

QMI Lesson 16: Derivative of Logarithmic and Exponential Functions

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Using the table below, we see that $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$

x	0.1	0.01	0.001	-0.1	-0.01	-0.001
$\frac{e^h - 1}{h}$	1.0517	1.0050	1.0005	0.9516	0.9950	0.9995

$$\text{So, } \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x$$

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In the same manner as in the previous slide, we can calculate

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If f is differentiable at x , then $\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$.

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The proof of this theorem is just a direct application of the chain rule!

Examples

Find the derivative of the following:

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- $\frac{d}{dx}[x^2 e^x]$

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- $\frac{d}{dx}[x^2 e^x] = \frac{d}{dx}[x^2]e^x + x^2 \frac{d}{dx}[e^x]$

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Find the derivative of the following:

$$\blacksquare \frac{d}{dx}[x^2 e^x] = \frac{d}{dx}[x^2]e^x + x^2 \frac{d}{dx}[e^x] = 2xe^x + x^2 e^x$$

Examples

Find the derivative of the following:

- $\frac{d}{dx}[x^2 e^x] = \frac{d}{dx}[x^2]e^x + x^2 \frac{d}{dx}[e^x] = 2xe^x + x^2 e^x = xe^x(2 + x).$
- $\frac{d}{dx}[(e^x + 2)^{\frac{3}{2}}]$

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i.e. when $x = \pm \frac{1}{\sqrt{2}}$,

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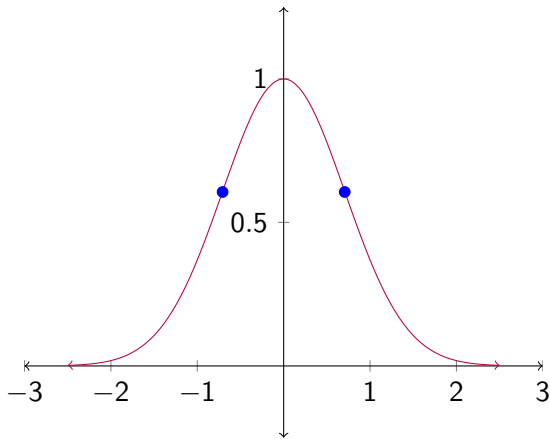
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i.e. when $x = \pm \frac{1}{\sqrt{2}}$, and the inflection points are $\left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$

Graph of $f(x) = 2^x$



Exponential Growth

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Well, $Q'(t) = kQ_0 e^{kt}$, so

$$\frac{Q'(t)}{Q(t)} = k,$$

as desired.

Example from Section 5.3

Blakely Properties owns a building in the commercial district of Birmingham. The present value of the market price of the property is given by $P(t) = 300000e^{-0.09t+0.5\sqrt{t}}$, where P is in dollars and t in years from present (for the next ten years). Find the optimal present value of the building's market price.

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$$0 = -0.09 + \frac{1}{4\sqrt{t}}$$

$$0.09(4) = \frac{1}{\sqrt{t}}$$

$$\frac{1}{0.09(4)} = \sqrt{t}$$

$$7.72 \approx t.$$

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We check the critical points $x = 0, 7.72, 10$ and see that the maximum present value of market price is \$600,779.

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What's more, we have the following theorem for logarithms of any base.

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If $f(x) = \log_b(|x|)$, then when $x \neq 0$ we have $f'(x) = \frac{1}{x \ln(b)}$.

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As an exercise, you might try to show why these derivatives work!

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Find the following derivatives.

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- $\frac{d}{dx}(x \ln x)$

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- $\frac{d}{dx}(x \ln x) = (x)'(\ln x) + x(\ln x)'$

Examples

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The Chain Rule for Logarithmic Functions

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Theorem (The Chain Rule for Logarithmic Functions)

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Clearly, it is easier to differentiate logarithmic functions when they are written in an expanded form. We can exploit the ease of this process to simplify the differentiation of other functions.

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Clearly, it is easier to differentiate logarithmic functions when they are written in an expanded form. We can exploit the ease of this process to simplify the differentiation of other functions. In particular, **logarithmic differentiation** can make functions that are products of other functions easier to differentiate.

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- 2 Differentiate both sides of the expression with respect to x .
- 3 Solve for $\frac{dy}{dx}$, i.e. for y' .

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Find the derivatives of the given function

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The population of a town t months after the opening of an auto assembly plant in the surrounding area is given by the function $P(t) = 18000e^{-(\ln 9)e^{-0.1t}}$. What is the relative rate of growth of the population 6 months after the opening of the auto assembly plant?

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So, 6 months after the auto assembly plant opens, the population will be growing at a relative rate of approximately 12.1%.

Assignment

Read 5.6. Do problems 8, 16, 28, 32, 46, 76 in 5.4 and 16, 30, 40, 48, 58, 78, 84 in 5.5.