Commun. Math. Phys. 199, 397 – 416 (1998)

Communications in
Mathematical Physics

Wild Attractors of Polymodal Negative Schwarzian Maps*

Alexander Blokh[1,**], **Michał Misiurewicz**[2]

[1] Department of Mathematics, University of Alabama in Birmingham, University Station, Birmingham, AL 35294-2060 USA. E-mail: ablokh@math.uab.edu

[2] Department of Mathematical Sciences, IUPUI, 402 N. Blackford Street, Indianapolis, IN 46202-3216, USA. E-mail: mmisiure@math.iupui.edu

Received: 15 December 1997 / Accepted: 13 May 1998

Abstract: We study "wild attractors" of polymodal negative Schwarzian interval maps and prove that they are *super persistently recurrent* (a polymodal version of *persistent recurrence*). We also prove that if a map has an attractor which is a cycle of intervals then at almost every point of this cycle the map has properties similar to the Markov property introduced by Martens. Thus, the lack of super persistent recurrence at a critical point c can be considered as a mild topological expanding property, and this expansion prevents $\omega(c)$ from being a wild attractor (in the previous paper we have shown that it also prevents the map from being C^2-stable).

1. Introduction

In his paper [Mi] Milnor suggested a new approach to the dynamics based on the notion of attractor. He showed that a smooth dynamical system has a unique so-called *global attractor* and posed a problem of decomposing it into minimal attractors, closely related to that of describing ω-limit sets of almost all points.

Since then many papers have appeared dealing with the problem (see our list of references, which is of course far from being complete). We continue this study and consider piecewise monotone (polymodal) negative Schwarzian maps of an interval. The results can be extended to one-dimensional branched manifolds, but to avoid complications we restrict our attention to the interval case (and thus give definitions only in this case, although some of them are more general). Precise and full definitions of some notions, as well as a lot of standard definitions, are given later.

---

* The first author was partially supported by NSF grant DMS 9626303 and Humboldt Foundation and would also like to thank G. Keller for inviting him to Erlangen and for useful discussions of the results of the paper. The second author was partially supported by NSF grant DMS 9704422.

** Current address: Mathematisches Institut, Universität Erlangen-Nürnberg, 91054 Erlangen, Germany

Mostly, we consider two spaces of maps. The space $\mathcal{C}^2_{\mathrm{nf}}$ consists of all maps of $[0,1]$ into itself of class C^2 with finite number of critical points, all of them non-flat. The space $\mathcal{S}$ consists of those maps from $\mathcal{C}^2_{\mathrm{nf}}$ that have negative Schwarzian. We denote the ω-limit set of a point x by $\omega(x)$ and call it simply the *limit* set of x.

Let $f \in \mathcal{C}^2_{\mathrm{nf}}$. Then for a set $A \subset [0,1]$, the set $\mathrm{rl}(A) = \{x : \omega(x) \subset A\}$ is called the *realm of attraction* of the set A. Also, the set $\mathrm{RL}(A) = \{x : \omega(x) = A\}$ is called the *realm of exact attraction* of the set A. Denote the Lebesgue measure of a set X by $|X|$. A closed invariant set A is called an *attractor* if

(1) $|\mathrm{rl}(A)| > 0$;
(2) $|\mathrm{rl}(A) \setminus \mathrm{rl}(A')| > 0$ for any proper closed invariant set $A' \subset A$.

Clearly, if $|\mathrm{RL}(A)| > 0$ then A is an attractor; such attractor is called *primitive*. An attractor A is called *global* if $|[0,1] \setminus \mathrm{rl}(A)| = 0$. In [Mi] it is shown that f has a unique global attractor. It is denoted by $A(f)$. The same holds for the restriction of f to a closed invariant set K.

Let us describe types of primitive attractors which can be considered natural. The first is rather simple. A point x is called a *(one-sided) periodic sink* if there exists $n > 0$ and a (one-sided) neighborhood U of x such that $f^n(x) = x$, $f^n(U) \subset U$ and the diameter of $f^k(U)$ tends to 0 as $k \to \infty$. The orbit of a periodic sink gives an example (perhaps the easiest one) of a primitive attractor.

To introduce the next type of primitive attractors we need more definitions. A closed interval I is called *periodic (of period n)* if the interiors of the intervals $I, \dots, f^{n-1}(I)$ are disjoint, while $f^n(I) \subset I$. Then the union $\bigcup_{i=0}^{n-1} f^i(I)$ is called a *cycle of intervals* and denoted by $\mathrm{cyc}(I)$. This includes also the case of $n = 1$; then $\mathrm{cyc}(I) = I$. Clearly, if $J \subset I$ and both I, J are periodic then the period of J is a multiple of the period of I (yet these periods may well coincide). Let $I_0 \supset I_1 \supset \dots$ be a nested sequence of periodic intervals of periods $m_0 < m_1 < \dots$. Then the intersection $X = \bigcap_{i=0}^{\infty} \mathrm{cyc}(I_i)$ is called a *solenoidal set* and the cycles of intervals $\mathrm{cyc}(I_i)$ are called *X-generating*.

The dynamics on X are well known (see, e.g., [B]) even when f is just a continuous interval map. In the smooth case it can be specified even further because of the absence of *wandering* intervals (an interval $J \subset [0,1]$ is called *wandering* for f if its images $f^n(J)$, $n \geq 0$, are pairwise disjoint and do not converge to a periodic orbit). The following theorem was proven in a series of papers ([G, L1, BL1, MMS]).

Theorem 1.1. *Maps from $\mathcal{C}^2_{\mathrm{nf}}$ have no wandering intervals.*

Theorem 1.1 implies that the map on X is conjugate to a minimal translation in a compact infinite zero-dimensional Abelian group. In this case for every point x absorbed by all X-generating cycles of intervals we have $\omega(x) = X$ (a point x is *absorbed* by an invariant set D if $f^m(x) \in D$ for some m).

Let us sketch the proof of the fact that any solenoidal set of f is indeed a primitive attractor. By a theorem of Martens, de Melo and van Strien [MMS] for any f there exists a number N such that all attracting or neutral periodic points of f have periods less than N. Now, if S is a solenoidal set of f then we can choose a generating cycle of intervals $\mathrm{cyc}(I)$ of period greater than N, so that there will be no attracting/neutral periodic orbits in $\mathrm{cyc}(I)$. Also, if C' is the set of all critical points of f belonging to S then we can also assume that C' is the set of all critical points of f belonging to $\mathrm{cyc}(I)$. Let us now apply a theorem of Mañé [Man], according to which almost all points of $\mathrm{cyc}(I)$ contain a critical point from C' in their limit sets. Thus, $\omega(x) = S$ for a.e. $x \in \mathrm{cyc}(I)$, which proves our claim.

It may also happen that there exist a cycle of intervals cyc(I) such that $f|_{\text{cyc}(I)}$ is transitive. This case plays an important role in one-dimensional dynamics. If the set of points $x \in$ cyc(I) such that $\omega(x)$ = cyc(I) is of positive Lebesgue measure then cyc(I) is a primitive attractor.

These three examples may be considered natural for the following reason: they all are also topological attractors in the sense that the set RL(A) for them is topologically big (of type G_δ, dense in some intervals). In fact it is proven in [B] that if a continuous interval map has no wandering intervals then for a dense G_δ set of points their limit set is either a periodic orbit, or a solenoidal set, or a cycle of intervals, on which the map is transitive. Hence, if there is a limit set D such that the set of points x attracted by D (i.e. such that $\omega(x) = D$) is G_δ and dense in some interval then D is necessarily of one of these types.

However, an amazing fact is that for Milnor attractors there is a fourth possibility. A primitive attractor which does not belong to any of the three classes described above is called a *wild attractor*. In other words, a wild attractor is an infinite nowhere dense and non-solenoidal primitive attractor. In [BKNS] an example of a wild attractor for a unimodal map was given.

In the series of papers ([BL2, BL3] for polymodal negative Schwarzian maps, [BL4] for unimodal negative Schwarzian maps, and [L2] for polymodal C^2-maps) the following theorem was proven.

Theorem 1.2. *The global attractor $A(f)$ of a map $f \in \mathcal{C}^2_{\text{nf}}$ is the union of all sinks of f and finitely many infinite primitive attractors A_i which are either solenoidal sets, or cycles of intervals on which a map is transitive, conservative and ergodic (with respect to Lebesgue measure) or wild attractors (on which f is minimal). Each set A_i contains a critical point of f and intersections between two of them are possible only if they are cycles of intervals with a few common boundary points.*

Unlike other primitive attractors, wild attractors are not well understood other than for the unimodal negative Schwarzian maps. Our work was motivated by this, and is an attempt to study wild attractors of polymodal negative Schwarzian maps. Hence, first we describe that case in more detail.

Let $f : [0,1] \to [0,1]$ be a piecewise monotone map. For $x \in [0,1]$ let us denote by $H_n(x)$ the maximal interval containing x on which f^n is monotone and let $f^n(H_n(x)) = M_n(x)$. Let $r_n(x)$ be the minimal distance between $f^n(x)$ and the endpoints of $M_n(x)$. If f^n has a local extremum at x, there is an ambiguity in the choice of $H_n(x)$ and $M_n(x)$, but $r_n(x) = 0$ independently of this choice. Moreover, in that case $r_m(x) = 0$ for all $m \geq n$. Also, if $x = 0$ or 1, then $r_n(x) = 0$ for all n. Thus either for some m we have $r_m(x) = 0$ (and then $r_n(x) = 0$ for all $n \geq m$) or $r_n(x) \neq 0$ for any n, in which case x is neither a preimage of a turning point nor $0, 1$. A recurrent critical point $c \in [0,1]$ of a unimodal map is called *persistently recurrent* if $r_n(f(c)) \to 0$. Now we summarize some information known about unimodal negative Schwarzian maps; by Theorem 1.2 in this case f has at most one infinite primitive attractor $A = \omega(c)$.

We say that a map is *purely dissipative* if it is not conservative on any set of positive measure (we use the terms "conservative" and "dissipative" with respect to the Lebesgue measure).

Theorem 1.3. *Let $f \in \mathcal{S}$ be unimodal with the critical point c. Then the following holds.*

(1) ([BL4, GJ, Ma]) *If A is a wild attractor of f then c is persistently recurrent. Moreover, there exists a cycle of intervals* cyc(I) *such that $A \subset$* cyc(I), *$f|_{\text{cyc}(I)}$ is purely dissipative and $r_n(x) \to 0$ as $n \to \infty$ for a.e. $x \in$* cyc(I).

(2) ([Ma]) *Let* cyc(I) *be an attractor. Then there is* $\varepsilon > 0$ *such that* $\limsup r_n(x) > \varepsilon$ *for a.e.* x.

Our main result generalizes Theorem 1.3 to the polymodal case. To state it we need more notions. To shorten the introduction we do this in brief, at least with respect to well-known notions (precise definitions will be given later).

First we need the notion of a chain introduced in [L1] for polymodal negative Schwarzian maps. In that paper they helped to prove non-existence of wandering intervals for such maps. Later chains were used to prove an analogous result for smooth polymodal interval maps (see [BL1, MMS]) and became a popular tool in one-dimensional dynamics. A sequence $(G_i)_{i=0}^l$ of closed intervals is called a *chain* if G_i is a maximal interval such that $f(G_i) \subset G_{i+1}, i = 0, \dots, l-1$. Given a point x and an interval $I \ni f^n(x)$ we construct a chain of intervals $(G_i)_{i=0}^n$ whose last interval G_n is equal to I and whose first interval G_0 contains x. If such a chain exists, it is unique. We call it the *pull-back chain of* I *along* $x, \dots, f^n(x)$. The number of intervals of the chain containing critical points of f is called the *order* of the chain.

For a map f of class C^1 with finitely many critical points let Cr(f) = Cr be its set of critical points. For every point x and $\varepsilon > 0$ we construct the pull-back chain of $[f^n(x) - \varepsilon, f^n(x) + \varepsilon]$ along $x, \dots, f^n(x)$. We define $r_n^k(x)$ as the supremum of all ε such that we get a chain of order k or less. Let $E_{k,\varepsilon}(f)$ be the set of all points y with $\limsup_{n\to\infty} r_n^k(y) > \varepsilon$. We call a point x such that for every k we have $r_n^k(x) \to 0$ as $n \to \infty$ *critically super persistent* or Cr-*super persistent*. If x is additionally recurrent, we call it *critically super persistently recurrent* or Cr-*super persistently recurrent* (cf. [BM1]). An important property of limit sets of Cr-super persistently recurrent points is that they are minimal ([BM2], see also Theorem 2.5 below). Also, if $r_n(x) \not\to 0$ then we call x *critically reluctant* or Cr-*reluctant*. Now we can state our main theorem.

Theorem 5.3. *For every* $f \in \mathcal{S}$ *and a primitive attractor* A *that is neither a periodic orbit nor a solenoidal set, one of the following holds.*

(1) *The attractor* A *is wild. Then* $A = \omega(c)$ *for some* Cr-*super persistently recurrent critical point* c. *Furthermore,* A *is contained in a basic set* $B($cyc$(I))$ *such that* $f|_B$ *is purely dissipative,* $|A| = 0$, *and almost all points of* B *are* Cr-*super persistent.*

(2) *The attractor* A *equals* $B($cyc$(I))$ = cyc(I) *and if* I *is of period* m *then* $f^m|_{f^j(I)}$ *is exact for* $0 \le j \le m-1$, $f|_{\mathrm{cyc}(I)}$ *is ergodic, conservative and* cyc$(I) \subset E_{0,\varepsilon}$ *for some* $\varepsilon > 0$ *up to a set of measure zero (almost all points of* A *are* Cr-*reluctant). Moreover, there exists a finite set* $Y \subset I$ *such that:*
   (a) Cr$(f) \cap I \not\subset Y$;
   (b) *for any two intervals* $U \subset$ int(V) *disjoint from* Y *and for almost every* $x \in I$ *there is an arbitrarily large* n *and two intervals* $x \in W' \subset W''$ *such that* $f^n|_{W''}$ *has no critical points,* $f^n(W'') = V$ *and* $f^n(W') = U$.

In Sect. 4 prove Theorem 4.6 which is a version of Theorem 5.3 with a milder statement (2) and then in Sect. 5 we strengthen it in Proposition 5.2 by establishing mild expanding properties for polymodal negative Schwarzian maps on their attractors which are cycles of intervals. These properties are similar to the ones proved by [Ma] for the unimodal case. It is this expansion that prevents the attractor from being wild. Similarly, such mild expansion along the trajectory of a critical point causes C^2 instability of the map.

Note that just like we define Cr-super persistent points, we can define Cr-*persistent points*. To avoid trivial cases we assume that x is not an eventual preimage of a critical point of f and call it Cr-*persistent* if $r_n(x) \to 0$ and Cr-*persistently recurrent* if x is also recurrent.

While it could seem that for the polymodal maps the natural thing is to use Cr-persistent recurrence (as for unimodal maps), it is not so. Sometimes it is crucial that the Cr-super persistent recurrence and related notions are used instead. One such place is the proof of Theorem 4.6, where we have to use the set $E(f)$, and not a similar set defined for Cr-persistent recurrence.

Actually, the results of this paper hold if we replace in the definition of $\mathcal{S}$ the assumption that f is of class C^2 by the assumption that it is of class C^1. The reason for this is that the assumption on negative Schwarzian is quite strong. However, with this weaker assumption (that f is of class C^1) we would not be able to invoke several results that we need and that are proved in the literature for functions of class C^2. While we could reprove them for functions of class C^1 with negative Schwarzian (by repeating existing proofs with some estimates changed), this would make this paper considerably longer and the results only slightly stronger. Therefore we choose not to do it.

2. Topological Properties of Chains

In this section we first summarize well-known facts about chains. Then we introduce some new notions and state new results, the main one establishing the minimality of the limit sets of super persistently recurrent points of f with arbitrarily small nice smart neighborhoods (see definitions below). This section contains almost no proofs. They can be found elsewhere, mainly in [BM2].

Throughout this section we assume that $f; [0, 1] \to [0, 1]$ is a piecewise monotone continuous map, strictly monotone on any lap. We call the local extrema of f (except 0 and 1) *turning points*. Let K_f be the closure of the convex hull of the union of the trajectories of the turning points of f. Clearly, K_f is a closed invariant interval. This is where the important things from the dynamical point of view happen. We want to have some extra space around K_f, so we assume that $0, 1 \notin K_f$. We call such f *loosely packed*. This assumption is not restrictive at all, since one can extend any f to a loosely packed map on a slightly larger interval, preserving smoothness and negative Schwarzian if necessary. This means that the properties of limit sets established with an additional assumption that f is loosely packed, hold also without this assumption.

Thus, from now on we assume that f is a loosely packed map. Also, we fix a finite set of points $C \subset K_f$ containing all turning points of f, call these points *exceptional* (cf. [BM2]) and assume that together with a map f there always comes the set C of exceptional points. In the smooth case C is usually chosen as the set Cr of all critical points of f. However, we would like to emphasize that the results of this section hold for any set $C \subset K_f$ containing all turning points of f, mainly because the definitions and arguments are topological.

If c is a turning point of f, let us take the largest interval $[a, b]$ such that $a < c < b$, $f(a) = f(b)$ and f is monotone on each of the intervals $[a, c]$ and $[c, b]$. Then there is a unique continuous function $\tau_c : [a, b] \to [a, b]$ such that $f \circ \tau = f$ and $f(x) \neq x$ for $x \neq c$. This function is an involution, that is τ_c^2 is the identity.

Although we are not dealing with smooth functions in this section, it is convenient to give some definitions related to them. We call an interval map f a C^2*-map with non-flat critical points* (and denote the class of such maps by $\mathcal{C}^2_{\mathrm{nf}}$) if f is of class C^2 with critical points $\{c_k\}$ such that the inequalities $A_1|x - c_k|^{\beta_k} \leq |f'(x)| \leq A_2|x - c_k|^{\beta_k}$ hold in neighborhoods of the points c_k with positive A_1, A_2, β_k (the inequalities are called *non-flatness inequalities*). Clearly, this implies that f has finitely many critical points and for some constant $R > 0$ we have $1/R > |\tau_c'| > R$ in the corresponding neighborhood of any turning point c.

Now for given loosely packed f and C we choose a positive constant $\varkappa$ such that

(A1) the distance between any two exceptional points of f is greater than $\varkappa$,
(A2) for any turning point c of f, the $\varkappa$-neighborhood of c is contained in the domain of τ_c,
(A3) for two exceptional points b, c either $f(b) = c$ or $|f(b) - c| > \varkappa$,
(A4) $K_f \subset (\varkappa, 1 - \varkappa)$,
(A5) if $f \in \mathcal{C}^2_{\mathrm{nf}}$ then non-flatness inequalities hold in $\varkappa$-neighborhoods of c_k.

Clearly, any sufficiently small $\varkappa$ satisfies the above conditions.

Now we define a chain modifying traditional definitions (see [L1, BL1, MMS]) to serve our purposes (for instance, we add (B3) below).

We call an interval *smart* if it does not contain any set of the form $f^k(V)$, where $k \geq 0$ and V is a one-sided $\varkappa$-neighborhood of an exceptional point of f. Note that any subinterval of a smart interval is also smart.

A sequence $(G_i)_{i=0}^{l}$ of closed intervals is called a *chain* if

(B1) G_i is a maximal interval such that $f(G_i) \subset G_{i+1}$, $i = 0, \dots, l-1$,
(B2) $G_0 \cap K_f \neq \emptyset$,
(B3) G_l is smart.

The number l is called the *length* of the chain, G_0 is called the *first* interval of a chain, and G_l is called the *last* interval of the chain. The typical situation in which we deal with a chain of intervals is the following. Given a point x and an interval $I \ni f^n(x)$ we construct a chain of intervals $(G_i)_{i=0}^{n}$ whose last interval G_n is equal to I and whose first interval G_0 contains x. If such chain exists, it is unique. We call it the *pull-back chain of I along $x, \dots, f^n(x)$* or just the *pull-back chain of I*. Any G_i is called a *pull-back* of I.

Construction of a pull-back is straightforward. Once we have G_i, we choose as G_{i-1} the component of $f^{-1}(G_i)$ containing $f^{i-1}(x)$. The only obstructions in the construction may be that (B2) or (B3) are not satisfied. However, condition (B2) is satisfied if $x \in K_f$ and this will be always the case in the sequel. Condition (B3) says that I is smart. This is not a great problem, because of the following lemma.

Lemma 2.1 ([BM2]**).** *Assume that f has no wandering intervals. Then every non-periodic point has a smart neighborhood.*

When we have a chain $(G_i)_{i=0}^{l}$, we cannot avoid the situation when G_i contains exceptional points of f. However, we have the following lemma.

Lemma 2.2 (see, e.g., [BM2]**).** *An interval G_i from a chain contains at most one exceptional point c of f, and if so then c is neither* 0 *nor* 1. *Moreover, if a turning point c of f belongs to G_i and $i < l$ then $\tau_c(G_i) = G_i$.*

The intervals of a chain that contain elements of C play a special role. Their number in a chain is called the *C-order* (or just *order*) of the chain.

The next lemma follows immediately from Lemma 2.2 and the definition of a chain. To state it we need the following definition. Let I be an interval, I' be a component of $f^{-1}(I)$ such that either $f|_{I'}$ is monotone and $f(I') = I$ or $f|_{I'}$ is unimodal, both endpoints of I' are mapped into one endpoint of I and (in the case of C^1-map f) non-flatness inequalities are satisfied in I'. Then we say that I' is a *regular* component of the preimage of I.

Lemma 2.3 (see, e.g., [BM2]**).** *The interval G_i is a regular component of the preimage of G_{i+1}.*

Let us now repeat with more details the definition of super persistent recurrence. Let us fix the set C of exceptional points of f (recall that C must contain all turning points of f) and consider the following construction. Fix a point $x \in K_f$. For every $\varepsilon > 0$ we try to construct the pull-back chain of $[f^n(x) - \varepsilon, f^n(x) + \varepsilon]$ along $x, \dots, f^n(x)$ and denote by $m_{x,n}(\varepsilon)$ its order. Clearly, $m_{x,n}(\varepsilon)$ grows monotonically with ε. If there are no exceptional points among $x, f(x), \dots, f^n(x)$ then for sufficiently small ε we have $m_{x,n}(\varepsilon) = 0$, otherwise even for arbitrarily small ε we have $m_{x,n}(\varepsilon) > 0$. We define $r_n^k(x)$ as the supremum of all ε such that $m_{x,n}(\varepsilon)$ exists and is smaller than or equal to k. In other words, ε is the biggest number such that for every $\varepsilon' < \varepsilon$ the ε'-neighborhood of $f^n(x)$ can be pulled back along $x, \dots, f^n(x)$ with order at most k. Note that $r_n^k(x)$ depends on f and C, yet for the sake of simplifying notation we avoid referring to them.

We call a point x such that for every k we have $r_n^k(x) \to 0$ as $n \to \infty$ *C-super persistent*. If x is additionally recurrent, we call it *C-super persistently recurrent* (cf. [BM1]). If we only claim the existence of a set C of exceptional points for which x is C-super persistently recurrent, but do not fix it, we call x simply a *super persistently recurrent* point. Finally, if the map f is smooth then Cr(f) = Cr denotes its set of critical points, so we get Cr-super persistent and Cr-super persistently recurrent points which will be the main focus of our study.

Let $E(f) = \bigcup_{k,\varepsilon} E_{k,\varepsilon}(f)$; recall that $E_{k,\varepsilon}(f)$ is the set of all points y with $\limsup r_n^k(y) > \varepsilon$ (where C = Cr). Thus, the set of Cr-super persistent points is $[0,1] \setminus E(f)$.

We will call an interval $I \subset [0,1]$ *nice* if for every $n > 0$ and an endpoint a of I the point $f^n(a)$ does not belong to the closure of I. In other words, the positive orbits of both endpoints of I miss the closure of I. A set $A \subset [0,1]$ is called *minimal* if $f|_A$ is minimal.

Theorem 2.4 ([BM2]**).** *Let x be a super persistently recurrent point of f having arbitrarily small smart nice neighborhoods. Then $\omega(x)$ is minimal.*

We also need some facts about so-called basic sets (see [B]) which will be used later on. Let M = cyc(I) be a cycle of intervals. Consider a set $\{x \in M :$ for any relative neighborhood U of x in M the orbit of U is dense in $M\}$; it is easy to see that this is a closed invariant set. It is called a *basic set* and denoted by $B(M)$ (or simply by B) provided it is infinite.

Let $F : I \to I$ and $G : J \to J$ be two interval maps, let $\varphi : I \to J$ be a (non-strict) monotone semiconjugacy between F and G and let $B \subset I$ be an F-invariant closed set such that $\varphi(B) = J$ and $\varphi^{-1}(x) \cap B = \partial\varphi^{-1}(x)$ for any $x \in J$. Then we say that φ *almost conjugates* $F|_B$ to G. Here ∂Z is the *boundary* of a set Z.

Now we can list some of the properties of basic sets of interval maps.

Theorem 2.5 ([B]**).** *Let* cyc(I) *be a cycle of intervals of period n of a continuous interval map f and let $B = B($*cyc$(I))$ *be the corresponding basic set. Then the following holds.*

(1) *There exists a mixing map $g : [0,1] \to [0,1]$ and a monotone map $\varphi : I \to [0,1]$ such that φ almost conjugates $f^n|_{(B\cap I)}$ to g. In particular, f maps complementary to B intervals one into another and also their boundaries one into another.*

(2) *The map* $f|_B$ *is transitive.*
(3) *The set* B *is perfect.*
(4) *The set* B *is contained in the closure of the set of periodic points of* f.
(5) *If* K *is a compact set contained in the interior of* $f^j(I)$ *and* U *is an open set intersecting* $B \cap I$ *then there exists a number* l *such that* $f^{mn+j}(U) \supset K$ *for all* $m \geq l$.
(6) *Any limit set is contained in either a periodic orbit, or a solenoidal set, or a basic set.*

The main properties here are (1) and (6). The rest can be deduced from it, yet in order to have convenient references we also include Properties (2)–(5).

Theorem 2.5 implies the following corollary for maps without wandering intervals (here by *(pre)periodic* we mean points which are periodic or preperiodic).

Corollary 2.6. *Let* f *be a map without wandering intervals, and let* $B = B(\text{cyc}(I))$ *be its basic set. Then the following holds.*

(1) *If* J *is a complementary to* B *interval then it is eventually mapped into a periodic complementary to* B *interval and its endpoints are (pre)periodic.*
(2) *If* $\omega(x) \subset B$ *is infinite then* $f^k(x) \in B$ *for some* k *and, moreover, for no* k *is* $f^k(x)$ *an endpoint of a complementary to* B *interval.*
(3) *If* x *is not an endpoint of a complementary to* B *interval and* ε *is less than the minimal length of an interval from* $\text{cyc}(I)$ *then the maximal length of a neighborhood* U *of* x *such that* $|f^N(U)| \leq \varepsilon$ *converges to* 0 *as* $N \to \infty$.

Proof. (1) Let $J = (a, b)$ be a complementary to B interval. Then we may assume that $J \subset I$. Suppose for the sake of definiteness that the period of I is m and consider the almost conjugacy φ between $f^m|_I$ and a mixing map $g : [0, 1] \to [0, 1]$. Then $\varphi(J) = \{x\}$ is a point. If x is not (pre)periodic for g then J is wandering, a contradiction. So, x is (pre)periodic for g which implies that J is eventually mapped into a periodic complementary to B interval K. Since endpoints of complementary to B intervals are mapped into endpoints of complementary to B intervals by Theorem 2.5 (1) we see that endpoints of J are (pre)periodic.

(2) Suppose that $\omega(x) \subset B(\text{cyc}(I))$ is infinite. Then for some k we have $f^k(x) \in I$. If $f^k(x) \in J$ where J is a complementary to B interval then by (1) x is eventually mapped into a periodic complementary to B interval. Since the limit set of x is contained in B then $\omega(x)$ is contained in the union of the endpoints of the intervals from $\text{cyc}(K)$ which contradicts the fact that $\omega(x)$ is infinite. Also, by (1) all endpoints of complementary to B intervals are (pre)periodic, so $f^k(x)$ cannot be such an endpoint because $\omega(x)$ is infinite.

(3) If the claim fails then there is a semi-neighborhood V of x and a sequence of integers $N_i \to \infty$ such that $|f^{N_i}(V)| \leq \varepsilon$. Choose closed subintervals of $f^j(I)$ for all $0 \leq j \leq n$ so that the length of any such subinterval is greater than ε. By Theorem 2.5 (5) for some k and all $m \geq k$ the set $f^m(V)$ contains at least one of these subintervals and therefore has length greater than ε; a contradiction. □

Theorem 2.7 (cf. [BM2]**).** *If* f *has no wandering intervals then any super persistently recurrent point* c *of* f *has minimal limit set.*

Remark. In [BM2] we prove the same statement under the assumption that the point c is a turning point of f. The main reason for such restriction was that we only needed the result for turning points; in what follows we get rid of this restriction for maps without wandering intervals.

Proof of Theorem 2.7. First of all notice that if c is a periodic point or belongs to a solenoidal set then $\omega(c)$ is indeed minimal. Thus we may assume from the very beginning that c has an infinite limit set which is not solenoidal. By Theorem 2.5 (6) then $\omega(c) \subset B(\text{cyc}(I))$ for some basic set $B(\text{cyc}(I))$. Since c is recurrent we conclude that $c \in B$. Moreover, by Corollary 2.6 (2) no point of the orbit of c is an endpoint of a complementary to B interval.

We claim that c can be approximated from both sides by the preimages of any point a from the interior of I. Indeed, since $c \in B$ is not an endpoint of a complementary to B interval then any semi-neighborhood of c is non-disjoint from B. By Theorem 2.5 (5) images of this semi-neighborhood cover a which proves our claim.

Our aim is to construct an arbitrarily small nice neighborhood of c. First observe that considering small neighborhoods U of c we may assume that there is a lot of periodic points with pairwise disjoint orbits which do not enter U. Indeed, by Theorem 2.5 (4) B is contained in the closure of the set of periodic points of f, so there are a lot of periodic points in B. We can choose several such points and then choose U to be disjoint from the union of their orbits. Now, take a periodic point $a \in I$. By the previous paragraph for n sufficiently large there are points of $\bigcup_{i=0}^{n} f^{-i}(a)$ in U at both sides of c very close to c. Choose the closest ones from both sides, $x < c < y$; we may assume that the orbit of a is disjoint from $[x, y]$.

The neighborhood (x, y) of c is nice unless one of the points x, y is an image (under some iterate of f) of the other one. In this case choose a periodic point b whose orbit is disjoint from both the orbit of a and the set $[x, y]$. Take the minimal k such that $f^{-k}(b)$ intersects (x, y). If $f^{-k}(b)$ intersects (x, c), replace x by the element of $f^{-k}(b)$ closest to c; similarly for (c, y) and y. The new neighborhood of c is nice. This is clear if we replaced only one of the points x, y. If we replaced both of them, the endpoints of the new neighborhood belong to $G^{-k}(b)$ with the same k. However, in this case none of them can be an image of the other one under any iterate of f because otherwise the orbit of b would not be disjoint from (x, y).

On the other hand it is proven in [BM2] that c has arbitrarily small smart neighborhoods. We complete the proof by applying Theorem 2.4 to the point c. $\square$

A set A will be called *C-super persistently recurrent* if for some C-super persistently recurrent point $x \in A$ we have $\omega(x) = A$. In view of the next proposition, it does not matter which point $x \in A$ we choose.

Proposition 2.8 ([BM2]**).** *Let f have no wandering intervals and C be a set of exceptional points. Let $X \subset [0, 1]$ be an infinite minimal set for f. Then either every point of X is C-super persistently recurrent or no point of X is C-super persistently recurrent.*

So far in Sect. 2 we have stated standard facts concerning chains or useful for us results from [BM2]. The next lemma establishes invariance of the sets $E(f)$ and $E_{k,\varepsilon}(f)$. By $\mathbf{Cr}(f)$ we denote the union of all big orbits of the set of all exceptional points of f (that is of all their images and preimages).

Lemma 2.9. *The set $E_{k,\varepsilon}(f) \setminus \mathbf{Cr}(f)$ is invariant and the set $E(f)$ is fully invariant.*

Proof. This follows from the fact that if $(G_i)_{i=0}^{l}$ is a chain of intervals of order m then the order of the chain $(G_i)_{i=1}^{l}$ is either m or $m - 1$. $\square$

3. Distortion Lemmas

Normally, one defines the *Schwarzian* (or *Schwarzian derivative*) of a function f of class C^3 as $Sf = f'''/f' - (3/2)(f''/f')^2$. It is defined outside the set $\mathrm{Cr}(f)$. Then *negative Schwarzian* means $Sf < 0$ at all non-critical points. As can be easily checked, this implies strict convexity of the function $1/\sqrt{|f'|}$ on each component of the complement of the set $\mathrm{Cr}(f)$, which requires only C^1 smoothness. Moreover, it is well known that distortion properties similar to those of maps f of class C^3 with $Sf < 0$ away from critical points hold for the maps of class C^1 described above as well. Thus we adopt this property as the definition of negative Schwarzian maps as was done in [BM2]: a *negative Schwarzian map* is a map of class at least C^1 such that the function $1/\sqrt{|f'|}$ is strictly convex on each component of the complement of the set $\mathrm{Cr}(f)$. The space of all negative Schwarzian maps from $\mathcal{C}^2_{\mathrm{nf}}$ will be denoted by $\mathcal{S}$. Note that we allow critical inflection points.

Some uniform estimates on distortion can be made for all negative Schwarzian diffeomorphisms. Also, iterates of negative Schwarzian maps have negative Schwarzian as well. This allows one to estimate distortion of an iterate of a negative Schwarzian map restricted to an interval where it has no critical points. Estimates of distortion for one iterate of a map in the presence of a critical point are necessary too. Since the "one-step" estimates can be made without negative Schwarzian assumptions, in Lemmas 3.1–3.3 we consider general maps from the class $\mathcal{C}^2_{\mathrm{nf}}$.

We need some notions. The *density* of a set X in an interval I is

$$\rho(X|I) = \frac{|X \cap I|}{|I|}.$$

In the probability theory it is called *conditional measure*, but we prefer a more geometrical name *density*.

As in [BM1], we introduce a function $r : [0,1]^2 \to \mathbb{R} \cup \{\infty\}$ as follows:

$$r(x,y) = \frac{|f(x) - f(y)|}{|x-y|\,|f'(x)|}$$

if $x \neq y$, and $r(x,x)$=1 and call the infimum of $r(x,y)$ over the pairs of points x, y from the same lap the *shrinkability* of f. We denote it ${\scriptstyle s}(f)$ ("${\scriptstyle s}$" is a shrunken "s"). It is proven in [BM1] that maps from $\mathcal{C}^2_{\mathrm{nf}}$ have non-zero shrinkability.

Let us now state our first distortion lemma.

Lemma 3.1. *Assume that $f \in \mathcal{C}^2_{\mathrm{nf}}$, X is a measurable set and I is an interval such that f is monotone on I. Then*

$$\rho(X|I) \geq \rho(f(X)|f(I))\,{\scriptstyle s}(f).$$

Proof. If $I = [a,b]$ and $a < x < b$ then by the definition of ${\scriptstyle s}(f)$ we have

$$|f(b) - f(x)| \geq {\scriptstyle s}(f)|f'(x)|\,|b-x|$$

and

$$|f(x) - f(a)| \geq {\scriptstyle s}(f)|f'(x)|\,|x-a|.$$

Hence,

$$\frac{|f(I)|}{|I|} = \frac{|f(b) - f(a)|}{|b-a|} \geq {\scriptstyle s}(f)|f'(x)|.$$

This holds for every $x \in (a, b)$, so taking into account that $|f(X \cap I)|/|X \cap I|$ is the mean value of $|f'(x)|$ over $X \cap I$, we get

$$\frac{|f(I)|}{|I|} \geq \mathfrak{s}(f)\, \frac{|f(X \cap I)|}{|X \cap I|}\,.$$

This inequality is equivalent to the one we wanted to prove. $\square$

The next lemma relies upon Lemma 3.1. To state it we need the following definitions. A point $x \in I$ is said to be *η-centered* in an interval I if the distance of x from the boundary of I is $\eta|I|$ or larger. An interval $K \supset I$ is said to be a *δ-scaled neighborhood* of I if the distance of both endpoints of K from I is at least $\delta|I|$.

Lemma 3.2. *Let $f \in \mathcal{C}^2_{\mathrm{nf}}$. Then there exists a positive constant ξ (depending only on f) such that if I' is a regular component of the preimage of an interval I, $x \in I'$, and $f(x)$ is η-centered in I then:*

(1) *the point x is $\xi\eta$-centered in I';*
(2) *if a set A has density at least α in both components of $I \setminus \{f(x)\}$ then $f^{-1}(A)$ has density at least $\xi\eta\alpha$ in both components of $I' \setminus \{x\}$.*

Proof. If f is monotone on I', by Lemma 3.1 both (1) and (2) hold with $\xi = \mathfrak{s}(f)$ (observe that $\eta \leq 1/2$).

Assume now that f is unimodal on I'. Let $I = [a, b]$ and $I' = [a', b']$. Without loss of generality we may assume that $f(a') = f(b') = a$ and that there is $c' \in (a', b')$ such that f is increasing on $[a', c']$ and decreasing on $[b', c']$. Set $c = f(c')$.

Recall that for $\tau = \tau_{c'}$ we have $1/R \leq |\tau'| \leq R$, where $R = R(f)$ depends only on f. Since $\tau([a', c']) = [c', b']$ and $\tau([c', b']) = [a', c']$, we get $1/R \leq |c' - a'|/|b' - c'| \leq R$. Therefore $|c' - a'|/|I'| \geq 1/(R + 1)$ and similarly $|b' - c'|/|I'| \geq 1/(R + 1)$.

Assume that $x \in I'$ and $f(x)$ is η-centered in I. Without loss of generality we may assume that $x \in [a', c']$. We have $|f(x) - a| \geq \eta|I| \geq \eta|c - a|$, so by Lemma 3.1 and the preceding paragraph we get

$$|x - a'| \geq \mathfrak{s}(f)\eta|c' - a'| \geq \frac{\mathfrak{s}(f)}{R + 1}\, \eta|I'|.$$

On the other hand (since $\mathfrak{s}(f) \leq 1$ and $\eta \leq 1$),

$$|x - b'| \geq |b' - c'| \geq \frac{1}{R + 1}\, |I'| \geq \frac{\mathfrak{s}(f)}{R + 1}\, \eta|I'|.$$

This means that (1) holds in this case with $\xi = \mathfrak{s}(f)/(R + 1)$.

Assume now that a set A has density at least α in $[a, f(x)]$. Then by Lemma 3.1 $f^{-1}(A)$ has density at least $\mathfrak{s}(f)\alpha$ in both $[a', x]$ and $[\tau(x), b']$. Since $f(\tau(x)) = f(x)$, by the preceding paragraph the point $\tau(x)$ is $\eta\, \mathfrak{s}(f)/(R + 1)$-centered in I', and hence

$$\frac{|b' - \tau(x)|}{|b' - x|} \geq \frac{|b' - \tau(x)|}{|I'|} \geq \frac{\eta\, \mathfrak{s}(f)}{R + 1}\,.$$

Therefore $f^{-1}(A)$ has density at least $\alpha\eta(\mathfrak{s}(f))^2/(R + 1)$ in $[x, b']$. This means that (2) holds in this case with $\xi = (\mathfrak{s}(f))^2/(R + 1)$.

Thus, the whole lemma holds with $\xi = (\mathfrak{s}(f))^2/(R + 1)$. $\square$

Lemma 3.3. *Let $f \in \mathcal{C}^2_{\rm nf}$. Then there exists a positive constant $\zeta < 1$ (depending only on f) such that if I' is a regular component of the preimage of the interval I, J is an interval such that I is its δ-scaled neighborhood with $\delta \leq 1$, and J' is a component of $(f|_{I'})^{-1}(J)$, then I' is a $\zeta\delta$-scaled neighborhood of J'.*

Proof. In the monotone case from Lemma 3.1 it follows that I' is a δ'-scaled neighborhood of J', where

$$\frac{\delta'}{1+\delta'} = s(f)\,\frac{\delta}{1+\delta}\,.$$

Since $\delta \leq 1$, we get $\delta' \geq s(f)\delta/2$, so we can take $\zeta = s(f)/2$ in this case.

Assume now that f is unimodal on I' and use the same assumptions and notation as in the preceding proof. Suppose first that $f(c')$ does not belong to J'. Then we may assume that $J' = [d', e'] \subset [a', c']$ and then we get (as in the monotone case)

$$\frac{|d'-a'|}{|J'|} \geq \frac{s(f)}{2}\,\delta.$$

On the other hand,

$$\frac{|b'-e'|}{|J'|} \geq \frac{|b'-c'|}{|c'-a'|} = \frac{|\tau([c',a'])|}{|[c',a']|} \geq \frac{1}{R} \geq \frac{\delta}{R}\,.$$

Hence, in this case we can take $\zeta = \min(s(f)/2, 1/R)$.

Suppose now that $f(c')$ belongs to $J' = [d', e']$. Then $f([d', c']) = f([c', e']) = J$, so by the preceding case I' is a $\min(s(f)/2, 1/R)\delta$-scaled neighborhood of both $[d', c']$ and $[c', e']$. However, $|c'-d'| \geq |J'|/(R+1)$ and $|e'-c'| \geq |J'|/(R+1)$, so I' is a $\min(s(f)/2, 1/R)\delta/(R+1)$-scaled neighborhood of J'. Hence we can take in this case (and in all cases) $\zeta = \min(s(f)/2, 1/R)/(R+1)$. $\square$

So far we have proven one step distortion lemmas which apply to f but not to its iterates. The famous Koebe Lemma fills this gap. Unlike Lemmas 3.1–3.3, it applies only to maps of negative Schwarzian and only if the interval contains no critical points in its interior. However, the estimates of the distortion do not depend on the map, which allows us to apply them to iterates of maps and makes the lemma very important. We state it in a form equivalent to the one from [BM2].

Koebe Lemma. *If I is an interval, $h : I \to \mathbb{R}$ is a negative Schwarzian map without critical points in the interior of I, and $J \subset I$ is an interval such that $h(I)$ is a δ-scaled neighborhood of $h(J)$ then:*

(1) *for every points $x, y \in J$ we have*

$$\frac{h'(x)}{h'(y)} \leq \left(\frac{1+\delta}{\delta}\right)^2;$$

(2) *the interval I is a $\delta^3/(2(3\delta+2)^2)$-scaled neighborhood of J.*

The next lemma shows what consequences similar to Lemma 3.2 can be drawn from the Koebe Lemma.

Lemma 3.4. *If I is an interval, $h : I \to \mathbb{R}$ a negative Schwarzian map without critical points in the interior of I, and $J \subset I$ an interval such that $h(I)$ is a δ-scaled neighborhood of $h(J)$ then:*

(1) *if $x \in J$ and $h(x)$ is η-centered in $h(J)$ then x is $(\delta/(1+\delta))^2\eta$-centered in J;*
(2) *for any measurable set A,*

$$\rho(f^{-1}(A)|J) \geq \left(\frac{\delta}{1+\delta}\right)^2 \rho(A|f(J)).$$

Proof. From the Koebe Lemma it follows that $s(f|_J) \geq (\delta/(1+\delta))^2$. Now (2) follows from this and Lemma 3.1 and (1) follows from (2). □

The next lemma is a distortion lemma for chains, which we will use in the next section. The Koebe Lemma shows that within segments of a chain which consist of intervals containing no critical points the distortion of the map remains bounded on smaller intervals. Therefore to estimate the distortion in the entire chain it is important to know how many intervals containing critical points it includes. Thus from now on we always consider maps f with the set of exceptional points $C = \mathrm{Cr}(f)$, and estimate the distortion of a map along a chain provided that the order of the chain is known. Note that due to our definition of chain and Lemma 2.1 we do not have to include 0 and 1 in the set of exceptional points. If chains $(G_i)_{i=0}^l$ and $(H_i)_{i=0}^l$ are such that $G_i \supset H_i$ for every i then we say that the chain $(G_i)_{i=0}^l$ *contains* the chain $(H_i)_{i=0}^l$ and denote this by $(G_i)_{i=0}^l \supset (H_i)_{i=0}^l$.

Lemma 3.5. *Assume that $f \in \mathcal{S}$, ν is a natural number, $\delta \leq 1$, α and η are positive numbers. Then there exist positive numbers $\gamma \leq 1$, ϑ and β (all of them depending on f, ν and δ; additionally ϑ depends on η, and β depends on η and α), such that whenever $(G_i)_{i=0}^l \supset (H_i)_{i=0}^l$ are chains of order ν or smaller and G_l is a δ-scaled neighborhood of H_l, the following holds:*

(1) *G_0 is a γ-scaled neighborhood of H_0;*
(2) *if $x \in H_0$ and $f^l(x)$ is η-centered in H_l then x is ϑ-centered in H_0;*
(3) *if x is as above and the density of a set A in both components of $H_l \setminus \{f^l(x)\}$ is at least α then the density of $f^{-l}(A)$ in both components of $H_0 \setminus \{x\}$ is at least β.*

Proof. We decompose our chain $(G_i)_{i=0}^l$ into $2\nu + 1$ (or less) pieces. Each piece corresponds either to f restricted to some G_i that contains one exceptional point (Case 1) or to an iterate f^j restricted to G_i such that there is no exceptional point of f^j in G_i and $f^j|_{G_i}$ has negative Schwarzian (Case 2).

We go back along the chain piece by piece, using inductively appropriate statements of Lemmas 3.2 and 3.3 in Case 1, and of the Koebe Lemma and Lemma 3.4 in Case 2. In this way when we get to G_0 and H_0, we obtain (1)–(3) for some constants γ, β and ϑ. In order to have these constants independent of the choice of chains (for given chains we may have a decomposition into less than $2\nu + 1$ pieces) we have to apply alternately Case 2 and Case 1, together $2\nu + 1$ times, starting with Case 2. □

4. Super Persistent Recurrence of Wild Attractors

In this section we consider maps from our class $\mathcal{S}$ with the set of exceptional points coinciding with the set of critical points $\mathrm{Cr}(f)$. We study wild attractors and specify the dynamics on them. The main result of this section is that they are Cr-super persistently recurrent.

We begin with a simple observation. By Theorem 2.5 (6) limit sets of points for interval maps are either periodic orbits, or are contained in solenoidal sets, or are contained in basic sets. The definitions imply that wild attractors are subsets of basic sets. Since by Theorem 1.2 a wild attractor A is the limit set of a recurrent critical point c we may assume that $c \in A \subset B$, where B is a basic set.

Now we need a number of standard definitions from ergodic theory which we include here for the sake of completeness. Let T be a nonsingular map on $(X, \mathcal{B}, \mu)$ (the measure μ here is finite but not assumed to be invariant). The set $D \subset X$ is called *fully invariant* if $T^{-1}(D) = D$. The map T is said to be *ergodic* if all its fully invariant sets have either measure 0 or full measure. The map T is called *conservative* if for any set R of positive measure there exists n such that $T^n(R) \cap R \neq \emptyset$; the map T is *conservative on its invariant set* D if $T|_D$ is conservative in the above sense. The map T is said to be *purely dissipative* if there are no invariant subsets on which it is conservative. The map T is said to be *exact* if $\bigcap_{n\geq 0} T^{-n}(\mathcal{B})$ contains only sets of measure 0 or $\mu(X)$. Clearly, if a map is exact then it is ergodic.

A useful tool in studying exactness is *lim sup full* sets and maps introduced by Julia Barnes in [Ba]. Let T be a nonsingular map on $(X, \mathcal{B}, \mu)$. A set $Y \subset X$ is called *lim sup full* provided $\limsup T^n(Y) = \mu(X)$. The map T is said to be *lim sup full* if every subset of positive measure is lim sup full. In her paper [Ba] Barnes proved the following theorem (the theorem is proven under the assumption that the map T is a non-singular d-to-1 map but the same argument goes through in a more general situation).

Theorem 4.1 ([Ba]**).** *Let $T : X \to X$ be a non-singular surjective map that is lim sup full with respect to a finite measure μ. Moreover, suppose that there exists a partition of X into finitely many subsets on each of which T is one-to-one. Then T is exact (and therefore ergodic) and conservative.*

Remark. If $X = \bigcup_{i=0}^{m-1} X_i$, $T(X_i) = X_{i+1}$ for all i and pairwise intersections of the sets $\{X_i\}$ have zero μ-measure then one can consider the question of whether $T^m|_{X_i}$ is lim sup full for all i. It is easy to see (we leave verification to the reader) that then if $T^m|_{X_i}$ is lim sup full for some i then $T^m|_{X_i}$ is lim sup full for all i and exact, while $T : X \to X$ is ergodic and conservative.

In the sequel T will be our interval map f, $\mathcal{B}$ will be the σ-field of all Borel subsets of $[0, 1]$, and μ will be the Lebesgue measure.

In the following lemma we establish a sufficient condition for the restriction of a map f to its basic set to be lim sup full. A point y is said to be a point of *semi-density* of a set X if for components I^-, I^+ of the set $I \setminus \{y\}$, where I is an interval centered at y we have $\max(\rho(X|I^-), \rho(X|I^+)) \to 1$ as $|I| \to 0$.

Lemma 4.2. *Let $B = B(\mathrm{cyc}(I))$ be a basic set of a map $f \in \mathcal{S}$, where I is of period m. Then the following holds.*

(1) *Suppose that a point $y \in E(f) \cap B \cap I$ is a point of semi-density of some set $X \subset B$. Then X is lim sup full for $f^m|_I$ (in particular if X is invariant then $X = \mathrm{cyc}(I)$ up to a set of measure zero).*
(2) *Suppose that the set $Y = E(f) \cap B$ is of positive measure. Then $B = \mathrm{cyc}(I)$, $f^m|_{f^j(I)}$ is lim sup full for any j, and thus $f^m|_{f^j(I)}$ is exact and $f|_{\mathrm{cyc}(I)}$ is ergodic and conservative. Moreover, then there exist k and $\varepsilon > 0$ such that $E_{k,\varepsilon}(f) \cap \mathrm{cyc}(I) = \mathrm{cyc}(I)$ up to a set of measure zero.*

Proof. (1) We may assume that $X \subset I$. Since $y \in E(f)$, there exists a sequence of integers $n_i \to \infty$ and a number $\varepsilon > 0$ such that the pull back chain of the 4ε-neighborhood of $f^{n_i}(y)$ along $y, \ldots, f^{n_i}(y)$ has order at most k. Choosing a subsequence we may assume that $f^{n_i}(y) \to z$ and $|f^{n_i}(y) - z| < \varepsilon$ for all i. Then the pull back chains of the 3ε-neighborhood G of z along $y, \ldots, f^{n_i}(y)$ have order at most k. Together with these chains we consider the pull back chains of the 2ε-neighborhood H of z along $y, \ldots, f^{n_i}(y)$, whose first interval we denote by H_0^i. Observe that ε above may be chosen sufficiently small; in particular we can choose it smaller than the one half of the minimal length of the intervals from cyc(I). Then Corollary 2.6 (3) applies, so we see that $|H_0^i| \to 0$ as $n_i \to \infty$.

First we prove that for some semi-neighborhood U of z and a subsequence of iterates n_i the measure of the sets $U \setminus f^{n_i}(X)$ converges to 0. Let us check if appropriate claims of Lemma 3.5 can be applied to the two constructed pull back chains. Observe that G is a $1/4$-scaled neighborhood of H and that the orders of the pull back chains of G along $y, \ldots, f^{n_i}(y)$ are at most k. Also, all points $f^{n_i}(y)$ are $1/4$-centered in H. Hence we can apply Lemma 3.5 with $\nu = k$, $\delta = 1/4$ and $\eta = 1/4$. We set $A_i = [0, 1] \setminus f^{n_i}(X)$ and suppose that there is $\alpha > 0$ such that for all i the set A on both components of $H \setminus \{f^{n_i}(y)\}$ has density greater than α.

By Lemma 3.5 there are positive constants ϑ and β such that for all i the point y is ϑ-centered in H_0^i and the density of $f^{-n_i}(A_i)$ in both components of $H_0^i \setminus \{y\}$ is at least β. Let $s_i \le t_i$ be the lengths of the components of $H_0^i \setminus \{y\}$. Set $W_i = [y - t_i, y + t_i]$. Since y is ϑ-centered in H_0^i, we have $s_i/(s_i + t_i) \ge \vartheta$, and hence $s_i/t_i \ge \vartheta/(1 - \vartheta)$. The density of $f^{-n_i}(A_i)$ in the component of W_i which is also a component of $H_0^i \setminus \{y\}$ is at least β; in the other component it is at least $\beta s_i/t_i \ge \beta\vartheta/(1 - \vartheta)$. Since $\vartheta \le 1/2$, we have $\vartheta/(1 - \vartheta) \le 1$, so the density of $f^{-n_i}(A_i)$ in each half of W_i is at least $\beta\vartheta/(1 - \vartheta)$. The set $f^{-n_i}(A_i)$ is disjoint from X. Since $|H_0^i| \to 0$, we have $t_i \to 0$, and this contradicts the fact that y is the point of semi-density of X.

Hence, after choosing a subsequence and without loss of generality we may assume that the density of $f^{n_i}(X)$ on the left component $V = [a, f^{n_i}(y)]$ of $H \setminus \{f^{n_i}(y)\}$ approaches 1. Since $f^{n_i}(y) \to z$ we see that the density of $f^{n_i}(X)$ on $[a, z]$ approaches 1. This proves our claim for $U = [a, z]$.

Now, by Theorem 2.5 (5) there exists j_0 such that $|I \setminus f^{mj}(U)|$ is arbitrarily small for all $j \ge j_0$. This means that the measure of $f^{mj+n_i}(X)$ can be made arbitrarily close to that of I with the appropriate choice of j and i (choose j first and i next), and thus that X is lim sup full for $f^m|_I$. This completes the proof of (1).

(2) Let us apply (1) to the set $X = Y$. Then $Y = \text{cyc}(I)$ up to a set of measure zero, and since $Y \subset B$ we have $B = \text{cyc}(I)$. Moreover, $E(f)$ has full measure in I. We can now apply (1) to an arbitrary subset X of $E(f) \cap I$ of positive measure. We see that X is lim sup full for $f^m|_I$. Since this holds for all positive measure sets X, we conclude that the map $f^m|_I$ is lim sup full. Therefore by Theorem 4.1 and the remark after it $f^m|_{f^j(I)}$ is exact for $0 \le j \le m - 1$ and $f|_{\text{cyc}(I)}$ is ergodic and conservative. The rest of the statement follows now easily from the invariance of sets $(E_{k,\varepsilon}(f) \setminus \mathbf{Cr}(f)) \cap \text{cyc}(I)$ (Lemma 2.9) and ergodicity of $f|_{\text{cyc}(I)}$. $\square$

It remains to consider a basic set B with $|E(f) \cap B| = 0$, which is exactly the situation where wild attractors appear. We do this in a sequence of lemmas.

Lemma 4.3. *Let A be a wild attractor contained in a basic set $B = B(\text{cyc}(I))$. Then for very $\varepsilon > 0$ there exists an invariant nowhere dense set $X \subset B$, contained in the ε-neighborhood of A, such that $|X| > 0$ and all points z with $\omega(z) = A$ are eventually mapped into X.*

Proof. We may assume that $\varepsilon > 0$ is so small that the compact ε-neighborhood U of A does not cover cyc(I). To see that, recall that wild attractors are always nowhere dense by the definition. Consider the set $X = \{x : x \in B$, the orbit of x is contained in U and $\omega(x) = A\}$. For any point z such that $\omega(z) = A$ there is j_0 such that $f^j(z) \in U$ for all $j \geq j_0$. By Corollary 2.6 (2) there is $j \geq j_0$ such that $f^j(z) \in B$. Therefore $f^j(z) \in X$. Since the realm of exact attraction of A has positive measure, it follows that $|X| > 0$. Clearly, X is invariant.

Let us show that $X \subset B$ is nowhere dense in cyc(I). This is obvious if B itself is a nowhere dense subset of cyc(I). Otherwise B = cyc(I) and by Theorem 2.5 (2) $f|_{\mathrm{cyc}(I)}$ is transitive. Hence, if $x \in X$ then the orbit of every neighborhood of x intersects cyc$(I) \setminus U$. Therefore X is nowhere dense. □

We need an important estimate on the density of X.

Lemma 4.4 ([L2]**).** *Let x be a point of density of an invariant set X absorbed by a basic set B. Then any point of $\omega(x)$ is a point of semi-density of X.*

Now we are ready to prove the main result of this section.

Theorem 4.5. *Let A be a wild attractor of a map $f \in \mathcal{S}$. Then A is* Cr-*super persistently recurrent.*

Proof. By Theorem 1.2 we may assume that $A = \omega(c)$, where c is a recurrent critical point. Let X be the set from Lemma 4.3. By Lemma 4.4 c is a point of semi-density of X. Thus, if c is not Cr-super persistently recurrent then by Lemma 4.2 (1) X = cyc(I) up to a set of measure zero, while on the other hand it is nowhere dense in cyc(I) by Lemma 4.3, a contradiction. □

The following theorem unites Lemma 4.2, Theorem 4.5 and some new arguments, thus giving a fuller description of Milnor attractors which are neither periodic orbits nor solenoidal sets. Note that if an attractor A is contained in a basic set B then since $|\,\mathrm{rl}(A)| > 0$ and by Corollary 2.6 (2) the measure of B is positive.

Theorem 4.6. *For every $f \in \mathcal{S}$ and a primitive attractor A that is neither a periodic orbit nor a solenoidal set, one of the following holds.*

(1) *The attractor A is wild. Then $A = \omega(c)$ for some* Cr-*super persistently recurrent critical point c. Furthermore, A is contained in a basic set B(cyc(I)) such that $f|_B$ is purely dissipative, $|A| = 0$, and almost all points of B are* Cr-*super persistent.*
(2) *The attractor A is equal to B(cyc(I)) = cyc(I) and if I is of period m then $f^m|_{f^j(I)}$ is exact for $0 \leq j \leq m-1$ and $f|_{\mathrm{cyc}(I)}$ is ergodic and conservative. Moreover, there exist k and $\varepsilon > 0$ such that $E_{k,\varepsilon} \supset$ cyc(I) up to a set of measure zero.*

Proof. Let A be an attractor which is neither a periodic orbit nor a solenoidal set. By Theorem 2.5 (6) there is a basic set $B = B$(cyc(I)) such that $A \subset B$. If $|E(f) \cap B| > 0$ then by Lemma 4.2 (2) we have the case (2).

Suppose that $|E(f) \cap B| = 0$ (that is, almost every point of B is Cr-super persistent) and show that this corresponds to the case (1) of the theorem. First we claim that in this case the set of recurrent points of $f|_B$ whose limit set is not minimal is of zero measure. Indeed, if the set of such recurrent points is of positive measure then almost all of them are not in $E(f)$ and thus are Cr-super persistently recurrent which by Theorem 2.7 implies that their limit sets are minimal, a contradiction. In particular, the set of points with the orbit dense in B is of zero measure (by Theorem 2.5 (1), B cannot be minimal). Therefore A cannot coincide with B and is a wild attractor.

Let us prove the rest of the statements of claim (1) of the theorem. The fact that $|A| = 0$ follows from the results of [Va] (note that we actually need a weaker version of results of [Va], since we apply them only in the negative Schwarzian case for which a significant part of the arguments from [Va] can be omitted).

It remains to show that $f|_B$ is purely dissipative, i.e. that there are no invariant subsets of B of positive measure on which f is conservative. To do this notice that by Theorem 1.2 there are finitely many primitive attractors $A_i, 0 \leq i \leq k$ such that for almost every point $x \in B$ the set $\omega(x)$ is one of them. Since $|E(f) \cap B| = 0$, all of them are wild. Now, suppose that $D \subset B$ is a set such that $f|_D$ is conservative and show that then it is contained in $\bigcup A_i$ modulo a set of measure zero. Indeed, otherwise there exist a number $\varepsilon > 0$ and a set $D' \subset D$ of positive measure disjoint from the ε-neighborhood of $\bigcup A_i$. By choosing a subset we may assume that D' consists of points z such that $\omega(z) = A_0$. Consider the set X constructed in Lemma 4.3 for this ε. By Lemma 4.3 all points of D' are eventually mapped into X and therefore will only be mapped into D' finitely many times. On the other hand by the Poincaré Recurrence Theorem if a map $f|_D$ is conservative then almost all points of D' return to D' infinitely many times. Thus $D \subset \bigcup A_i$. However then by the results of [Va] quoted above we have $|D| = 0$, a contradiction. □

5. Markov Maps

Consider in more detail the case (2) of Theorem 4.6. We want to strengthen the last property of the attractor $A = \text{cyc}(I)$ according to which $\text{cyc}(I) \subset E_{k,\varepsilon}$ up to a set of measure zero. This means that for almost every point of cyc(I) we can find a large n and a neighborhood V of $f^n(x)$, "big" on both sides of $f^n(x)$, which can be pulled back with small order. Our stronger version of this property says that we can find V which is independent of x and n, and the order of the pull-back is 0. Moreover, we can choose V as a neighborhood of any given point outside some finite set. Since in the weaker version $f^n(x)$ is the midpoint of V, we mimic this property in the stronger version. Thus, we will not only specify V, but also its subinterval U and require that $f^n(x) \in U$. This allows us to construct *Markov maps* introduced in the unimodal case by Martens in [Ma] and shows that the results of that paper related to Markov maps can be deduced from ours.

To simplify notation, assume that the period of I is 1. The arguments can be repeated for any period almost literally. A map $g : I \to I$ is called *topologically exact* if for every interval $J \subset I$ there is n with $g^n(J) = I$. Theorem 2.5 (5) implies that $f|_I$ is topologically exact since the endpoints of I are images of some critical points from the interior of I (otherwise there would be an invariant proper subinterval of I). We start with a simple lemma.

Lemma 5.1. *Let $f : I \to I$ be topologically exact. Then for every $\xi > 0$ there exists $N(\xi)$ such that every subinterval of I of length ξ is mapped by $f^{N(\xi)}$ onto I.*

Proof. If the lemma is false then we can find (by compactness of I) a subinterval of I that is not mapped onto I by arbitrarily large iterates of f, contrary to the exactness of f. □

To state the next proposition we need the following notation. Let $C_N(f)$ be the set of all critical points of f^N and the endpoints of the interval on which f is defined. Let $Y_N(f) = f^N(C_N(f))$.

Proposition 5.2. *Let $f \in \mathcal{S}$ be topologically exact on an invariant interval I. Assume also that $I \subset E_{k,\varepsilon'}$ for some k and $\varepsilon' > 0$ up to a set of measure zero. Then $I \subset E_{0,\varepsilon}$ for some $\varepsilon > 0$. Moreover, there is N such that for every pair of intervals U, V disjoint from the finite set $Y = Y_N(f|_I) \subset I$ and such that U is contained in the interior of V, for almost every $x \in I$ there exists an arbitrarily large n such that $f^n(x) \in U$ and the pull-back of V along $x, \ldots, f^n(x)$ has order 0.*

Proof. By Lemma 5.1 there exists $N = N(\varepsilon/2^{k+1})$ such that every subinterval I of length $\varepsilon/2^{k+1}$ is mapped by f^N onto I. Fix intervals U, V disjoint from Y and such that U is contained in the interior of V. If J is a subinterval of I of length $\varepsilon/2^{k+1}$ then it is mapped by f^N onto I, and thus there is an interval $K \subset J$ such that $f^N(K) = V$. Note that since V is disjoint from Y, we can choose K that is contained in the interior of J.

Denote by D_n the set of all points $x \in I$ for which there exists $l > n$ such that $f^l(x) \in U$ and the pull-back of V along $x, \ldots, f^l(x)$ has order 0. The set D_n is measurable. We show that it has full measure in I. Otherwise, there exists $y \in E_{k,\varepsilon}$ which is a point of density of the complement of D_n. Thus, it is enough to show that there are arbitrarily small neighborhoods of every $y \in E_{k,\varepsilon}$ in which the density of D_n is larger than some fixed $\beta > 0$.

Fix $y \in E_{k,\varepsilon}$. There is an arbitrarily large $t > n$ such that $G_t = [f^t(y) - \varepsilon, f^t(y) + \varepsilon]$ can be pulled back along $y, \ldots, f^t(y)$ with order at most k. Let the corresponding chain be $(G_0, \ldots, G_t)$. Also, let $G'_t = [f^t(y) - \varepsilon/2, f^t(y) + \varepsilon/2]$. The map f^t has at most $2^k - 1$ critical points in G_0. Hence, there are intervals $W_1 \subset [f^t(y) - \varepsilon/2, f^t(y)]$ and $W_2 \subset [f^t(y), f^t(y) + \varepsilon/2]$ of length $\varepsilon/2^{k+1}$ whose interiors are disjoint from $Y_t(f|_{G_0})$. Thus, every component of $f^{-t}(W_i)$ contained in G_0 is mapped by f^t onto W_i and there are no critical points of f^t in its interior. Also, since the endpoints of G_0 are mapped by f^t to the endpoint(s) of G_t, each point of $f^{-t}(W_i) \cap G_0$ is contained in one of these components.

Now we can choose intervals R_i contained in the interior of W_i (for $i = 1, 2$) which are mapped by f^N onto V. Since V is disjoint from $Y = f^N(C)$, there are no critical points of f^N in R_1 or R_2. Let Q_i be the subinterval of R_i that is mapped by f^N onto U for $i = 1, 2$. The sets Q_1 and Q_2 are two of the finitely many components of $f^{-N}(U)$. Thus, $|Q_i| \geq \delta$, where δ is the minimum length of these components. Let $(G'_0, \ldots, G'_t)$ be the pull-back chain of G'_t along $y, \ldots, f^t(y)$. Since G'_t is $1/2$ centered in G_t, the point $f^t(y)$ is $1/2$ centered in G'_t and the density of $Q = Q_1 \cup Q_2$ in G'_t is at least $2\delta/\varepsilon$, by Lemma 3.5 there exists $\beta > 0$, depending only on δ and ε, such that the density of $f^{-t}(Q)$ in G'_0 is at least β. By the construction, for every $x \in f^{-t}(Q)$ we have $f^{t+N}(x) \in U \subset V$ and the pull-back of V along $x, \ldots, f^{t+N}(x)$ has order 0. Thus, $f^{-t}(Q) \subset D_n$. By Lemma 5.1, $|G'_0|$ can be made arbitrarily small by taking sufficiently large t. This completes the proof that D_n has full measure.

The intersection of all sets D_n, $n \geq 0$ has also full measure and every point x from this intersection has the desired property. $\square$

Note that f in Proposition 5.2 has no periodic critical points. Therefore there is at least one critical point which is not an eventual critical image. Denote this point by c. Then $c \notin Y$ and therefore we can choose as intervals U, V small neighborhoods of c. By Proposition 5.2 for almost every point $x \in U$ there exists $n(x)$ and a pair of neighborhoods $W'(x) \subset W''(x)$ such that $f^{n(x)}$ maps $W''(x)$ onto V, has no critical points in the interior of $W''(x)$ and $f^{n(x)}(W'(x)) = U$. By the Koebe Lemma the quotient $|(f^n)'(y)|/|(f^n)'(z)|$ is bounded from above for any $y, z \in W'(x)$ by a constant which depends only on U and V. Choosing nice neighborhoods of c we see that the map T defined as $f^{n(x)}$ on intervals $W'(x)$ is exactly the so-called *Markov map* defined in [Ma] in the unimodal case. Therefore the results of [Ma] related to Markov maps indeed follow from Theorem 4.6 and Proposition 5.2. Replacing the second part of Theorem 4.6 by Proposition 5.2 we finally get Theorem 5.3.

Theorem 5.3. *For every $f \in \mathcal{S}$ and a primitive attractor A that is neither a periodic orbit nor a solenoidal set, one of the following holds.*

(1) *The attractor A is wild. Then $A = \omega(c)$ for some* Cr-*super persistently recurrent critical point c. Furthermore, A is contained in a basic set $B(\mathrm{cyc}(I))$ such that $f|_B$ is purely dissipative, $|A| = 0$, and almost all points of B are* Cr-*super persistent.*
(2) *The attractor A equals $B(\mathrm{cyc}(I)) = \mathrm{cyc}(I)$ and if I is of period m then $f^m|_{f^j(I)}$ is exact for $0 \leq j \leq m-1$, $f|_{\mathrm{cyc}(I)}$ is ergodic, conservative and $\mathrm{cyc}(I) \subset E_{0,\varepsilon}$ for some $\varepsilon > 0$ up to a set of measure zero (almost all points of A are* Cr-*reluctant). Moreover, there exists a finite set $Y \subset I$ such that:*
   (a) $\mathrm{Cr}(f) \cap I \not\subset Y$;
   (b) *for any two intervals $U \subset \mathrm{int}(V)$ disjoint from Y and for almost every $x \in I$ there is an arbitrarily large n and two intervals $x \in W' \subset W''$ such that $f^n|_{W''}$ has no critical points, $f^n(W'') = V$ and $f^n(W') = U$.*

References

[Ba] Barnes, J.: Conservative exact rational maps of the sphere. Preprint (1997)

[B] Blokh, A.: The “spectral” decomposition for one-dimensional maps. Dynamics Reported **4**, 1–59 (1995)

[BL1] Blokh, A. and Lyubich, M.: Nonexistence of wandering intervals and structure of topological attractors of one-dimensional dynamical systems. II. The smooth case. Ergod. Th. & Dynam. Sys. **9**, 751–758 (1989)

[BL2] Blokh, A. and Lyubich, M.: Attractors of maps of the interval. In: *Dynamical Systems and Ergodic Theory (Warsaw 1986)*. Banach Center Publ. **23**, Warsaw: PWN, 1989, pp. 427–442

[BL3] Blokh, A. and Lyubich, M.: On decomposition of one-dimensional dynamical systems into ergodic components. The case of negative Schwarzian. Leningr. Math. J. **1**, 137–155 (1990)

[BL4] Blokh, A. and Lyubich, M.: Measurable dynamics of S-unimodal maps of the interval. Ann. Sci. Ecole Norm. Sup. (4) **24**, 545–573 (1991)

[BM1] Blokh, A. and Misiurewicz, M.: Collet–Eckmann maps are unstable. Commun. Math. Phys. **191**, 61–70 (1998)

[BM2] Blokh, A. and Misiurewicz, M.: Typical limit sets of critical points for smooth interval maps. Preprint (1997)

[BKNS] Bruin, H., Keller, G., Nowicki, T. and van Strien, S.: Wild Cantor attractors exist. Ann. Math. **143**, 97–130 (1996)

[G] Guckenheimer, J.: Sensitive dependence to initial conditions for one-dimensional maps. Commun. Math. Phys. **70**, 133–160 (1979)

[GJ] Guckenheimer, J. and Johnson, S.: Distortion of S-unimodal maps. Ann. Math. **132**, 71–130 (1990)

[L1] Lyubich, M.: Nonexistence of wandering intervals and structure of topological attractors of one-dimensional dynamical systems. I. The case of negative Schwarzian derivative Ergod. Th. & Dynam. Sys. **9**, 737–749 (1989)

[L2] Lyubich, M.: Ergodic theory for smooth one-dimensional dynamical systems. SUNY at Stony Brook, preprint #1991/11

[Man] Mañé, R.: Hyperbolicity, sinks and measure in one-dimensional dynamics. Commun. Math. Phys. **100**, 495–524 (1985;)(Erratum: vol. **112** , 721–724 (1987))

[Ma] Martens, M.: Distortion results and invariant Cantor sets of unimodal maps. Ergod. Th. & Dynam. Sys. **14**, 331–349 (1994)

[MMS] Martens, M., de Melo, W. and van Strien, S.: Julia–Fatou–Sullivan theory for real one-dimensional dynamics. Acta Math. **168**, 273–318 (1992)

[Mi] Milnor, J.: On the concept of attractor. Commun. Math. Phys. **99**, 177–195 (1985)(Correction and remarks: vol. **102**, 517–519 (1985))

[Va] Vargas, E.: Measure of minimal sets of polymodal maps. Ergod. Th. & Dynam. Sys. **16**, 159–178 (1996)

Communicated by Ya. G. Sinai