PERIODS IMPLYING ALMOST ALL PERIODS FOR TREE MAPS

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Abstract. Let $X$ be a compact tree, $f : X \to X$ be a continuous map and $\text{End}(X)$ be the number of endpoints of $X$. We prove the following

Theorem 1. Let $X$ be a tree. Then the following holds.

(1) Let $n > 1$ be an integer with no prime divisors less than or equal to $\text{End}(X) + 1$. If a map $f : X \to X$ has a cycle of period $n$, then $f$ has cycles of all periods greater than $2\text{End}(X)(n - 1)$. Moreover, $h(f) \geq \frac{\ln 2}{n\text{End}(X) - 1}$.

(2) Let $1 \leq n \leq \text{End}(X)$ and $E$ be the set of all periods of cycles of some interval map. Then there exists a map $f : X \to X$ such that the set of all periods of cycles of $f$ is $\{1\} \cup nE$, where $nE \equiv \{nk : k \in E\}$.

This implies that if $p$ is the least prime number greater than $\text{End}(X)$ and $f$ has cycles of all periods from 1 to $2\text{End}(X)(p - 1)$, then $f$ has cycles of all periods (for tree maps this verifies Misiurewicz’s conjecture, made in Bratislava in 1990). Combining the spectral decomposition theorem for graph maps (see [3-5]) with our results, we prove the equivalence of the following statements for tree maps:

(1) there exists $n$ such that $f$ has a cycle of period $mn$ for any $m$;

(2) $h(f) > 0$.

Note that Misiurewicz’s conjecture and the last result are true for graph maps ([6,7]); the alternative proof of the last result may be also found in [11].

0. Introduction

Let us call one-dimensional compact branched manifolds graphs; we call them trees if they are connected and contractible. Note that by the definition we deal with the finite trees. In what follows we consider only continuous tree maps. One of the

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well-known and impressive results about dynamical properties of one-dimensional maps is Sharkovskii’s theorem [12] about the co-existence of periods of cycles for interval maps. To formulate it let us introduce the following *Sharkovskii ordering* for positive integers:

\[(*) \quad 3 < 5 < 7 < \cdots < 2 \cdot 3 < 2 \cdot 5 < 2 \cdot 7 < \cdots < 8 < 4 < 2 < 1\]

Denote by \( S(k) \) the set of all integers \( m \) such that \( k \prec m \) or \( k = m \) and by \( S(2^\infty) \) the set \( \{1, 2, 4, 8, \ldots\} \). Also denote by \( P(\varphi) \) the set of periods of cycles of a map \( \varphi \).

**Theorem[12].** Let \( g : [0, 1] \to [0, 1] \) be a continuous map. Then there exists \( k \in \mathbb{N} \cup 2^\infty \) such that \( P(g) = S(k) \). Moreover, for any such \( k \) there exists a map \( g : [0, 1] \to [0, 1] \) with \( P(g) = S(k) \).

Generalizations of Sharkovskii’s theorem were studied in [1] for maps of the triod (a tree in the shape of the letter Y) and for general \( n \)-od in [2]).

Sharkovskii’s theorem implies that if a map \( f : \mathbb{R} \to \mathbb{R} \) has a cycle of period 3 then it has cycles of all periods. The following conjecture, formulated by M. Misiurewicz at the Problem Session at Czecho-Slovak Summer Mathematical School near Bratislava in 1990, is related to the aforementioned property of interval maps.

**Misiurewicz’s Conjecture.** For a graph \( X \) there exists an integer \( L = L(X) \) such that for a map \( f : X \to X \) the inclusion \( P(f) \supset \{1, 2, \ldots, L\} \) implies that \( P(f) = \mathbb{N} \).

We verify Misiurewicz’s conjecture when \( X \) is a tree. The general verification of this conjecture for arbitrary continuous graph maps may be found in [6,7]. Note that all results of the paper are true in the same formulations for finite unions of connected trees; the corresponding extension is left to the reader.

Fix a tree \( X \). We use the terms “vertex”, “edge” and “endpoint” in the usual sense. Denote the number of endpoints of \( X \) by \( \text{End}(X) \). We prove the following

**Theorem 1.** Let \( X \) be a tree. Then the following holds.

(1) Let \( n > 1 \) be an integer with no prime divisors less than or equal to \( \text{End}(X) \).

If a map \( f : X \to X \) has a cycle of period \( n \), then \( f \) has cycles of all periods greater than \( 2\text{End}(X)(n-1) \). Moreover, \( h(f) \geq \frac{\ln 2}{\text{End}(X)} \).


(2) Let \( 1 \leq n \leq \text{End}(X) \) and \( E \) be the set of all periods of cycles of some interval map. Then there exists a map \( f : X \to X \) such that the set of all periods of cycles of \( f \) is \( \{1\} \cup nE \), where \( nE = \{nk : k \in E\} \).

For interval maps this implies that when \( n \) is odd and \( f \) has a point of period \( n \), then \( f \) has all periods greater than \( 4(n-1) \). This statement is slightly weaker than Sharkovskii’s theorem.

Let us formulate some corollaries of Theorem 1.

**Corollary 1 (cf. [9]).** Let \( f : X \to X \) be a cycle of period \( n = pk \) where \( p > 1 \) has no prime divisors less than \( \text{End}(X) + 1 \). Then \( h(f) \geq \frac{\ln 2}{k[p\text{End}(X) - 1]} > \frac{\ln 2}{n\text{End}(X) - n} \).

*Proof.* It is enough to consider the map \( f^k \) and apply Theorem 1. \( \square \)

The next corollary verifies for tree maps Misiurewicz’s conjecture.

**Corollary 2.** Let \( p \) be the least prime number greater than \( \text{End}(X) \). If \( f : X \to X \) has cycles of all periods from 1 to \( 2\text{End}(X)(p-1) \) then \( f \) has cycles of all periods.

Theorem 1 and the spectral decomposition theorem for graph maps ([3-5]) imply

**Corollary 3.** The following two statements are equivalent:

1. there exists \( n \) such that \( f \) a cycle of period \( mn \) for any \( m \);
2. \( h(f) > 0 \).

In fact Corollary 3 is true for arbitrary graph maps ([6,7]; the different proof may be found in [11]). The preprint [8] contains a preliminary version of this paper.

**Notation**

- \( f^n \) is the \( n \)-fold iterate of a map \( f \);
- \( \text{orb} x \equiv \{f^n x \}_{n=0}^{\infty} \) is the orbit (trajectory) of \( x \);
- \( \text{Per} f \) is the set of all periodic points of a map \( f \);
- \( P(f) \) is the set of all periods of periodic points of a map \( f \);
- \( h(f) \) is the topological entropy of a map \( f \).
1. Preliminary lemmas

Let $X$ be a tree (see the definition in Introduction). Any closed connected subset of $X$ is also a tree and will be called a subtree of $X$. Let $A \subset X$; then $[A]$, the connected hull of $A$, is the smallest subtree containing $A$. We will use the following easy

**Property A.** If $\{A_1, A_2, \ldots, A_n\}$ are sets and $B = \bigcup_{i=1}^{n} [A_i]$ is connected then $B = [\bigcup_{i=1}^{n} A_i]$.

For two points $a, b \in X$ the connected hull of the set $\{a, b\}$ is denoted by $[a, b]$. If these points are distinct, $[a, b]$ in inner topology is homeomorphic to a closed interval; we also use the following notations: $(a, b) \equiv [a, b] \setminus \{a\}, [a, b) \equiv [a, b] \setminus \{b\}, (a, b) \equiv [a, b] \setminus \{a, b\}$. All the sets $[a, b], (a, b), [a, b), (a, b)$ are called intervals. Given a point $a$ and points $x, y$, we say that $x$ is closer to $a$ than $y$ iff $[a, x] \subset [a, y]$. Given subsets $C$ and $D$, we say that $C$ is closer to $a$ than $D$ iff for any $c \in C$ and $d \in D$, $c$ is closer to $a$ than $d$. In what follows we consider a continuous map $f : X \to X$.

**Lemma 1.** Let $[a, b], [c, d]$ be intervals and $f[a, b] \supseteq [c, d]$, $(fa, c) \cap (c, d) = \emptyset$, $(d, fb) \cap (c, d) = \emptyset$. Suppose also that $I_0, I_1, \ldots, I_k \subset [c, d]$ are intervals with pairwise disjoint interiors containing no vertices of $X$ and that $I_{i+1}$ is further from $c$ than $I_i$ for $0 \leq i \leq k - 1$. Then there exist intervals $J_0, J_1, \ldots, J_k \subset [a, b]$ with pairwise disjoint interiors such that $J_{i+1}$ is further from $a$ than $J_i$ for $0 \leq i \leq k - 1$ and $fJ_i = I_i$, $0 \leq i \leq k$.

**Proof.** Clearly, for any $0 \leq i \leq k$ there exist intervals $L \subset [a, b]$ such that $fL = I_i$. Indeed, let $I_i = [x, y]$ where $x$ is closer to $c$ than $y$. Choose the closest to $a$ preimage of $y$ and denote it by $y_{-1}$. Then choose the preimage of $x$ closest to $y_{-1}$ in $[a, y_{-1}]$, and denote it by $x_{-1}$. It is easy to see that $f[x_{-1}, y_{-1}] = [x, y]$. Say that an interval $L$ is good if $fL = I_i$ for some $i$ and for any interval $M$ the inclusion $M \nsubseteq L$ implies that $fM \neq I_i$. Choose for $0 \leq i \leq k$ the closest to $a$ good interval $J_i$ such that $fJ_i = I_i$. The relations $(fa, c) \cap (c, d) = \emptyset, (d, fb) \cap (c, d) = \emptyset$ easily imply now that $J_i$ is closer to $a$ than $J_{i+1}$ for $0 \leq i \leq k - 1$ which completes the proof. □
Lemma 2. Let \( J_0 = [c_0,d_0], J_1 = [c_1,d_1], \ldots, J_k = [c_k,d_k], J_{k+1} = J_0 \) be intervals, \( c_{k+1} = c_0, d_{k+1} = d_0 \) and \( 0 = n_0 < n_1 < \cdots < n_{k+1} \) be integers. Suppose that for any \( 0 \leq i \leq k \) we have \( f^{n_{i+1}-n_i}J_i \supset J_{i+1}, (d_{i+1}, f^{n_{i+1}-n_i}d_i) \cap (c_{i+1}, d_{i+1}) = \emptyset \). Then there exists \( z \in J_0 \) such that \( f^{n_i}z \in J_i(0 \leq i \leq k) \) and \( f^{n_{k+1}}z = z \).

Proof. Let us show that there exist intervals \( L_0, L_1, \ldots, L_M \subset J_0 \) such that \( \{f^{n_i}L_j(0 \leq i \leq k, 0 \leq j \leq M)\} \) are intervals with pairwise disjoint interiors, \( f^{n_{k+1}}L_0 \cup f^{n_{k+1}}L_1 \cup \cdots \cup f^{n_{k+1}}L_M = J_0 \) and for \( 0 \leq i \leq M - 1 \) the interval \( L_i \) is closer to \( c_0 \) than \( L_{i+1} \) and the interval \( f^{n_{k+1}}L_i \) is closer to \( c_0 \) than \( f^{n_{k+1}}L_{i+1} \).

First choose intervals \( N_0, N_1, \ldots, N_m \) so that their union is \( J_0 \), their interiors are pairwise disjoint and do not contain vertices of \( X \); we may assume that \( N_i \) is closer to \( c_0 \) than \( N_{i+1} \) for \( 0 \leq i \leq m - 1 \). Choose a point \( x_k \in J_k \) such that \( f^{n_{k+1}-n_k}x_k = c_0 \). By Lemma 1 we can find intervals \( T_0, T_1, \ldots, T_s \subset [x_k, d_k] \) with pairwise disjoint interiors so that \( f^{n_{k+1}-n_k}T_i = N_i, 0 \leq i \leq s \), and \( T_i \) is closer to \( x_k \) than \( T_{i+1}, 0 \leq i \leq s - 1 \). Let us divide the intervals \( T_i \) into subintervals with pairwise disjoint interiors which do not contain vertices of \( X \) and are ordered on the interval \([x_k, d_k]\). Repeating the construction and using Lemma 1, we will find the required intervals \( L_0, L_1, \ldots, L_M \).

Let us now show that there exists a point \( z \in \bigcup_{i=0}^{M} L_i \) such that \( f^{n_{k+1}}z = z \). Denote \( f^{n_{k+1}} \) by \( g \). Assume that \( J_0 = [0,1] \) and intervals \( L_0, L_1, \ldots, L_M \) and \( gL_0, gL_1, \ldots, gL_M \) increase in the usual sense. Since \( \bigcup_{i=1}^{M} gL_i = [0,1] \supset \bigcup_{i=1}^{M} L_i \) then \( \sup g|L_M = 1 \geq L_M, \inf g|L_0 = 0 \leq L_0 \); let us show that there exists \( i \) such that \( \sup g|L_i \geq L_i \) and \( \inf g|L_i \leq L_i \). Indeed, the fact that \( \inf g|L_{j+1} > L_{j+1} \) implies that \( \sup g|L_j > L_j \) (for the intervals \( \{L_j\} \) are ordered by increasing and \( \bigcup_{i=1}^{M} gL_i = [0,1] \)). Let \( i \) be the maximal such that \( \inf g|L_i \leq L_i \). If \( i = M \) then \( \inf g|L_M \leq L_M \) and \( \sup g|L_M \geq L_M \); if \( i < M \) then \( \inf g|L_{i+1} > L_{i+1} \), so \( \sup g|L_i > L_i \) and \( \inf g|L_i \leq L_i \). In any case \( gL_i \supset L_i \) which completes the proof. \( \square \)

Lemma 3. Let \( X \) be a tree, \( Y \subset X \) be a subtree and \( f : Y \to X \) be a continuous map such that if \( a \in Y \) then \( (a,fa) \cap Y \neq \emptyset \). Then there exists \( z \in Y \) such that \( fz = z \).

Proof. Let us construct a map \( h : X \to X \) in the following way. First define a map \( h : X \to X \) so that if \( x \in X \) then \( hx = x \) and if \( x \notin X \) then \( hx = y \) where \( y \in X \), all \( h \) are continuous and do not contain vertices of \( X \). Let \( f^i a \) for positive integers \( i \) be a subtree and \( J_0 = [0,1] \) be a continuous map such that if \( a \in X \) then \( (a,fa) \cap Y \neq \emptyset \). Then there exists \( z \in Y \) such that \( f^i z = z \).
is the unique point with \((y, x) \cap Y = \emptyset\). Now consider a map \(g = f \circ h : X \to X\). Then there exists \(z \in X\) such that \(gz = z\). If \(z \in Y\) then \(hz = z = fz\) and we are done. Let \(z \notin Y\). Then \(hz = y\) where \(y \in Y\) and \((y, z) \cap Y = \emptyset\); at the same time \(gz = f(hz) = fy = z\), so \((y, fy) \cap Y = \emptyset\) which is a contradiction. □

**Lemma 4.** Let \(Y \subset X\) be a subtree, \(f : X \to X\) be a continuous map. Then there exists a point \(y \in Y\) such that for any \(z \in Y\) the relation \(fz \in Y\) implies the inclusion \(f[y, z] \supset [y, fz]\) and either \(fy = y\) or \(fy \notin Y\) and \((y, fy) \cap Y = \emptyset\).

**Proof.** Consider the case when there is no fixed point in \(Y\). Then by Lemma 3 \((y, fy) \cap Y = \emptyset\) for some \(y \in Y\); since \(f(z) \in Y\) we now have \(f([y, z]) \supset [f(z), fy] \supset [f(z), y]\). □

In what follows we call the point \(y \in Y\) existing by Lemma 4 a basic point for \((f, Y)\).

### 2. Proofs of Theorem 1 and Corollary 3

Let \(x \in X\); we call points \(a, b \in orb x\) neighboring if \((a, b) \cap orb x = \emptyset\).

**Proof of Theorem 1.** Let \(x\) be a periodic point of period \(n > 1\) where \(n\) has no prime divisors less than \(\text{End}(X) + 1\). Let \(y\) be a basic point for \((f, [orb x])\); then \(y \in [orb x] \setminus orb x\). Consider the connected component \(Z\) of \([orb x] \setminus orb x\) such that \(y \in Z\). If \(z_1, z_2, \ldots, z_l\) are endpoints of \(Z\) then \(z_i \in orb x\) and \((y, z_i) \cap orb x = \emptyset\), \(1 \leq i \leq l\). Denote by \(Z_i\) the connected component of the set \([orb x] \setminus Z\) containing \(z_i\) and let \(Y_i = Z_i \cap orb x\). Note that \(l \leq \text{End}(X)\) and \(n \geq 3\). We divide the rest of the proof by steps.

**Step 1.** There exist two neighboring points \(a, b \in orb x\) such that \(b \in (a, y)\) and \(y \in f^{l-1}(a, b)\).

Let us describe the following procedure. Let \(F_1, \ldots, F_m\) be pairwise disjoint subsets of \(orb x = \bigcup_{i=1}^{m} F_i\) such that \([F_1], \ldots, [F_m]\) are pairwise disjoint subtrees of \(X\); denote \(\bigcup_{i=1}^{m} [F_i]\) by \(D_0\). Now consider the set \(D_1 = \bigcup_{i=1}^{m} ([F_i] \cup [F_i])\); let \(G_1, \ldots, G_u\) be the connected components of \(D_1\). Denoting \(H_1 = G_1 \cap orb x, \ldots, H_u = G_u \cap orb x\), we can easily see that \(G_i = [H_i], 1 \leq i \leq u\). Indeed, let \(A_1\) be the family of all sets of type \(f^r F_i, 1 \leq i \leq m, r = 0, 1\). Consider the set \(G_j\). By the definition there is a subfamily \(B_j \subset A\), such that \(G_j = \bigcup B_j\), \(H_j = \bigcup B_j \cap F\) and by Property
A we have $G_j = [H_j]$. Thus the procedure of constructing the pairwise disjoint subtrees may go on.

Let us show that if we start the procedure in question with $m \leq \text{End}(X)$ subtrees then after at most $m - 1$ steps we get the set $[\text{orb } x]$ (in other words we are going to show that $D_{m-1} = [\text{orb } x]$). By assumption $m$ and $n$ are relatively prime. Hence on the first step of the procedure there is at least one set, say $F_1$, such that $fF_1$ intersects with at least two of the sets $F_1, \ldots, F_m$ and so the number of connected components of $D_1$ is less than or equal to $m - 1$. Repeating this argument we get the conclusion.

It is quite easy to give the exact formula for sets $D_i$. However we need here only to show that $D_j \subset \bigcup_{i=1}^{m} \bigcup_{s=0}^{j} f^s[F_i] \equiv S_j$. Clearly, it is true for $j = 0, 1$. Suppose that it is the case for some $j$; we show that $D_{j+1} \subset \bigcup_{i=1}^{m} \bigcup_{s=0}^{j+1} f^s[F_i]$. Indeed, by the construction $D_{j+1} \subset D_j \cup fD_j \subset S_j \cup fS_j = S_{j+1}$ and we are done. Finally we have that $[\text{orb } x] = D_{m-1} \subset \bigcup_{i=1}^{m} \bigcup_{s=0}^{m-1} f^s[F_i]$. Now let us start our procedure with the sets $[Y_1] = Z_1, \ldots, [Y_l] = Z_l$; then after $l - 1$ steps we get the set $[\text{orb } x]$. In other words, $[\text{orb } x] \subset \bigcup_{i=1}^{l} \bigcup_{s=0}^{l-1} f^sZ_i$. Thus there exist $s \leq l - 1$ and two neighboring points $a, b \in \text{orb } x$ such that $b \in (a, y)$ and $y \in f^s(a, b)$; by the properties of basic points (see Lemma 4) this implies Step 1.

Choose a point $\zeta \in (a, b)$ such that $f^{l-1}\zeta = y$; let for definiteness $f^{l-1}[a, \zeta] \supset [y, z_1]$. Note that by the choice of $\zeta$ the sets $[a, \zeta]$ and $Z$ are disjoint.

**Step 2.** There exist integers $p, q, r$ such that $f^p[y, z_1] \supset [y, z_q], f^r[y, z_q] \supset [y, z_q]$ where $1 \leq r, p + r \leq l \leq \text{End}(X)$.

Lemma 4 implies for all $j \leq l$ the existence of an integer $s(j)$ such that $[y, f(z_j)] \supset [y, z_{s(j)}]$. Let $p$ be the smallest integer for which $q = s^p(1)$ is a periodic point of $s$. Denote by $r$ its period. Then $p + r \leq l$.

Denote by $D$ the set $\text{orb}_s(q) = \{q, s(q), \ldots, s^{r-1}(q)\}$.

**Step 3.** For any $v \geq (n - 1)r$ and $t \in D$ we have $f^v[y, z_t] \supset [\text{orb } x]$.

Clearly, if $B_j = f^{rj}[y, z_t] \cap \text{orb } x$ then $B_j \cup f^rB_j \subset B_{j+1}$ ($\forall j$). Thus $\bigcup_{j=0}^{n-1} f^{rj}z_t \subset f^{(n-1)r}[y, z_t]$. But $r \leq \text{End}(X)$ and hence $r$ and $n$ have no common divisors. Therefore $\bigcup_{j=0}^{n-1} f^{rj}z_t = \text{orb } x$ which proves Step 3.
Denote \( \text{End}(X) \) by \( c \) and assume that \( N \geq 2c(n-1) \). We will use Lemma 2 to show that \( f \) has a point of period \( N \). Let \( k = N - (n-1)r - l + 2 \). Consider the following sequence of intervals and integers (points \( \zeta, a \) have been chosen in Step 1):

0) \( J_0 = [\zeta, a], n_0 = 0; \)
1) \( J_1 = [y, z_1], n_1 = l - 1; \)
2) \( J_2 = [y, z_{s(1)}], n_2 = l; \)

\( \vdots \)

k) \( J_k = [y, z_{s_{k-1}(1)}], n_k = N - (n-1)r; \)

k+1) \( n_{k+1} = N. \)

It is easy to see that the inequalities \( n \geq 3, N \geq 2c(n-1), r \geq 1 \) and \( c \geq l \geq p+r \) imply that \( k = N - (n-1)r - l + 2 \geq (2c - r)(n - 1) - l + 2 \geq 2(l + p) - l + 2 \geq l. \) Hence \( s^{k-1}(1) \in D \) and by Step 3, \( f^{(n-1)r}[y, z_{s^{k-1}(1)}] \supset [\text{orb } x] \supset [\zeta, a] = J_0. \) So by Lemma 2 there is a point \( \alpha \in [\zeta, a] \) such that \( f^{n_i} \alpha \in J_i \) (0 \( \leq i \leq k \), \( f^N \alpha = \alpha. \)

Let us prove that \( N \) is a period of \( \alpha \). Indeed, otherwise \( \alpha \) has a period \( m \) which is a divisor of \( N \). Consider all iterates of \( \alpha \) of type \( f^{n_i} \alpha, 1 \leq i \leq k \). Clearly, \( \frac{N}{3} \geq \frac{2c(n-1)}{3} \geq \frac{4l}{3} > l - 1 = n_1 \) since \( c \geq l \) and \( n \geq 3 \). Furthermore, \( n_k = N - (n-1)r \geq \frac{N}{2} \) because \( N \geq 2c(n-1) \geq 2r(n-1). \) So \( l - 1 = n_1 \leq \frac{N}{3} < \frac{N}{2} \leq n_k = N - (n-1)r. \) At the same time, there exists \( i \) such that \( n_1 \leq \frac{N}{3} \leq m_i \leq \frac{N}{2} \leq n_k. \) Hence \( f^{m_i} \alpha \in [\zeta, a], \) but on the other hand, \( f^{m_i} \alpha \in \bigcup_{j=1}^{k}[y, z_j] = S \) where \( Z \cap [\zeta, a] = \emptyset \) (see the note before Step 2). This contradiction shows that \( \alpha \) has a period \( N. \)

Let \( l - 1 + p + r(n-1) = u. \) To estimate \( h(f) \) observe that \( f^u[\zeta, a] \supset [\zeta, a] \cup Z, \)

\( f^u Z \supset [\zeta, a] \cup Z \) and \( [\zeta, a], Z \) are disjoint compact sets. Let us show then \( h(f^u) \geq \ln 2. \) Indeed, consider the compact set \( S \) of all the points \( x \in [\zeta, a] \cup Z \) such that their \( f^u \)-orbits belong to \( [\zeta, a] \cup Z. \) Taking an open covering of \( S \) by the sets \( [\zeta, a] \cap S \) and \( Z \cap S \) we see directly by the definition of the topological entropy that \( h(f^u) \geq \ln 2 \) and so \( h(f) \geq \frac{\ln 2}{u}. \) The inequality \( u = l - 1 + p + r(n-1) \leq nc - 1 \) now implies that \( h(f) \geq \frac{\ln 2}{nc - 1}. \)

Let us pass to statement 2) of Theorem 1. First we show that if there is an interval map \( g' \) such that \( P(g') = E \) then there is \( g : [0, 1] \rightarrow [0, 1] \) such that \( g(0) = 0, g(1) = 1, P(g) = E. \) We may assume that \( g' \) is a map of the interval \([1/3, 2/3]\).
into itself. Define $g : [0,1] \to [0,1]$ so that it coincides with $g'$ on $[1/3, 2/3]$, $g(0) = 0, g(1) = 1$ and $g$ is linear on $[0,1/3]$ and on $[2/3,1]$. It is easy to see now that $P(g) = P(g') = E$, so $g$ has the required properties.

Let $Y$ be a tree, $y_1, y_2, \ldots, y_k$ be its endpoints and $m \leq k$. Let us construct a map $\phi : Y \to Y$ such that $P(\phi) = \{m,1\}$ and $\phi(y_i) = y_{i+1}(1 \leq i \leq m-1), \phi(y_m) = y_1$. Indeed, choose small nondegenerate neighborhoods $[y_i, y'_i]$ of points $y_1, \ldots, y_m$ containing no vertices of $Y$ distinct from endpoints. Let $B = Y \setminus \bigcup_{i=1}^{m} [y_i, y'_i], x \in B$ and $\phi(z) = x (\forall z \in B)$. Let us define $\phi|[y_i, y'_i], 1 \leq i \leq m$ so that $\phi(y_i) = y_{i+1}(1 \leq i \leq m-1), \phi(y_m) = y_1, \phi(y'_i) = x$ and also $\phi|[y_i, y'_i](1 \leq i \leq m)$ is injective. Then it is easy to see that $P(\phi) = \{m,1\}$.

Now let $1 \leq m \leq End(X)$ and $g : [0,1] \to [0,1]$ be a map with $P(g) = E, g(0) = 0, g(1) = 1$. Let us construct a map $f : X \to X$ such that $P(f) = 1 \cup mE$ where $mE = \{mk : k \in E\}$. First fix $m$ endpoints $z_1, \ldots, z_m$ of $X$ and their small neighborhoods $[z_i, y_i]$ containing no vertices of $Y$ distinct from endpoints. Let $Y = X \setminus \bigcup_{i=1}^{m} [z_i, y_i]$. Applying the result from the previous paragraph we can find a map $\phi : Y \to Y$ such that $\phi(y_i) = y_{i+1}(1 \leq i \leq m-1), \phi(y_m) = y_1$ and $P(\phi) = \{m,1\}$. Let us define $f : X \to X$ so that it coincides with $\phi$ on $Y$, $f|[z_i, y_i]$ is a homeomorphism onto $[z_{i+1}, y_{i+1}]$ and $f(z_i) = z_{i+1}, f(y_i) = y_{i+1}$ for $1 \leq i \leq m-1$. Moreover, define $f|[z_m, y_m]$ so that $f(z_m) = z_1, f(y_m) = y_1$, $f([z_m, y_m]) = [z_1, y_1]$ and $f^m|[z_1, y_1]$ is topologically conjugate to $g$. The choice of $g$ guarantees that the construction is possible and that $P(f) = 1 \cup mE$. This completes the proof. \[\square\]

Corollary 3 follows from Theorem 1 and the spectral decomposition theorem for graph maps (see [3-5]).

**Corollary 3.** Let $f : X \to X$ be continuous. Then the following two statements are equivalent:

1. there exists $n$ such that $f$ has a cycle of period $mn$ for any $m$;
2. $h(f) > 0$. 

Remark. Note that Corollary 3 is true for arbitrary continuous graph maps [6,7]; see also [11] for the alternative proof.

Proof. Statement 1) implies statement 2) by Corollary 1. The inverse implication follows from the spectral decomposition theorem for graph maps (see [3-5]) and some properties of maps with the specification property.

First we need the following definition: a graph map $\varphi : M \to N$ is called monotone if for any connected subset of $N$ its $\varphi$-preimage is a connected subset of $M$. We also need the definition of the specification property. Namely, let $T : X \to X$ be a map of a compact infinite metric space $(X, d)$ into itself. A dynamical system $(X, T)$ is said to have the specification property [10] if for any $\varepsilon > 0$ there exists such integer $M = M(\varepsilon)$ that for any $k > 1$, for any $k$ points $x_1, x_2, \ldots, x_k \in X$, for any integers $a_1 \leq b_1 < a_2 \leq b_2 < \ldots < a_k \leq b_k$ with $a_i - b_{i-1} \geq M$, $2 \leq i \leq k$ and for any integer $p$ with $p \geq M + b_k - a_1$ there exists a point $x \in X$ with $T^p x = x$ such that $d(T^n x, T^n x_i) \leq \varepsilon$ for $a_i \leq n \leq b_i, 1 \leq i \leq k$.

By the results of [3-5], the fact that the map $f : X \to X$ has a positive topological entropy implies that there exist a subtree $Y \subset X$, an integer $n$, a tree $Z$, a continuous map $g : Z \to Z$ with the specification property and a monotone map $\varphi : Y \to Z$ such that $f^n Y = Y$ and $f^n | Y \circ \varphi = \varphi \circ g$ (i.e. $\varphi$ monotonically semiconjugates $f^n | Y$ to $g$). Moreover, for any $1 \leq i \leq n - 1$ the set $Y \cap f^i Y$ either is empty or has a $g$-fixed point as a $\varphi$-image. This implies that if $z \in Z$ is a $g$-periodic point of period $s > 1$ then $\varphi^{-1}(z)$ contains an $f$-periodic point of period $sn$. Indeed, by monotonicity of $\varphi$ the set $\varphi^{-1}(z)$ is connected, so the fixed point property for trees implies that there is an $f^{-sn}$-fixed point $\zeta \in \varphi^{-1}(z)$. Let the $f$-period of $\zeta$ is $k < sn$. Then $k$ cannot be a multiplier of $n$ since $g$-period of $z$ is exactly $s$, so $f^k \zeta = \zeta \in Y \cap f^k Y$ which by the just mentioned properties implies that $\varphi(\zeta) = z$ is a $g$-fixed point contradicting the choice of $s$. Hence the period of $\zeta$ is $sk$.

The specification property of $g$ implies that $g$ has all sufficiently big periods. The arguments from the preceding paragraph now show that $f$ has all the periods which are sufficiently big multipliers of $n$ thus completing the proof. □

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