QMI Lesson 20: The Fundamental Theorem of Calculus

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5 November 2014

Motivation

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Theorem (The Fundamental Theorem of Calculus)

Let f be continuous on $[a, b]$. Then

$$\int_a^b f(x)\ dx = F(b) - F(a),$$

where F is **any** *antiderivative of f; that is, $F'(x) = f(x)$.*

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Example

Let R be the region between the graph of $f(x) = x$ and the x-axis and between the lines $x = 1$ and $x = 3$. Find the area of this R.

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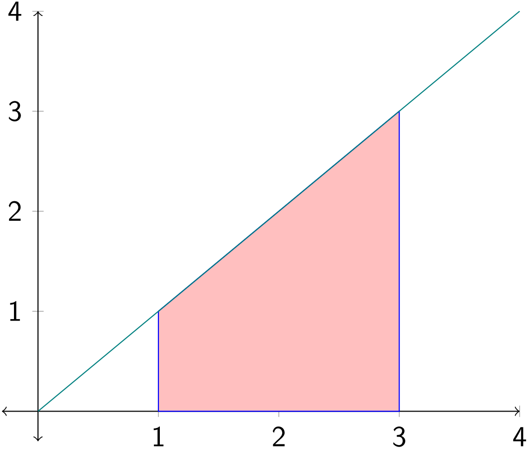
Let's first look at the graph of this region.

Area Under the Curve of $f(x) = x$ on [1,3]



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Notice, you should always pick the antiderivative with constant $C = 0$.

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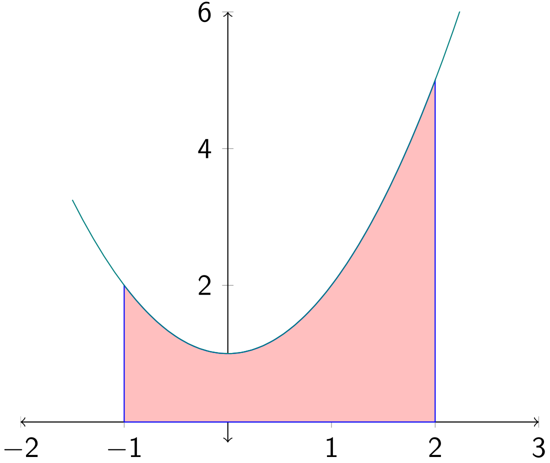
See the area on the next slide.

Area Under the Curve of $f(x) = x^2 + 1$ on [-1,2]



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Thus, we can relate the net change to the instantaneous rate(s) of change through the definite integral:

$$\int_a^b P'(x)\ dx = P(b) - P(a).$$

Example

As the 20th Century ended, Clark County in Nevada - home of Las Vegas - was one of the fastest growing metro areas in the US. From 1970 through 2000, the population was growing at the rate of

$$R(t) = 133680t^2 - 178788t + 234633 \qquad (0 \leq t \leq 3)$$

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So the total change in population during that decade was 278371.

Formula for Net Change

Theorem (Net Change Formula)

The net change in a function f on an interval $[a, b]$ is given by

$$f(a) - f(b) = \int_a^b f'(x)dx$$

provided that f' is continuous on $[a, b]$.

Example

The daily marginal cost function associated with producing a battery-operated pencil sharpener is given by

$$C'(x) = 0.000006x^2 - 0.006x + 4$$

where $C'(x)$ is measured in dollars per unit and x denotes the number of units produced. The daily fixed cost was determined to be \$100. Find the daily cost of producing (a) the first 100 units and (b) the 201st to the 400th units.

Example

a Producing the first unit involves the fixed daily cost! So to calculate the cost of producing the first 100 units, we need to calculate $100 + C(100) - C(0)$, i.e.

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b Producing the 201st to the 400th units costs $C(400) - C(200)$, i.e.

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Example

A certain city's rate of electricity consumption is expected to grow exponentially with growth rate constant $k = 0.04$. If the present rate of consumption is 40 million kilowatt-hours per year, what should be the total production of electricity over the next three years in order to meet the projected demand?

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Well, we have that $R(t) = 40e^{0.04t}$, where t is in years from now and R is in millions of kilowatt-hours. If $C(t)$ denotes the total consumption of electricity of the city in t years, then $C'(t) = R(t)$.

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$$1000(e^{0.12} - 1) = 127.5.$$

Thus, 127.5 million kWh must be produced over the next three years to meet demand.

The Validity of the Fundamental Theorem of Calculus

Let $A(t)$ denote the area of the region R under the graph of $f(x)$ from $x = a$ to $x = t$, then if h is a small positive number, we have the area of the region under the curve from $x = t$ to $x = t + h$ is

$$A(t+h) - A(t) \approx f(t) \cdot h \implies \lim_{h \to 0} \frac{A(t+h) - A(t)}{h} = f(x).$$

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Thus, $A'(t) = f(t)$. Hence, $A(x) = F(x) + C$, where F is some antiderivative of f. But since $A(a) = 0$, we have $A(x) = F(x) - F(a)$. Therefore,

$$F(b) - F(a) = A(b) = \int_a^b f(x)\ dx.$$

And this shows the validity of the Fundamental Theorem of Calculus.

Assignment

Read 6.5. Do problems 4, 16, 18, 32, 38, 40, 42, 48, 52, 56 in 6.4. (Due November 10.)