TREES WITH SNOWFLAKES AND ZERO ENTROPY MAPS

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Abstract. We introduce the notion of a snowflake (related to Block's simple periodic orbits [Bl]) and show that the dynamics of a zero entropy forest map is determined by the corresponding family of snowflakes. This provides information about the sets of periods and limit sets for zero entropy forest maps.

0. Introduction

Let us call one-dimensional branching manifolds with finitely many branching points *graphs*. A connected contractible graph is a *tree*, a finite disjoint union of trees with disjoint compactifications is a *forest*. *We do not assume forests to be compact but by definition they are always finite; also, we consider only continuous maps.* Continuous self-mappings of such graphs like interval or circle are studied in a number of papers and books; maps of other graphs have attracted some attention too (see [IK], [ALM], [AM], [Ba], [LM], [B2-B4]). One of the reasons is that one-dimensionality allows to get surprising results and see how topological (and quite elementary in this case) properties of spaces influence dynamics. The description of sets of periods of a map is a good example (see [ALM], [Ba], [LM]); it originates in the Sharkovskii's paper [S] and fully shows the specifics of one-dimensional maps. Let $\mathcal{Z}$ be the set of zero entropy interval maps; another question is that of the description of sets of periods for maps from $\mathcal{Z}$ asked by Bowen [Bo2] and answered by Misiurewicz [M1] (see also [MS]) who proved that the maps from $\mathcal{Z}$ have sets of periods of the form $\{2^i : i < n\}, n \le \infty$ (important information about periodic orbits and infinite ω-limit sets of maps from $\mathcal{Z}$ may be found in [Bl],[M2], [B1, B5]). Some of the results may be generalized for graphs; e.g., for a graph map f the set $P(f)$ coincides up to a finite set with a finite union of sets of the form $k\mathbb{N}$ and $\{2^i m : i < \infty$ ([B3]), the entropy is zero if and only if there are no sets of the form $k\mathbb{N}$ in the union ([B3], [LM]). Our aim is to specify the description

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of the sets of periods for zero entropy *forest* maps thus extending the results of [MS],[M1-M2],[Bl].

We begin with definitions. Let Z be a forest, $\{Y_i\}_{i=0}^{n-1}$ be pairwise disjoint connected subsets of Z; then $\overline{Y_i}$ and $\overline{Y_j}$ have no more than one common point if $i \neq j$. For any $i < n$ the set $Y_{i+1 \mod n}$ is called the *next* to Y_i and denoted $nxt(Y_i)$. The sequence of sets $\{Y_i\}_{i=0}^{n-1}$ is a *z-cycle of sets (of period n)* if for any sets $A_0, A_1, \dots, A_k$ from the sequence such that $\bigcap_{i=0}^k \overline{A_i} \neq \emptyset$ we have $\bigcap_{i=0}^k \overline{nxt(A_i)} \neq \emptyset$; the union $\bigcup_{i=0}^{n-1} Y_i$ is also called a *cycle of sets (of period n)* without causing ambiguity. Usually z-cycles of sets are generated by a map $g : Z \longrightarrow Z$ such that $gY_i \subset Y_{i+1}$, $gY_{n-1} \subset Y_0$. Then we call $\{Y_i\}_{i=0}^{n-1}$ (and the union $\bigcup_{i=0}^{n-1} Y_i$) a *g-cycle* or simply *cycle of sets (of period n)*. If $Y, gY, \dots, g^{n-1}Y$ is a g-cycle of sets we call Y *a g-periodic set (of period n)*. In fact a cycle of sets is obtained when we forget the map defined on it but keep the sequence in which the map permutes its components; if we then forget the way the cycle of sets was obtained we get z-cycle of sets ("z" is the first letter of the Russian for "forget" which explains the appearance of "z" before this and some other terms).

If A and B are z-cycles of sets we say that A *contains* B (denoted by $A \sqsupset B$) if (1) $A \supset B$ in the set-theoretical sense and(2) for any components A' of A and B' of B if $A' \supset B'$ then $nxt(A') \supset nxt(B')$. Clearly if $A \sqsupset B$ are of periods n and m respectively then n is a multipler of m. A *z-tower* is a nested sequence (finite or infinite) $\mathbf{G} = \{G^0 \sqsupset G^1 \sqsupset G^2 \dots\}$ of z-cycles of sets such that if each G^i is of period n_i then $n_0 < n_1 < \dots$; clearly n_{i+1} is a multipler of n_i for all i. The set G^i and its components are of *level* i in $\mathbf{G}$ and $\mathbf{G}$ is of *type* $T'(\mathbf{G}) = \{n_0 < n_1 < \dots\}$; the intersection of G^{i+1} with a component of G^i is a *slice of level* $i+1$. The number of levels $\mathbf{h}(\mathbf{G})$ in $\mathbf{G}$ is the *height* of $\mathbf{G}$; $\mathbf{G}$ is *finite* or *infinite* depending on $\mathbf{h}(\mathbf{G})$. The period $p(\mathbf{G}) = p(T'(\mathbf{G}))$ of the z-cycle of sets of the last level of $\mathbf{G}$ is the *period of* $\mathbf{G}$. For a set $\mathcal{B}$ of z-towers the set of their types is $T'(\mathcal{B})$, the set of their periods is $p(\mathcal{B})$ and the set of numbers involved in their types is $T(\mathcal{B})$. Cycles of sets of a forest map give rise to towers just like z-cycles of sets give rise to z-towers. Note that if an f-cycle of sets A contains another f-cycle of sets B in the set-theoretical sense than $A \sqsupset B$ as z-cycles of sets; thus we write $A \supset B$ in case of cycles of sets generated by the same map. We denote (z-)towers by bold capital letters.

In Section 1 we do not assume maps to have zero entropy. We describe the dynamics on a tower (Theorem 1); it is close to that of a minimal translation in a special compact Abelian zero-dimensional group which depends on the tower. The fact that a point x enters cycles of sets in a tower gives information about its orbit; to get information about more points we study maximal by inclusion cycles of sets and towers. Then in Section 2 we study a special kind of their disposition important for the dynamics of zero entropy maps. Let X be a tree. A closed connected subset of X is called a *subtree*. Let $A \subset X$; then $[A]$, the *hull of* A, is the smallest subtree containing A. If $[A] \setminus A$ is connected we call the set A *surrounding* (e.g., on the interval the only surrounding sets are those with one or two components). If a z-cycle of sets as a set is surrounding we call it a *surrounding z-cycle of sets*. If $\mathbf{Y}$ is a z-tower in a forest Z, the zero level z-cycle of sets has surrounding intersections with each component of Z and each slice of $\mathbf{Y}$ is surrounding then we call $\mathbf{Y}$ a *z-snowflake*. A surrounding z-cycle of sets and a z-snowflake generated by a map are called a *surrounding cycle of sets* and a *snowflake*. Everything defined for (z-)towers may be defined for (z-)snowflakes with the corresponding extension of results.

Towers give information about periods of periodic points of a map: due to the fixed point property of compact trees one would expect that for a tower $\mathbf{Y}$ of type $m_0 < m_1 < m_2 < \dots$ there is a periodic orbit P of period m_i in the cycle of sets of level i in $\mathbf{Y}$ for any i. Indeed this holds if all cycles of sets in $\mathbf{Y}$ have compact components; otherwise it may fail (see example in Section 1 after the proof of Lemma 9). One could skip the cycles of sets which contain no periodic points of their periods, but as a result the tower could lose some properties, e.g. to be a snowflake. Fortunately, the latter is not the case, so from now on we consider only snowflakes with cycles of sets containing periodic points of their periods (*basic snowflakes*); if $\mathbf{Y}$ is a basic f-snowflake of type $m_0 < m_1 < \dots$ then f has periodic points of periods m_i, $(\forall i)$. In Theorem 2 we show that maximal towers of zero entropy maps are snowflakes which allows to see how topology of a graph influences periods of periodic orbits of its zero entropy maps; here we state Corollary 6, a direct application of Theorem 2 to maps of compact forests.

Corollary 6. *Let $f : X \longrightarrow X$ be a zero entropy map of a compact forest X. Then any maximal f-tower is a snowflake and for any $x \in X$ there exists a unique snowflake $\mathbf{L}_f(\omega(x))$ of period $\mathrm{card}\{\omega(x)\}$ maximal among all snowflakes $\mathbf{Y}$ such that $\omega(x)$ belongs to all cycles of sets in $\mathbf{Y}$ and if $\omega(x)$ is infinite then $\mathrm{orb}\, x$ eventually enters all cycles of sets in $\mathbf{Y}$. Moreover, if $\mathcal{M}(f)$ is the family of maximal f-towers then $P(f) = T(\mathcal{M}(f))$.*

We illustrate the picture on interval maps. Then the only non-connected surrounding sets Z are those with two components. Let $f : [0, 1] \longrightarrow [0, 1]$ be continuous and $\{\bigcup_{r=0}^{m_i-1} Y_r^i\}_{i=0}^k$ be a snowflake of type $m_0 = 1 < m_1 < \dots$ (perhaps $k = \infty$). Then every Y_r^i is an interval, $\bigcup_{r=0}^{m_1-1} Y_r^1$ is surrounding, so $m_1 = 2$ and Y_0^1, Y_1^1 are intervals interchanged by f. The picture on each level is the same: $m_{i+1} = 2m_i$ for any $0 \le i < k$, the intervals $Y_t^{i+1}, Y_{t+m_i}^{i+1}$ are the only intervals of level $i+1$ in $Y_t^i, 0 \le t < m_i$ and they are interchanged by f^{m_i}. Hence $m_i = 2^i$ and an interval snowflake is of type $\{1 < 2 < 4 < \dots\}$ (the number of powers of 2 may be infinite). If $k < \infty$ is the maximal number of levels of a maximal basic f-snowflake then $P(f) = \{2^i\}_{i=0}^k$ and any point converges to a periodic orbit. If there are maximal basic snowflakes with arbitrary large periods but no infinite snowflakes then $P(f) = \{2^i\}_{i=0}^\infty$ and $\omega(x)$ is a periodic orbit for any x. If there is a maximal basic infinite snowflake then $P(f) = \{2^i\}_{i=0}^\infty$ and for some x the set $\omega(x)$ is infinite. Thus our results extend the results of [Bl],[M1-M2],[MS] onto the forest case.

We now specify for forests the description of sets of periods of zero entropy graph maps ([B3],[LM]). The terms "edge" and "endpoint" have the usual sense; the number of edges of Z is $Edg(Z)$, the number of endpoints of Z is $End(Z)$ and the number of components of Z is $Comp(Z)$.

Corollary 7. *Let X be a forest with components having no more than r endpoints. Then the following statements are equivalent:*

(1) *$h(f) = 0$;*

(2) *for every $x \in Per\, f$ there is a snowflake $\mathbf{Y}$ of period $\mathrm{card}\,(\mathrm{orb}\, x)$ such that the cycle of sets of the last level in $\mathbf{Y}$ contains $\mathrm{orb}\, x$;*

(3) *any $k \in P(f)$ is of form $k = 2^j tn$ where $tn \le Edg(X)$ is odd, $t \le Comp(X)$ and all prime divisors of n are less than or equal to r.*

In particular if $f : X \longrightarrow X$ is a zero entropy map of an r-star (i.e. a graph with r edges coming out of a branching point) then an f-periodic point is of period

$2^j n, n \le r$.

Theorem 3. *Let X be a forest with components having no more than r endpoints. Then there is a finite family $\mathcal{G}(X) = \{G_1 \subset H_1, \dots, G_N \subset H_N\}$ of pairs of finite sets of integers $G_i = \{n_j^{(i)}\}_{j=1}^{l_i} \subset H_i = \{n_j^{(i)}\}_{j=1}^{m_i}$ such that if n is one of the integers than $n = 2^l tq \le 4Edg(X) - 2End(X)$, $t \le Comp(X)$, $tq \le Edg(X)$, q is an odd integer with all prime divisors less than r and the following holds.*

(1) *$h(f) = 0$ if and only if there is $i \le N$, a set of numbers $\{t_j\}_{j=l_i+1}^{m_i}$, $0 \le t_j \le \infty$ with $P(f) = (\bigcup_{j=1}^{l_i} n_j^{(i)}) \cup (\bigcup_{j=l_i+1}^{m_i} \bigcup_{k=0}^{t_j} 2^k n_j^{(i)})$ and a set $Q \subset \{l_i+1, \dots, m_i\}$ (perhaps empty) such that $t_j = \infty$ for any $j \in Q$, any infinite limit set of f belongs to an f-tower of type $\{n_j^{(i)} < 2n_j^{(i)} < \dots\}, j \in Q$, and such limit sets exist for any $j \in Q$.*

(2) *For any $i \le N$, any set of integers $\{t_j\}_{j=l_i+1}^{m_i}$, $0 \le t_j \le \infty$ and any set $Q \subset \{l_i+1, \dots, m_i\}$ (perhaps empty) such that $t_j = \infty$ for any $j \in Q$ there is a zero entropy map $g : X \to X$ such that $P(g) = (\bigcup_{j=1}^{l_i} n_j^{(i)}) \cup (\bigcup_{j=l_i+1}^{m_i} \bigcup_{k=0}^{t_j} 2^k n_j^{(i)})$, any infinite limit set of g belongs to a g-tower of type $\{n_j^{(i)} < 2n_j^{(i)} < \dots\}, j \in Q$, and such limit sets exist for any $j \in Q$.*

Let X be an r-star X with the branching point C and show that $\mathcal{G}(X)$ is the family of pairs $\{G_i \subset H_i\}$ where $G_i \equiv \{1\}$ and H_i runs through the family of all subsets of $\{1, 2, \dots, r\}$ containing $\{1\}$. Let cycles of sets of the first level in our snowflakes be non-connected. Then a snowflake living on X has an interval among its k components of the first level; thus the snowflake is of type $\{k < 2k < \dots < 2^i k < \dots\}$, $i < n$ for some $n \le \infty$. If $f : X \to X$ is of zero entropy then by Theorem 2 numbers of components of the first level of all f-snowflakes form a finite set $H' = \{n_j\}_{j=1}^m \subset \{2, \dots, r\}$; set $H = H' \cup \{1\}$. For any j let t_j be the supremum of heights of all snowflakes with n_j components on the first level. Also, if there is an infinite snowflake with n_j components on the first level then include n_j into Q. Clearly the choice of the sets $G = \{1\} \subset H$, the numbers t_j and the set Q complies with the statement (1) of Theorem 3.

Let us show that for any $G = \{1\} \subset H = \{n_j\}_{j=1}^m \subset \{1, 2, \dots, r\}$, any $\{t_j\}_{j=1}^m$, $0 \le t_j \le \infty$ and any set $Q \subset \{1, \dots, m\}$ (perhaps empty) such that $t_j = \infty$ for any $j \in Q$ there is a zero entropy map $g : X \to X$ such that $P(g) = \{1\} \cup (\bigcup_{j=1}^m \bigcup_{k=0}^{t_j} 2^k n_j)$, any infinite limit set of g belongs to a g-tower of type $\{n_j < 2n_j < \dots\}, j \in Q$, and such limit sets exist for any $j \in Q$. Let $n_1 < \dots < n_m$, the clockwise numbered edges of X be $A_1, \dots, A_r$. For any j choose n_j intervals $L_1^j \subset A_1, L_2^j \subset A_2, \dots, L_{n_j}^j \subset A_{n_j}$, all L_i^j, $(1 \le j \le m, 1 \le i \le n_j)$ pairwise disjoint and not containing C. Let for any $j < m$ the interval L_i^j be closer to C than L_i^{j+1}. We construct g so that $gC = C$, $fL_i^j = L_{i+1}^j$ (i.e. $\{L_i^j\}_{i+1}^{n_j}$ is a g-cycle of sets) and g is monotone on any interval complementary to $\bigcup L_i^j$. Since each $\{L_i^j\}_{i+1}^{n_j}$ is in fact a cycle of *intervals* it is easy to construct g on them so that the rest of the conditions from the beginning of this paragraph is satisfied.

The present paper is an extended and revised version of a part of the preprint [B4].

Notation

f^n is the n-fold iterate of a map f;

$\overline{Z}$ is the closure of Z;
$int\, Z$ is the interior of Z;
$\operatorname{orb} x \equiv \{f^n x\}_{n=0}^{\infty}$ is the *orbit (trajectory)* of x;
$\operatorname{Per} f$ is the set of all periodic points of a map f;
$P(f)$ is the set of all periods of periodic points of a map f;
$h(f)$ is the topological entropy of a map f.

1. Preliminary lemmas and properties of towers

In Section 1 we consider a forest map $f : Z \to Z$ without the zero entropy assumption. We need some definitions. A forest Y has its well-defined compactification $\hat{Y}$ which is a compact forest with the same number of components; we refer to endpoints of a tree Y which may belong to $\hat{Y}$ (e.g. we consider neighborhoods (d, c) where c in an endpoint of Y and call them *neighborhoods of endpoints of* Y). We describe the tower dynamics in Theorem 1 [B2] (the proof here is given for the sake of completeness). Let $D = \{m_i\}_{i=0}^{\infty}$ be a sequence of integers and $m_{i+1} > m_i$ be a multiple of m_i for all i. Consider a group $H(D) \subset \mathbb{Z}_{m_0} \times \mathbb{Z}_{m_1} \times \dots$, defined by $H(D) \equiv \{(r_0, r_1, \dots) : r_{i+1} \equiv r_i \pmod{m_i} (\forall i)\}$. The group operation is trivially defined; let τ be the minimal translation in $H(D)$ by the element $(1, 1, \dots)$. By *monotone* we mean a continuous map such that the preimage of any point is connected.

Theorem 1[B2]. *Let* $\mathbf{Y} = \{\bigcup_{r=0}^{m_i-1} Y_r^i\}_{i=0}^{\infty}$ *be an infinite* f*-tower of type* $D = \{m_0 < m_1 < m_2 < \dots\}$, $Q = Q(\mathbf{Y}) = \bigcap \overline{\bigcup_{r=0}^{m_i-1} Y_r^i}$. *Then there is a monotone map* $\varphi : Q \to H(D)$ *which semiconjugates* $f|Q$ *and* $\tau|H(D)$. *Moreover, the following holds:*

(1) *there is a unique minimal set* $S \subset Q$ *such that* $\omega(x) = S$ *for all points* $x \in Q$;

(2) *for any* b *if* $\omega(b) \cap Q \neq \emptyset$ *then* $S \subset \omega(b) \subset (Q \cap \Omega(f))$, φ *is surjective at most* 2*-to-*1 *on the set* $Q \cap \omega(b)$, *at most* $End(Z)$*-to-*1 *on the set* $Q \cap \Omega(f)$ *and injective on* $Q \cap \Omega(f)$ *outside an at most countable set.*

Proof. Assume that $Y_0^0 \supset Y_0^1 \supset \dots$ and that for big i the set $\overline{Y_0^i}$ is compact. Define the map φ as follows: for any $x \in Q$ let $\varphi(x)$ be the sequence $(r_0, r_1, \dots) \in H(D)$ such that $x \in \bigcap_{i=0}^{\infty} \overline{Y_{r_i}^i} \equiv I_x$. Let us prove that φ is well defined and has the required properties. First we show that if $x \in Q$ then $x \notin Per\, f$. Indeed, if $x \in Per\, f$ is of period n then there exists i such that $m_i > nEdg(X)$. The closures of the sets $Y_{r_0}^i, Y_{r_0+n}^i, \dots, Y_{r_0+nEdg(X)}^i$ contain x and at the same time the sets $Y_{r_0}^i, Y_{r_0+n}^i, \dots, Y_{r_0+nEdg(X)}^i$ are connected and pairwise disjoint which is impossible. So for any k the sets $Q \cap \overline{Y_j^k}, 0 \le j < m_k$ are pairwise disjoint and the map φ is well defined and continuous.

Let us show now that φ is surjective; to this end it is enough to prove that for any $(r_0, \dots) \in H(D)$ we have $A = \bigcap_{i=0}^{\infty} \overline{Y_{r_i}^i} \neq \emptyset$. First note that since Z may be non-compact then some sets $\overline{Y_j^k}$ may be non-compact too; however every non-compact $\overline{Y_j^k}$ must contain a neighborhood of at least one endpoint of Z, so the number of non-compact sets Y_j^k cannot exceed $End(Z)$ for a fixed k. Now, if there is i such that $Y_{r_i}^i$ does not contain a neighborhood of an endpoint of Y then its closure is compact and $A \neq \emptyset$. Let for all sufficiently big i the set $Y_{r_i}^i$

set $f^{m_i}\overline{Y^i_{r_i}}$ belongs to the image of some compact set of the form $\overline{Y^i_j}$ under the corresponding iteration of f, so $f^{m_i}\overline{Y^i_{r_i}}$ is a compact subset of $\overline{Y^i_{r_i}}$ and is disjoint from neighborhoods $(d_1, c_1), \dots, (d_l, c_l)$. Then for any $j > i$ we have that $f^{m_j}\overline{Y^j_{r_j}} \subset f^{m_i}\overline{Y^i_{r_i}}$ and hence $f^{m_j}\overline{Y^j_{r_j}}$ is disjoint from $(d_1, c_1), \dots, (d_l, c_l)$; since $f^{m_j}\overline{Y^j_{r_j}} \subset \overline{Y^j_{r_j}}$ and $\overline{Y^j_{r_j}}$ is a connected set containing some neighborhoods of endpoints $c_1, \dots, c_l$ we see that in fact $\overline{Y^j_{r_j}}$ contains $(c_1, d_1), \dots, (c_l, d_l)$ and hence $A = \bigcap_{s=0}^{\infty} \overline{Y^s_{r_s}} \neq \emptyset$.

Moreover, the same arguments show that the set Q may be non-compact only if some of the components of Q are non-degenerate non-compact connected sets containing neighborhoods of endpoints of Z; let B be such a component of Q. By the construction for any i there is a unique component B'' of Q such that $f^i B'' \subset B$. Take the smallest j such that if B' is a component of Q with $f^j B' \subset B$ then B' is compact. Replace all components of Q containing $fB', f^2B', \dots, f^jB'$ by the compact sets $fB', f^2B', \dots, f^jB'$ and then do the same with all non-compact components of Q and denote the resulting set by Q'. By the construction Q' is compact, invariant and $\varphi|Q'$ is surjective.

Let us show that $\Omega(f) \cap Q' = \Omega(f) \cap Q$ and so for any a we have $\omega(a) \cap Q' = \omega(a) \cap Q$. Indeed, it is enough to show that if $x \in Q \setminus Q'$ then $x \notin \Omega(f)$. By the construction $x \in Q \setminus Q'$ implies that there are components B', B'' of Q and a number $i > 0$ such that B' is compact, $f^i B' \subset B''$ and $x \in B'' \setminus f^i B'$. Let $\varphi(B') = (r_0, r_1, \dots)$; then $B' = \bigcap_{i \geq 0} \overline{Y^j_{r_j}}$. Moreover, if j is large enough then $x \notin \bigcup_{k=0}^{m_j - 1} f^k \overline{Y^j_{r_j}} = A_j$ and A_j is compact (since B' is compact). At the same time since $f^{m_j - i}x \in \overline{Y^j_{r_j}}$ and $x \notin Per(f)$ there is a number l such that $f^l x \in int\, A_j$; hence $x \notin \Omega(f)$. Replacing if necessary Q by Q' we may now assume that Q is compact; then the fact that φ semiconjugates $f|Q$ and $\tau|H(D)$ follows from the definitions. Note that by the construction sets of the form I_z are components of Q and preimages of the points of $H(D)$ under φ, the forward iterates of any such set are pairwise disjoint and so the diameter of forward iterates of any such set tends to zero.

Let us prove statement (1). If $W = \omega(b) \cap Q \neq \emptyset$ then the set W is invariant, infinite and for any i there is a point of W in the interior of Y^i_0. Hence there is an iterate of b in $\overline{Y^i_0}$ for any i and so $\omega(b) \subset Q$. Since τ is minimal and φ semiconjugates $f|Q$ to τ then φ is surjective on any closed invariant set, in particular on $Q \cap \omega(b)$. Let us show that $\varphi|(Q \cap \Omega(f))$ is at most $End(X)$-to-1 and $\varphi|\omega(b)$ is at most 2-to-1. Indeed, the set $I_z \cap \Omega(f)$ belongs to the set of all endpoints of I_z for any $z \in Q$ which implies the former statement. To prove the latter one observe that $f|\omega(b)$ is surjective, so for any $z \in Q$ the number of points in $\omega(b) \cap I_z$ is less than or equal to the minimum number of endpoints of a set I_ζ over all preimages $\zeta \in \omega(b)$ of z under all iterations of f. Since there are intervals among the sets I_ζ this minimum is 2 and $\varphi|\omega(b)$ is at most 2-to-1. Finally, the family of all non-degenerate sets I_z is at most countable and outside this set $\varphi|Q$ is injective.

It remains to prove statement (2). Denote by S the set of all limit points of the set $Q \cap \Omega(f)$ and show that $\omega(x) = S$ for any $x \in Q$. Indeed, if $x \in Q$ then there is a point $y \in (I_x \cap \Omega(f))$. Since the diameter of iterates of I_x tends to zero then $\omega(x) = \omega(y)$. At the same time I_x has pairwise disjoint iterates, so by definition $\omega(y) = \omega(x) \subset S$. Now if $z \in S$ then there exists a sequence of pairwise distinct points $z_i \in Q \cap \Omega(f)$ with $z_i \to z$. We may assume that sets I_{z_i} are pairwise disjoint;

thus for any sequence of points $\zeta_i \in I_{z_i}$ we have $\zeta_i \to z$. The surjectivity of φ on $\omega(x)$ implies $I_{z_i} \cap \omega(x) \neq \emptyset$ for any i so one can take $\zeta_i \in \omega(x)$. Thus $z \in \omega(x)$ and $S = \omega(x)$ for any $x \in Q$. $\square$

Let X be a tree. For two points $a, b \in X$ the hull of the set $\{a, b\}$ is denoted by $[a, b]$ and called an *interval*. We use the following notations: $(a, b] \equiv [a, b] \setminus \{a\}, [a, b) \equiv [a, b] \setminus \{b\}, (a, b) \equiv [a, b] \setminus \{a, b\}$; all these sets are also called *intervals*. Given points a, x, y we say that x *is closer to* a *than* y iff $[a, x] \subset [a, y]$. For a compact subtree $Z \subset X$ let r_Z be the natural retraction on Z.

Lemma 1. *Let $Y = [c, d] \subset Z$, $f : Y \to Z$ be continuous, $f[c, d] \supset [c, d]$, $[c, d) \cap (d, fd] = \emptyset$. Then there is $z \in [c, d]$ such that $fz = z$.*

Proof. Consider a preimage $c_1 \in [c, d]$ of c, then a preimage $c_2 \in [c_1, d]$ of c_1 etc.; clearly $\lim c_i = z \in [c, d]$ and $fz = z$. $\square$

Lemma 2. *Let Z be connected, $Y \subset Z$ be connected and compact, $f : Y \to Z$ be continuous. If $(a, fa] \cap Y \neq \emptyset\, (\forall a \in Y)$ then there is $z \in Y$ such that $fz = z$.*

Proof. Let $g = r_Y \circ f$ and $b \in Y$ be g-fixed point. If $fb \notin Y$ then $[fb, b) \cap Y = \emptyset$ which contradicts the assumption. So $fb = b$ which completes the proof. $\square$

Lemma 3. *Let $Y \subset Z$ be connected and $f : Y \to Z$ be continuous. Then one of the following possibilities holds:*

(1) *there is a fixed point $a \in Y$;*

(2) *there is a point $b \in Y$ such that $b \neq fb$, $(b, fb] \cap Y = \emptyset$;*

(3) *there is a unique endpoint c of Y such that if $[d, c)$ is the unique edge in Y ending in c then for any $x \in [d, c)$ we have $(x, fx] \cap (d, x) = \emptyset$ and so $f[x, c) \cap Y \subset (x, c)$.*

Proof. Suppose that neither (1) nor (2) holds and prove (3). Indeed, if there are no endpoints of Y with the properties from (3) then for any endpoint c of Y, corresponding edge $(d_c, c) \subset Y$ and some point $a_c \in (d_c, c)$ we have $(a_c, fa_c] \cap (d_c, a_c) \neq \emptyset$. By the assumption (2) does not hold, so by Lemma 2 we have that there is a fixed point in the hull of all a_c which is a contradiction. So there is an endpoint of Y with the required properties. Suppose b and c are two such points. Take $e_b \in (d_b, b)$ and $e_c \in (d_c, c)$ and consider $f|[e_b, e_c]$. By Lemma 1 there is a fixed point in $[e_c, e_b]$ which is a contradiction. $\square$

Let $Y \subset Z$ be connected, $f : Y \to Z$ be continuous. We call any fixed point of f a *basic point for* (f, Y). If f has no fixed points then any point $y \in Y$ with $(y, fy] \cap Y = \emptyset$ is called a *basic point for* (f, Y) too. The definition implies Property 1 stated without proof.

Property 1. *Basic points have the following properties.*

1. *If a is a basic point for $f : Y \to Z$, $y \in Y$ and $fy \in Y$ then $f[a, y] \supset [a, fy]$.*
2. *If f is defined on Z and $f^i y \in Y\, (0 \leq i \leq n)$ then $f^i[a, y] \supset [a, f^i y]$ for $0 \leq i \leq n$.*
3. *If b is a basic point which is not fixed then $fb \notin \overline{Y}$.* $\square$

Lemma 3 implies some corollaries; we begin with the following

Corollary 1. *In the situation of Lemma 3 the following holds.*

(1) *If Y is compact then the case (1) or (2) from Lemma 3 holds and so there is a basic point for (f, Y).*

(2) *If any endpoint e of Y has a neighborhood U_e in Y such that $fU_e \subset Y$ and there are no basic points for (f, Y) then there is a unique endpoint c of Y, $c \notin Y$, such that f may be extended to $Y \cup \{c\}$ as a continuous map with c an attractive fixed point and if $\hat{A} \subset Y$ is a surrounding cycle of sets then $\hat{A} \cup \{c\}$ is surrounding too.*

(3) *If $f : Y \longrightarrow Y$ does not have fixed points then there is a unique endpoint c of Y, $c \notin Y$, such that f may be extended to $Y \cup \{c\}$ as a continuous map with c an attractive fixed point and if $\hat{A} \subset Y$ is a surrounding cycle of sets then $\hat{A} \cup \{c\}$ is surrounding too.*

Proof. (1) If the case (3) of Lemma 3 holds, the case (2) of Lemma 3 does not hold and Y is compact then the endpoint of Y from the case (3) of Lemma 3 is a fixed point.

(2) The first statement of this part of Corollary 1 follows from Lemma 3. Now let $\hat{A}$ be a surrounding cycle of sets in Y and show that $\hat{A} \cup \{c\}$ is a surrounding set. Suppose $\hat{A} \cup \{c\}$ is not surrounding. There is an interval $[d, c)$ such that $\hat{A} \cap [d, c) = \emptyset$ and by Lemma 3 we may assume that all points from $[d, c)$ are attracted by c. Since $\hat{A} \cup \{c\}$ is not surrounding there are disjoint components A_1, A_2 of $\hat{A}$ and points $a_1 \in A_1$ and $a_2 \in A_2$ such that $a_1 \in (d, a_2)$. Since $\hat{A}$ is surrounding and of period greater than 1 (the latter follows from the assumption) then $[d, a_1) \cap (a_1, fa_1) = \emptyset$. Together with $fd \in [d, c)$ it implies $[d, a_1] \subset f[d, a_1]$ and by Lemma 1 there is a fixed point in $[d, a_1]$ which is a contradiction.

(3) Follows from (2). $\square$

For the rest of this Section we assume without loss of generality that $Y = \bigcup_{i=0}^{n-1} Y_i$ is a forest with connected components $\{Y_i\}_{i=0}^{n-1}$ and $f : Y \longrightarrow Y$ cyclically permutes them. In Corollaries 2 and 3 $B \subset Y$ is a cycle of sets of period $m > n$; denote $B \cap Y_j$ by B_j.

Corollary 2. *There are basic points for $(f^n, [B_j]), 0 \le j < n$; none of them lie in B. Furthermore, if B is a cycle of sets $\{G_i\}_{i=0}^{m-1}$ then these sets are components of B.*

Proof. Consider only the case $n = 1$. If there are no basic points for $(f, [B])$ then by Corollary 1 there is the endpoint c of $[B]$ and arbitrary close to c interval $I \subset [B]$ such that $fI \subset I$; however, one can choose I to be a subset of a component of B, so $fI \subset I$ is impossible since $m > 1 = n$. Thus there are basic points; by definition they do not lie in B. If the number k of components of B equals m then the components coincide with the sets $\{G_i\}_{i=0}^{m-1}$, so it is enough to consider the case when $k < m$. Replacing f by its power we can assume that $k = 1$; in other words, we can assume that B is connected, i.e. $B = [B]$. By the first statement there is a basic point b for $(f, [B])$; however since $B = [B]$ is invariant b must be a fixed point which contradicts the fact that B is a cycle of sets of period $m > n = 1$. Thus $k = m$ and $\{G_i\}_{i=0}^{m-1}$ are the components of B. $\square$

As we remarked in Introduction to get information about more points it is reasonable to study maximal by inclusion cycles of sets and towers. Say that a z-tower $\{G^i\}$ contains a z-tower $\{F^i\}$ if for any cycle of sets F^j there is a z-cycle of sets G^i

of the same period containing F^j; obviously the definition can be literally repeated for towers generated by a forest map. Let $\mathcal{A}(f,Y)$ be the ordered by inclusion family of all f-cycles of sets of periods greater than n. It is natural to expect that maximal cycles are closed and maximal towers have closed cycles of sets (otherwise one could replace a non-closed cycle of sets by its closure). This only fails when the closure of a cycle of sets in a tower has less components than the cycle itself which is described in Corollary 3.

Corollary 3. *Let $B = \bigcup_{i=0}^{m-1} G_i$ be a cycle of sets of period m and let the set $\overline{B}$ have $l < m$ connected components $\{A_i\}_{i=0}^{l-1}$. Then $\overline{B}$ is a cycle of sets of period l, there is a unique periodic orbit $P \subset \overline{B} \setminus B$ of period l such that $P \cap A_j = \{a_j\}$ is one point for any j and any A_s intersects (actually contains) exactly m/l components of B. Moreover, for any i with $G_i \subset A_j$ we have $a_j \in \overline{G}_i \setminus G_i$ and there is a unique edge $R_i \ni a_j$ such that $G_i \cap R_i \neq \emptyset$.*

Proof. By definition the set $\overline{B}$ is a cycle of sets of period l. Changing Y to $\overline{B}$ we may assume that $l = n$; let us now restrict ourselves to the case $n = l = 1$. Then $\overline{B}$ is connected and $\bar{[B]} = \overline{B} = f\overline{B}$. By Corollary 2 there is a basic point $a \in [B]$ for $(f, [B])$. If a is not fixed then by Property 1.(3) $fa \notin \overline{[B]}$ which is a contradiction. So $fa = a \in [B] \subset \overline{B}$; now the fact that $m > n = l = 1$ implies the rest of Corollary 3. $\square$

Let us call (z-)cycles of sets with properties from Corollary 3 *contacting (of periods $l < m$)*; if $l = 1$ we call B a *simple contacting (z-)cycle of sets.* A (z-)cycle of sets which is simple contacting or has all components closed is called *almost closed.* A (z-)tower of type $m_0 < m_1 < \dots$ such that any (z-)cycle of sets of level j is contacting of periods $m_{j-1} < m_j$ or has closed components is called *almost closed.* In Corollary 4 and Lemmas 4,5 we study properties of the ordered by inclusion family $\mathcal{A}(f,Y)$ of all f-cycles of sets of periods greater than n.

Corollary 4. *If $\{\hat{R}_\beta\}_{\beta \in B}$ is a family of cycles of sets from $\mathcal{A}(f,Y)$ then $\hat{R} = \bigcup_{\beta \in B} \hat{R}_\beta$ is not a cycle of sets of period n; hence if $\hat{G}$ and $\hat{F}$ are non-disjoint elements of $\mathcal{A}(f,Y)$ then $\hat{H} = \hat{G} \cup \hat{F}$ is an element of $\mathcal{A}(f,Y)$ too.*

Proof. Consider only the case of connected Y. If $\hat{R}$ is connected then it is invariant and does not contain fixed points. Thus by Corollary 1.(3) there is an endpoint c of $\hat{R}$, $c \notin \hat{R}$ and a small interval $I = (d, c) \subset \hat{R}$ such that $f^n z \to c$ for $z \in I$. Then I has non-empty intersection with some set $\hat{R}_\beta$ and so since components of sets $\hat{R}_\beta$ are connected and because of the dynamics near c we may assume that the whole interval I belongs to a component of $\hat{R}_\beta$ which contradicts the fact that the period of $\hat{R}_\beta$ is greater than $n = 1$. The second statement follows from the first one. $\square$

Lemma 4. *The family $\mathcal{A}(f,Y)$ satisfies the Zorn lemma and its maximal elements are pairwise disjoint. Moreover, the maximality of the set $B \in \mathcal{A}(f,Y)$ is equivalent to that of $B \cap Y_j$ in $\mathcal{A}(f^n, Y_j)$ for any $0 \le j < n$.*

Proof. Follows from Corollary 4. $\square$

We need more definitions. Let $A \in \mathcal{A}(f,Y)$ be a cycle of sets of period s. Clearly, all the sets $A_i = A \cup Y_i$ are f^n-cycles of sets of period s/n. Let $pr(f,A) \equiv pr(A) = \bigcup_{j=0}^{n-1} \bigcup_{i=0}^{\infty} f^i[A_j]$ (so $pr(A)$ is the smallest invariant set containing all the sets $[A_j]$) and $pe(f,A) = pe(A) = pr(A) \setminus A$ (pr stands for "prolongation" and pe for "realm" although no precise meaning is intended in these abbreviations). Observe that $[A_{j+i \pmod n}] \setminus A_{j+i \pmod n} \subset f^i([A_j] \setminus A_j)$ for any $0 \le i$ and $0 \le j < n$; in particular $[A_j] \setminus A_j \subset f^n([A_j] \setminus A_j)$.

Lemma 5. *Let $A \in \mathcal{A}(f, Y)$. Then the following holds.*

(i) $f^m pr(f^n, A_j) \setminus A_{j+m \pmod n} = f^m re(f^n, A_j) \setminus A_{j+m \pmod n} = re(f^n, A_{j+m \pmod n})$ *(in particular, $pr(f^n, A_j) \supset f^n re(f^n, A_j) \supset re(f^n, A_j)$) and $pr(f, A) = \bigcup_{j=0}^{n-1} pr(f^n, A_j)$, so $pr(A)$ is a d-cycle of sets of period n.*

(ii) *If A is a maximal cycle of sets then:*

(1) *if C is connected and strictly contains a component of A (so $C \setminus A = D \neq \emptyset$) then $orb\, C$ and $orb\, C \cup A = orb\, D \cup A = R \supset pr(A)$ are cycles of sets of period n, $orb\, D \supset re(A)$ and there is l such that $\bigcup_{i=0}^{l} f^i D \supset [A_j] \setminus A_j$ for any $0 \le j < n$ and so for any basic point b of $(f^n, [A_j])$ we have $b_j \in f^{ni} D$ for all $i \ge l$;*

(2) *if A is not closed then $\overline{A_j}$ is connected for any j, all the conclusions of Corollary 3 hold, A is almost closed and additionally for any $b \in \overline{A} \setminus A$ there exists k such that $f^k(b) \in P$ where $P \subset \overline{A}$ is the periodic orbit of period n existing by Corollary 3.*

Proof. (i) Let $C_j = \bigcup_{i=0}^{\infty} f^{in}([A_j] \setminus A_j)$. Since A_j is f^n-invariant then $re(f^n, A_j) = C_j \setminus A_j$. Moreover, by the above made observation $f^s C_j = C_{j+s \pmod n}$. This implies (i).

(ii) Consider only the case $n = 1$ and Y connected.

(1) Observe that $orb\, C \cup A = orb\, D \cup A = R$ since A is invariant; clearly, R is invariant too. If R is not connected then it is a cycle of sets of period greater 1 strictly containing B which contradicts the maximality of A. So R is connected, contains $[A]$ and hence $R \supset pr(A)$; since A is invariant we have $orb\, D \supset [A] \setminus A$ and so we also have $orb\, D \supset re(A)$. If b is a basic point for $(f, [A])$ then $b \in [A] \setminus A \subset orb\, D$. Let $b \in f^r D \subset f^r C$. Since $f^i C$ contains points from A for any i then by Property 1 $b \in f^j C$ and in fact $b \in f^j D$ for any $j \ge r$. Hence $\bigcup_{i=0}^{r+m} f^i D \supset [A] \setminus A$. Now the fact that $orb\, C$ is connected and invariant follows from the construction and what we have proved.

(2) Consider the set $\overline{A}$. If it is not connected then by the maximality of A we have $\overline{A} = A$ which is a contradiction. So $\overline{A}$ is connected and all but the last statement of Lemma 5(ii)(2) follow from Corollary 3. The last statement follows from Lemma 5(ii)(1). Indeed, if $b \in \overline{A} \setminus A$ then $b \notin A$; so one can apply Lemma 5(ii)(1) to $C = G \cup b$ where G is a component of A such that $G \cup b$ is connected. $\square$

It turns out that the Zorn lemma holds for towers; the maximality of a tower is equivalent to that of its cycles of sets of all levels in families similar to $\mathcal{A}(f, Y)$. We also show that a maximal tower is almost closed. Denote the set of all towers of (f, Y) by $\mathcal{T}(f, Y)$, the set of all towers with the period less than or equal to m by $\mathcal{T}_m(f, Y)$, the set of all towers with the property that all their cycles contain a set Z by $\mathcal{T}(f, Y, Z)$.

Lemma 6. *The Zorn lemma holds for $\mathcal{T}(f, Y), \mathcal{T}_m(f, Y)$ $(\forall m)$, $\mathcal{T}(f, Y, Z)$. Moreover, let $\mathbf{Y} = Y_0^0 \supset \bigcup_{i=0}^{m_1-1} Y_i^1 \supset \dots$ be a tower from one of these families (denote this family by $\mathcal{T}$). Then $\mathbf{Y}$ is a maximal tower in $\mathcal{T}$ iff $Y_0^0 = Y$ and the following properties hold:*

(1) $\bigcup_{i=0}^{m_{j+1}-1} Y_i^{j+1}$ *is a maximal cycle of sets in $\mathcal{A}(f, \bigcup_{i=0}^{m_j-1} Y_i^j)$ for any j;*

(2) *if* $\mathbf{Y}$ *has* $N < \infty$ *levels then there is no tower of period greater than* m_N *in* $\mathcal{T}$ *having* $\bigcup_{i=0}^{m_N - 1} Y_i^N$ *as one of its cycle of sets.*

Furthermore, any maximal tower $\mathbf{Y}$ *is almost closed.*

Proof. The Zorn lemma for towers follows from that for cycles of sets (see Lemma 4). Now let $\hat{A}$ be a cycle of sets of a maximal tower $\mathbf{Y}$, $\hat{B} \subset \hat{A}$ be the cycle of sets of $\mathbf{Y}$ of the next level. If $\hat{B}$ is not maximal in $\mathcal{A}(f, \hat{A})$ then by Lemma 4 there is a unique maximal in $\mathcal{A}(f, \hat{A})$ cycle of sets $\hat{C}$ such that $\hat{A} \supsetneqq \hat{C} \supsetneqq B$. Let us construct a tower $\mathbf{Z} \neq \mathbf{Y}$ which contains $\mathbf{Y}$. Indeed, if the period of $\hat{C}$ is bigger than that of $\hat{B}$ then one can insert $\hat{C}$ in $\mathbf{Y}$ between $\hat{A}$ and $\hat{B}$ and get the required tower $\mathbf{Z}$. If the period of $\hat{C}$ is equal to that of $\hat{B}$ one can replace $\hat{B}$ in $\mathbf{Y}$ by $\hat{C}$ and again obtain the required tower $\mathbf{Z}$. The rest of Lemma 6 follows from the definitions, Lemma 4 and Corollary 3. □

Corollary 5 follows immediately from Lemmas 4-6; before we state it we need a few definitions. Two towers are *separate* if their cycles of sets of the first $k \geq 0$ levels coincide and their cycles of sets of levels bigger than k are pairwise disjoint. A set of towers is called *separate* if they are pairwise separate. Also a tower $\mathbf{G}'$ containing all cycles of sets from $\mathbf{G}$ of levels less than or equal to m is called an *(m-)section of* $\mathbf{G}$.

Corollary 5. *The following properties hold.*

(1) *Let* $\mathcal{T}$ *be* $\mathcal{T}(f, Y)$ *or* $\mathcal{T}_M(f, Y)$. *Then the family of maximal towers from* $\mathcal{T}$ *is separate.*

(2) *If* $\mathbf{Y}$ *is a maximal tower from* $\mathcal{T}(f, Y)$, $\mathcal{T}_M(f, Y)$ *or* $\mathcal{T}(f, Y, Z)$ *and* $\mathbf{Y}$ *has type* $m_0 < m_1 < \dots$ *then for* r *less than or equal to the number of levels of* $\mathbf{Y}$ *the* r*-section of* $\mathbf{Y}$ *is a maximal tower in both* $\mathcal{T}_{m_r}(f, Y)$ *and* $\mathcal{T}(f, Y, Y^r)$ *where* Y^r *is the cycle of sets of level* r *in* $\mathbf{Y}$.

(3) *If* $\mathcal{T}(f, Y, Z) \neq \emptyset$ *then there is a unique maximal tower in* $\mathcal{T}(f, Y, Z)$.

Let us prove analogs of Lemmas 4 and 6 for snowflakes. Let $\mathcal{D}(f, Y)$ be the family of all f-cycles of sets D such that the intersection $D \cap Y_i$ is a surrounding set with more than one component for any i. Let $\mathcal{S}_\infty(f, Y) = \mathcal{S}(f, Y)$ be the family of all snowflakes of f and $\mathcal{S}_k(f, Y)$ be the family of all snowflakes of f of period less than or equal to k.

Lemma 7. *The family* $\mathcal{D}(f, Y)$ *satisfies the Zorn lemma and its maximal elements are pairwise disjoint. Moreover, any maximal cycle of sets* $\hat{V} \in \mathcal{D}(f, Y)$ *is almost closed.*

Proof. Consider only the case of connected Y. The definition and Lemma 4 imply that $\mathcal{D}(f, Y)$ satisfies the Zorn lemma. Let us prove that if $\hat{G}$ and $\hat{F}$ are distinct maximal cycles of sets from $\mathcal{D}(f, Y)$ then they are disjoint. Indeed, otherwise by Corollary 4 $\hat{H} = \hat{G} \cup \hat{F}$ is a cycle of sets of period greater than $n = 1$. Let us show that $\hat{H}$ is surrounding. Indeed, otherwise there is an interval $[a, b]$ and a point $c \in (a, b)$ such that a, b, c belong to different components of $\hat{H}$ which we denote by H_a, H_b, H_c. Let $c \in \hat{G}$ and choose points $d \in H_a \cap \hat{G}$ and $e \in H_b \cap \hat{G}$. Consider intervals $[d, c)$ and $[e, c)$. Since X is a tree then the fact that c belongs to the interval $[a, b]$ implies that $[e, c)$ and $[d, c)$ are disjoint; at the same time they belong to $[\hat{G}]$. Take points $d' \in [d, c) \cap [\hat{G}] \setminus \hat{G}$ and $e' \in [e, c) \cap [\hat{G}] \setminus \hat{G}$. By the definition of a surrounding set the interval $[e', d']$ cannot intersect $\hat{G}$; however $[e', d'] = [e', c] \cup [c, d']$

and $c \in \hat{G}$ which is a contradiction. The fact that any maximal cycle of sets from $\mathcal{D}(f, Y)$ is almost closed follows from Lemma 5(ii)(2). $\square$

The proof of the next lemma is left to the reader.

Lemma 8. *If A is surrounding and B is connected then $A \cap B$ is surrounding.* $\square$

Lemma 9. *Let $\mathbf{Y} = Y_0^0 \supset \bigcup_{i=0}^{m_1-1} Y_i^1 \supset \dots$ be a snowflake of period $M \le \infty$; $\mathbf{Y}$ is maximal in $\mathcal{S}_M(f, Y)$ iff $Y_0^0 = Y$ and the cycle of sets $\bigcup_{i=0}^{m_{j+1}-1} Y_i^{j+1}$ is a maximal cycle of sets in $\mathcal{D}(f^{m_j}, Y_r^j)$ for any r. Moreover, any maximal snowflake is almost closed.*

Proof. Let $\hat{A}$ be a cycle of sets of a maximal snowflake $\mathbf{Y}$, $\hat{B} \subset \hat{A}$ be the cycle of sets in $\mathbf{Y}$ of the next level. If $\hat{B}$ is not maximal in $\mathcal{D}(f, \hat{A})$ then by Lemma 7 there is a unique maximal in $\mathcal{D}(f, \hat{A})$ cycle of sets $\hat{C}$ and $\hat{A} \supsetneqq \hat{C} \supsetneqq B$. Let us construct a snowflake $\mathbf{Z} \ne \mathbf{Y}$ containing $\mathbf{Y}$. If the period of $\hat{C}$ is bigger than that of $\hat{B}$ one can insert $\hat{C}$ in $\mathbf{Y}$ between $\hat{A}$ and $\hat{B}$ and get (by Lemma 8) the required snowflake $\mathbf{Z}$. If the period of $\hat{C}$ is equal to that of $\hat{B}$ one can replace $\hat{B}$ in $\mathbf{Y}$ by $\hat{C}$ and obtain the required snowflake $\mathbf{Z}$ which contradicts the maximality of $\mathbf{Y}$. The fact that $\mathbf{Y}$ is almost closed follows from Lemma 7. $\square$

As we mention in Introduction non-compact cycles of sets in a tower may contain no periodic orbits of the corresponding period. To illustrate this possibility let us consider the following example (see Fig. 1).

Let $H \subset \mathbb{R}^2$ be the following tree: $H = [a, b] \cup [c, d] \cup [p, q]$, $a = (-1, +1)$, $b = (-1, -1)$, $c = (+1, +1)$, $d = (+1, -1)$, $p = (-1, 0)$, $q = (+1, 0)$; let also $(0, 0) = z$. Define a map $f : H \to H$ so that $f(z) = z, f(a) = d, f(b) = c, f(c) = a, f(d) = b, f([a, b, z] = [c, d, z], f([c, d, z] = [a, b, z]$; then $[a, b, z]$ and $[c, d, z]$ are invariant for f^2. Moreover, let all points but a, b, c, d converge to z and $f^{-1}(z) = \{z\}$. Then $h(f) = 0$, the unique maximal tower for f is $\mathbf{Y} = X \supset ([a, b, z] \cup [c, d, z] \supset \{a, b, c, d\}$ of type $1 < 2 < 4$, but f has only periods 1 and 4. Let us omit from $\mathbf{Y}$ a cycle of sets $[a, b, z] \cup [c, d, z]$ which does not contain any 2-periodic orbit. The new tower remains a snowflake and its type corresponds to the periods of f. The general fact is proven in Lemma 10, but first we define *extended* forest maps. Let $f : Z \to Z$ be a forest map and C be the maximal subset of the set of endpoints of Z such that f may be extended to a continuous map $\hat{f} : Z \cup C \to Z \cup C$; $\hat{f}$ is called the *extension of f* and if a map g coincides with its extension we call g an *extended map*. Clearly any map of a compact forest is extended.

Lemma 10. *Let $f : Z \to Z$ be an extended tree map, $\mathbf{Y}$ be a maximal snowflake of type $m_0 < m_1 < m_2 < \dots$ and $\mathbf{Z}$ be a new tower consisting of the cycles of sets in $\mathbf{Y}$ of periods m_j which contain periodic orbits of periods m_j. Then $\mathbf{Z}$ is an almost closed snowflake.*

Proof. Let us show that $\mathbf{Z}$ is a snowflake. Let $\bigcup_{i=0}^{m_j-1} Y_i^j$ be a cycle of sets of level j not containing a periodic orbit of period m_j. Then by Corollary 1 there is a uniquely defined set of endpoints $C = \bigcup_{i=0}^{m_j-1} c_i$ of components of $\bigcup_{i=0}^{m_j-1} Y_i^j$ such that f may be extended onto $\bigcup_{i=0}^{m_j-1} Y_i^j \cup c_i$ and for any i the point c_i will be an attractive point for $f^{m_j}|Y_i^j$. Since f itself is an extended map we may assume that $\bigcup_{i=0}^{m_j-1} c_j$ is an f-periodic orbit of some period $k < m_j$ and m_j is a multiple of k. We have already seen that $\mathbf{Y}$ is an almost closed snowflake. Hence $\bigcup_{i=0}^{m_j-1} Y_i^j$ is either closed or contacting cycle of periods $m_{j-1} < m_j$. In the first case $C \subset \bigcup_{i=0}^{m_j-1} Y_i^j$ which contradicts the assumption. In the second case the facts that $k < m_j$ and $C \subset \overline{\bigcup_{i=0}^{m_j-1} Y_i^j}$ imply that $k = m_{j-1}$; indeed, in this case closures of different components of $\bigcup_{i=0}^{m_j-1} Y_i^j$ may have in common only the points which belong to one periodic orbit of period m_{j-1}. So the period of the periodic orbit C is m_{j-1} and $\bigcup_{i=0}^{m_j-1} Y_i^j$ is a contacting cycle of sets of periods $m_{j-1} < m_j$. Thus the cycles of sets which will be omitted in the new tower are the contacting cycles of sets of periods $m_{j-1} < m_j$ which contain no periodic orbits of period m_j.

Let us prove that if we omit the cycle of sets of level j from $\mathbf{Y}$ the resulting tower remains an almost closed snowflake. The situation is as follows: $\bigcup_{i=0}^{m_j-1} Y_i^j$ is a contacting cycle of periods $m_{j-1} < m_j$ which does not contain a point of period m_j and should be omitted in the new tower $\mathbf{Z}$. Let us show that the cycle of sets of level $j+1$ from $\mathbf{Y}$ is not contacting. Indeed, otherwise since $\mathbf{Y}$ is almost closed this cycle is contacting of periods $m_j < m_{j+1}$ and so $\bigcup_{i=0}^{m_j-1} Y_i^j$ contains a periodic orbit of period m_j which contradicts the assumption. Thus the cycle of sets of level $j+1$ is closed and will not be omitted from $\mathbf{Y}$ which proves that the new tower $\mathbf{Z}$ is almost closed. Let us show that $\mathbf{Z}$ is a snowflake. Take a component of the cycle of sets of level $j-1$, say, Y_0^{j-1}; then using the notation from the preceding paragraph we have that there is a unique point from C, say, c_0 which belongs to Y_0^{j-1}. Let $Y_0^{j-1} \cap (\bigcup_{i=0}^{m_{j+1}-1} Y_i^{j+1}) = Z_0$; by definition Z_0 is a slice of the new tower $\mathbf{Z}$ and we need to show that Z_0 is surrounding. First let us note that each component of level j from the snowflake $\mathbf{Y}$ belonging to Y_0^{j-1} intersects Z_0; since the cycle of sets of level j in $\mathbf{Y}$ is contacting of periods $m_{j-1} < m_j$ then $c_0 \in [Z_0]$. At the same time if Y_i^j is a component of level j which belongs to Y_0^{j-1} then by Corollary 1.(3) the slice of level $j+1$ in Y_i^j together with the point c_0 form a surrounding set, i.e. the difference between the hull of the union of this slice and c_0 and the union itself is connected. Thus $[Z_0] \setminus Z_0$ is the union of the point c_0 with $\dfrac{m_{j+1}}{m_j}$ connected sets corresponding to the slices of $\mathbf{Y}$ of level $j+1$. Clearly each of these connected sets has c_0 as its endpoint which means that $[Z_0] \setminus Z_0$ is connected and Z_0 is surrounding which completes the proof. $\square$

Snowflakes whose cycles of sets contain periodic points of the corresponding periods are called *basic d-snowflakes* (so if $\mathbf{Y}$ is a basic f-snowflake of type $m_0 < m_1 < \dots$ then f has periodic points of periods m_i, $\forall i$); Lemma 10 shows that if we omit the cycles of sets not containing periodic orbits of the corresponding periods then the resulting tower remains a snowflake which allows us to talk about basic snowflakes and maximal basic snowflakes.

2. Snowflakes and zero entropy maps

In Section 2 we obtain the main results of the paper. Note that the topological entropy $h(f)$ for maps of non-compact spaces was defined in [Bo1]. We use the following property: if for a map F there are two disjoint compact sets A, B and iterations m, n of F such that $F^m A \cap f^n B \supset A \cup B$ then $h(F) > 0$. In case of forest maps the same holds if A, B are non-degenerate intervals with a common point (see, e.g., [LM]). All our conclusions are based on the assumption that f does not have the aforementioned pair of compact sets.

Proposition 1. *Let $f : X \to X$ be a zero entropy forest map cyclically permuting components $X_0, \dots, X_{n-1}$ of X and for any $C \in \mathcal{A}(f, X)$ the intersections of C with $X_0, \dots, X_{n-1}$ be $C_0, \dots, C_{n-1}$. If $B \in \mathcal{A}(f, X)$ is a maximal cycle of sets then B_j are surrounding sets and $re(B_j)$ are connected.*

Proof. Consider only the case of connected X. If B is not connected then by Lemma 5(ii)(2) $[B] \setminus B = \{a\}$ is a fixed point and all the statements hold. Let B be closed, y be a basic point for $(f, [B])$; by Corollary 2 $y \notin B$. Let Z be the connected component of $re(B)$ containing y; we show that $A = pr(B) = B \cup Z$.

Indeed, let G be a component of B neighboring to Z and E be the maximal component of $A \setminus Z$ containing G; clearly it is enough to show that $E = G$. Let $E \supsetneqq G$; then there is a point $x \in E \setminus B$ such that $[x, y] \cap G = [b, c]$ and $[x, y] = [x, b] \cup [b, c] \cup [c, y]$ where $[x, b) \cap (c, y] = \emptyset$. We construct a sort of "symbolic dynamics" for the map f which guarantees that $h(f) > 0$. Indeed, by Lemma 5(ii)(1) $B \cup orb\,(c, y] \supset A$. Thus there is a point $u \in (c, y]$ and an integer L such that $f^L u = x$. It implies by Property 1 that $f^L[u, y] \supset [y, x]$. On the other hand by Lemma 5(ii)(1) $B \cup orb\,[x, b) \supset A$, so there is a point $v \in [x, b)$ and an integer K such that $f^K[v, b] \supset [y, x]$. Thus $f^L[u, y] \supset [y, x] \supset [y, u] \cup [b, v]$, $f^K[v, b] \supset [y, x] \supset [y, u] \cup [b, v]$ and $[b, v] \cap [y, u] = \emptyset$ which implies that $h(f) > 0$; this contradiction completes the proof. $\square$

Proposition 1 and Lemma 6 immediately imply that a maximal tower of any kind of a zero entropy forest map is an almost closed snowflake. We specify this in the following

Theorem 2. *Let $f : X \to X$ be a zero entropy forest map. Then any maximal tower of f is an almost closed snowflake and for any $x \in X$ there are two possibilities:*

(1) $\omega(x) = \emptyset$ and if $\hat{f}$ is the extension of f then $\omega_{\hat{f}}(x)$ is an $\hat{f}$-periodic orbit consisting of endpoints of X;

(2) $\omega(x)$ is a compact subset of X, there exists a unique snowflake $\mathbf{L}_f(\omega(x))$ of period $\operatorname{card}\{\omega(x)\}$ maximal among all snowflakes $\mathbf{Y}$ such that $\omega(x)$ belongs to all cycles of sets in $\mathbf{Y}$ and if $\omega(x)$ is infinite then $orb\, x$ eventually enters all cycles of sets in $\mathbf{Y}$.

Moreover, if $\mathcal{M}(f)$ is the family of maximal towers of f then $P(\hat{f}) = T(\mathcal{M}(f))$.

Proof. Let X be connected, fix a point x and show that if some iterates of x approach an endpoint of X, say, c, which does not belong to X then $\omega_f(x) = \emptyset$ and for the extended map $\hat{f}$ the set $\omega_{\hat{f}}(x)$ is a periodic orbit consisting of some endpoints of X. We may assume that $[x, c)$ does not contain vertices of X and there is a number N such that $f^N x \in (x, c)$. If there is no f^N-fixed point in (x, c) then all points in (x, c) are mapped by f^N towards c and the statement in question holds. Indeed, $\bigcap_{i=0}^{\infty} f^{iN}[x, c) = \emptyset$ since otherwise there is an f^N-fixed point in $[x, c)$; thus $\bigcap_{i=0}^{\infty} f^{iN+k}[x, c) = \emptyset$ for any k as well. Let $b_j \in f^j[x, c)$. Then for any k there is a unique limit point c_k of the sequence $b_{iN+k}, i \to \infty$ and since $f^{iN+k}[x, c) \supset f^N(f^{iN+k}[x, c))$ then $c_k = c_{k+N}$, $(\forall k)$. It remains to observe that if $c_k \in X$ for some k then $\overline{f^{iN+k}[x, c)} \subset X$ is compact for big i and so there is an f^N-fixed point in $[x, c)$ which is a contradiction.

Suppose there is an f^N-fixed point $d \in (x, c)$. By the assumption there are infinitely many iterates of x in (d, c); so replacing x by its appropriate iterate we may assume that for some n which is a multiple of N we have $d < x < f^n x < c$. Moreover, replacing if necessary the point d by the closest to x f^n-fixed point we may assume that $d < z < f^n z < c$ for all $z \in (d, x]$; note that $f^N d = f^n d = d$. Let us show that $f^{ni} x \in (d, c)$ for any i. Indeed, otherwise let m be the minimal number such that $f^{nm} x \notin (d, c)$ and j be the minimal number such that $f^{nj} x \in [f^{n(m-1)} x, c)$. Clearly $j > 0$; so $d < f^{n(j-1)} x < f^{n(m-1)} x \le f^{nj} x$. Let $[d, f^{n(j-1)} x] = I$, $[f^{n(j-1)} x, f^{n(m-1)} x] = J$. Then $f^n I \cap f^n J \supset I \cup J$ and so $h(f) > 0$ which is a contradiction. Hence $f^{ni} x \in (d, c)$ for any i. There are iterates of x under f^n which approach c since otherwise $\omega_{f^n}(x)$ is a compact subset of $[d, c)$, $\omega_f(x)$ is a compact subset of X and iterates of x under f do not approach c which is a contradiction. Let us prove that $f^n[d, c) \subset [d, c)$. Indeed, otherwise for some $z \in (d, c)$ we have $f^n z = d$. Take the minimal j such that $d < z \le f^{nj} x < c$. Then $j > 0$ and $d < f^{n(j-1)} x < z \le f^{nj} x < c$. If $[d, f^{n(j-1)} x] = I$, $[f^{n(j-1)} x, z] = J$ we have $f^n I \cap f^n J \supset I \cup J$ and so $h(f) > 0$ which is a contradiction.

Consider local properties of f^n in a small neighborhood of c which does not contain x. First let us show that there is no interval of the form (a, c) such that all points in (a, c) are mapped by f^n away from c. Indeed, otherwise $f^n[d, a] \subsetneqq [d, c)$ is an f^n-invariant compact interval containing x which contradicts the assumption. On the other hand if there is an interval (a, c) such that all points in (a, c) are mapped towards c then as it was shown in the first paragraph of the proof the extended map $\hat{f}$ has c as its $\hat{f}^n$-fixed point, $\omega_{\hat{f}}(x) = orb_{\hat{f}} c$ and the statement in question is proven.

Now let there be no neighborhoods of c in which points are mapped by f^n towards or away from c. Then there is a sequence of f^n-fixed points $d_i \to c$ such that for any i there is $k = k(i)$ with $f^{nk} x \in (d_i, d_{i+1})$. The arguments similar to those from the preceding paragraphs show that then $f^{n(k+1)} x < f^{nk} x < c$ is impossible. Indeed, otherwise there is a fixed point $d' \in (f^{nk} x, c)$ such that there are no fixed points in $(f^{nk} x, d')$. At the same time some f^n-iterates of x approach c. Now the mere repetition of the aforementioned arguments show that this implies $h(f) > 0$. Hence $f^{nk} x < f^{n(k+1)} x\, (\forall k)$. Repeating the arguments from the second paragraph of the proof we see that $[d_i', c)$ is an f^n-invariant set for all i. Thus by the arguments from the first paragraph of the proof we see that the extended map $\hat{f}$ has c as its $\hat{f}^n$-fixed point and $\omega_{\hat{f}}(x) = orb_{\hat{f}} c$; since $c \notin X$ then all points in $orb_{\hat{f}} c$ are endpoints of X not belonging to X which completes the consideration of the case when some iterates of x approach an endpoint of X. From now on we assume that this is not the case and $\omega_f(x) \neq \emptyset$ is a compact subset of X.

Fix a point x and consider the family $\mathcal{T}$ of the towers such that their cycles of sets contain $\omega(x)$. Let $\mathbf{Y}$ be the unique maximal tower in $\mathcal{T}$ existing by Corollary 5. If $\omega(x)$ is finite then by Proposition 2 $\mathbf{Y}$ is a snowflake of period $card\{\omega(x)\}$, so it remains to consider the case when $\omega(x)$ is infinite and show that $\mathbf{Y}$ is infinite. This fact follows from the spectral decomposition for graph maps (see [B2]) which implies that if $h(f)=0$ then all infinite limit sets of f belong to infinite towers (limit sets of this kind are called in [B2] *solenoidal sets*); we give here an alternative proof.

The first step is to show that if $\omega(x)$ is infinite then there is a cycle of sets of period greater than 1 containing $\omega(x)$. Let $A=[\omega(x)]$; then A is compact and connected. Let $a\in A$ be a basic point for (f,A). Since $\omega(x)$ is infinite there exist an edge $\mathbf{r}=[z,y]$ and points s,s',p,q such that the following properties hold: 1)$z<s<s'<p<q<y$; 2)$[z,y]\subset[z,a)$; 3) $s,p,q\in(z,a)\cap\omega(x)$. Take neighborhoods U of p and V of q so that their closures are disjoint and $s'\notin\overline{U}$. Since U and V are not wandering then $orb\,U$ and $orb\,V$ are cycles of sets. Let us study their disposition on X. First of all, since $h(f)=0$ then there are no integers N,M such that $f^N\overline{U}\supset\overline{U}\cup\overline{V}$, $f^M\overline{V}\supset\overline{U}\cup\overline{V}$. Let for the definiteness $f^n\overline{U}\not\supset\overline{U}\cup\overline{V}\ (\forall n)$. Then $a\notin orb\,U$ since otherwise all large iterates of U contain a and also there are large iterates of U containing points close enough to s which implies that $f^n\overline{U}\supset[s',a]\supset\overline{U}\cup\overline{V}$ and contradicts the fact that $h(f)=0$. Hence $orb\,U$ is not connected, i.e. $orb\,U=B=\bigcup_{i=0}^{n-1}G_i$ where G_i are the components of B, $n>1$. The construction implies that B contains all but finite number of iterates of x; so B contains all but finite number of points from $\omega(x)$ and $\overline{B}\supset\omega(x)$. If $\overline{B}$ is not connected then it is the required cycle of sets; if $B\supset\omega(x)$ then B is the required cycle of sets. It remains to consider the case when $\overline{B}$ is connected and $B\not\supset\omega(x)$; note that $\overline{B}\supset[\omega(x)]=A$.

By Corollary 3 there is a unique fixed point $c\in\overline{B}$. Then $c\in A$ and by definition of a basic point $c=a$. Let us show that $a\notin\omega_f(x)$. Points s',p,q belong to the same component of B, say, to G_0; by Corollary 3 a is an endpoint of $\overline{G}_0$. Set $g=f^n$ and assume that $\omega_g(x)=\overline{G}_0\cap\omega_f(x)$, G_0 contains all but finite number of points from $\omega_g(x)$. Let us show that $a\notin\omega_g(x)$. Suppose that $a\in\omega_g(x)$. Take a point $b\in G_0$ such that $[b,a)$ contains no vertices of X, $[s',q]$ and $[b,a]$ are disjoint. If there is a g-fixed point $e\in[b,a)$ then $e\in f^NU$ for some N and $g^{kN}U\supset[s',e]\supset\overline{U}\cup\overline{V}$ for some k which is a contradiction. So g maps points in $[b,a)$ either towards or away from a.

If they are mapped towards a then for any $y'\in[b,a]$ we have $\omega_g(y')=\{a\}$; hence no iterates of x enter $[b,a]$ and $a\notin\omega_g(x)$ which implies that $a\notin\omega_f(x)$. Let points on $[b,a)$ be mapped away from $a\in\omega_g(x)$ and show that it leads to a contradiction. Consider some cases. If a has a g-preimage in $\omega_g(x)$ distinct from a then since $g|\omega_g(x)$ is surjective a has infinitely many preimages under different iterations of g in $\omega_g(x)$. Since G_0 is g^n-invariant and contains all but finite number of points from $\omega_g(x)$ we see that $a\in G_0$ which is a contradiction. Hence $g^{-1}(a)\cap\omega_g(x)=\{a\}$ and so if $F=(\overline{G_0}\setminus(b,a])\cap\omega_g(x)$ then $a\notin gF$. Thus there are points d',d such that $b<d'<d<a$ and $F\cup gF$ is disjoint from $[d',a]$ which implies that there is an open $W\supset F$ such that $W\cup gW$ is disjoint from $[d,a]$. By the definition of an ω-limit set we may assume that all g-iterates of x outside $[b,a]$ belong to W. Together with the fact that all points in $[b,a)$ are mapped by g away from a it implies that $orb_g x\cap[d,a]=\emptyset$ and thus $a\notin\omega_g(x)$ which again implies that $a\notin\omega_f(x)$.

Consider sets $G'_i=G_i\cup(\omega_f(x)\cap\overline{G}_i)$. Since $a\notin\omega_f(x)$ then $\bigcup_{i=0}^{n-1}G'_i$ is the required cycle of sets of period greater than 1 containing $\omega_f(x)$. Suppose now that $\mathbf{Y} \supset \omega(x)$ is the maximal tower among all towers containing $\omega(x)$. Let $\mathbf{Y}$ be finite and $\bigcup_{i=0}^{m-1} Y_i$ be the cycle of sets of the last level in $\mathbf{Y}$. Let $f^m = g$; we may assume that $x \in Y_0$ and $\omega_g(x) \subset Y_0$ is infinite. Then by what we have proved there is a g-cycle of sets of period greater than 1 which contains $\omega_g(x)$. Hence $\mathbf{Y}$ is not a maximal tower and so $\mathbf{Y}$ must be infinite. Thus $\mathbf{Y}$ is always of period $card\,\{\omega(x)\}$ and if $\omega(x)$ is infinite then obviously x eventually enters all d-cycles of sets from $\mathbf{Y}$. Now let us consider the basic snowflake $\mathbf{L}_f(\omega(x))$, corresponding to $\mathbf{Y}$ by Lemma 10; $\mathbf{L}_f(\omega(x))$ is the required basic snowflake. The final statement of the theorem follows from what we have shown and Lemma 10. $\square$

Corollary 6 is a direct application of Theorem 2 to compact forests.

Corollary 6. *Let $f : X \to X$ be a zero entropy map of a compact forest X. Then any maximal f-tower is a snowflake and for any $x \in X$ there exists a unique snowflake $\mathbf{L}_f(\omega(x))$ of period card $\{\omega(x)\}$ maximal among all snowflakes $\mathbf{Y}$ such that $\omega(x)$ belongs to all cycles of sets in $\mathbf{Y}$ and if $\omega(x)$ is infinite then $\operatorname{orb} x$ eventually enters all cycles of sets in $\mathbf{Y}$. Moreover, if $\mathcal{M}(f)$ is the family of maximal f-towers then $P(f) = T(\mathcal{M}(f))$.*

Let us now specify for forests the description of sets of periods of zero entropy graph maps given in [B3] (see also [LM]). If $\mathbf{Y}$ is a (z-)snowflake and i is its lowest level such that the corresponding (z-)cycle of sets has an interval component then we call the i-section of $\mathbf{Y}$ the *interval section* of $\mathbf{Y}$. Say that a number n is of *interval section type* for X if there exists a map $f : X \to X$ and an f-snowflake $\mathbf{Y}$ such that its interval section has the period n; equivalently one can say that n is of interval section type for X if there is a z-snowflake such that its interval section is of period n. We prove the following

Proposition 2. *If X has s components each of which has less than r endpoints and n is of interval section type for X then $n = 2^l tm \le 2Edg(X) - End(X)$ where $t \le s$, $tm \le Edg(X)$ and m is an odd integer with all prime divisors less than r.*

Proof. Assume that $\mathbf{A}$ is a snowflake of a map $f : X \to X$ which coincides with its interval section and has the period n and k levels. Let t be the period of the cycle of sets which is formed by components of X and contains the zero level cycle of sets in $\mathbf{A}$; then $t \le s, n = tq$ and the definition implies that all prime divisors of q are less than r. Let us show that $n \le 2Edg(X) - End(X)$. Indeed, none of the components of the cycle of sets D of level $k-1$ is an interval, so any edge contains at most two endpoints of components of D and the edges coming out of the endpoints of X contain at most one such endpoint. Thus the number of these endpoints is not bigger than $2Edg(X) - End(X)$, and so by the definition of a snowflake $n \le 2Edg(X) - End(X)$. It remains to show that if $q = 2^l m$ and m is odd then $tm \le Edg(X)$. Since $n \le 2Edg(X) - End(X)$ we assume that $l = 0$ and $q = m$ is odd; replacing f by f^t we assume that X is a tree, $t = s = 1$ and $n = m$. Let $\mathbf{A} = \bigcup_{i=0}^{m_0-1} A_i^0 = \hat{A}^0 \supset \bigcup_{i=0}^{m_1-1} A_i^1 = \hat{A}^1 \supset \dots$ be of type $(m_0, m_1, \dots, m_k = n)$. We show that there is no edge of X intersecting more than one set from the last level cycle of sets D in $\mathbf{A}$. Indeed, otherwise there exist an edge $[x, y]$ and endpoints $a, b \in [x, y]$ of distinct components of D such that $(a, b) \cap D = \emptyset$. Let $a \in A_0^j, j \le k$ and i be such that $b \notin A_0^i$ and $b \in A_0^{i-1}$. Since the slice of the level i in the set A_0^{i-1} is surrounding it has two components contradicting the assumption that $n = m$ is odd. So $n \le Edg(X)$. $\square$

Corollary 7 (cf [B4]). *Let X be a forest with components having no more than r endpoints. Then the following statements are equivalent:*

(1) $h(f) = 0$;

(2) *for every $x \in Per\, f$ there is a snowflake $\mathbf{Y}$ of period $card\,(orb\, x)$ such that the cycle of sets of the last level in $\mathbf{Y}$ contains $orb\, x$;*

(3) *any $k \in P(f)$ is of form $k = 2^j tn$ where $tn \le Edg(X)$ is odd, $t \le Comp(X)$ and all prime divisors of n are less than or equal to r.*

Proof. By Theorem 2 (1) implies (2). Let us show that (2) implies (3). Let $x \in Per\, f$ be of period k. Consider a snowflake $\mathbf{Y}$ of period k such that cycle of sets of the last level in $\mathbf{Y}$ contains $orb\, x$; we may assume that this cycle of sets is the orbit of x. Let the interval section of $\mathbf{Y}$ be $\mathbf{Y}'$ having the period n. The properties of interval maps imply that there exists j such that $k = 2^j n$; at the same time by definition n is of interval section type for X. Thus due to Proposition 2 (2) implies (3). Finally by [B3] (see also [LM]) $h(f) > 0$ for a graph map iff $P(f)$ contains a subset of the form $k\mathbb{N}$; hence (3) implies (1). $\square$

Theorem 3. *Let X be a forest with components having no more than r endpoints. Then there is a finite family $\mathcal{G}(X) = \{G_1 \subset H_1, \dots, G_N \subset H_N\}$ of pairs of finite sets of integers $G_i = \{n_j^{(i)}\}_{j=1}^{l_i} \subset H_i = \{n_j^{(i)}\}_{j=1}^{m_i}$ such that if n is one of the integers than $n = 2^l tq \le 4Edg(X) - 2End(X)$, $t \le Comp(X)$, $tq \le Edg(X)$, q is an odd integer with all prime divisors less than r and the following holds.*

(1) *$h(f) = 0$ if and only if there is $i \le N$, a set of numbers $\{t_j\}_{j=l_i+1}^{m_i}$, $0 \le t_j \le \infty$ with $P(f) = (\bigcup_{j=1}^{l_i} n_j^{(i)}) \cup (\bigcup_{j=l_i+1}^{m_i} \bigcup_{k=0}^{t_j} 2^k n_j^{(i)})$ and a set $Q \subset \{l_i + 1, \dots, m_i\}$ (perhaps empty) such that $t_j = \infty$ for any $j \in Q$, any infinite limit set of f belongs to an f-tower of type $\{n_j^{(i)} < 2n_j^{(i)} < \dots\}, j \in Q$, and such limit sets exist for any $j \in Q$.*

(2) *For any $i \le N$, any set of integers $\{t_j\}_{j=l_i+1}^{m_i}$, $0 \le t_j \le \infty$ and any set $Q \subset \{l_i + 1, \dots, m_i\}$ (perhaps empty) such that $t_j = \infty$ for any $j \in Q$ there is a zero entropy map $g : X \to X$ such that $P(g) = (\bigcup_{j=1}^{l_i} n_j^{(i)}) \cup (\bigcup_{j=l_i+1}^{m_i} \bigcup_{k=0}^{t_j} 2^k n_j^{(i)})$, any infinite limit set of g belongs to a g-tower of type $\{n_j^{(i)} < 2n_j^{(i)} < \dots\}, j \in Q$, and such limit sets exist for any $j \in Q$.*

Proof. For a zero entropy map $f : X \to X$ let $\mathcal{I}(f) = \mathcal{I}$ be the family of all interval sections of maximal basic towers of f. Let $\mathbf{Y} \in \mathcal{I}$ be of type $\{k_0 < k_1 < \dots < k_s\}$ and R be an interval component of the cycle of sets B from $\mathbf{Y}$ of the level s. If the cycle of sets of the level i has a degenerate component then set $G(\mathbf{Y}) = \{k_1, k_2, \dots, k_s\}, M(\mathbf{Y}) = \emptyset$. If not, and the endpoints of R do not belong to the same $2k_s$-periodic orbit belonging to the boundary of B then set $G(\mathbf{Y}) = \{k_1, k_2, \dots, k_{s-1}\}, M(\mathbf{Y}) = \{k_s\}$; if not and otherwise set $G(\mathbf{Y}) = \{k_1, k_2, \dots, k_s\}, M(\mathbf{Y}) = \{2k_s\}$. By definition k_s is of interval section type; also the union of sets $G(\mathbf{Y})$, $\mathbf{Y} \in \mathcal{I}$ is finite and we denote it by $G(f)$. Similarly the union of sets $G(\mathbf{Y}) \cup M(\mathbf{Y})$, $\mathbf{Y} \in \mathcal{I}$ is finite; denote it by $H(f)$. Clearly, $G(f) \subset H(f)$; the family $\mathcal{G}$ of all such pairs of sets for all zero entropy maps is also finite. The definition and the choice of $\mathbf{Y}$ show that in fact all the numbers $\{k_1, k_2, \dots, k_s\}$ are of interval section type for X (for k_s it follows from the definition, for k_i it follows from the fact that one can make a component of the cycle of sets of level i in $\mathbf{Y}$ smaller and replace it by an interval keeping it a z-snowflake). Thus all the numbers which appear in the construction are of interval section type or twice as big; together with Proposition 2 this explains the properties of the numbers from sets from $\mathcal{G}$ claimed in Theorem 3.

Let us show that the theorem holds with this family $\mathcal{G}$. If $h(f) = 0$ then the needed pair of sets from $\mathcal{G}$ is $G(f) = G_i \subset H(f) = H_i$; the definition and properties of zero entropy interval maps following from Theorem 2 show that there exist numbers $\{t_j\}_{j=l_i+1}^{m_i}$, $0 \le t_j \le \infty$ and a set $Q \subset \{l_i+1, \dots, m_i\}$ (perhaps empty) such that $t_j = \infty$ for any $j \in Q$ with all the properties from Theorem 3. Let us prove that if $G_i \subset H_i$ is a pair from $\mathcal{G}$, $\{t_j\}_{j=l_i+1}^{m_i}$ are numbers, $0 \le t_j \le \infty$, and $Q \subset \{l_i+1, \dots, m_i\}$ is such that $t_j = \infty$ for $j \in Q$ then there is a zero entropy map $g : X \to X$ with all the properties from Theorem 3. Indeed, let $G_i = G(f)$ and $H_i = H(f)$ for a zero entropy map $f : X \to X$. We describe how one can change f to get a map g with the required properties. Let $\mathbf{Y}$ be an interval section of a basic snowflake of f; we change f on its last level cycle of sets K depending on the properties of $\mathbf{Y}$. If K has a degenerate component we will not change f on it. Otherwise K has at least one interval component, say, $[a, b]$, and no degenerate components.

Let $K = \bigcup_{i=0}^{k-1} T_i$ be of period k ($T_0 = [a, b], \dots, T_{k-1}$ are its components), R be the set of all endpoints of components of K. Let $\bigcup_{i=0}^{k-1} f^i R = S$. Choose pairwise disjoint interval neighborhoods of points from R containing no vertices in their interiors so that their union U has the following property: for any $x \in R$ the point fx belongs either to R or to $K \setminus \overline{U}$. Let $T_i \setminus U = V_i$. Then for any $s \le \infty$ one can define a map $g|W = \bigcup_{i=0}^{k-1} V_i$ so that W is a g-cycle of sets of period k with periodic points of periods $\{2^i k\}_{i=0}^{s}$ only, and if $s = \infty$ we can define g so that it has infinite limit sets belonging to towers of type $\{k < 2k < \dots\}$. Moreover, we may assume that the positive orbits of all endpoints of sets V_i belong to $int\,(W)$. Clearly one can now extend g to the map defined on K so that all points from U are eventually mapped into W and $g|R = f|R$. Let $\mathcal{B}$ be the set of periodic orbits belonging to R, P' be the set of their periods. Then $g|K$ has periodic orbits of periods $P' \cup \{k, 2k, \dots, 2^s k\}$. If points a, b do not belong to the same $2k$-periodic orbit from $\mathcal{B}$ then $P' = \emptyset$ or $P' = k$; in this case $k = n_j^{(i)}$ for some $l_i < j \le m_i$, thus taking $s = t_j$ we will construct g so that $g|K$ has periods $n_j^{(i)}, 2n_j^{(i)}, \dots, 2^{t_j} n_j^{(i)}$. If points a, b belong to the same $2k$-periodic orbit from $\mathcal{B}$ then $P' = \{k, 2k\}$ and $2k = n_j^{(i)}$ for some $l_i < j \le m_i$. In this case we set $s = t_j + 1$ which gives a map $g|K$ with periods $\{k, 2k, \dots, 2^{t_j+1} k\} = \{(1/2) \cdot n_j^{(i)}, n_j^{(i)}, 2n_j^{(i)}, \dots, 2^{t_j} n_j^{(i)}\}$. Note that by the construction $k = (1/2) \cdot n_j^{(i)} \in G(f)$ and so the set of periods of $g|K$ belongs to $\bigcup_{j=1}^{l_i} n_j^{(i)}) \cup (\bigcup_{j=l_i+1}^{m_i} \bigcup_{k=0}^{t_j} 2^k n_j^{(i)})$. Finally, if the chosen j belongs to Q then $t_j = \infty$ and one can construct g so that $g|K$ has an infinite limit set belonging to a tower of type $\{n_j^{(i)} < 2n_j^{(i)} < \dots\}$. Now it is clear that if we change f similarly on all last level cycles of sets of all interval sections of basic snowflakes the resulting map g has zero entropy and the required properties. This completes the proof. $\square$

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