord having a common endpoint with an edge of \mathfrak{U} but otherwise located outside \mathfrak{U} . Suppose that $\mathfrak{M} = \overline{\theta_1 + \frac{1}{3}\theta_2 + \frac{2}{3}}$. By properties of senior gaps of periodic type, we may assume that $\alpha = \theta_1 + \frac{1}{3}$ and $\theta_1 + \frac{1}{3} < \beta < \theta_2 + \frac{2}{3}$ while, by our assumptions, no image of ℓ has an endpoint $\theta_1 + \frac{2}{3}$ or an endpoint $\theta_2 + \frac{1}{3}$. This implies that an eventual image $\tilde{\ell}$ of ℓ will have one endpoint $\theta_1 + \frac{1}{3}$ and the other endpoint belonging to $(\theta_2 + \frac{1}{3}, \theta_1 + \frac{2}{3})$. It follows that $\sigma_3^{3^k}(\tilde{\ell})$ will have an endpoint $\theta_1 + \frac{1}{3}$ and the other endpoint in $(\theta_2 + \frac{2}{3}, \theta_1 + \frac{1}{3})$, which implies that either (1) or (2) must hold, a contradiction. Thus we may assume that eventual images of ℓ are disjoint from \mathfrak{U} .

We claim that then (3) holds. Indeed, suppose first that ℓ is eventually mapped inside \mathfrak{V} . Then the claim follows from the fact that

$\sigma_3^{\circ k}|_{\mathfrak{W}}$ is semiconjugate to σ_2 and the well-known (and easy) fact that an eventual σ_2 -image of any non-degenerate chord separates 0 and $\frac{1}{2}$ or coincides with $\overline{0\frac{1}{2}}$. Suppose that ℓ is never mapped inside \mathfrak{W} . Combining this and the fact that all images of ℓ are disjoint from \mathfrak{U} we see that ℓ crosses a leaf ℓ_1 of $\mathcal{L}_{\mathfrak{U}}$. If no image of ℓ separates \mathfrak{M} and \mathfrak{M}' , then this implies that the images of ℓ continue crossing the corresponding images of ℓ_1 . However, ℓ_1 is eventually mapped to \mathfrak{M} , a contradiction. \square

Lemma 7.9 studies periodic cuts of f .

Lemma 7.9. *Suppose that $f \in \mathcal{C}_\lambda$ is immediately renormalizable, and that $\mathfrak{U}(f) = \mathfrak{U}$ is of periodic type with major $\mathfrak{M} = \overline{\theta_1 + \frac{1}{3}\theta_2 + \frac{2}{3}}$ of period k . Suppose that there exists a periodic cut $\Gamma(\ell_1) = \Gamma$ with vertex $\neq 0$ corresponding to a chord ℓ_1 that does not belong to the orbit of \mathfrak{M} . Then there is a cut $\Gamma(\ell_2)$ such that ℓ_2 is separated from \mathfrak{U} by \mathfrak{M} and such that $\sigma_3^{\circ k}(\ell_2)$ is separated from \mathfrak{U} by ℓ_2 .*

Proof. Let us show that all images of ℓ_1 is disjoint from \mathfrak{U} . Assume otherwise. Then by Corollary 7.5, the iterated images of ℓ_1 do not cross edges of \mathfrak{U} . Hence an eventual image ℓ'_1 of ℓ_1 has either $\theta_1 + \frac{1}{3}$ or $\theta_2 + \frac{2}{3}$ as its endpoint. On the other hand, by Corollary 7.5, ℓ'_1 cannot cross \mathfrak{M}^* , the sibling of \mathfrak{M} disjoint from \mathfrak{U} . By Theorem 7.7 except for $\theta_1 + \frac{1}{3}$ or $\theta_2 + \frac{2}{3}$ there are no other periodic points of period k between \mathfrak{M} and \mathfrak{M}^* . Hence $\ell'_1 = \mathfrak{M}$, a contradiction.

Since ℓ_1 and all its images are disjoint from \mathfrak{U} , then, by Corollary 7.8, there is an image of ℓ_1 separating \mathfrak{M} and \mathfrak{M}^* . We may assume that ℓ_1 separates \mathfrak{M} and \mathfrak{M}^* . Recall that k is the period of \mathfrak{M} . Let $\bar{\alpha}_1$ and $\bar{\beta}_1$ be the endpoints of ℓ_1 chosen so that

$$\theta_1 + \frac{1}{3} < \alpha_1 < \beta_1 < \theta_2 + \frac{2}{3}.$$

Since both arcs of \mathbb{S}^1 between \mathfrak{M} and \mathfrak{M}^* are mapped onto the major hole of \mathfrak{M} homeomorphically under $\sigma_3^{\circ k}$, there is a unique angle α_2 in $(\theta_1 + \frac{1}{3}, \alpha_1)$ with $3^k \alpha_2 = \alpha_1$, and there is a unique angle β_2 in $(\beta_1, \theta_2 + \frac{2}{3})$ with $3^k \beta_2 = \beta_1$. Consider the chord $\ell_2 = \overline{\alpha_2 \beta_2}$; then we have $\sigma_3^{\circ k}(\ell_2) = \ell_1$. By Corollary 7.5, no pullback of ℓ_1 can cross an edge of $\mathfrak{U}(f)$. Hence the cut that maps onto $\Gamma(\ell_1)$ under $f^{\circ k}$ and contains $R_f(\alpha_2)$ coincides with $\Gamma(\ell_2)$. Hence ℓ_2 is the desired pullback of ℓ_1 . \square

Lemma 7.10. *Let $f \in \mathcal{C}_\lambda \setminus \mathcal{CU}_\lambda$, where $|\lambda| \leq 1$. Moreover, suppose that there is one cycle of parabolic domains at 0 and two cycles of external rays landing at 0. Let \mathfrak{g} be the gap whose vertices are arguments of all external rays landing on 0. Then \mathfrak{g} is of type D. There exists a unique major $\mathfrak{M} = \overline{\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3}}$ of \mathfrak{g} such that the quadratic invariant gap*

defined by \mathfrak{M} coincides with $\mathfrak{U}(f)$, and the polynomial f belongs to a special wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$

Proof. The fact that \mathfrak{g} is of type D is immediate. Moreover, by the assumptions 0 is parabolic. Since $f \in \mathcal{C}_\lambda \setminus \mathcal{CU}_\lambda, |\lambda| \leq 1$, then f is immediately renormalizable. It follows that $\mathfrak{U}(f)$ contains arguments of all external (pre)periodic rays landing on points from the cycle of parabolic domains at 0. Denote by $\mathfrak{M} = \overline{\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3}}$ the major of \mathfrak{g} coming from the cycle of edges of \mathfrak{g} corresponding to the cycle of planar wedges at 0 that do not contain parabolic domains at 0. Then the wedge $W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$ contains no points of $J(f^*)$. It follows that the quadratic invariant gap defined by \mathfrak{M} coincides with $\mathfrak{U}(f)$. Moreover, by definition the polynomial f belongs to a special wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ as desired. \square

We are ready to prove a lemma describing dynamics of polynomials $f \in \mathcal{C}_\lambda \setminus \mathcal{CU}_\lambda, |\lambda| \leq 1$.

Lemma 7.11. *Let $f \in \mathcal{C}_\lambda \setminus \mathcal{CU}_\lambda$, where $|\lambda| \leq 1$. Then f is immediately renormalizable, $\mathfrak{U}(f)$ is of periodic type, and, if $\mathfrak{M} = \overline{\theta_1 + \frac{1}{3}\theta_2 + \frac{2}{3}}$ is the major of $\mathfrak{U}(f)$, then the rays $R_f(\theta_1 + \frac{1}{3}), R_f(\theta_2 + \frac{2}{3})$ land at the same point $z \in K(f^*)$. Moreover, either $z \neq 0$ is repelling, or $z = 0$ is parabolic with exactly one cycle of parabolic Fatou domains at 0 and two cycles of external rays landing on 0. In particular, f belongs to one of the parameter wakes.*

Proof. Since $f \in \mathcal{C}_\lambda \setminus \mathcal{CU}_\lambda$, then, by Corollary 2.6, the polynomial f is immediately renormalizable. Consider first the case when there exists a periodic cut $\Gamma(\ell_1)$ with vertex $x \neq 0$. By Corollary 7.5, the gap $\mathfrak{U}(f)$ is of periodic type. Assume that $\mathfrak{M} = \overline{\theta_1 + \frac{1}{3}\theta_2 + \frac{2}{3}}$ is the major of $\mathfrak{U}(f)$. We claim that the rays $R_f(\theta_1 + \frac{1}{3}), R_f(\theta_2 + \frac{2}{3})$ land at the same point. By way of contradiction suppose otherwise. Then the cut $\Gamma(\ell_1)$ satisfies conditions of Lemma 7.9. This implies that there exists a cut $\Gamma(\ell_2)$ with properties from that lemma. Consider the first $k-1$ iterated σ_3 -pullbacks of the chord ℓ_2 whose endpoints are located in holes of $\mathfrak{U}(f)$ behind the periodic edges. The corresponding cuts together with a suitably chosen equipotential bound a Jordan disk such that the restriction of f to this Jordan disk is a quadratic-like map. Let \widehat{J} be the corresponding quadratic-like Julia set. Then, by Lemma 6.1, we have $\widehat{J} = J(f^*)$, and so the landing point z_1 of $R_f(\theta_1 + \frac{1}{3})$ and the landing point z_2 of $R_f(\theta_2 + \frac{2}{3})$ belong to $J(f^*)$.

Suppose that $z_1 \neq z_2$. Consider the component W of $\mathbb{C} \setminus [R_f(\theta_1 + \frac{1}{3}) \cup R_f(\theta_2 + \frac{2}{3}) \cup J(f^*)]$ containing external rays with arguments from

$(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$. Since $z_1 \neq z_2$, there are preperiodic points of $J(f^*)$ that belong to $\text{Bd}(W)$ and such that some quadratic-like rays landing at these points (or at least parts of these quadratic-like rays near $K(f^*)$) are contained in W . Then, by Theorem 6.7, there are repelling preperiodic points $y \in J(f^*) \cap \text{Bd}(W)$ at which the corresponding external rays with arguments from $(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$ land. However, by definition of \widehat{X} , this is impossible. Thus $z_1 = z_2 = z \in K(f^*)$, as desired.

Let us prove the claims of the lemma concerning the point z . First assume that $z \neq 0$. Then, if z is parabolic, it would imply that points of a parabolic domain at z will belong to $K(f^*)$, a contradiction. Hence z is repelling as stated in the lemma. Observe that by definition in this case f belongs to the parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ that is non-special.

Assume now that $z = 0$. Then by definition z must be parabolic. We claim that there is exactly one cycle of parabolic domains at 0. Indeed, if there are two cycles of parabolic Fatou domains at 0 then, clearly, there are no quadratic-like Julia sets containing 0, which implies that f is not immediately renormalizable, a contradiction. Thus, there is exactly one cycle, say, \mathcal{P} , of parabolic domains at 0. By way of contradiction suppose that there is exactly one cycle of external rays landing at 0. Then each wedge formed by these rays contains one parabolic domain from this cycle. However the wedge associated to the major $\mathfrak{M} = \overline{\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3}}$ (i.e., formed by the external rays with arguments $\theta_1 + \frac{1}{3}$ and $\theta_2 + \frac{2}{3}$) has $z = 0$ as its vertex and by construction cannot contain any parabolic domains from \mathcal{P} , a contradiction. Hence there is exactly one cycle of parabolic Fatou domains at 0 and there are exactly two cycles of external rays landing on 0 as claimed. Then the rest follows from Lemma 7.10.

Assume now that there are no periodic cuts with non-zero vertex. In particular, there are no periodic cuts whose vertex is a repelling periodic point. Since $f \notin \mathcal{CU}_\lambda$, it follows that there exist two distinct periodic non-repelling points of multipliers not equal to 1. In particular, there exists a non-repelling periodic point $y \neq 0$ of multiplier not equal to 1. If y were parabolic then a periodic cut with non-zero vertex would exist, a contradiction. Hence y is not parabolic. By Lemma 7.3 (Kiwi's Separation Lemma) there are periodic cuts separating 0 (or possibly existing parabolic domains at 0) from points of the orbit of y . Since the only periodic cuts of f are cuts with vertex 0, it follows that 0 is parabolic, and cuts at 0 separate parabolic domains at 0 from the points of the orbit of y . This is only possible if there are two cycles of external rays landing on 0 and forming two cycles of wedges at 0: one

cycle contains a cycle of parabolic domains at 0 and the other cycle of wedges contains the entire orbit of y . Then, again, the desired claims follow from Lemma 7.10. \square

Lemma 7.11 implies a few claims from our main theorems, which are not proven yet. To begin with, recall that the last claim of Theorem B states that if f_{root} is the root point of a parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$, then f_{root} belongs to \mathcal{CU}_λ . Indeed, otherwise f_{root} would have properties listed in Lemma 7.11, and it clearly does not.

Now, the first claim of Theorem C is that the set \mathcal{CU}_λ is disjoint from all parameter wakes, unless $\lambda = 1$. Indeed, suppose that $\lambda \neq 1$ and $f \in \mathcal{CU}_\lambda$ belongs to a parameter wake. Since $f \in \mathcal{CU}_\lambda$ then by definition f has no periodic cutpoints of its Julia set. Moreover, since $f \in \mathcal{CU}_\lambda$ then $J(f)$ is connected. Hence by Lemma 7.1 f has a non-repelling periodic point $x \neq 0$ with multiplier not equal to 1. Since $\lambda \neq 1$, f has at least two non-repelling periodic points with multiplier not equal to 1, a contradiction with $f \in \mathcal{CU}_\lambda$.

To complete the proof of Theorem C, it remains to prove that \mathcal{CU}_λ , where $|\lambda| \leq 1$, is a full continuum. The set \mathcal{C}_λ is a full continuum [BuHe01]; this is very similar to the fact that the standard Mandelbrot set is a full continuum [DH8485]. By Theorem 7.2 and Lemma 7.11, the set \mathcal{CU}_λ is obtained from the full continuum \mathcal{C}_λ by removing all limbs. Note that, if we remove finitely many limbs from \mathcal{C}_λ , then we are left with a full continuum; indeed, a limb does not separate \mathcal{C}_λ . Being the intersection of a nested sequence of full continua, the set \mathcal{CU}_λ is also a full continuum. This concludes the proof of Theorem C.

7.3. Root points of non-special wakes. We now complete the characterization of non-special wakes by providing conditions on polynomials necessary and sufficient for being root points of non-special wakes. Recall the following notation: a unique quadratic invariant gap with major $\overline{0\frac{1}{2}}$ contained in the upper half of the unit disk (**a**bove $\overline{0\frac{1}{2}}$) is denoted by FG_a while a unique quadratic invariant gap with major $\overline{0\frac{1}{2}}$ contained in the lower half of the unit disk (**b**elow $\overline{0\frac{1}{2}}$) is denoted by FG_b . We begin with a laminational claim concerning majors of quadratic invariant gaps of σ_3 .

Lemma 7.12. *Let \mathfrak{M} be a major of a quadratic invariant gap. Suppose that \mathfrak{M} is of period q . Then the only major of a quadratic invariant gap of σ_3 in the orbit of \mathfrak{M} is \mathfrak{M} itself. Moreover, if there exists a finite stand alone periodic gap G of period q with edge \mathfrak{M} , then G is a triangle, and the only edge of G whose σ_3 -orbit contains a major of a quadratic invariant gap of σ_3 is \mathfrak{M} itself.*

Recall that by a finite stand alone gap G of period q we mean a gap that maps onto itself under $\sigma_3^{\circ q}$ so that $G, \sigma_3(G), \dots, \sigma_3^{\circ q-1}(G)$ are pairwise disjoint.

Proof. The first claim is immediate if \mathfrak{M} is invariant (i.e., if $\mathfrak{M} = \overline{0\frac{1}{2}}$). Suppose that \mathfrak{M} is not invariant (i.e., $q > 1$). Then, by [BOPT13], the chord \mathfrak{M} is the only leaf in the orbit of \mathfrak{M} that divides the circle into two arcs of length $> \frac{1}{3}$. Since, by [BOPT13], every major of an invariant quadratic gap of σ_3 divides the circle into two arcs of length $> \frac{1}{3}$, we see that \mathfrak{M} is the only major of an invariant quadratic gap of σ_3 in the orbit of \mathfrak{M} .

Suppose now that there exists a stand alone gap G of period q containing \mathfrak{M} . Clearly, this is impossible if $\mathfrak{M} = \overline{0\frac{1}{2}}$, thus we may assume that $\mathfrak{M} \neq \overline{0\frac{1}{2}}$ and $q > 1$. Then all vertices of G are of period q . By [Kiw02], this implies that G is a triangle of period q such that \mathfrak{M} is one of its sides, and all vertices of G are of period q . By the previous paragraph, the only major in the orbit of \mathfrak{M} is \mathfrak{M} itself. Denote by ℓ' and ℓ'' the two remaining edges of G ; also, denote by \mathfrak{U} the quadratic invariant gap with major \mathfrak{M} (since $\mathfrak{M} \neq \overline{0\frac{1}{2}}$, the gap \mathfrak{U} is unique).

Let us show that the orbit of ℓ' does not contain the major of an invariant quadratic gap (similar arguments show that neither does ℓ''). Suppose that there is some i , $0 \leq i < q$, such that $\sigma_3^{\circ i}(\ell')$ is the major of a quadratic invariant gap of σ_3 . Then $\sigma_3^{\circ i}(\ell')$ is the unique leaf in the orbit of ℓ' that divides the circle in two arcs of length $> \frac{1}{3}$. By definition, we have one of the following two cases:

- (1) the leaf ℓ' and all its images are contained in \mathfrak{U} , or
- (2) the leaf ℓ' and all its images are contained in the closures of the holes of \mathfrak{U} behind the corresponding images of \mathfrak{M} .

Let us show that in either case $i > 0$. Indeed, consider case (1). Then the quadratic invariant gap generated by ℓ' is located on the same side of \mathfrak{M} and hence must be strictly contained in \mathfrak{U} . This contradicts the fact that on both quadratic invariant gaps in question the map σ_3 is exactly two-to-one and the fact that the set of all preimages of every point is dense in the basis of the corresponding gap. Similarly one can consider case (2). Hence $i > 0$; consider now cases (1) and (2) separately using the fact that $i > 0$. Observe that non-major leaves from the orbit of ℓ' are contained in one of the two components into which $\sigma_3^{\circ i}(\ell')$ divides the closed unit disk, namely, the component containing ℓ' .

(1) Use the “projection” ψ that collapses all edges of \mathfrak{U} to points and semiconjugates $\sigma_3|_{\mathfrak{U}}$ to $\sigma_2|_{\mathbb{S}^1}$. Then the ψ -image of G equals the ψ -image of ℓ' . Moreover, the orbit of ℓ' (equivalently, of G) “projects” to a σ_2 -periodic orbit consisting of q leaves $\psi(\ell'), \sigma_2(\psi(\ell')) = \psi(\sigma_3(\ell')), \dots$. Observe that since, by [BOPT13], the gap \mathfrak{U} contains no concatenations of edges, and the set $\mathfrak{U} \cap \mathbb{S}^1$ is a Cantor set, then $\psi(\ell')$ and all its σ_2 -images are non-degenerate leaves.

Moreover, the fact that all triangles in the σ_3 -orbit of G are disjoint implies that the leaves $\psi(\ell'), \sigma_2(\psi(\ell')) = \psi(\sigma_3(\ell')), \dots$ are disjoint. Indeed, it is clear that these leaves are pairwise unlinked. If two of them are concatenated, then a triangle $\sigma_3^{\circ k}(G)$ would have one of its sides coinciding with $\sigma_3^{\circ k}(\ell')$ while the remaining vertex would be an endpoint of another image of ℓ' . This would contradict the claim that all distinct images of G are pairwise disjoint.

Recall that since $\sigma_3^{\circ i}(\ell')$ is a major of some quadratic invariant gap, it is the unique leaf in the orbit of ℓ' that divides the circle in two arcs of length $> \frac{1}{3}$. In particular, the arc not containing ℓ' must be longer than $\frac{1}{3}$. Moreover, the rest of the orbit of ℓ' is contained in one of the two components into which $\sigma_3^{\circ i}(\ell')$ divides the closed unit disk, namely the component containing ℓ' . It follows that $\psi(\sigma_3^{\circ i}(\ell'))$ is a σ_2 -periodic leaf of period q whose entire orbit is contained in one component of $\mathbb{D} \setminus \psi(\sigma_3^{\circ i}(\ell'))$, namely in the component bounded by $\psi(\sigma_3^{\circ i}(\ell'))$ and an arc of length strictly less than $\frac{1}{2}$. However, then it follows from well-known properties of σ_2 that leaves $\psi(\ell'), \sigma_2(\psi(\ell')) = \psi(\sigma_3(\ell')), \dots$ cannot be pairwise disjoint, a contradiction.

(2) Recall that in this case ℓ' and all its σ_3 -images are contained in the closures of the holes of \mathfrak{U} behind the corresponding images of \mathfrak{M} . Since all holes of \mathfrak{U} behind σ_3 -images of \mathfrak{M} are of length $< \frac{1}{3}$ except for the major hole behind \mathfrak{M} , then the only potential major in the σ_3 -orbit of ℓ' is ℓ' itself (recall that the major of a quadratic σ_3 -invariant gap has to divide \mathbb{S}^1 into two arcs of length at least $\frac{1}{3}$), contradicting the fact that $i > 0$.

Thus, in either case the orbit of ℓ' does not contain the major of an invariant quadratic gap in its orbit; similar arguments show that neither does ℓ'' . This completes the proof. \square

We are ready to prove the theorem describing root polynomials of non-special wakes.

Theorem 7.13. *Suppose that $\mathcal{W}_\lambda(\theta_1, \theta_2)$ is a non-special wake, $f \in \mathcal{F}_\lambda$. Then the following holds.*

- (1) Suppose that $(\theta_1, \theta_2) \neq (\frac{2}{3}, \frac{5}{6})$ and $(\theta_1, \theta_2) \neq (\frac{1}{6}, \frac{1}{3})$. Then the dynamic rays $R_f(\theta_1 + \frac{1}{3})$, $R_f(\theta_2 + \frac{2}{3})$ land at the same periodic parabolic point $z \neq 0$ of multiplier 1 if and only if f is the root point of $\mathcal{W}_\lambda(\theta_1, \theta_2)$.
- (2) Suppose that either $(\theta_1, \theta_2) = (\frac{2}{3}, \frac{5}{6})$ or $(\theta_1, \theta_2) = (\frac{1}{6}, \frac{1}{3})$. Then the dynamic rays $R_f(0)$ and $R_f(\frac{1}{2})$ land at the same periodic parabolic point $z \neq 0$ of multiplier 1 if and only if f is the root point of either $\mathcal{W}_\lambda(\frac{2}{3}, \frac{5}{6})$ or $\mathcal{W}_\lambda(\frac{1}{6}, \frac{1}{3})$.

Observe that in case (1) the arc (θ_1, θ_2) is in one-to-one correspondence with the major $\mathfrak{M} = \overline{(\theta_1 + \frac{1}{3})(\theta_2 + \frac{2}{3})} \neq \overline{0\frac{1}{2}}$ of a quadratic invariant gap \mathfrak{U} not equal FG_a or FG_b . However, in case (2) both arcs $(\theta_1, \theta_2) = (\frac{2}{3}, \frac{5}{6})$ or $(\theta_1, \theta_2) = (\frac{1}{6}, \frac{1}{3})$ are associated with the same major $\overline{0\frac{1}{2}}$ that can serve as the major of either FG_a or FG_b .

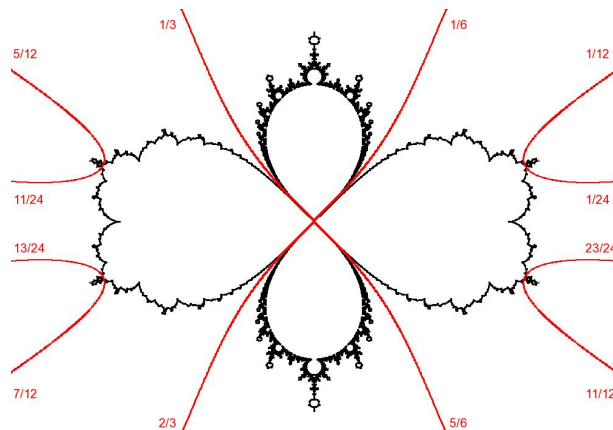
Proof. The “if” part in both cases follows from Theorem 4.10. We need to prove the “only if” part, i.e. to prove that if the dynamic rays $R_f(\theta_1 + \frac{1}{3})$, $R_f(\theta_2 + \frac{2}{3})$ land at the same periodic parabolic point $z \neq 0$ of multiplier 1, then f is the root point of $\mathcal{W}_\lambda(\theta_1, \theta_2)$ (in case (1)) or the root point of either $\mathcal{W}_\lambda(\frac{2}{3}, \frac{5}{6})$ or $\mathcal{W}_\lambda(\frac{1}{6}, \frac{1}{3})$ (in case (2)). The arguments are alike, so we separate the two cases only later in the proof.

Suppose that the dynamic rays $R_f(\theta_1 + \frac{1}{3})$, $R_f(\theta_2 + \frac{2}{3})$ land at the same periodic parabolic point $z \neq 0$ of multiplier 1. Let q be the minimal period of z . Since the multipliers of the periodic points of $g \in \mathcal{F}_\lambda$ are branches of certain multivalued analytic functions of g , there exists a stable component $\mathcal{U} \subset \mathcal{F}_\lambda$ bounded by a real analytic curve such that,

- we have $f \in \text{Bd}(\mathcal{U})$;
- for all $g \in \mathcal{U}$, there is an attracting periodic point z_g of period q depending analytically on g ;
- we have $z_g \rightarrow z$ as $g \rightarrow f$.

By Theorem C, the domain \mathcal{U} is contained in a parameter wake $\mathcal{W}(\theta_1^*, \theta_2^*)$ and f belongs to the closure of $\mathcal{W}(\theta_1^*, \theta_2^*)$. Then either $f \in \mathcal{W}_\lambda(\theta_1^*, \theta_2^*)$ and the dynamic rays $R_f(\theta_1^* + \frac{1}{3})$, $R_f(\theta_2^* + \frac{2}{3})$ land at the same *repelling* periodic point z^* , or f is the root point of $\mathcal{W}_\lambda(\theta_1^*, \theta_2^*)$ and the dynamic rays $R_f(\theta_1^* + \frac{1}{3})$, $R_f(\theta_2^* + \frac{2}{3})$ land at the same *parabolic* periodic point z^* . Consider cases.

First, let $(\theta_1, \theta_2) = (\theta_1^*, \theta_2^*)$. Then $f \notin \mathcal{W}_\lambda(\theta_1, \theta_2)$, as otherwise, since $\mathcal{W}_\lambda(\theta_1, \theta_2)$ is non-special, the external rays $R_f(\theta_1 + \frac{1}{3})$ and $R_f(\theta_2 + \frac{2}{3})$ land at a repelling periodic point, a contradiction. Hence then f is the root polynomial of $\mathcal{W}_\lambda(\theta_1, \theta_2)$. This corresponds to case (1).

FIGURE 3. Parameter slice \mathcal{F}_1 with some rays

Suppose now that $(\theta_1, \theta_2) \neq (\theta_1^*, \theta_2^*)$. Set $\mathfrak{M} = \overline{(\theta_1 + \frac{1}{3})(\theta_2 + \frac{2}{3})}$ and $\mathfrak{M}^* = \overline{(\theta_1^* + \frac{1}{3})(\theta_2^* + \frac{2}{3})}$, and again consider various cases.

(a) The polynomial f is the root polynomial of a non-special wake $\mathcal{W}_\lambda(\theta_1^*, \theta_2^*)$. Then $z^* \neq 0$ is parabolic. By the Fatou–Shishikura inequality, z and z^* belong to the same orbit. By Lemma 7.12, we have $\mathfrak{M} = \mathfrak{M}^*$. Since $(\theta_1, \theta_2) \neq (\theta_1^*, \theta_2^*)$, it follows from [BOPT13] that the only way it can happen is when $(\theta_1, \theta_2) = (0, \frac{1}{2})$ and $(\theta_1^*, \theta_2^*) = (\frac{1}{2}, 0)$ or vice versa. This corresponds to case (2).

(b) The polynomial f is the root polynomial of a special wake $\mathcal{W}_\lambda(\theta_1^*, \theta_2^*)$. Then, by Theorem 4.7, there are two cycles of parabolic domains at 0, which contradicts the assumption about f and z .

(c) Suppose now that z^* is repelling. Then f is immediately renormalizable, and $\mathfrak{U}(f)$ is the quadratic invariant gap with major \mathfrak{M}^* . It follows from [BOPT13] that \mathfrak{M} crosses the interior of $\mathfrak{U}(f)$, hence z separates $K(f)$. In particular, $z \in K(f)$, a contradiction with the fact that $K(f)$ contains no parabolic points different from 0 (see [BOPT16, Theorems A and B]). \square

7.4. The slice \mathcal{F}_1 . The case $\lambda = 1$ is an exception in Theorem C. Thus it requires separate attention. The slice \mathcal{F}_1 consists of polynomials $f = z^3 + bz^2 + z$. The point 0 is a fixed parabolic point with multiplier 1 of every polynomial in \mathcal{F}_1 . Figure 3 shows the parameter slice \mathcal{F}_1 in which several parameter rays and several wakes are shown.

The mutual position of \mathcal{CU}_1 and the wakes in \mathcal{F}_1 is described below:

Theorem 7.14. *Parameter wakes in \mathcal{F}_1 of period greater than 1 are disjoint from \mathcal{CU}_1 . A polynomial $f \in \mathcal{F}_1$ belongs to the set*

$$\left(\mathcal{W}_1 \left(\frac{1}{6}, \frac{1}{3} \right) \cup \mathcal{W}_1 \left(\frac{2}{3}, \frac{5}{6} \right) \right) \cap \mathcal{CU}_1$$

if and only if f is affinely conjugate to the root point of some wake $\mathcal{W}_{\lambda^}(\frac{1}{6}, \frac{1}{3})$ with $|\lambda^*| \leq 1$ and $\lambda^* \neq 1$.*

Proof. Let $\mathcal{W}_1(\theta_1, \theta_2)$ be a parameter wake of period $k > 1$. Take $f \in \mathcal{W}_\lambda(\theta_1, \theta_2)$. If $K(f)$ is disconnected, then, by definition, $f \notin \mathcal{CU}_1$. Suppose that $K(f)$ is connected. Then by Lemma 7.1 there are two possibilities at least of which holds. First, there may exist a repelling periodic cutpoint of $K(f)$. Then, by definition, $f \notin \mathcal{CU}_1$, as desired. Second, there may exist a non-repelling k -periodic point $x \neq 0$. Hence there are at least two non-repelling periodic points, x and $f(x)$, whose multiplier is different from 1, and so again $f \notin \mathcal{CU}_1$ as desired.

Suppose now that $f \in \mathcal{F}_1$ is affinely conjugate to the root point f_{root} of some wake $\mathcal{W}_{\lambda^*}(\frac{1}{6}, \frac{1}{3})$, where $|\lambda^*| \leq 1$ and $\lambda^* \neq 1$. By Lemma 7.11 and the description of the root points of wakes given in Theorems 4.10, 4.7, we have $[f_{root}] \in \mathcal{CU}$, hence $f \in \mathcal{CU}_1$.

Finally, let us prove that if $f \in \mathcal{F}_1$ belongs to the set

$$\left(\mathcal{W}_1 \left(\frac{1}{6}, \frac{1}{3} \right) \cup \mathcal{W}_1 \left(\frac{2}{3}, \frac{5}{6} \right) \right) \cap \mathcal{CU}_1,$$

then f is affinely conjugate to the root point of some wake $\mathcal{W}_{\lambda^*}(\frac{1}{6}, \frac{1}{3})$ with $|\lambda^*| \leq 1$ and $\lambda^* \neq 1$. Note that the rays $R_f(0)$ and $R_f(\frac{1}{2})$ land at 0 (the common landing point is a parabolic fixed point of multiplier 1, hence it must be 0). Since $f \in \mathcal{CU}_1$, there are no repelling periodic cutpoints in $K(f)$. Hence, by Lemma 7.1, the polynomial f has a non-repelling fixed point $x \neq 0$ with multiplier $\lambda^* \neq 1$. Consider a translation of \mathbb{C} that moves x to 0, and let f^* be the cubic polynomial obtained as the conjugation of f by this translation. Then $f^* \in \mathcal{F}_{\lambda^*}$, and $x^* = -x$ is a fixed point of f^* with multiplier 1. Since the rays $R_f(0)$ and $R_f(\frac{1}{2})$ land at 0, the rays $R_{f^*}(0)$ and $R_{f^*}(\frac{1}{2})$ land at x^* . It follows from Theorem 7.13 that f^* is the root point of the wake $\mathcal{W}_{\lambda^*}(\frac{1}{6}, \frac{1}{3})$ or of the wake $\mathcal{W}_{\lambda^*}(\frac{2}{3}, \frac{5}{6})$. \square

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