QMI Lesson 5:
One-Sided Limits & Continuity

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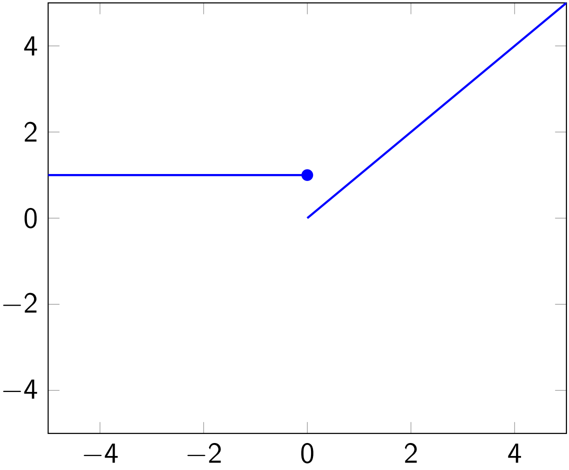
Samford University Brock School of Business

10 September 2014

One-Sided Limits

Recall from Section 2.5 that the function below had no limit as $x \to 0$.



One-Sided Limits

$$f(x) = \begin{cases} 1 & x = 2 \\ 2x - 1 & x \neq 1 \end{cases}$$

However, this function does have a limit as $x \to 0^+$ and as $x \to 0^-$, i.e. as x approaches 0 from the right and left respectively.

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However, this function does have a limit as $x \to 0^+$ and as $x \to 0^-$, i.e. as x approaches 0 from the right and left respectively. We call these limits one-sided limits.

One-Sided Limits

Definition (One-Sided Limits)

The function f has the **right-hand** limit L as x approaches a from the right, written $\lim_{x \to a^+} f(x) = L$, if the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to (but not equal to) a and greater than a.

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The function f has the **left-hand** limit L as x approaches a from the left, written $\lim_{x \to a^-} f(x) = L$, if the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to (but not equal to) a and less than a.

Relationship Between Limits and One-Sided Limits

Theorem

If f is a function defined for all values x close to $x = a$ (with the possible exception of a itself), then

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$$

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Basically, this means that the existence of a limit means that the left- and right-limits exist and that they equal one another.

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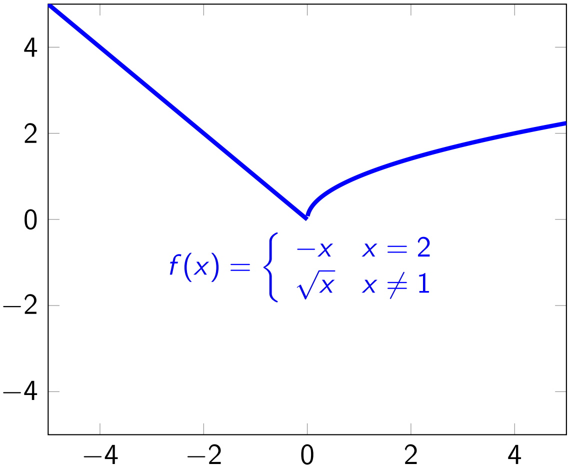
Basically, this means that the existence of a limit means that the left- and right-limits exist and that they equal one another. And the converse is true as well.

Example



Does $\lim_{x \to 0^+} f(x)$ exist? What about $\lim_{x \to 0^-} f(x)$? What is $f(0)$?

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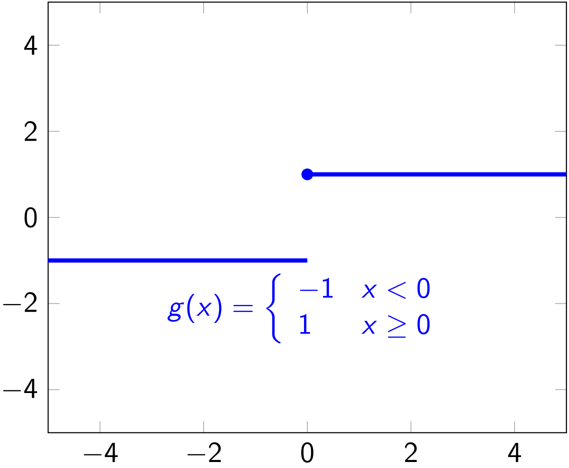

Does $\lim_{x \to 0^+} g(x)$ exist? What about $\lim_{x \to 0^-} g(x)$? What is $g(0)$?

Continuity

Definition (Continuity)

A function f is continuous at a if all of the following conditions hold.

1. $f(a)$ is defined.
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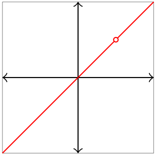
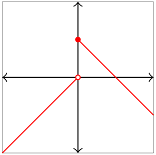
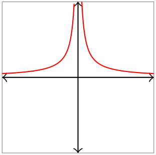
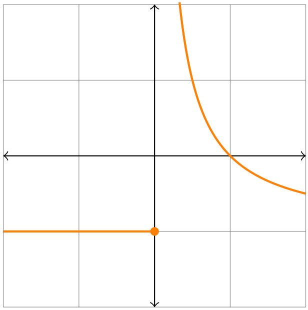
These conditions may be broken when a graph of a function has a **hole/puncture**,a **jump**,or a **vertical asymptote**.

Examples of Discontinuity

- Hole: 
- Jump: 
- Vertical asymptote: 

Intervals of Continuity: Example

On what interval(s) is $f(x) = \begin{cases} -1 & x < 0 \\ \frac{1}{x} - 1 & x \geq 0 \end{cases}$ continuous?



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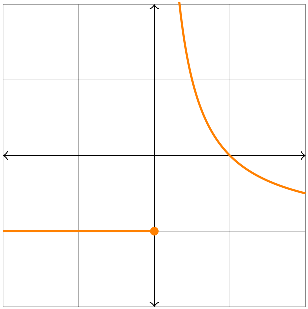

$(-\infty, 0) \cup (0, \infty)$

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A function must be defined to be continuous! So $f(x) = \sqrt{x-1}$ cannot be continuous on $(-\infty, 1)$!

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Theorem (Properties of Continuous Functions)

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- $\frac{f}{g}$, *provided* $g(a) \neq 0$.

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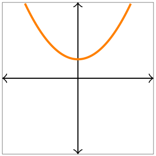

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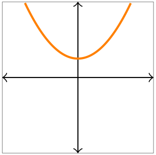

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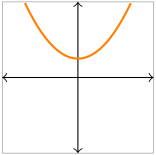

Clearly, the denominator is always positive, i.e. it's never zero. So the function has no discontinuities.

Root Theorem and the Intermediate Value Theorem: Motivation

Consider a hiker climbing from the bottom of Death Valley (elevation: -86m) to the top of nearby Telescope Peak (elevation 3368m).

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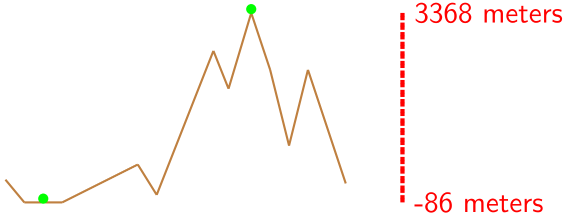

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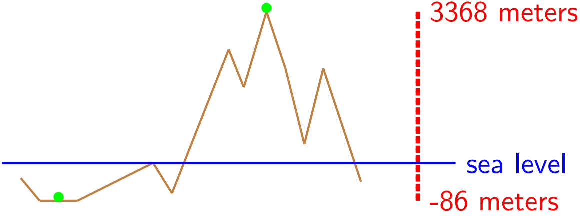

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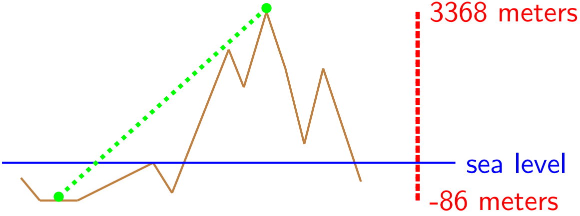

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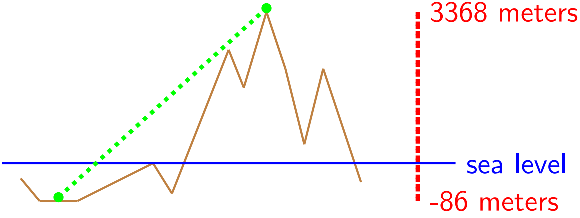

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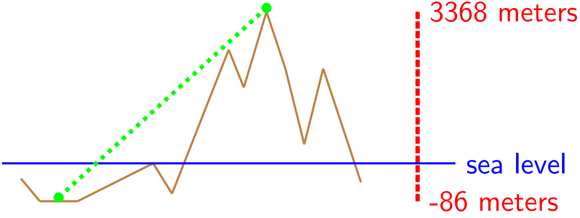

Clearly, the hiker must "cross" sea level, i.e. her elevation must be exactly zero at some point.

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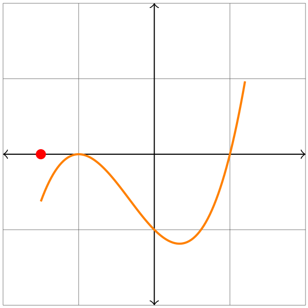
We can mathematically formalize this notion. The reason the hiker's elevation must be zero at some point is because motion is a continuous action. You cannot get to 3368 meters from -86 meters without passing every elevation on the way. In particular, you cannot "skip" the zero-level elevation. This idea is called the **Root Theorem** because it says, essentially, that a continuous function that takes both a negative and a positive value must have a root somewhere in the interval between the points at which it is negative and positive.

Root Theorem

Theorem (Existence of Roots of a Continuous Function)

If a continuous function f on a closed interval $[a, b]$ takes values $f(a)$ and $f(b)$ such that $f(a) \cdot f(b) < 0$, then there is at least one solution to the equation $f(x) = 0$ in the interval (a, b).



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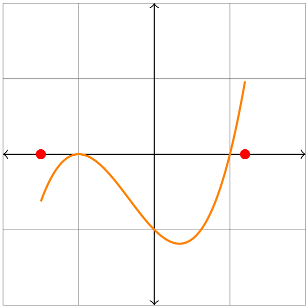

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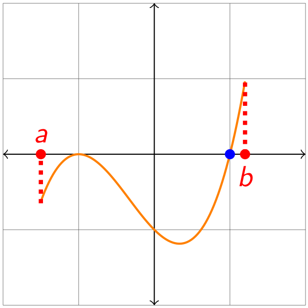

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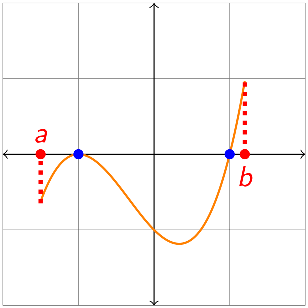

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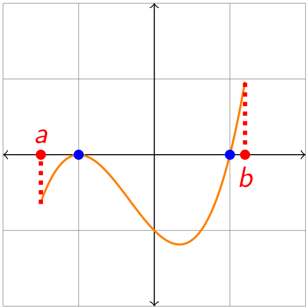

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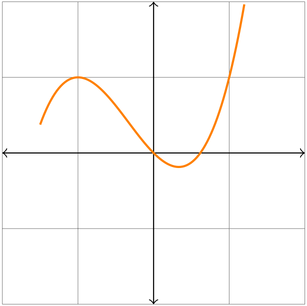

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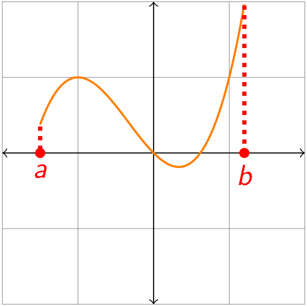

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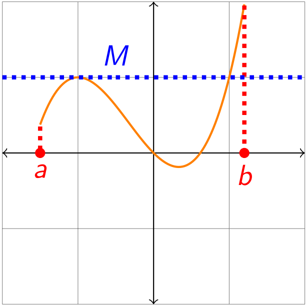

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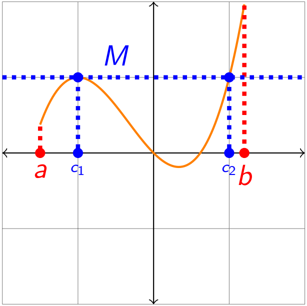

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Well, $f(0) = 3 > 0$, and $f(1) = -1 < 0$. Therefore, since f is continuous on $[0, 1]$ and since $f(1) < 0$ and $f(0) > 0$, we know that there exists some $c \in (0, 1)$ such that

Example

Find an interval on which $f(x) = x^4 - 5x + 3$ must have a root.

Well, $f(0) = 3 > 0$, and $f(1) = -1 < 0$. Therefore, since f is continuous on $[0, 1]$ and since $f(1) < 0$ and $f(0) > 0$, we know that there exists some $c \in (0, 1)$ such that $f(c) = 0$.

Example

Do the conclusions of the Intermediate Value Theorem hold if f is not a continuous function?

Example

Do the conclusions of the Intermediate Value Theorem hold if f is not a continuous function? **No!** Draw an example.

Assignment

Read 2.6. Do problems 6, 10, 18, 34, 42, 52, 60, 72, 88, 94, 96 in 2.5.