PERIODS IMPLYING ALMOST ALL PERIODS FOR TREE MAPS

A. M. Blokh

Department of Mathematics, Wesleyan University
Middletown, CT 06459-0128, USA

August 1991, revised May 1992

Abstract. Let X be a compact tree, $f : X \longrightarrow X$ be a continuous map and $End(X)$ be the number of endpoints of X. We prove the following

Theorem 1. *Let X be a tree. Then the following holds.*

(1) *Let $n > 1$ be an integer with no prime divisors less than or equal to $End(X) + 1$. If a map $f : X \longrightarrow X$ has a cycle of period n, then f has cycles of all periods greater than $2End(X)(n-1)$. Moreover, $h(f) \geq \dfrac{\ln 2}{nEnd(X) - 1}$.*
(2) *Let $1 \leq n \leq End(X)$ and E be the set of all periods of cycles of some interval map. Then there exists a map $f : X \longrightarrow X$ such that the set of all periods of cycles of f is $\{1\} \cup nE$, where $nE \equiv \{nk : k \in E\}$.*

This implies that if p is the least prime number greater than $End(X)$ and f has cycles of all periods from 1 to $2End(X)(p-1)$, then f has cycles of all periods (for tree maps this verifies Misiurewicz's conjecture, made in Bratislava in 1990). Combining the spectral decomposition theorem for graph maps (see [3-5]) with our results, we prove the equivalence of the following statements for tree maps:

(1) there exists n such that f has a cycle of period mn for any m;
(2) $h(f) > 0$.

Note that Misiurewicz's conjecture and the last result are true for graph maps ([6,7]); the alternative proof of the last result may be also found in [11].

0. Introduction

Let us call one-dimensional compact branched manifolds *graphs*; we call them *trees* if they are connected and contractible. Note that by the definition we deal with the finite trees. *In what follows we consider only continuous tree maps.* One of the

1991 *Mathematics Subject Classification.* 54H20, 58F11.

Key words and phrases. Tree map, periodic point, topological entropy.

This paper was partly written while I was visiting MPI für Mathematik in Bonn and SUNY at Stony Brook; I would like to thank both institutions for their kind hospitality

well-known and impressive results about dynamical properties of one-dimensional maps is Sharkovskii's theorem [12] about the co-existence of periods of cycles for interval maps. To formulate it let us introduce the following *Sharkovskii ordering* for positive integers:

$$(*) \qquad 3 \prec 5 \prec 7 \prec \cdots \prec 2\cdot 3 \prec 2\cdot 5 \prec 2\cdot 7 \prec \cdots \prec 8 \prec 4 \prec 2 \prec 1$$

Denote by $S(k)$ the set of all integers m such that $k \prec m$ or $k = m$ and by $S(2^\infty)$ the set $\{1, 2, 4, 8, \dots\}$. Also denote by $P(\varphi)$ the set of periods of cycles of a map φ.

Theorem[12]. *Let $g : [0,1] \longrightarrow [0,1]$ be a continuous map. Then there exists $k \in \mathbb{N} \cup 2^\infty$ such that $P(g) = S(k)$. Moreover, for any such k there exists a map $g : [0,1] \longrightarrow [0,1]$ with $P(g) = S(k)$.*

Generalizations of Sharkovskii's theorem were studied in [1] for maps of the triod (a tree in the shape of the letter Y) and for general n-od in [2]).

Sharkovskii's theorem implies that if a map $f : \mathbb{R} \longrightarrow \mathbb{R}$ has a cycle of period 3 then it has cycles of all periods. The following conjecture, formulated by M. Misiurewicz at the Problem Session at Czecho-Slovak Summer Mathematical School near Bratislava in 1990, is related to the aforementioned property of interval maps.

Misiurewicz's Conjecture. *For a graph X there exists an integer $L = L(X)$ such that for a map $f : X \longrightarrow X$ the inclusion $P(f) \supset \{1, 2, \dots, L\}$ implies that $P(f) = \mathbb{N}$.*

We verify Misiurewicz's conjecture when X is a tree. The general verification of this conjecture for arbitrary continuous graph maps may be found in [6,7]. Note that all results of the paper are true in the same formulations for finite unions of connected trees; the corresponding extension is left to the reader.

Fix a tree X. We use the terms "vertex", "edge" and "endpoint" in the usual sense. Denote the number of endpoints of X by $End(X)$. We prove the following

Theorem 1. *Let X be a tree. Then the following holds.*

(1) *Let $n > 1$ be an integer with no prime divisors less than or equal to $End(X)$. If a map $f : X \longrightarrow X$ has a cycle of period n, then f has cycles of all periods greater than $2End(X)(n-1)$. Moreover, $h(f) \geq \dfrac{\ln 2}{End(X) - 1}$.*

(2) *Let* $1 \leq n \leq End(X)$ *and* E *be the set of all periods of cycles of some interval map. Then there exists a map* $f : X \longrightarrow X$ *such that the set of all periods of cycles of* f *is* $\{1\} \cup nE$*, where* $nE \equiv \{nk : k \in E\}$.

For interval maps this implies that when n is odd and f has a point of period n, then f has all periods greater than $4(n-1)$. This statement is slightly weaker than Sharkovskii's theorem.

Let us formulate some corollaries of Theorem 1.

Corollary 1 (cf. [9]). *Let* $f : X \longrightarrow X$ *be a cycle of period* $n = pk$ *where* $p > 1$ *has no prime divisors less than* $End(X) + 1$. *Then* $h(f) \geq \frac{\ln 2}{k[pEnd(X) - 1]} > \frac{\ln 2}{nEnd(X) - n}$.

Proof. It is enough to consider the map f^k and apply Theorem 1. □

The next corollary verifies for tree maps Misiurewicz's conjecture.

Corollary 2. *Let* p *be the least prime number greater than* $End(X)$. *If* $f : X \longrightarrow X$ *has cycles of all periods from* 1 *to* $2End(X)(p-1)$ *then* f *has cycles of all periods.*

Theorem 1 and the spectral decomposition theorem for graph maps ([3-5]) imply

Corollary 3. *The following two statements are equivalent:*

(1) *there exists* n *such that* f *a cycle of period* mn *for any* m*;*

(2) $h(f) > 0$.

In fact Corollary 3 is true for arbitrary graph maps ([6,7]; the different proof may be found in [11]). The preprint [8] contains a preliminary version of this paper.

Notation

f^n is the n-fold iterate of a map f;

$\text{orb}\, x \equiv \{f^n x\}_{n=0}^{\infty}$ is the *orbit (trajectory)* of x;

$\text{Per}\, f$ is the set of all periodic points of a map f;

$P(f)$ is the set of all periods of periodic points of a map f;

$h(f)$ is the topological entropy of a map f;

1. Preliminary lemmas

Let X be a tree (see the definition in Introduction). Any closed connected subset of X is also a tree and will be called a *subtree* of X. Let $A \subset X$; then $[A]$, the *connected hull of* A, is the smallest subtree containing A. We will use the following easy

Property A. *If* $\{A_1, A_2, \dots, A_n\}$ *are sets and* $B = \bigcup_{i=1}^n [A_i]$ *is connected then* $B = [\bigcup_{i=1}^n A_i]$.

For two points $a, b \in X$ the connected hull of the set $\{a, b\}$ is denoted by $[a, b]$. If these points are distinct, $[a, b]$ in inner topology is homeomorphic to a closed interval; we also use the following notations: $(a, b] \equiv [a, b] \setminus \{a\}, [a, b) \equiv [a, b] \setminus \{b\}, (a, b) \equiv [a, b] \setminus \{a, b\}$. All the sets $[a, b], (a, b], [a, b), (a, b)$ are called *intervals*. Given a point a and points x, y, we say that x *is closer to* a *than* y iff $[a, x] \subset [a, y]$. Given subsets C and D, we say that C *is closer to* a *than* D iff for any $c \in C$ and $d \in D$, c is closer to a than d. In what follows we consider a continuous map $f : X \to X$.

Lemma 1. *Let* $[a, b], [c, d]$ *be intervals and* $f[a, b] \supset [c, d]$, $(fa, c) \cap (c, d) = \emptyset$, $(d, fb) \cap (c, d) = \emptyset$. *Suppose also that* $I_0, I_1, \dots, I_k \subset [c, d]$ *are intervals with pairwise disjoint interiors containing no vertices of* X *and that* I_{i+1} *is further from* c *than* I_i *for* $0 \le i \le k-1$. *Then there exist intervals* $J_0, J_1, \dots, J_k \subset [a, b]$ *with pairwise disjoint interiors such that* J_{i+1} *is further from* a *than* J_i *for* $0 \le i \le k-1$ *and* $fJ_i = I_i$, $0 \le i \le k$.

Proof. Clearly, for any $0 \le i \le k$ there exist intervals $L \subset [a, b]$ such that $fL = I_i$. Indeed, let $I_i = [x, y]$ where x is closer to c than y. Choose the closest to a preimage of y and denote it by y_{-1}. Then choose the preimage of x closest to y_{-1} in $[a, y_{-1}]$, and denote it by x_{-1}. It is easy to see that $f[x_{-1}, y_{-1}] = [x, y]$. Say that an interval L is *good* if $fL = I_i$ for some i and for any interval M the inclusion $M \subsetneqq L$ implies that $fM \ne I_i$. Choose for $0 \le i \le k$ the closest to a good interval J_i such that $fJ_i = I_i$. The relations $(fa, c) \cap (c, d) = \emptyset, (d, fb) \cap (c, d) = \emptyset$ easily imply now that J_i is closer to a than J_{i+1} for $0 \le i \le k-1$ which completes the proof. □

Lemma 2. *Let $J_0 = [c_0, d_0], J_1 = [c_1, d_1], \dots, J_k = [c_k, d_k], J_{k+1} = J_0$ be intervals, $c_{k+1} = c_0, d_{k+1} = d_0$ and $0 = n_0 < n_1 < \dots < n_{k+1}$ be integers. Suppose that for any $0 \le i \le k$ we have $f^{n_{i+1}-n_i} J_i \supset J_{i+1}, (d_{i+1}, f^{n_{i+1}-n_i} d_i) \cap (c_{i+1}, d_{i+1}) = \emptyset$. Then there exists $z \in J_0$ such that $f^{n_i} z \in J_i (0 \le i \le k)$ and $f^{n_{k+1}} z = z$.*

Proof. Let us show that there exist intervals $L_0, L_1, \dots, L_M \subset J_0$ such that $\{f^{n_i} L_j \subset J_i : 0 \le i \le k+1, 0 \le j \le M\}$ are intervals with pairwise disjoint interiors, $f^{n_{k+1}} L_0 \cup f^{n_{k+1}} L_1 \cup \dots \cup f^{n_{k+1}} L_M = J_0$ and for $0 \le i \le M-1$ the interval L_i is closer to c_0 than L_{i+1} and the interval $f^{n_{k+1}} L_i$ is closer to c_0 than $f^{n_{k+1}} L_{i+1}$.

First choose intervals $N_0, N_1, \dots, N_m$ so that their union is J_0, their interiors are pairwise disjoint and do not contain vertices of X; we may assume that N_i is closer to c_0 than N_{i+1} for $0 \le i \le m-1$. Choose a point $x_k \in J_k$ such that $f^{n_{k+1}-n_k} x_k = c_0$. By Lemma 1 we can find intervals $T_0, T_1, \dots, T_s \subset [x_k, d_k]$ with pairwise disjoint interiors so that $f^{n_{k+1}-n_k} T_i = N_i, 0 \le i \le s$, and T_i is closer to x_k than $T_{i+1}, 0 \le i \le s-1$. Let us divide the intervals T_i into subintervals with pairwise disjoint interiors which do not contain vertices of X and are ordered on the interval $[x_k, d_k]$. Repeating the construction and using Lemma 1, we will find the required intervals $L_0, L_1, \dots, L_M$.

Let us now show that there exists a point $z \in \bigcup_{i=0}^M L_i$ such that $f^{n_{k+1}} z = z$. Denote $f^{n_{k+1}}$ by g. Assume that $J_0 = [0,1]$ and intervals $L_0, L_1, \dots, L_M$ and $gL_0, gL_1, \dots, gL_M$ increase in the usual sense. Since $\bigcup_{i=1}^M gL_i = [0,1] \supset \bigcup_{i=1}^M L_i$ then $\sup g|L_M = 1 \ge L_M$, $\inf g|L_0 = 0 \le L_0$; let us show that there exists i such that $\sup g|L_i \ge L_i$ and $\inf g|L_i \le L_i$. Indeed, the fact that $\inf g|L_{j+1} > L_{j+1}$ implies that $\sup g|L_j > L_j$ (for the intervals $\{L_j\}$ are ordered by increasing and $\bigcup_{i=1}^M gL_i = [0,1]$). Let i be the maximal such that $\inf g|L_i \le L_i$. If $i = M$ then $\inf g|L_M \le L_M$ and $\sup g|L_M \ge L_M$; if $i < M$ then $\inf g|L_{i+1} > L_{i+1}$, so $\sup g|L_i > L_i$ and $\inf g|L_i \le L_i$. In any case $gL_i \supset L_i$ which completes the proof. □

Lemma 3. *Let X be a tree, $Y \subset X$ be a subtree and $f : Y \to X$ be a continuous map such that if $a \in Y$ then $(a, fa] \cap Y \ne \emptyset$. Then there exists $z \in Y$ such that $fz = z$.*

Proof. Let us construct a map $g : X \to X$ in the following way. First define a map $h : X \to Y$ so that if $x \in Y$ then $hx = x$ and if $x \notin Y$ then $hx = y$ where $y \in Y$

is the unique point with $(y, x] \cap Y = \emptyset$. Now consider a map $g = f \circ h : X \to X$. Then there exists $z \in X$ such that $gz = z$. If $z \in Y$ then $hz = z = fz$ and we are done. Let $z \notin Y$. Then $hz = y$ where $y \in Y$ and $(y, z] \cap Y = \emptyset$; at the same time $gz = f(hz) = fy = z$, so $(y, fy] \cap Y = \emptyset$ which is a contradiction. □

Lemma 4. *Let $Y \subset X$ be a subtree, $f : X \to X$ be a continuous map. Then there exists a point $y \in Y$ such that for any $z \in Y$ the relation $fz \in Y$ implies the inclusion $f[y, z] \supset [y, fz]$ and either $fy = y$ or $fy \notin Y$ and $(y, fy] \cap Y = \emptyset$.*

Proof. Consider the case when there is no fixed point in Y. Then by Lemma 3 $(y, fy] \cap Y = \emptyset$ for some $y \in Y$; since $f(z) \in Y$ we now have $f([y, z]) \supset [f(z), f(y)] \supset [f(z), y]$. □

In what follows we call the point $y \in Y$ existing by Lemma 4 *a basic point for* (f, Y).

2. Proofs of Theorem 1 and Corollary 3

Let $x \in X$; we call points $a, b \in orb\, x$ *neighboring* if $(a, b) \cap orb\, x = \emptyset$.

Proof of Theorem 1. Let x be a periodic point of period $n > 1$ where n has no prime divisors less than $End(X) + 1$. Let y be a basic point for $(f, [orb\, x])$; then $y \in [orb\, x] \setminus orb\, x$. Consider the connected component Z of $[orb\, x] \setminus orb\, x$ such that $y \in Z$. If $z_1, z_2, \dots, z_l$ are endpoints of Z then $z_i \in orb\, x$ and $(y, z_i) \cap orb\, x = \emptyset, 1 \le i \le l$. Denote by Z_i the connected component of the set $[orb\, x] \setminus Z$ containing z_i and let $Y_i = Z_i \cap orb\, x$. Note that $l \le End(X)$ and $n \ge 3$. We divide the rest of the proof by steps.

Step 1. *There exist two neighboring points $a, b \in orb\, x$ such that $b \in (a, y)$ and $y \in f^{l-1}(a, b)$.*

Let us describe the following procedure. Let $F_1, \dots, F_m$ be pairwise disjoint subsets of $orb\, x = \bigcup_{i=1}^m F_i$ such that $[F_1], \dots, [F_m]$ are pairwise disjoint subtrees of X; denote $\bigcup_{i=1}^m [F_i]$ by D_0. Now consider the set $D_1 = \bigcup_{i=1}^m ([fF_i] \cup [F_i])$; let $G_1, \dots, G_u$ be the connected components of D_1. Denoting $H_1 = G_1 \cap orb\, x, \dots, H_u = G_u \cap orb\, x$, we can easily see that $G_i = [H_i], 1 \le i \le u$. Indeed, let $\mathcal{A}_1$ be the family of all sets of type $f^r F_i, 1 \le i \le m, r = 0, 1$. Consider the set G_j. By the definition there is a subfamily $\mathcal{B}^j \subset \mathcal{A}_1$ such that $G_j = \bigcup [F], H_j = \bigcup F$ and by Property

A we have $G_j = [H_j]$. Thus the procedure of constructing the pairwise disjoint subtrees may go on.

Let us show that if we start the procedure in question with $m \le End(X)$ subtrees then after at most $m-1$ steps we get the set $[orb\, x]$ (in other words we are going to show that $D_{m-1} = [orb\, x]$). By assumption m and n are relatively prime. Hence on the first step of the procedure there is at least one set, say F_1, such that fF_1 intersects with at least two of the sets $F_1, \dots, F_m$ and so the number of connected components of D_1 is less than or equal to $m-1$. Repeating this argument we get the conclusion.

It is quite easy to give the exact formula for sets D_i. However we need here only to show that $D_j \subset \bigcup_{i=1}^m \bigcup_{s=0}^j f^s[F_i] \equiv S_j$. Clearly, it is true for $j = 0, 1$. Suppose that it is the case for some j;we show that $D_{j+1} \subset \bigcup_{i=1}^m \bigcup_{s=0}^{j+1} f^s[F_i]$. Indeed, by the construction $D_{j+1} \subset D_j \cup fD_j \subset S_j \cup fS_j = S_{j+1}$ and we are done. Finally we have that $[orb\, x] = D_{m-1} \subset \bigcup_{i=1}^m \bigcup_{s=0}^{m-1} f^s[F_i]$. Now let us start our procedure with the sets $[Y_1] = Z_1, \dots, [Y_l] = Z_l$; then after $l-1$ steps we get the set $[orb\, x]$. In other words, $[orb\, x] \subset \bigcup_{i=1}^l \bigcup_{s=0}^{l-1} f^s Z_i$. Thus there exist $s \le l-1$ and two neighboring points $a, b \in orb\, x$ such that $b \in (a, y)$ and $y \in f^s(a, b)$; by the properties of basic points (see Lemma 4) this implies Step 1.

Choose a point $\zeta \in (a, b)$ such that $f^{l-1}\zeta = y$; let for definiteness $f^{l-1}[a, \zeta] \supset [y, z_1]$. Note that by the choice of ζ the sets $[a, \zeta]$ and Z are disjoint.

Step 2. *There exist integers p, q, r such that $f^p[y, z_1] \supset [y, z_q]$, $f^r[y, z_q] \supset [y, z_q]$ where $1 \le r$, $p + r \le l \le End(X)$.*

Lemma 4 implies for all $j \le l$ the existence of an integer $s(j)$ such that $[y, f(z_j)] \supset [y, z_{s(j)}]$. Let p be the smallest integer for which $q = s^p(1)$ is a periodic point of s. Denote by r its period. Then $p + r \le l$.

Denote by D the set $orb_s(q) = \{q, s(q), \dots, s^{r-1}(q)\}$.

Step 3. *For any $v \ge (n-1)r$ and $t \in D$ we have $f^v[y, z_t] \supset [orb\, x]$.*

Clearly, if $B_j = f^{rj}[y, z_t] \cap orb\, x$ then $B_j \cup f^r B_j \subset B_{j+1}$ $(\forall j)$. Thus $\bigcup_{j=0}^{n-1} f^{rj} z_t \subset f^{(n-1)r}[y, z_t]$. But $r \le End(X)$ and hence r and n have no common divisors. Therefore $\bigcup_{j=0}^{n-1} f^{rj} z_t = orb\, x$ which proves Step 3.

Denote $End(X)$ by c and assume that $N \geq 2c(n-1)$. We will use Lemma 2 to show that f has a point of period N. Let $k = N - (n-1)r - l + 2$. Consider the following sequence of intervals and integers (points ζ, a have been chosen in Step 1):

0) $J_0 = [\zeta, a], n_0 = 0$;

1) $J_1 = [y, z_1], n_1 = l - 1$;

2) $J_2 = [y, z_{s(1)}], n_2 = l$;

$\vdots$

k) $J_k = [y, z_{s^{k-1}(1)}], n_k = N - (n-1)r$;

k+1) $n_{k+1} = N$.

It is easy to see that the inequalities $n \geq 3, N \geq 2c(n-1), r \geq 1$ and $c \geq l \geq p+r$ imply that $k = N-(n-1)r-l+2 \geq (2c-r)(n-1)-l+2 \geq 2(l+p)-l+2 \geq l$. Hence $s^{k-1}(1) \in D$ and by Step 3, $f^{(n-1)r}[y, z_{s^{k-1}(1)}] \supset [orb\, x] \supset [\zeta, a] = J_0$. So by Lemma 2 there is a point $\alpha \in [\zeta, a]$ such that $f^{n_i}\alpha \in J_i$ $(0 \leq i \leq k), f^N\alpha = \alpha$.

Let us prove that N is a period of α. Indeed, otherwise α has a period m which is a divisor of N. Consider all iterates of α of type $f^{n_i}\alpha, 1 \leq i \leq k$. Clearly, $\frac{N}{3} \geq \frac{2c(n-1)}{3} \geq \frac{4l}{3} > l - 1 = n_1$ since $c \geq l$ and $n \geq 3$. Furthermore, $n_k = N - (n-1)r \geq \frac{N}{2}$ because $N \geq 2c(n-1) \geq 2r(n-1)$. So $l - 1 = n_1 \leq \frac{N}{3} < \frac{N}{2} \leq n_k = N - (n-1)r$. At the same time, there exists i such that $n_1 \leq \frac{N}{3} \leq mi \leq \frac{N}{2} \leq n_k$. Hence $f^{mi}\alpha = \alpha \in [\zeta, a]$, but on the other hand, $f^{mi}\alpha \in \bigcup_{j=1}^{l}[y, z_j] = S$ where $Z \cap [\zeta, a] = \emptyset$ (see the note before Step 2). This contradiction shows that α has a period N.

Let $l - 1 + p + r(n-1) = u$. To estimate $h(f)$ observe that $f^u[\zeta, a] \supset [\zeta, a] \cup Z$, $f^u Z \supset [\zeta, a] \cup Z$ and $[\zeta, a]$, Z are disjoint compact sets. Let us show then $h(f^u) \geq \ln 2$. Indeed, consider the compact set S of all the points $x \in [\zeta, a] \cup Z$ such that their f^u-orbits belong to $[\zeta, a] \cup Z$. Taking an open covering of S by the sets $[\zeta, a] \cap S$ and $Z \cap S$ we see directly by the definition of the topological entropy that $h(f^u) \geq \ln 2$ and so $h(f) \geq \frac{\ln 2}{u}$. The inequality $u = l - 1 + p + r(n-1) \leq nc - 1$ now implies that $h(f) \geq \frac{\ln 2}{nc - 1}$.

Let us pass to statement 2) of Theorem 1. First we show that if there is an interval map g' such that $P(g') = E$ then there is $g : [0, 1] \to [0, 1]$ such that $g(0) = 0, g(1) = 1, P(g) = E$. We may assume that g' is a map of the interval $[1/3, 2/3]$

into itself. Define $g : [0,1] \longrightarrow [0,1]$ so that it coincides with g' on $[1/3, 2/3]$, $g(0) = 0, g(1) = 1$ and g is linear on $[0, 1/3]$ and on $[2/3, 1]$. It is easy to see now that $P(g) = P(g') = E$, so g has the required properties.

Let Y be a tree, $y_1, y_2, \dots, y_k$ be its endpoints and $m \le k$. Let us construct a map $\phi : Y \longrightarrow Y$ such that $P(\phi) = \{m, 1\}$ and $\phi(y_i) = y_{i+1}(1 \le i \le m-1), \phi(y_m) = y_1$. Indeed, choose small nondegenerate neighborhoods $[y_i, y_i']$ of points $y_1, \dots, y_m$ containing no vertices of Y distinct from endpoints. Let $B = Y \setminus \bigcup_{i=1}^m [y_i, y_i']$, $x \in B$ and $\phi(z) = x \, (\forall z \in B)$. Let us define $\phi|[y_i, y_i'], 1 \le i \le m$ so that $\phi(y_i) = y_{i+1}(1 \le i \le m-1), \phi(y_m) = y_1$, $\phi(y_i') = x$ and also $\phi|[y_i, y_i'](1 \le i \le m)$ is injective. Then it is easy to see that $P(\phi) = \{m, 1\}$.

Now let $1 \le m \le End(X)$ and $g : [0,1] \longrightarrow [0,1]$ be a map with $P(g) = E, g(0) = 0, g(1) = 1$. Let us construct a map $f : X \longrightarrow X$ such that $P(f) = 1 \cup mE$ where $mE \equiv \{mk : k \in E\}$. First fix m endpoints $z_1, \dots, z_m$ of X and their small neighborhoods $[z_i, y_i]$ containing no vertices of Y distinct from endpoints. Let $Y = X \setminus \bigcup_{i=1}^m [z_i, y_i)$. Applying the result from the previous paragraph we can find a map $\phi : Y \longrightarrow Y$ such that $\phi(y_i) = y_{i+1}(1 \le i \le m-1), \phi(y_m) = y_1$ and $P(\phi) = \{m, 1\}$. Let us define $f : X \longrightarrow X$ so that it coincides with ϕ on Y, $f|[z_i, y_i]$ is a homeomorphism onto $[z_{i+1}, y_{i+1}]$ and $f(z_i) = z_{i+1}, f(y_i) = y_{i+1}$ for $1 \le i \le m-1$. Moreover, define $f|[z_m, y_m]$ so that $f(z_m) = z_1, f(y_m) = y_1$, $f([z_m, y_m]) = [z_1, y_1]$ and $f^m|[z_1, y_1]$ is topologically conjugate to g. The choice of g guarantees that the construction is possible and that $P(f) = 1 \cup mE$. This completes the proof. □

Corollary 3 follows from Theorem 1 and the spectral decomposition theorem for graph maps (see [3-5]).

Corollary 3. *Let $f : X \longrightarrow X$ be continuous. Then the following two statements are equivalent:*

(1) *there exists n such that f has a cycle of period mn for any m;*
(2) $h(f) > 0$.

Remark. Note that Corollary 3 is true for arbitrary continuous graph maps [6,7]; see also [11] for the alternative proof.

Proof. Statement 1) implies statement 2) by Corollary 1. The inverse implication follows from the spectral decomposition theorem for graph maps (see [3-5]) and some properties of maps with the specification property.

First we need the following definition: a graph map $\varphi : M \longrightarrow N$ is called *monotone* if for any connected subset of N its φ-preimage is a connected subset of M. We also need the definition of the specification property. Namely, let $T : X \to X$ be a map of a compact infinite metric space (X, d) into itself. A dynamical system (X, T) is said to have *the specification property* [10] if for any $\varepsilon > 0$ there exists such integer $M = M(\varepsilon)$ that for any $k > 1$, for any k points $x_1, x_2, \dots, x_k \in X$, for any integers $a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_k \leq b_k$ with $a_i - b_{i-1} \geq M$, $2 \leq i \leq k$ and for any integer p with $p \geq M + b_k - a_1$ there exists a point $x \in X$ with $T^p x = x$ such that $d(T^n x, T^n x_i) \leq \varepsilon$ for $a_i \leq n \leq b_i, 1 \leq i \leq k$.

By the results of [3-5], the fact that the map $f : X \longrightarrow X$ has a positive topological entropy implies that there exist a subtree $Y \subset X$, an integer n, a tree Z, a continuous map $g : Z \longrightarrow Z$ with the specification property and a monotone map $\varphi : Y \longrightarrow Z$ such that $f^n Y = Y$ and $f^n | Y \circ \varphi = \varphi \circ g$ (i.e. φ monotonically semiconjugates $f^n | Y$ to g). Moreover, for any $1 \leq i \leq n-1$ the set $Y \cap f^i Y$ either is empty or has a g-fixed point as a φ-image. This implies that if $z \in Z$ is a g-periodic point of period $s > 1$ then $\varphi^{-1}(z)$ contains an f-periodic point of period sn. Indeed, by monotonicity of φ the set $\varphi^{-1}(z)$ is connected, so the fixed point property for trees implies that there is an f^{sn}-fixed point $\zeta \in \varphi^{-1}(z)$. Let the f-period of ζ is $k < sn$. Then k cannot be a multipler of n since g-period of z is exactly s, so $f^k \zeta = \zeta \in Y \cap f^k Y$ which by the just mentioned properties implies that $\varphi(\zeta) = z$ is a g-fixed point contradicting the choice of s. Hence the period of ζ is sk.

The specification property of g implies that g has all sufficiently big periods. The arguments from the preceding paragraph now show that f has all the periods which are sufficiently big multiplers of n thus completing the proof. □

Acknowledgments. I would like to thank E. M. Coven for informing me about Baldwin's paper [2] and helpful discussions of the results and J. Milnor and

S. Sutherland for looking through this paper and providing useful comments. I am also grateful to the referee for useful remarks.

References.

[**1**] Alsedá L, Llibre, J and Misiurewicz M 1989 Periodic orbits of maps of Y *Trans. Amer. Math. Soc.* **313** 475-538

[**2**] Baldwin S 1991 An Extension of Sharkovskii Theorem to the n-od *Erg. Th. and Dyn. Syst.* **11** 249-71

[**3**] Blokh A 1982 On the limit behavior of one-dimensional dynamical systems.1,2 (in Russian) *Preprints no. 1156-82, no. 2704-82* VINITI, Moscow

[**4**] Blokh A 1983 Decomposition of Dynamical Systems on an Interval *Russ.Math.Surv.* **38** no. 5 133-34

[**5**] Blokh A 1986 On Dynamical Systems on One-Dimensional Branched Manifolds (in Russian) 1 *Theory of Functions, Functional Analysis and Applications* Kharkov **46** 8-18; 2 *Theory of Functions, Functional Analysis and Applications* Kharkov **47** 67-77; 3 *Theory of Functions, Functional Analysis and Applications* Kharkov **48** 32-46

[**6**] Blokh A 1990 The Spectral Decomposition, Periods of Cycles and Misiurewicz Conjecture for Graph Maps (submitted to *"Proceedings of the Conference on Dynamical Systems in Güstrow"* , to appear in *Lecture Notes in Mathematics*)

[**7**] Blokh A 1991 On Some Properties of Graph Maps: Spectral Decomposition, Misiurewicz Conjecture and Abstract Sets of Periods *Preprint no. 35* Max-Planck-Institut für Mathematik

[**8**] Blokh A 1991 Periods implying almost all periods, trees with snowflakes and zero entropy maps *Preprint no. 1991/13* Institute for Mathematical Sciences, SUNY at Stony Brook

[**9**] Bowen R and Franks J 1976 The Periodic Points of Maps of the Disk and the Interval *Topology* **15** 337-42

[**10**] Denker M, Grillenberger, C and Sigmund K 1976 *Ergodic Theory on Compact Spaces (Lecture Notes in Mathematics* **527** *)* (Berlin:Springer)

[**11**] Llibre J and Misiurewicz M 1991 Excess of Gods Implies Chaos, to appear

[**12**] Sharkovskii A N 1964 Coexistence of Cycles of a Continuous Map of a Line into itself *Ukr. Math. Journal* **16** 61-71

Department of Mathematics, Wesleyan University, Middletown, CT 06459-0128, USA

E-mail address: ablokh@jordan.math.wesleyan.edu