QMI Lesson 22:
Area Between Two Curves

C C Moxley

Samford University Brock School of Business

12 November 2014

A Motivating Example

Suppose that the petroleum consumption of a certain state is given by some function $f(t)$, where f is in millions of barrels and t is time in years.

A Motivating Example

Suppose that the petroleum consumption of a certain state is given by some function $f(t)$, where f is in millions of barrels and t is time in years. Suppose further that f is a good measure for the next five years.

A Motivating Example

Suppose that the petroleum consumption of a certain state is given by some function $f(t)$, where f is in millions of barrels and t is time in years. Suppose further that f is a good measure for the next five years. The state legislature has proposed a new nuclear power plant and a hybrid car tax incentive that they expect will result in a new consumption function of $g(t)$

A Motivating Example

Suppose that the petroleum consumption of a certain state is given by some function $f(t)$, where f is in millions of barrels and t is time in years. Suppose further that f is a good measure for the next five years. The state legislature has proposed a new nuclear power plant and a hybrid car tax incentive that they expect will result in a new consumption function of $g(t) < f(t)$ over the next five years.

A Motivating Example

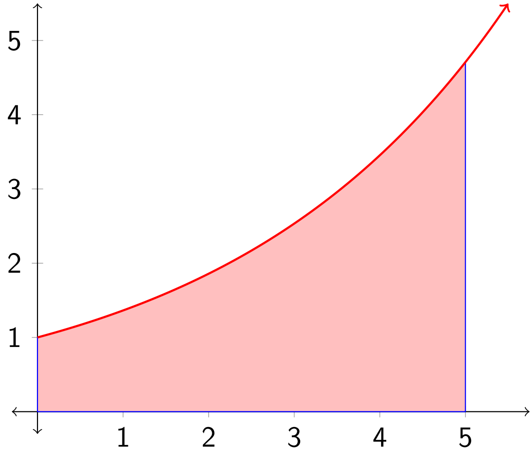
Suppose that the petroleum consumption of a certain state is given by some function $f(t)$, where f is in millions of barrels and t is time in years. Suppose further that f is a good measure for the next five years. The state legislature has proposed a new nuclear power plant and a hybrid car tax incentive that they expect will result in a new consumption function of $g(t) < f(t)$ over the next five years. How could we represent the total petroleum consumption saved by these measures?

Area Between Two Curves

The graph of g is in blue. And the graph of f is in red.

Area Between Two Curves

The graph of g is in blue. And the graph of f is in red.

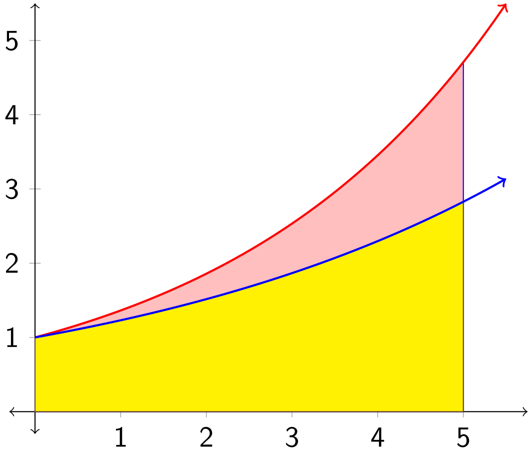


Area Between Two Curves

The graph of g is in blue. And the graph of f is in red.

Area Between Two Curves

The graph of g is in blue. And the graph of f is in red.

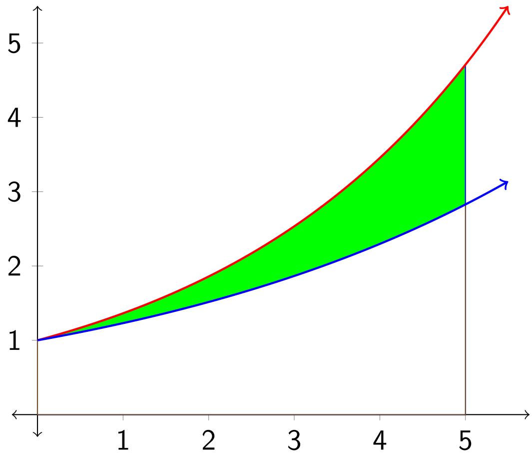


Area Between Two Curves

The graph of g is in blue. And the graph of f is in red.

Area Between Two Curves

The graph of g is in blue. And the graph of f is in red.



Area Between Two Curves

The area of the green region on the previous graph represents the difference in areas of the pink and yellow regions.

Area Between Two Curves

The area of the green region on the previous graph represents the difference in areas of the pink and yellow regions. That is, the area of the green region is simply

Area Between Two Curves

The area of the green region on the previous graph represents the difference in areas of the pink and yellow regions. That is, the area of the green region is simply

$$\int_0^5 f(x)\ dx - \int_0^5 g(x)\ dx.$$

Area Between Two Curves

The area of the green region on the previous graph represents the difference in areas of the pink and yellow regions. That is, the area of the green region is simply

$$\int_0^5 f(x)\ dx - \int_0^5 g(x)\ dx.$$

But by the linearirt of the integral, this area is just

Area Between Two Curves

The area of the green region on the previous graph represents the difference in areas of the pink and yellow regions. That is, the area of the green region is simply

$$\int_0^5 f(x)\ dx - \int_0^5 g(x)\ dx.$$

But by the linearirt of the integral, this area is just

$$\int_0^5 f(x) - g(x)\ dx.$$

Deriving a Formula for the Area Between Two Curves

Suppose that on some interval $[a, b]$ we have $f(x) \geq g(x) \geq 0$, then we have a situation just like before, and it's clear that the area between the graph of f and g on $[a, b]$ is simply

Deriving a Formula for the Area Between Two Curves

Suppose that on some interval $[a, b]$ we have $f(x) \geq g(x) \geq 0$, then we have a situation just like before, and it's clear that the area between the graph of f and g on $[a, b]$ is simply

$$\begin{aligned}\text{Area of region between } f \text{ and } g = & \text{ Area of region under the graph of } f \\ & - \text{Area of region under the graph of } g.\end{aligned}$$

Deriving a Formula for the Area Between Two Curves

Suppose that on some interval $[a, b]$ we have $f(x) \geq g(x) \geq 0$, then we have a situation just like before, and it's clear that the area between the graph of f and g on $[a, b]$ is simply

$$\begin{aligned}\text{Area of region between } f \text{ and } g = & \text{ Area of region under the graph of } f \\ & - \text{Area of region under the graph of } g.\end{aligned}$$

And this is just

$$\int_a^b f(x) - g(x)\ dx.$$

A Formula for the Area Between Two Curves

We have the following result.

A Formula for the Area Between Two Curves

We have the following result.

Theorem (Area of Between Two Curves)

If $f(x) \geq g(x)$ for all x in $[a, b]$ and if f and g are integrable on $[a, b]$, then the area of the region bounded above by f and below by g on $[a, b]$ is given by

$$\int_a^b f(x) - g(x)\ dx.$$

A Formula for the Area Between Two Curves

We have the following result.

Theorem (Area of Between Two Curves)

If $f(x) \geq g(x)$ for all x in $[a, b]$ and if f and g are integrable on $[a, b]$, then the area of the region bounded above by f and below by g on $[a, b]$ is given by

$$\int_a^b f(x) - g(x)\ dx.$$

Notice, this result holds even when f and/or g is negative on all or part of the interval $[a, b]$.

Example

Find the area of the region between $h(x) = 0$ and $k(x) = -x^2 + 4x - 8$ and between $x = -1$ and $x = 4$.

Example

Find the area of the region between $h(x) = 0$ and $k(x) = -x^2 + 4x - 8$ and between $x = -1$ and $x = 4$.

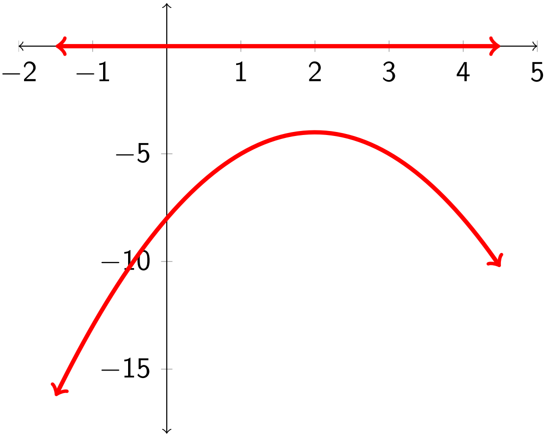
You will often need to graph these functions to see which lies above the other.

Example

Find the area of the region between $h(x) = 0$ and $k(x) = -x^2 + 4x - 8$ and between $x = -1$ and $x = 4$.

You will often need to graph these functions to see which lies above the other.

Example



Example

Find the area of the region between $h(x) = 0$ and $k(x) = -x^2 + 4x - 8$ and between $x = -1$ and $x = 4$.

Example

Find the area of the region between $h(x) = 0$ and $k(x) = -x^2 + 4x - 8$ and between $x = -1$ and $x = 4$.

You will often need to graph these functions to see which lies above the other.

Example

Find the area of the region between $h(x) = 0$ and $k(x) = -x^2 + 4x - 8$ and between $x = -1$ and $x = 4$.

You will often need to graph these functions to see which lies above the other. Because of the graph, we calculate

$$\int_{-1}^{4} 0 - (-x^2 + 4x - 8)\ dx = \int_{-1}^{4} x^2 - 4x + 8\ dx$$

Example

Find the area of the region between $h(x) = 0$ and $k(x) = -x^2 + 4x - 8$ and between $x = -1$ and $x = 4$.

You will often need to graph these functions to see which lies above the other. Because of the graph, we calculate

$$\int_{-1}^{4} 0 - (-x^2 + 4x - 8)\ dx = \int_{-1}^{4} x^2 - 4x + 8\ dx$$

$$\frac{x^3}{3} - 2x^2 + 8x \Big|_{-1}^{4} =$$

Example

Find the area of the region between $h(x) = 0$ and $k(x) = -x^2 + 4x - 8$ and between $x = -1$ and $x = 4$.

You will often need to graph these functions to see which lies above the other. Because of the graph, we calculate

$$\int_{-1}^{4} 0 - (-x^2 + 4x - 8)\ dx = \int_{-1}^{4} x^2 - 4x + 8\ dx$$

$$\frac{x^3}{3} - 2x^2 + 8x\Big|_{-1}^{4} = \frac{64}{3} - 32 + 32 - \left(\frac{-1}{3} - 2 - 8\right) = 31.\overline{66}.$$

Example

Find the area of the region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$ and between $x = 1$ and $x = 2$.

Example

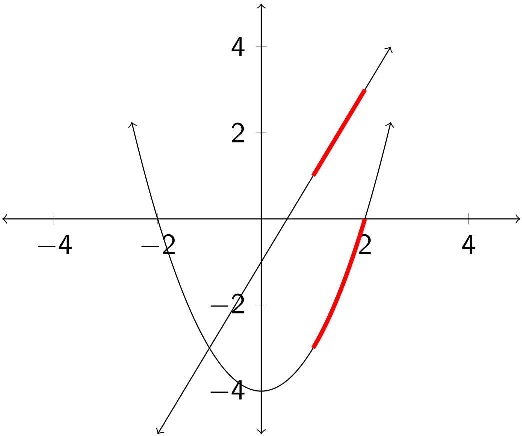
Find the area of the region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$ and between $x = 1$ and $x = 2$.

Example

Find the area of the region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$ and between $x = 1$ and $x = 2$.

Example

Find the area of the region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$ and between $x = 1$ and $x = 2$.



Example

Find the area of the region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$ and between $x = 1$ and $x = 2$.

Example

Find the area of the region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$ and between $x = 1$ and $x = 2$.

Example

Find the area of the region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$ and between $x = 1$ and $x = 2$.

Because of the graph, we calculate

$$\int_1^2 2x - 1 - (x^2 - 4)\ dx = \int_1^2 -x^2 + 2x + 3\ dx$$

Example

Find the area of the region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$ and between $x = 1$ and $x = 2$.

Because of the graph, we calculate

$$\int_1^2 2x - 1 - (x^2 - 4)\ dx = \int_1^2 -x^2 + 2x + 3\ dx$$

$$\left.\frac{-x^3}{3} + x^2 + 3x\right|_1^2 =$$

Example

Find the area of the region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$ and between $x = 1$ and $x = 2$.

Because of the graph, we calculate

$$\int_1^2 2x - 1 - (x^2 - 4)\ dx = \int_1^2 -x^2 + 2x + 3\ dx$$

$$\frac{-x^3}{3} + x^2 + 3x\Big|_1^2 = \frac{-8}{3} + 4 + 6 - \left(\frac{-1}{3} + 1 + 3\right) = \frac{11}{3}.$$

Example

Find the area of the entire region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$.

Example

Find the area of the entire region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$.

Example

Find the area of the entire region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$.

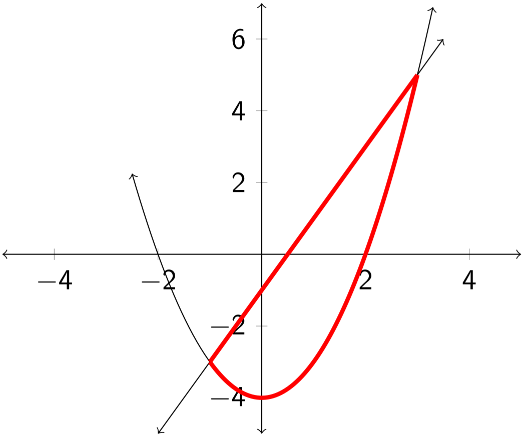
To do this, we must find the x-values that make $2x - 1 = x^2 - 4$.

Example

Find the area of the entire region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$.

To do this, we must find the x-values that make $2x - 1 = x^2 - 4$. This corresponds to the places that f and g intersect. We may again view the graph.

Example



Example

Find the area of the entire region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$.

Example

Find the area of the entire region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$.

Example

Find the area of the entire region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$.

To do this, we must find the x-values that make $2x - 1 = x^2 - 4$.

Example

Find the area of the entire region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$.

To do this, we must find the x-values that make $2x - 1 = x^2 - 4$. This corresponds to the places that f and g intersect. We may again view the graph. Solving, we get

$$x^2 - 2x + 3 = 0 \implies$$

Example

Find the area of the entire region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$.

To do this, we must find the x-values that make $2x - 1 = x^2 - 4$. This corresponds to the places that f and g intersect. We may again view the graph. Solving, we get

$$x^2 - 2x + 3 = 0 \implies (x-3)(x+1) = 0 \implies$$

Example

Find the area of the entire region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$.

To do this, we must find the x-values that make $2x - 1 = x^2 - 4$. This corresponds to the places that f and g intersect. We may again view the graph. Solving, we get

$$x^2 - 2x + 3 = 0 \implies (x-3)(x+1) = 0 \implies x = -1, 3.$$

Example

Find the area of the entire region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$.

To do this, we must find the x-values that make $2x - 1 = x^2 - 4$. This corresponds to the places that f and g intersect. We may again view the graph. Solving, we get

$$x^2 - 2x + 3 = 0 \implies (x-3)(x+1) = 0 \implies x = -1, 3.$$

Thus, we calculate

$$\int_{-1}^{3} -x^2 + 2x + 3 \, dx$$

Example

Find the area of the entire region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$.

To do this, we must find the x-values that make $2x - 1 = x^2 - 4$. This corresponds to the places that f and g intersect. We may again view the graph. Solving, we get

$$x^2 - 2x + 3 = 0 \implies (x-3)(x+1) = 0 \implies x = -1, 3.$$

Thus, we calculate

$$\int_{-1}^{3} -x^2 + 2x + 3 \; dx = \frac{-x^3}{3} + x^2 + 3x \bigg|_{-1}^{3} =$$

Example

Find the area of the entire region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$.

To do this, we must find the x-values that make $2x - 1 = x^2 - 4$. This corresponds to the places that f and g intersect. We may again view the graph. Solving, we get

$$x^2 - 2x + 3 = 0 \implies (x - 3)(x + 1) = 0 \implies x = -1, 3.$$

Thus, we calculate

$$\int_{-1}^{3} -x^2 + 2x + 3 \; dx = \frac{-x^3}{3} + x^2 + 3x \Big|_{-1}^{3} = 9 - \frac{-5}{3} =$$

Example

Find the area of the entire region between $f(x) = 2x - 1$ and $g(x) = x^2 - 4$.

To do this, we must find the x-values that make $2x - 1 = x^2 - 4$. This corresponds to the places that f and g intersect. We may again view the graph. Solving, we get

$$x^2 - 2x + 3 = 0 \implies (x - 3)(x + 1) = 0 \implies x = -1, 3.$$

Thus, we calculate

$$\int_{-1}^{3} -x^2 + 2x + 3 \; dx = \frac{-x^3}{3} + x^2 + 3x \Big|_{-1}^{3} = 9 - \frac{-5}{3} = \frac{32}{3}.$$

Example

Find the area of the region bounded by $f(x) = x^3$, the x-axis, and the vertical lines $x = -1$ and $x = 1$.

Example

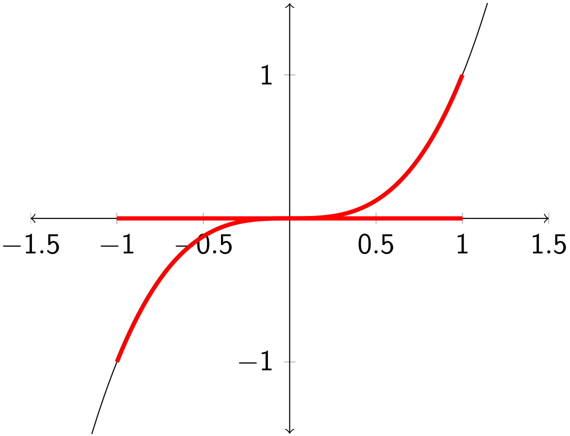
Find the area of the region bounded by $f(x) = x^3$, the x-axis, and the vertical lines $x = -1$ and $x = 1$.

Example

Find the area of the region bounded by $f(x) = x^3$, the x-axis, and the vertical lines $x = -1$ and $x = 1$.

Let's look at the graph of this region.

Example



Example

Find the area of the region bounded by $f(x) = x^3$, the x-axis, and the vertical lines $x = -1$ and $x = 1$.

Example

Find the area of the region bounded by $f(x) = x^3$, the x-axis, and the vertical lines $x = -1$ and $x = 1$.

Example

Find the area of the region bounded by $f(x) = x^3$, the x-axis, and the vertical lines $x = -1$ and $x = 1$.

Clearly, we need to calculate two different integrals because the function on top changes at $x = 0$, so we calculate

Example

Find the area of the region bounded by $f(x) = x^3$, the x-axis, and the vertical lines $x = -1$ and $x = 1$.

Clearly, we need to calculate two different integrals because the function on top changes at $x = 0$, so we calculate

$$\int_{-1}^{0} 0 - (x^3)\ dx + \int_{0}^{1} x^3 - 0\ dx =$$

Example

Find the area of the region bounded by $f(x) = x^3$, the x-axis, and the vertical lines $x = -1$ and $x = 1$.

Clearly, we need to calculate two different integrals because the function on top changes at $x = 0$, so we calculate

$$\int_{-1}^{0} 0 - (x^3)\ dx + \int_{0}^{1} x^3 - 0\ dx = -\left.\frac{x^4}{4}\right|_{-1}^{0} + \left.\frac{x^4}{4}\right|_{0}^{1} =$$

Example

Find the area of the region bounded by $f(x) = x^3$, the x-axis, and the vertical lines $x = -1$ and $x = 1$.

Clearly, we need to calculate two different integrals because the function on top changes at $x = 0$, so we calculate

$$\int_{-1}^{0} 0 - (x^3)\ dx + \int_{0}^{1} x^3 - 0\ dx = -\left.\frac{x^4}{4}\right|_{-1}^{0} + \left.\frac{x^4}{4}\right|_{0}^{1} = \frac{1}{4} + \frac{1}{4} =$$

Example

Find the area of the region bounded by $f(x) = x^3$, the x-axis, and the vertical lines $x = -1$ and $x = 1$.

Clearly, we need to calculate two different integrals because the function on top changes at $x = 0$, so we calculate

$$\int_{-1}^{0} 0 - (x^3)\ dx + \int_{0}^{1} x^3 - 0\ dx = -\frac{x^4}{4}\bigg|_{-1}^{0} + \frac{x^4}{4}\bigg|_{0}^{1} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Example

Find the area of the region bounded by $f(x) = x^3$, the x-axis, and the vertical lines $x = -1$ and $x = 1$.

Clearly, we need to calculate two different integrals because the function on top changes at $x = 0$, so we calculate

$$\int_{-1}^{0} 0 - (x^3)\ dx + \int_{0}^{1} x^3 - 0\ dx = -\left.\frac{x^4}{4}\right|_{-1}^{0} + \left.\frac{x^4}{4}\right|_{0}^{1} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

You could also obtain this result by using symmetry.

Example

Find the area of the region lying entirely between the graphs of $f(x) = x^3 - 3x + 3$ and $g(x) = x + 3$.

Example

Find the area of the region lying entirely between the graphs of $f(x) = x^3 - 3x + 3$ and $g(x) = x + 3$.

Example

Find the area of the region lying entirely between the graphs of $f(x) = x^3 - 3x + 3$ and $g(x) = x + 3$.

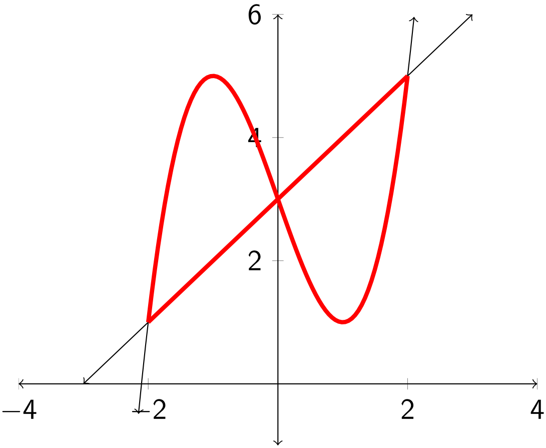
To do this, we must find the x-values that make $x^3 - 3x + 3 = x + 3$.

Example

Find the area of the region lying entirely between the graphs of $f(x) = x^3 - 3x + 3$ and $g(x) = x + 3$.

To do this, we must find the x-values that make $x^3 - 3x + 3 = x + 3$. This corresponds to the places that f and g intersect. We may again view the graph.

Example



Example

Find the area of the region lying entirely between the graphs of $f(x) = x^3 - 3x + 3$ and $g(x) = x + 3$.

Example

Find the area of the region lying entirely between the graphs of $f(x) = x^3 - 3x + 3$ and $g(x) = x + 3$.

Example

Find the area of the region lying entirely between the graphs of $f(x) = x^3 - 3x + 3$ and $g(x) = x + 3$.

To do this, we must find the x-values that make $x^3 - 3x + 3 = x + 3$.

Example

Find the area of the region lying entirely between the graphs of $f(x) = x^3 - 3x + 3$ and $g(x) = x + 3$.

To do this, we must find the x-values that make $x^3 - 3x + 3 = x + 3$. This corresponds to the places that f and g intersect. We may again view the graph. Solving, we get

$$x^3 - 4x = 0 \implies$$

Example

Find the area of the region lying entirely between the graphs of $f(x) = x^3 - 3x + 3$ and $g(x) = x + 3$.

To do this, we must find the x-values that make $x^3 - 3x + 3 = x + 3$. This corresponds to the places that f and g intersect. We may again view the graph. Solving, we get

$$x^3 - 4x = 0 \implies (x)(x-2)(x+2) = 0 \implies$$

Example

Find the area of the region lying entirely between the graphs of $f(x) = x^3 - 3x + 3$ and $g(x) = x + 3$.

To do this, we must find the x-values that make $x^3 - 3x + 3 = x + 3$. This corresponds to the places that f and g intersect. We may again view the graph. Solving, we get

$$x^3 - 4x = 0 \implies (x)(x - 2)(x + 2) = 0 \implies x = 0, 2, -2.$$

Example

Find the area of the region lying entirely between the graphs of $f(x) = x^3 - 3x + 3$ and $g(x) = x + 3$.

To do this, we must find the x-values that make $x^3 - 3x + 3 = x + 3$. This corresponds to the places that f and g intersect. We may again view the graph. Solving, we get

$$x^3 - 4x = 0 \implies (x)(x-2)(x+2) = 0 \implies x = 0, 2, -2.$$

Thus, we calculate

$$\int_{-2}^{0} -x^3 - 4x \; dx + \int_{0}^{2} x^3 + 4x \; dx$$

Example

Find the area of the region lying entirely between the graphs of $f(x) = x^3 - 3x + 3$ and $g(x) = x + 3$.

To do this, we must find the x-values that make $x^3 - 3x + 3 = x + 3$. This corresponds to the places that f and g intersect. We may again view the graph. Solving, we get

$$x^3 - 4x = 0 \implies (x)(x-2)(x+2) = 0 \implies x = 0, 2, -2.$$

Thus, we calculate

$$\int_{-2}^{0} -x^3 - 4x \;dx + \int_{0}^{2} x^3 + 4x \;dx =$$

$$-\frac{x^4}{4} - 2x\Big|_{-2}^{0} + \frac{x^4}{4} + 2x\Big|_{0}^{2} =$$

Example

Find the area of the region lying entirely between the graphs of $f(x) = x^3 - 3x + 3$ and $g(x) = x + 3$.

To do this, we must find the x-values that make $x^3 - 3x + 3 = x + 3$. This corresponds to the places that f and g intersect. We may again view the graph. Solving, we get

$$x^3 - 4x = 0 \implies (x)(x-2)(x+2) = 0 \implies x = 0, 2, -2.$$

Thus, we calculate

$$\int_{-2}^{0} -x^3 - 4x \; dx + \int_{0}^{2} x^3 + 4x \; dx =$$

$$-\frac{x^4}{4} - 2x\Big|_{-2}^{0} + \frac{x^4}{4} + 2x\Big|_{0}^{2} = 4 + 4 =$$

Example

Find the area of the region lying entirely between the graphs of $f(x) = x^3 - 3x + 3$ and $g(x) = x + 3$.

To do this, we must find the x-values that make $x^3 - 3x + 3 = x + 3$. This corresponds to the places that f and g intersect. We may again view the graph. Solving, we get

$$x^3 - 4x = 0 \implies (x)(x-2)(x+2) = 0 \implies x = 0, 2, -2.$$

Thus, we calculate

$$\int_{-2}^{0} -x^3 - 4x \; dx + \int_{0}^{2} x^3 + 4x \; dx =$$

$$-\frac{x^4}{4} - 2x \Big|_{-2}^{0} + \frac{x^4}{4} + 2x \Big|_{0}^{2} = 4 + 4 = 8.$$

Example

A nation's economist estimates the nation's rate of oil consumption to be $R(t) = 20e^{0.08t}$ for the next 5 years with t in years and R in millions of barrells per year. After implementing conservation measures, she estimates the nation's rate of oil consumption to be $R(t) = 20e^{0.05t}$. Calculate the total amount of oil saved in the next 5 years.

Example

We must calculate

$$\int_0^5 20e^{0.08t} - 20e^{0.05t}\ dt = 20\int_0^5 e^{0.08t} - e^{0.05t}\ dt =$$

Example

We must calculate

$$\int_0^5 20e^{0.08t} - 20e^{0.05t}\ dt = 20\int_0^5 e^{0.08t} - e^{0.05t}\ dt =$$

$$20\left[12.5e^{0.08t} - 20e^{0.05t}\Big|_0^5\right] \approx$$

Example

We must calculate

$$\int_0^5 20e^{0.08t} - 20e^{0.05t}\ dt = 20\int_0^5 e^{0.08t} - e^{0.05t}\ dt =$$

$$20\left[12.5e^{0.08t} - 20e^{0.05t}\Big|_0^5\right] \approx 20(-7.032699 + 7.5) \approx 9.3$$

Assignment

Read 6.7. Do problems 2, 4, 16, 24, 26, 30, 34, 42, 48, 58 in 6.6.