SHARKOVSKIĬ TYPE OF CYCLES

Alexander M. Blokh and Ethan M. Coven

Abstract. The Sharkovskiĭ type of a map of an interval is the Sharkovskiĭ-greatest integer t such that it has a periodic point of period t. The Sharkovskiĭ type of a cycle (i.e., a cyclic permutation) is the Sharkovskiĭ type of the "connect the dots" map determined by it. For $n \geq 2$, let $\mathcal{C}(n)$ denote the finite set of integers which are Sharkovskiĭ types of n-cycles. We give an internal characterization of $\mathcal{C}(n)$ and an n^4-time algorithm for determining the Sharkovskiĭ type of an n-cycle.

Introduction

This paper deals with "combinatorial dynamics on an interval" [1], specifically with some combinatorial aspects of A. N. Sharkovskiĭ's celebrated theorem [9]. To state it, we need the *Sharkovskiĭ order* on the positive integers:

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$$3 >_{\mathrm{S}} 5 >_{\mathrm{S}} 7 >_{\mathrm{S}} \cdots >_{\mathrm{S}} 2 \cdot 3 >_{\mathrm{S}} 2 \cdot 5 >_{\mathrm{S}} 2 \cdot 7 >_{\mathrm{S}} \cdots >_{\mathrm{S}} 2^2 \cdot 3$$
$$>_{\mathrm{S}} 2^2 \cdot 5 >_{\mathrm{S}} 2^2 \cdot 7 >_{\mathrm{S}} \cdots >_{\mathrm{S}} 2^3 >_{\mathrm{S}} 2^2 >_{\mathrm{S}} 2 >_{\mathrm{S}} 1.$$

Sharkovskiĭ's Theorem states that if a continuous map f of a compact interval to itself (a *map of an interval* for short) has a point of period n (a point p such that n is the least positive integer such that $f^n(p) = p$ – here the exponent on f denotes iterated composition), then for each $m <_{\mathrm{S}} n$ (m Sharkovskiĭ-less than n), it has a point of period m.

The *Sharkovskiĭ type*, denoted $\mathcal{S}(\cdot)$, of a map of an interval is the Sharkovskiĭ-greatest integer t such that it has a point of period t. No such t may exist. In that case, the map has points of period 2^k for all $k \geq 0$ and of no other period, and we say that it has Sharkovskiĭ type 2^∞. Define $m <_{\mathrm{S}} 2^\infty <_{\mathrm{S}} n$ if m is a power of 2 and n is not. P. Štefan's examples [10] show that if a map has a point of period n, then its Sharkovskiĭ type can be any "integer" (including 2^∞) Sharkovskiĭ-greater than or equal to n.

We extend the notion of Sharkovskiĭ type to cyclic permutations. For a cyclic permutation π of $\{1, \dots, n\}$, an *n-cycle* for short, the Sharkovskiĭ type $\mathcal{S}(\pi)$ of π is the Sharkovskiĭ type of the map $L_\pi : [1, n] \to [1, n]$ obtained by "connecting the dots" of $\{(i, \pi(i)) : i = 1, \dots, n\}$ by straight lines. Since there are finitely many n-cycles, the set $\mathcal{C}(n)$ of Sharkovskiĭ types of n-cycles is finite. We give an internal characterization of $\mathcal{C}(n)$ and an n^4-time algorithm for determining the Sharkovskiĭ type of an n-cycle.

For cycles, as for maps, we use exponentiation to denote iterated composition. In keeping with the opening sentence of the paper, most of the concepts will have purely combinatorial formulations.

1. Combinatorial preliminaries

The notion of the Sharkovskiĭ type of a cycle is connected with S. Baldwin's notion of forcing. An n-cycle π *forces* an m-cycle θ (n need not equal m) if every map of an interval which has a representative of π also has a representative of θ. A *representative* of π in f is a set $\{p_1, \dots, p_n\}$ such that $p_1 < \cdots < p_n$ and each $f(p_i) = p_{\pi(i)}$. The connection is that π forces θ if and only if the connect-the-dots map L_π has a representative of θ [2], and hence the Sharkovskiĭ type of π is the Sharkovskiĭ-greatest t such that π forces a t-cycle.

The *Markov graph* of π is the directed graph with vertices $[1, 2], \dots, [n-1, n]$, and an edge $[i, i+1] \to [j, j+1]$ if and only if $L_\pi[i, i+1] \supseteq [j, j+1]$ — equivalently, $\pi(i) \le j < \pi(i+1)$ or $\pi(i+1) \le j < \pi(i)$. A *path* is a sequence of edges, each starting where the previous one ends. Since Markov graphs do not have multiple edges, in this situation a path is also a sequence of vertices, each connected to the next by an edge. The *length* of a path is the number of (not necessarily distinct) edges in it. A closed path (one that starts and ends at the same vertex) is *nonrepetitive* if it is not the repetition of a shorter closed path.

The periods of the cycles forced by π are almost in one-to-one correspondence with the lengths of the nonrepetitive closed paths in the Markov graph G of π. The exact correspondence is described as follows. If

$$(\Lambda) \qquad [\lambda_0, \lambda_0 + 1] \to \cdots \to [\lambda_{k-1}, \lambda_{k-1} + 1] \to [\lambda_0, \lambda_0 + 1]$$

is a nonrepetitive closed path of length k in G, then there is an L_π-periodic point p of period k such that $L_\pi^j(p) \in [\lambda_j, \lambda_j + 1]$, $j = 0, \dots, k-1$ [4]. Conversely, if p is an L_π-periodic point, then there is a nonrepetitive closed path (Λ) in G such that $L_\pi^j(p) \in$

$[\lambda_j, \lambda_j+1]$, $j = 0, \dots, k-1$. Note that k need not equal $\#P$, the L_π-period of p. However, it is easy to see that $k = \#P$ or $\frac{1}{2}\#P$.

We describe the latter situation, as occurs later in the paper. For ease of notation, let $m = \#P$. Suppose that there is an $\frac{m}{2}$-cycle θ and an exactly two-to-one, increasing factor map $\varphi : \{1, \dots, m\} \to \{1, \dots, \frac{m}{2}\}$. ($\varphi$ is a *factor map* means that $\varphi \circ \pi = \theta \circ \varphi$.) Then the collection $\{[1,2], L_\pi[1,2], \dots, L_\pi^{m-1}[1,2]\}$ is nonoverlapping, i.e., the intersections have empty interior, and $L_\pi^m[1,2] = [1,2]$. It is easy to see that $L_\pi^m(1) = 2$ and $L_\pi^m(2) = 1$. Therefore, $L_\pi^{\frac{m}{2}}(x) = 3-x$ for all $x \in [1,2]$. Let P be the L_π-orbit of any point in $(1, 1\frac{1}{2})$. Since the path (Λ) corresponding to P is nonrepetetive, its length is $\frac{1}{2}\#P$. The situation described above is the only one which can occur when $k = \frac{1}{2}\#P$.

Theorem 1.1. *The Sharkovskiĭ type $\mathcal{S}(\pi)$ of a cycle π is the Sharkovskiĭ-greatest t such that the Markov graph of π contains a nonrepetitive closed path of length t, except if t is a power of 2, in which case $\mathcal{S}(\pi) = \#\pi$.*

Proof. Let t be Sharkovskiĭ-greatest such that the Markov graph of π contains a nonrepetitive closed path of length t.

Suppose that t is not a power of 2. There exists an L_π-periodic point of period t, and so $t \leq_{\mathrm{S}} \mathcal{S}(\pi)$. To show the opposite inequality, first note that $\mathcal{S}(\pi)$ is not a power of 2. There is an L_π-periodic orbit P with $\#P = \mathcal{S}(\pi)$. Let s be the length of the corresponding nonrepetitive closed path. Then $s = \mathcal{S}(\pi)$ or $\frac{1}{2}\,\mathcal{S}(\pi)$. In either case, $\mathcal{S}(\pi) \leq_{\mathrm{S}} s$. But $s \leq_{\mathrm{S}} t$, so $\mathcal{S}(\pi) \leq_{\mathrm{S}} t$.

Suppose that t is a power of 2. Then every nonrepetitive closed path in the Markov graph of π has length a power of 2, and so every L_π-periodic point has period a power of 2. In particular, $\#\pi$ is a power of 2, say 2^k. Furthermore, π is simple. (An n-cycle π is *simple* if $n = 1$, or n is even, $\pi\{1, \dots, \frac{n}{2}\} =$

$\{\frac{n}{2}+\ 1,\dots,n\}$, and $\pi^2|_{\{1,\dots,\frac{n}{2}\}}$ is a simple $\frac{n}{2}$-cycle.) Simple 2^k-cycles force only 1-, 2-, 4-, …, and 2^k-cycles [3, Theorem B]. Thus $\mathcal{S}(\pi)=\#\pi$. $\square$

2. The Sharkovskiĭ type of an n-cycle is in $\mathcal{C}(n)$

An n-cycle π has a *division* if n is even and $\pi\{1,\dots,\frac{n}{2}\}=\{\frac{n}{2}+1,\dots,n\}$.

Lemma 2.1. [8, Proposition 3.4] *A $2n$-cycle which has no division forces an n-cycle if n is odd, an $(n+1)$-cycle if n is even.*

For a map f of a space to itself, let $\mathrm{Per}(f)$ denote the set of periods of the periodic points of f.

Lemma 2.2. *If π has a division, then* $\mathrm{Per}(L_\pi)=2\,\mathrm{Per}(L_\pi^2)\cup\{1\}$*, and therefore* $\mathcal{S}(\pi)=2\,\mathcal{S}(L_\pi^2)$.

Proof. Let π be an n-cycle which has a division. L_π is monotone on $(\frac{n}{2},\frac{n}{2}+1)$ and permutes the intervals $[1,\frac{n}{2}]$ and $[\frac{n}{2}+1,n]$. Therefore $\mathrm{Per}(L_\pi|_{[1,\frac{n}{2}]\cup[\frac{n}{2}+1,n]})=2\,\mathrm{Per}(L_\pi^2|_{[1,\frac{n}{2}]\cup[\frac{n}{2}+1,n]})$. The period of any L_π-periodic point in $(\frac{n}{2},\frac{n}{2}+1)$ is either 1 or 2. Unless $n=\ 2$ and $\pi=(12)$, there is exactly one L_π-periodic point in $(\frac{n}{2},\frac{n}{2}+1)$ and it is a fixed point. In any case, $\mathcal{S}(\pi)=2\,\mathcal{S}(L_\pi^2)$. $\square$

For notational convenience, let

$$n_{\mathrm{odd}}=\begin{cases} n & \text{if } n \text{ is odd}\\ n+1 & \text{if } n \text{ is even.}\end{cases}$$

For $n\geq 2$, define a finite set $\mathcal{C}(n)$ by

(1) $\mathcal{C}(2)=\{2\}$
(2) $\mathcal{C}(n)=\{3,5,\dots,n\}$ if n is odd
(3) $\mathcal{C}(2n)=\mathcal{C}(n_{\mathrm{odd}})\cup 2\,\mathcal{C}(n)$.

To prove that the Sharkovskiĭ type of an n-cycle is in $\mathcal{C}(n)$, we use the following description of $\mathcal{C}(n)$. We omit the straightforward proof.

Lemma 2.3. *$\mathcal{C}(n) = \bigcup_{2^k \backslash n} 2^k\, \mathcal{C}'(\frac{n}{2^k})$, where $\mathcal{C}'(n) = \mathcal{C}(n)$ if $n = 2$ or n is odd, and $\mathcal{C}'(2n) = \mathcal{C}(n_{\mathrm{odd}})$.*

An n-cycle π has a *k-fold division* if 2^k is a factor of n and for $j = 1, \dots, k$, π (cyclically) permutes the 2^j sets $\widetilde{I}_{j,r} = \{(r-1)\frac{n}{2^j} + 1, \dots, r\frac{n}{2^j}\}$, $r = 1, \dots, 2^j$.

Lemma 2.4. *If π has a k-fold division, then for $j = 1, \dots, k$,*

(1) *L_π (cyclically) permutes the 2^j intervals $I_{j,r} = [(r-1)\frac{n}{2^j} + 1, r\frac{n}{2^j}]$, $r = 1, \dots, 2^j$.*
(2) *The periods of the L_π-periodic points in $[1, n] - \bigcup I_{j,r}$ are $1, 2, 4, \dots, 2^k$.*
(3) *$\mathcal{S}(\pi) = 2^k\, \mathcal{S}(L_\pi^{2^k})$.*
(4) *$\mathcal{S}(L_\pi^{2^k}) = \mathcal{S}(L_\pi^{2^k}|_{I_{k,r}})$, $r = 1, \dots, 2^k$.*

Proof. (1) is clear from the definitions. (2) follows from (1) and the fact that L_π is monotone on each component of $[1, n] - \bigcup I_{j,r}$. (3) follows from (1) and (2). (4) follows from the fact that the maps $L_\pi^{2^k}|_{I_{k,r}}$, $r = 1, \dots, 2^k$, are factors of each other, via compositions of the restrictions of L_π to appropriate intervals $I_{k,s}$. $\square$

Theorem 2.5. *The Sharkovskiĭ type of an n-cycle is in $\mathcal{C}(n)$.*

Proof. Let π be an n-cycle. If π has no division, the result follows from Lemma 2.1. Suppose then that π has a k-fold division, but not a $(k+1)$-fold division. Then some $\pi^{2^k}|_{\widetilde{I}_{k,r}}$ has no division. By Lemma 2.1, $\mathcal{S}(\pi^{2^k}|_{\widetilde{I}_{k,r}}) \in \{3, 5, \dots, (\frac{n}{2^{k+1}})_{\mathrm{odd}}\}$. It is easy to see that $\mathcal{S}(L_\pi^{2^k}|_{I_{k,r}}) \geq_{\mathrm{S}} \mathcal{S}(\pi^{2^k}|_{\widetilde{I}_{k,r}})$, so $\mathrm{Per}(L_\pi^{2^k}|_{I_{k,r}}) \supseteq$

$\mathrm{Per}(L_{\pi^{2^k}}|_{I_{k,r}})$. Thus $\mathcal{S}(L_\pi^{2^k}|_{I_{k,r}}) \in \{3, 5, \ldots, (\frac{n}{2^{k+1}})_{\mathrm{odd}}\}$. By Lemma 2.4(3), $\mathcal{S}(\pi) = 2^k\,\mathcal{S}(L_{\pi^{2^k}}|_{I_{k,r}})$, and by Lemma 2.3, $2^k\{3, 5, \ldots, (\frac{n}{2^{k+1}})_{\mathrm{odd}}\} \subseteq \mathcal{C}(n)$. $\square$

We close this section with the following closed-form description of $\mathcal{C}(n)$. It can be proved by induction on k.

Theorem 2.6.

(1) $\mathcal{C}(2) = \{2\}$.
(2) *If* $n \geq 3$ *is odd, then* $\mathcal{C}(n) = \{3, 5, \ldots, n\}$ *and* $\mathcal{C}(2n) = \{3, 5, \ldots, n\} \cup 2\{3, 5, \ldots, n\}$.
(3) *If* $n = 2^k m$, *where* $k \geq 2$ *and* m *is odd, then* $\mathcal{C}(n) = \{t : t \leq \frac{n}{2}, t$ *is not a power of* $2\} \cup \{\frac{n}{2} + 2^\ell : \ell = 0, \ldots, k-2\} \cup \{2^k r : r = 3, 5, \ldots, m\} \cup \{n\}$.

(If $m = 1$, $\{2^k r : r = 3, 5, \ldots, m\} = \varnothing$.*)*

3. Every $t \in \mathcal{C}(n)$ is the Sharkovskiĭ type of an n-cycle

To prove the title assertion of this section, we use the kneading theory of unimodal maps [7]. Recall that a map $f : I \to I$ of an interval is *unimodal* if $I = I_L \cup I_R$, where I_L and I_R are nondegenerate, compact intervals with one point in common, $x \leq y$ for every $x \in I_L$, $y \in I_R$, and f is increasing (resp. decreasing) on I_L (resp. I_R). A cycle π is *unimodal* if the connect-the-dots map L_π is unimodal.

The (f-)*itinerary* of $x \in I$ is the sequence $\nu_0\nu_1\ldots$, where $\nu_k = L$ if $f^k(x) \in I_L - I_R$, $\nu_k = R$ if $f^k(x) \in I_R - I_L$, and $\nu_k = C$ if $f^k(x) \in I_L \cap I_R$. Sequences on $\{L, C, R\}$ are ordered by the standard kneading order [7], denoted $\prec$. As usual, if $\nu_0\nu_1\ldots$ is an itinerary and k is least such that $\nu_k = C$, we write $\nu_0\ldots\nu_{k-1}C$ in place of $\nu_0\nu_1\ldots$. If π is a unimodal

n-cycle, the *kneading sequence* $\nu(\pi)$ of π is the L_π-itinerary of n $(=\ \max[1,n])$.

Lemma 3.1. [7, Theorem II.3.8] *$\nu_0 \dots \nu_n C$, where each $\nu_k = L$ or R, is the kneading sequence of a unimodal cycle if and only if $\nu_0 \dots \nu_n C \succ \nu_k \dots \nu_n C$ for $k = 1, \dots, n$.*

The Štefan *square root construction* [10], adapted to cycles, takes an n-cycle π and produces a $2n$-cycle $\sqrt{\pi}$ such that $L_{\sqrt{\pi}^2}|_{[1,n]} = L^2_{\sqrt{\pi}}|_{[1,n]}$ is conjugate to L_π. Let

$$\sqrt{\pi}(i) = \begin{cases} n + \pi(n+1-i) & \text{if} \quad 1 \le i \le n \\ 2n+1-i & \text{if} \quad n+1 \le i \le 2n. \end{cases}$$

Then $\sqrt{\pi}^2(i) = n+1-\pi(n+1-i)$, $i = 1, \dots, n$. If π is unimodal, then so is $\sqrt{\pi}$.

Lemma 3.2. $\mathcal{S}(\sqrt{\pi}) = 2\,\mathcal{S}(\pi)$.

Proof. Let π be an n-cycle. $\sqrt{\pi}$ has a division, so by Lemma 2.2, $\mathcal{S}(\sqrt{\pi}) = 2\,\mathcal{S}(L^2_{\sqrt{\pi}})$. As in the proof of Lemma 2.4, the only $L_{\sqrt{\pi}}$-periodic point in $(\frac{n}{2}, \frac{n}{2}+1)$ is the fixed point, and $L^2_{\sqrt{\pi}}|_{[1,\frac{n}{2}]}$ and $L^2_{\sqrt{\pi}}|_{[\frac{n}{2}+1,n]}$ are factors of each other. In fact, both are conjugate to L_π. Therefore, $\mathcal{S}(\sqrt{\pi}) = 2\,\mathcal{S}(\pi)$. $\square$

The *Štefan unimodal n-cycle* is defined as follows (see [10]). For $n = 1$, it is the unique n-cycle; for $n \ge 3$ odd, it is the unimodal cycle with kneading sequence $RLR^{n-3}C$; and for $n = 2^k m$, with m odd, it is obtained from the Štefan unimodal m-cycle by applying the square root construction k times. For $n \ge 3$, it is the unique unimodal n-cycle with Sharkovskiĭ type n. (See [10, Section E] for the case n is odd.)

Lemma 3.3. *Let θ be a unimodal cycle and let n be odd. Then θ has Sharkovskiĭ type 3 if and only if $\nu(\theta) \succeq RLC$; θ has Sharkovskiĭ type $n \geq 5$ if and only if $RLR^{n-3}C \preceq \nu(\theta) \prec RLR^{n-5}C$.*

Proof. For unimodal cycles, π forces θ if and only if $\nu(\pi) \succeq \nu(\theta)$ [7, Theorem II.3.8]. □

The following lemma is immediate from the definition of $\mathcal{C}(n)$.

Lemma 3.4. *Suppose t is not a power of 2. Then $2t \in \mathcal{C}(n)$ if and only if n is even and $t \in \mathcal{C}(\frac{n}{2})$.*

Theorem 3.5. *Every $t \in \mathcal{C}(n)$ is the Sharkovskiĭ type of an n-cycle.*

Proof. Write $t = 2^k u$, where u is odd. If $u = 1$, then $t = n$, and we let π be the Štefan unimodal n-cycle.

Suppose then that $u \geq 3$. Repeated applications of Lemma 3.4 show that 2^k is a factor of n and $u \in \mathcal{C}(\frac{n}{2^k})$. By Lemma 3.6 below, u is the Sharkovskiĭ type of some $\frac{n}{2^k}$-cycle θ. Applying the square root construction k times to θ, we obtain an n-cycle which, by Lemma 3.2, has Sharkovskiĭ type t. □

Remark. The n-cycle produced is unimodal, except in the trivial case $t = n = 2$.

Lemma 3.6. *Every odd $t \in \mathcal{C}(m)$ is the Sharkovskiĭ type of a unimodal m-cycle.*

Proof. For every odd $t \in \mathcal{C}(m)$, we exhibit a sequence $\nu = \nu_0 \ldots \nu_{m-2}C$, where each $\nu_k = L$ or R. One can check, using Lemmas 3.1 and 3.3, that ν is the kneading sequence of a unimodal m-cycle of Sharkovskiĭ type t.

If $t = 3$, let $\nu = RL^{m-2}C$; if $t = m$, let $\nu = RLR^{m-3}C$. Suppose then that $5 \leq t < m$, and let $\nu = RLR^{t-3}LR^{m-t-1}C$. m odd and m even must be considered separately. The relevant fact for using Lemma 3.3 is that if m is even, then $t \leq \frac{m}{2}+1$ (Lemma 2.1). $\square$

4. Determining the Sharkovskiĭ type of a cycle

In this section, we show how to efficiently (time n^4) determine the Sharkovskiĭ type of an n-cycle. The algorithm involves the adjacency matrix A of the Markov graph G of the cycle π, the $(n-1)\times(n-1)$ matrix given by the equivalent definition of G: $A_{ij}=1$ if $\pi(i)\leq j<\pi(i+1)$ or $\pi(i+1)\leq j<\pi(i)$, $A_{ij}=0$ otherwise.

First we give an easy-to-verify condition which is sufficient for the Sharkovskiĭ type of a cycle to be 3. It is the "cycle version" of [5, Lemma 3.3]. We will use it in the proof of Theorem 4.5.

Theorem 4.1. *Let π be a cycle and let A be the adjacency matrix of its Markov graph. If $\operatorname{Tr}(A)\geq 2$, then the Sharkovskiĭ type of π is 3.*

Proof. A map f of an interval has a *horseshoe* if there exist $x<y<z$ such that $f(x), f(z)\leq x$ and $f(y)\geq z$, or $f(x), f(z)\geq z$ and $f(y)\leq x$. It is well-known that a map which has a horseshoe must have a point of period 3. (For a connect-the-dots map, the Markov graph contains a nonrepetitive closed path of length 3.) We show that L_π has a horseshoe.

Since π is a cycle, it follows that L_π has at least two, but finitely many fixed points. Let $p<z$ be its two smallest fixed points. Since p is not an integer, there exists an integer i such that $i<p<i+1$. L_π is linear on $[i,i+1]$; if it is increasing, then p isn't the smallest fixed point of L_π. Thus L_π is decreasing on $[i,i+1]$.

Let $y\in(p,z)$ be such that $L_\pi(y)=\min L_\pi[p,z]$. Then y is an integer and $L_\pi(y)<p$. Now $L_\pi[L_\pi(y),z]\not\subseteq[L_\pi(y),z]$, for

otherwise $\pi^k(y) \leq z$ for all $k \geq 0$. This is impossible, since π is a cycle and y is an integer. Therefore, there exists x such that $L_\pi(y) < x < z$ and $L_\pi(x) \geq z$. Since p and z are the two smallest fixed points of L_π, it follows that $x < p$. $\square$

Using the lemmas in Section 2, we obtain

Lemma 4.2. *Let π be a cycle, let A be the adjacency matrix of its Markov graph, and let $k \geq 1$. The following statements are equivalent.*

(1) *π has a k-fold division.*
(2) *2^k is a factor of $\mathcal{S}(\pi)$.*
(3) $\mathrm{Tr}(A^{2^{k-1}}) = 2^k - 1$.

Using Lemma 4.2 and arguing in a manner similar to that of the proof of Theorem 4.1, we obtain

Theorem 4.3. *Let π be a cycle and let A be the adjacency matrix of its Markov graph. Let k be the greatest integer such that π has a k-fold division. Then $\mathrm{Tr}(A^{2^k}) \geq 2^{k+1} - 1$, and if $\mathrm{Tr}(A^{2^k}) \geq 2^{k+1}$, then the Sharkovskiĭ type of π is $3 \cdot 2^k$.*

The verification of the algorithm implicit in Theorem 4.5 uses the facts stated in Lemma 4.2 and the following lemma.

Lemma 4.4. *Let A be the adjacency matrix of the Markov graph of a cycle. Then there is a nonrepetitive closed path of length t in the graph if and only if $\mathrm{Tr}(A^t) > \mathrm{Tr}(A^{t'})$ for every factor $t' < t$ of t.*

Theorem 4.5. *Let π be a cycle and let A be the adjacency matrix of its Markov graph. Then the Sharkovskiĭ type of π is the Sharkovskiĭ-greatest t such that $\mathrm{Tr}(A^t) \geq 2^{\ell+1}$, where $t = 2^\ell v$ and v is odd. If no such t exists, then the Sharkovskiĭ type of π is $\#\pi$.*

Proof. Suppose first that $\mathcal{S}(\pi) = 2^k$. Then [4], π is a simple 2^k-cycle, and hence $\mathcal{S}(\pi) = \#\pi$. It is easy to check that for w odd,

$$\mathrm{Tr}(A^{2^j w}) = \begin{cases} 2^{j+1} - 1 & \text{if} \quad 0 \le j \le k-1 \\ 2^k - 1 & \text{if} \quad j \ge k. \end{cases}$$

Therefore, no t as in the statement of the theorem exists.

Suppose then that $\mathcal{S}(\pi)$ is not a power of 2, and write $\mathcal{S}(\pi) = 2^k u$, where $u \ge 3$ is odd. By Theorem 1.1, there is a nonrepetitive closed path of length $\mathcal{S}(\pi)$. We show that $\mathcal{S}(\pi) \le_{\mathrm{S}} t$ and $t \le_{\mathrm{S}} \mathcal{S}(\pi)$.

Suppose that $k = 0$. Then $\mathrm{Tr}(A^{\mathcal{S}(\pi)}) \ge \mathcal{S}(\pi) \ge 2 = 2^{0+1}$. If $\mathrm{Tr}(A) = 1$, then, since there are no nonrepetitive closed paths of lengths $3, 5, \ldots, \mathcal{S}(\pi) - 2$, we have by Theorem 1.1 that $\mathrm{Tr}(A) = \mathrm{Tr}(A^3) = \cdots = \mathrm{Tr}(A^{\mathcal{S}(\pi)-2}) = 1$. Hence $\mathcal{S}(\pi) \le_{\mathrm{S}} t$. If $\mathrm{Tr}(A) \ge 2$, then by Theorem 4.1, $\mathcal{S}(\pi) = t = 3$.

Suppose that $k \ge 1$. Since $2^k < \mathcal{S}(\pi)$ is a factor of $\mathcal{S}(\pi)$, it follows from Lemma 4.4 that $\mathrm{Tr}(A^{2^k}) < \mathrm{Tr}(A^{\mathcal{S}(\pi)})$. But, by Theorem 4.3, $\mathrm{Tr}(A^{2^k}) \le 2^{k+1} - 1$. Thus $\mathcal{S}(\pi) \le_{\mathrm{S}} t$.

To show that $t \le_{\mathrm{S}} \mathcal{S}(\pi)$, first notice that since $\mathcal{S}(\pi) \le_{\mathrm{S}} t$ and $\mathcal{S}(\pi)$ is not a power of 2, it follows that t is not a power of 2 either. So write $t = 2^\ell v$, where $\ell \le k$ and $v \ge 3$ is odd. We show that there is a nonrepetitive closed path of length t. If not, then by Lemma 4.4, $\mathrm{Tr}(A^{t'}) = \mathrm{Tr}(A^t)$ for some factor $t' < t$ of t.

Suppose that $t' = 1$. Then by Theorem 4.1, $\mathcal{S}(\pi) = t = 3$.

Suppose that $t' = 2^{\ell'}$, where $1 \le \ell' \le \ell$. If $\ell' \le k-1$, then by Lemma 4.2, $\mathrm{Tr}(A^{t'}) = 2^{\ell'+1} - 1$. This is impossible, since $\mathrm{Tr}(A^{t'}) = \mathrm{Tr}(A^t) \ge 2^{\ell+1} \ge 2^{\ell'+1}$. Suppose then that $\ell' = \ell = k$. Then $\mathrm{Tr}(A^{t'}) = \mathrm{Tr}(A^t) \ge 2^{k+1}$, which by Theorem 4.3 implies that $\mathcal{S}(\pi) = 3 \cdot 2^k = t$.

Suppose that $t' = 2^{\ell'} v'$, where $\ell' \le \ell$ and $v' \ge 3$ is odd. Then $\mathrm{Tr}(A^{t'}) = \mathrm{Tr}(A^t) \ge 2^{\ell+1} \ge 2^{\ell'+1}$, so $t' \ge_{\mathrm{S}} t$. By the Sharkovskiĭ-maximality of t, $t' = t$. Therefore, there is a nonrepetitive closed path of length t, and hence $t \le_{\mathrm{S}} \mathcal{S}(\pi)$. $\square$

Since, by Theorems 2.5 and 2.6, the Sharkovskiĭ type of an n-cycle cannot exceed n, Theorem 4.5 gives an algorithm for determining the Sharkovskiĭ type of a cycle. Finding the kth power of a matrix requires $\log k$ matrix multiplications. Using Strassen's algorithm [6], two $n \times n$ matrices may be multiplied using $n^{2+\alpha}$ multiplications, where $0 < \alpha < 1$. Thus (equating time with number of multiplications), for an n-cycle, the matrices $A^2, A^3, \ldots, A^{n-1}$, and hence their traces, may be determined in time $n^{3+\alpha} \log n < n^4$.

References

[1] Ll. Alsedà, J. Llibre, and M. Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, World Scientific, River Edge NJ, 1993.

[2] S. Baldwin, *Generalizations of a theorem of Sarkovskii on orbits of continuous real-valued functions*, Discrete Math. **67** (1987), 111–127.

[3] L. Block, *Simple periodic orbits of mappings of the interval*, Trans. Amer. Math. Soc. **254** (1979), 391–398.

[4] ———, *Periodic orbits of continuous mappings of the circle*, Trans. Amer. Math. Soc. **260** (1980), 555–562.

[5] L. Block and E. M. Coven, *Topological conjugacy and transitivity for a class of piecewise monotone maps of the interval*, Trans. Amer. Math. Soc. **300** (1987), 297–306.

[6] G. Brassard and P. Bratley, *Algorithmics: theory and practice*, Prentice Hall, Englewood Cliffs NJ, 1987.

[7] P. Collet and J.-P. Eckmann, *Iterated maps of the interval as dynamical systems*, Progress in Physics, vol. 1, Birkhäuser, Boston, 1980.

[8] T.-Y. Li, M. Misiurewicz, G. Pianigiani, and J. A. Yorke, *No division implies chaos*, Trans. Amer. Math. Soc. **273** (1982), 191–199.

[9] A. N. Sharkovskiĭ, *Coexistence of the cycles of a continuous mapping of the line into itself*, Ukrain. Mat. Zh. **16** (1964), 61–71. (Russian,

English summary)
[10] P. Štefan, *A theorem of Šarkovskii on the existence of periodic orbits of continuous endomorphisms of the real line*, Comm. Math. Phys. **54** (1977), 237–248.

Deparment of Mathematics, Wesleyan University, Middletown, Connecticut 06459-0128

Current address: Department of Mathematics, University of Alabama in Birmingham, University Station, Birmingham, Alabama 35294-0001

E-mail address: ablokh@math.uab.edu

Deparment of Mathematics, Wesleyan University, Middletown, Connecticut 06459-0128

E-mail address: ecoven@wesleyan.edu