QMI Lesson 21: Evaluating the Definite Integral

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Well, if we let $x^2+1=u$, then we can see that an indefinite integral of $6x(x^2+1)^2$ is simply $(x^2+1)^3=u^3$. But notice:

$$(x^2+1)^3\Big|_0^2 = 124 \neq u^3\Big|_0^2 = 8.$$

Therefore, you have to **change** the limits of integration or **back-substitute**!

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$$\lim_{x \to b} u(x) = b' \; \& \; \lim_{x \to a} u(x) = a'.$$

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Then you can simply calculate the new integral with the new upper and lower limits b' and a' respectively.

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And this is the correct integral!

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So,

$$\int_0^4 x\sqrt{9+x^2}\ dx = \frac{1}{3}(x^2+9)^{\frac{3}{2}}\Big|_0^4 = \frac{1}{3}(125) - \frac{1}{3}(27) = 32.$$

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Of course, we could have calculated

$$\int_0^4 x\sqrt{9+x^2}\ dx = \int_9^{25} \frac{1}{2}\sqrt{u}\ dx.$$

Notice the changes in the limits of the integral!

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Notice the changes in the limits of the integral! Then we would have gotten:

$$\frac{1}{3}u^{\frac{3}{2}}\Bigg|^{2} 5_9 = 32.$$

And this is the correct integral!

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$$\int_0^1 \frac{x^2}{x^3+1}\,dx = \int_1^2 \frac{1}{3}\frac{du}{u} = \frac{1}{3}\ln(u)\Big|_1^2 = \frac{1}{3}\ln 2.$$

Example

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The Average Value of a Function

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Now, the average value of a function at n points can be written similarly as

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

And we may use this to approximate the average value of a function on an interval $[a, b]$ from which the x_i's are taken.

The Average Value of a Function

Let's rewrite this last expression.

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \frac{b-a}{b-a}\left[\frac{f(x_1)}{n} + \frac{f(x_2)}{n} + \cdots + \frac{f(x_n)}{n}\right].$$

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And we can rewrite this as

$$\frac{1}{b-a}\left[\frac{b-a}{n}f(x_1) + \frac{b-a}{n}f(x_2) + \cdots + \frac{b-a}{n}f(x_n)\right] =$$

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Passing through the limit, this function is simply

$$\frac{1}{b-a}\int_a^b f(x)\ dx.$$

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If f is integrable on $[a, b]$, then the average value of f on $[a, b]$ is

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There is also a clear geometric interpretation of this!

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If a function's integral over $[a, b]$ is C, then the area under f is the same as the area under the line $y = \frac{C}{b-a} =$

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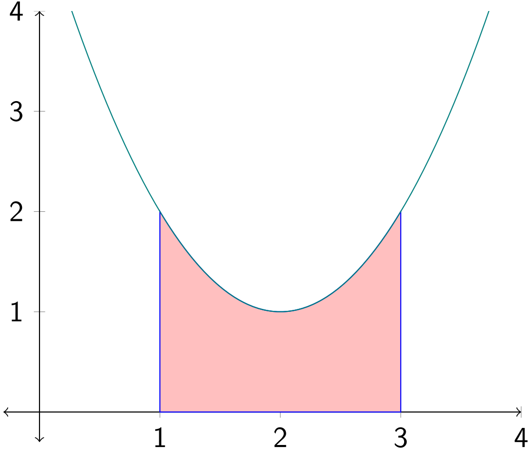

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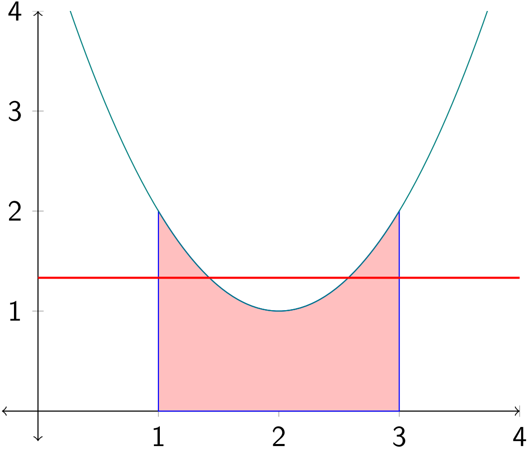

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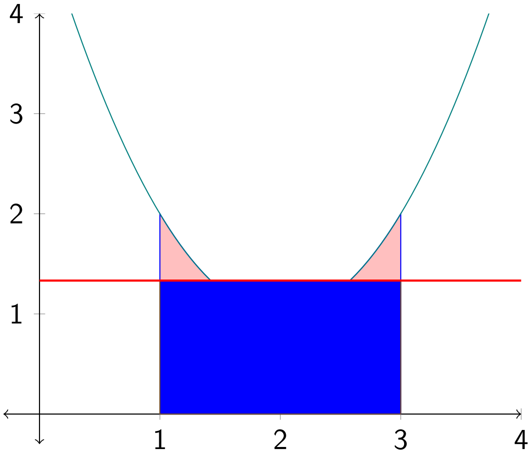

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Example

The amount of a certain drug in a patient's body t days after first taking it is given by $C(t) = 5e^{-0.2t}$ units. Determine the average amount of the drug present during the first four days after first taking it.

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We calculate

$$\frac{1}{4}\int_0^4 5e^{-0.2t}\ dt = -\frac{25}{4}e^{-0.2t}\Big|_0^4 \approx 3.44.$$

Assignment

Read 6.6. Do problems 2, 8, 14, 22, 28, 34, 44, 52, 64, 68, 72 in 6.5.