QMI Lesson 19: Integration by Substitution, Definite Integral, and Area Under Curve

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4. Evaluate the resulting integral.
5. Replace u by $g(x)$ to obtain the final solution as a function of x.

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Example

Naturally, no computers were sold at time $t = 0$, so to solve for C, we simply notice

$$2000(0) + 30000e^{-0.05(0)} + C = 0 \implies C = -30000.$$

So we get our expression for the number of computers sold after month t to be

$$2000t + 30000e^{-0.05t} - 30000.$$

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5. So, $\int 2x(x^2+3)^4\ dx = \frac{2(x^2+1)^{\frac{5}{2}}}{5} - \frac{2(x^2+1)^{\frac{3}{2}}}{3} + C$.

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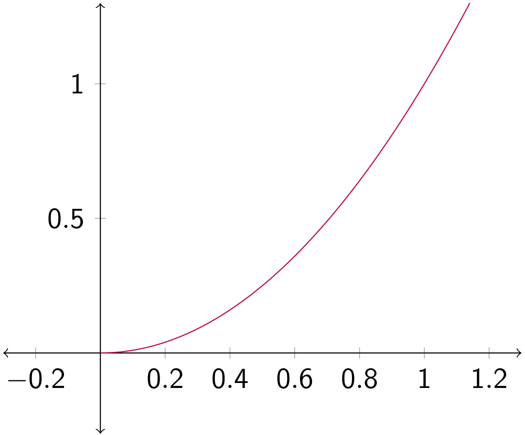
Recall when we talked about the **differential** that we interpreted $dy = f'(x)dx$ as an appoximation for Δy. On the graph of a function, we can see Δy and $\Delta x = dx$ forming an area related to the graph of a function. See the next slide for details.

Area Under the Curve of $f(x) = x^2$ on [0,1]



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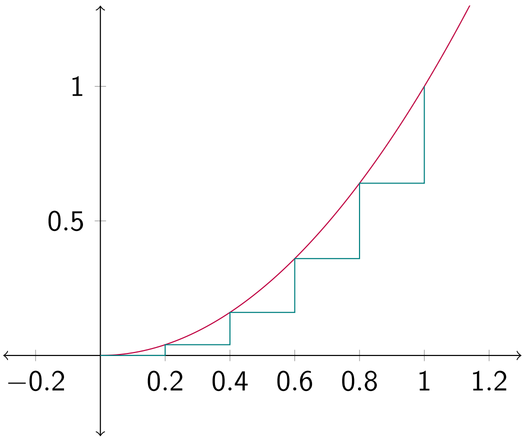

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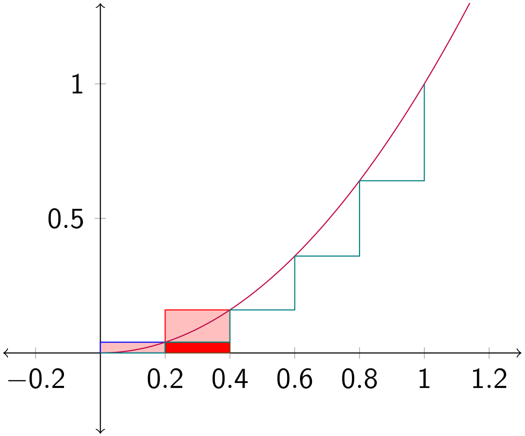

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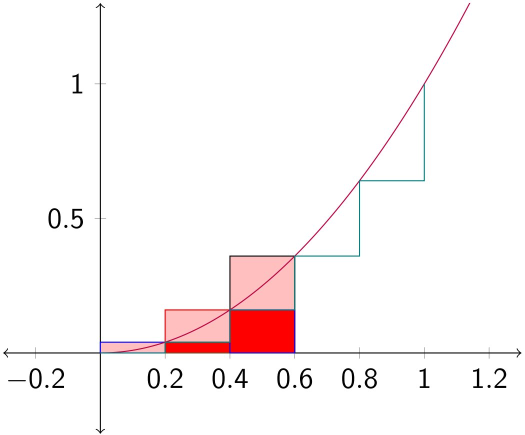

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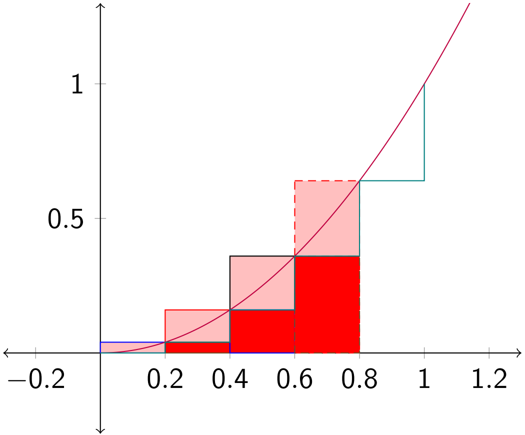

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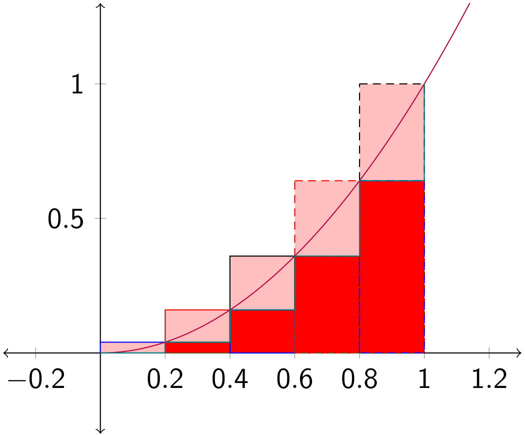
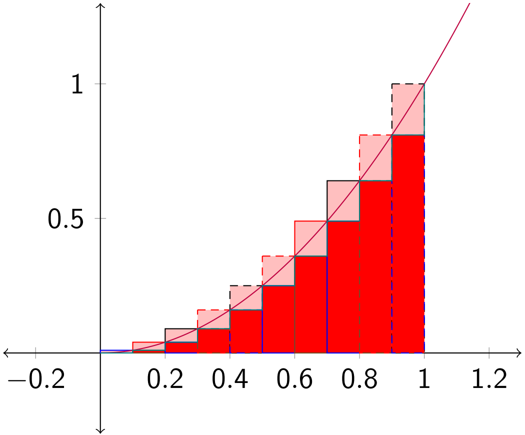


Improving Area Under the Curve of $f(x) = x^2$ on [0,1]



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And our second division of the area under $f(x)$ on [0,1] into ten intervals gave the approximation

$$\frac{1}{10}\left(f(0.1) + f(0.2) + \cdots + f(0.9) + f(1)\right) = 0.385.$$

Example

Let R be the region under the graph of $f(x) = 16 - x^2$ on the interval [1,3]. Find an approximation of the area of R by using four subintervals of equal length and picking the midpoint of each subinterval to evaluate the height of the approximating rectagle.

Well, our four subintervals are [1,1.5], [1.5,2], [2,2.5], and [2.5,3].

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$$\frac{1}{2}\left(f(1.25) + f(1.75) + f(2.25) + f(2.75)\right) = 23.375$$

Riemann Sums and the Area Under a Curve

We can generalize this process of approximation via Riemann sums and pass through the limit to calculate the actual area under the graph of a function.

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Let f be a nonnegative continuous function on $[a,b]$. Then the area A of the region under the graph of f is given by

$$A = \lim_{n \to \infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$

where x_1, x_2, $\ldots$, x_n are arbitrary points in the n subintervals of $[a,b]$ of equal width $\Delta x = \frac{b-a}{n}$.

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where x_1, x_2, ..., x_n are arbitrary points in the n subintervals of $[a,b]$ of equal width $\Delta x = \frac{b-a}{n}$.

If f is continuous on $[a, b]$, then this limit always exists, to the definition is not degenerate.

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exists and is the same for all choices of x_1, x_2, $\ldots$, x_n in the n subintervals of $[a,b]$ of equal width $\Delta x = \frac{b-a}{n}$, then this limit is called the definite integral of f from a to b and is denoted

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We call a the lower limit of integration and b the upper limit of integration.

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- if a function is non-negative, then its definite integral on an open interval is equal to the area under its curve.
- a definite integral is a number whereas an indefinite integral is a family of functions.

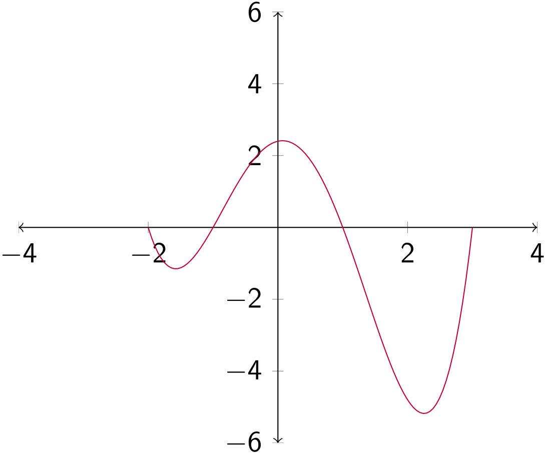
Geometric Interpretations of the Definite Integral

If f is non-negative on $[a, b]$, as we mentioned, the definite integral

$$\int_a^b f(x)\ dx$$

is the area between the curve and the x-axis. But if f takes both positive and negative values, then the definite intergral is equal to the area of the region below the graph but above the x-axis minus the area of the region blow the x-axis but above f. We can see this graphically.

The Definite Integral and Area Generalized



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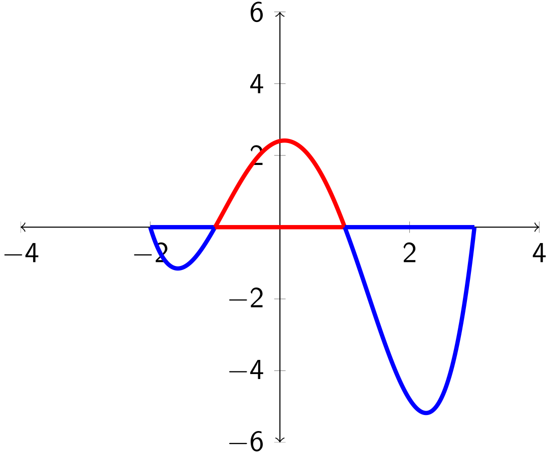

Assignment

Read 6.4. Do problems 14, 26, 36, 50, 54, 66 in 6.2 and 2, 10, 14, 16, 18 in 6.3.