QMI Lesson 9: Higher Order Derivatives, Implicit Differentiation, and Differentials

C C Moxley

Samford University Brock School of Business

24 September 2014

Higher Order Derivatives

A function $f(x)$ has a first-order derivative which we have discussed at length: $f'(x)$

Higher Order Derivatives

A function $f(x)$ has a first-order derivative which we have discussed at length: $f'(x)$ or $\frac{df}{dx}$

Higher Order Derivatives

A function $f(x)$ has a first-order derivative which we have discussed at length: $f'(x)$ or $\frac{df}{dx}$ or $D_x^1 f(x)$.

Higher Order Derivatives

A function $f(x)$ has a first-order derivative which we have discussed at length: $f'(x)$ or $\frac{df}{dx}$ or $D_x^1 f(x)$.

But it also has second-, third-, and n^{th}-order derivatives.

Higher Order Derivatives

A function $f(x)$ has a first-order derivative which we have discussed at length: $f'(x)$ or $\frac{df}{dx}$ or $D_x^1 f(x)$.

But it also has second-, third-, and n^{th}-order derivatives.

First Order:	$f'(x)$	$\frac{df}{dx}$	$D_x^1 f(x)$
Second Order:	$f''(x)$	$\frac{d^2 f}{dx^2}$	$D_x^2 f(x)$
Third Order:	$f'''(x)$	$\frac{d^3 f}{dx^3}$	$D_x^3 f(x)$
n^{th} Order:	$f^{(n)}(x)$	$\frac{d^n fn}{dx^n}$	$D_x^n f(x)$

Higher Order Derivatives: Example

The higher order derivatives give useful information about the function they describe.

Higher Order Derivatives: Example

The higher order derivatives give useful information about the function they describe. For instance, if $s(t) = 2t^2 - 3t + 20$ is a function giving position s with respect to time t, then

Higher Order Derivatives: Example

The higher order derivatives give useful information about the function they describe. For instance, if $s(t) = 2t^2 - 3t + 20$ is a function giving position s with respect to time t, then the first-derivative gives velocity while the second derivative gives acceleration.

Higher Order Derivatives: Example

The higher order derivatives give useful information about the function they describe. For instance, if $s(t) = 2t^2 - 3t + 20$ is a function giving position s with respect to time t, then the first-derivative gives velocity while the second derivative gives acceleration. Find the velocity and acceleration at $x = 2$.

Higher Order Derivatives: Example

The higher order derivatives give useful information about the function they describe. For instance, if $s(t) = 2t^2 - 3t + 20$ is a function giving position s with respect to time t, then the first-derivative gives velocity while the second derivative gives acceleration. Find the velocity and acceleration at $x = 2$.

Well, $f'(x) = 4t - 3$,

Higher Order Derivatives: Example

The higher order derivatives give useful information about the function they describe. For instance, if $s(t) = 2t^2 - 3t + 20$ is a function giving position s with respect to time t, then the first-derivative gives velocity while the second derivative gives acceleration. Find the velocity and acceleration at $x = 2$.

Well, $f'(x) = 4t - 3$, and $f''(x) = 4$.

Higher Order Derivatives: Example

The higher order derivatives give useful information about the function they describe. For instance, if $s(t) = 2t^2 - 3t + 20$ is a function giving position s with respect to time t, then the first-derivative gives velocity while the second derivative gives acceleration. Find the velocity and acceleration at $x = 2$.

Well, $f'(x) = 4t - 3$, and $f''(x) = 4$. So the velocity is $f'(2) = 5$,

Higher Order Derivatives: Example

The higher order derivatives give useful information about the function they describe. For instance, if $s(t) = 2t^2 - 3t + 20$ is a function giving position s with respect to time t, then the first-derivative gives velocity while the second derivative gives acceleration. Find the velocity and acceleration at $x = 2$.

Well, $f'(x) = 4t - 3$, and $f''(x) = 4$. So the velocity is $f'(2) = 5$, and the acceleration is $f''(2) = 4$.

Example: The CPI and Inflation

The Consumer Price Index (CPI) is a measure of prices of good across the economy. If $I(t)$ is a function giving the CPI at a certain time t, then how could you describe

Example: The CPI and Inflation

The Consumer Price Index (CPI) is a measure of prices of good across the economy. If $I(t)$ is a function giving the CPI at a certain time t, then how could you describe $\frac{I'(t)}{I(t)}$?

Example: The CPI and Inflation

The Consumer Price Index (CPI) is a measure of prices of good across the economy. If $I(t)$ is a function giving the CPI at a certain time t, then how could you describe $\frac{I'(t)}{I(t)}$?

This is the *relative rate of change in the consumer price index*,

Example: The CPI and Inflation

The Consumer Price Index (CPI) is a measure of prices of good across the economy. If $I(t)$ is a function giving the CPI at a certain time t, then how could you describe $\frac{I'(t)}{I(t)}$?

This is the *relative rate of change in the consumer price index*, i.e. it is the **inflation rate**.

Example: The CPI and Inflation

The Consumer Price Index (CPI) is a measure of prices of good across the economy. If $I(t)$ is a function giving the CPI at a certain time t, then how could you describe $\frac{I'(t)}{I(t)}$?

This is the *relative rate of change in the consumer price index*, i.e. it is the **inflation rate**. Because the CPI is always positive, the inflation rate takes the same sign as the rate of change of the CPI, i.e. of $I'(t)$.

Example: The CPI and Inflation

The Consumer Price Index (CPI) is a measure of prices of good across the economy. If $I(t)$ is a function giving the CPI at a certain time t, then how could you describe $\frac{I'(t)}{I(t)}$?

This is the *relative rate of change in the consumer price index*, i.e. it is the **inflation rate**. Because the CPI is always positive, the inflation rate takes the same sign as the rate of change of the CPI, i.e. of $I'(t)$.

The second derivative $I''(t)$ describes the acceleration of the CPI.

Example: The CPI and Inflation

The Consumer Price Index (CPI) is a measure of prices of good across the economy. If $I(t)$ is a function giving the CPI at a certain time t, then how could you describe $\frac{I'(t)}{I(t)}$?

This is the *relative rate of change in the consumer price index*, i.e. it is the **inflation rate**. Because the CPI is always positive, the inflation rate takes the same sign as the rate of change of the CPI, i.e. of $I'(t)$.

The second derivative $I''(t)$ describes the acceleration of the CPI. You can think of this as the rate of change of inflation. (But is it?)

Example: The CPI and Inflation

The Consumer Price Index (CPI) is a measure of prices of good across the economy. If $I(t)$ is a function giving the CPI at a certain time t, then how could you describe $\frac{I'(t)}{I(t)}$?

This is the *relative rate of change in the consumer price index*, i.e. it is the **inflation rate**. Because the CPI is always positive, the inflation rate takes the same sign as the rate of change of the CPI, i.e. of $I'(t)$.

The second derivative $I''(t)$ describes the acceleration of the CPI. You can think of this as the rate of change of inflation. (But is it?) What would it mean for $I'(t)$ to be positive with $I''(t)$ negative?

Example: The CPI and Inflation

The Consumer Price Index (CPI) is a measure of prices of good across the economy. If $I(t)$ is a function giving the CPI at a certain time t, then how could you describe $\frac{I'(t)}{I(t)}$?

This is the *relative rate of change in the consumer price index*, i.e. it is the **inflation rate**. Because the CPI is always positive, the inflation rate takes the same sign as the rate of change of the CPI, i.e. of $I'(t)$.

The second derivative $I''(t)$ describes the acceleration of the CPI. You can think of this as the rate of change of inflation. (But is it?) What would it mean for $I'(t)$ to be positive with $I''(t)$ negative? This would mean that CPI was **increasing at a decreasing rate**.

Example: The CPI and Inflation

The Consumer Price Index (CPI) is a measure of prices of good across the economy. If $I(t)$ is a function giving the CPI at a certain time t, then how could you describe $\frac{I'(t)}{I(t)}$?

This is the *relative rate of change in the consumer price index*, i.e. it is the **inflation rate**. Because the CPI is always positive, the inflation rate takes the same sign as the rate of change of the CPI, i.e. of $I'(t)$.

The second derivative $I''(t)$ describes the acceleration of the CPI. You can think of this as the rate of change of inflation. (But is it?) What would it mean for $I'(t)$ to be positive with $I''(t)$ negative? This would mean that CPI was **increasing at a decreasing rate**. This would be the case if you paid $400 more for goods this year than you did last year

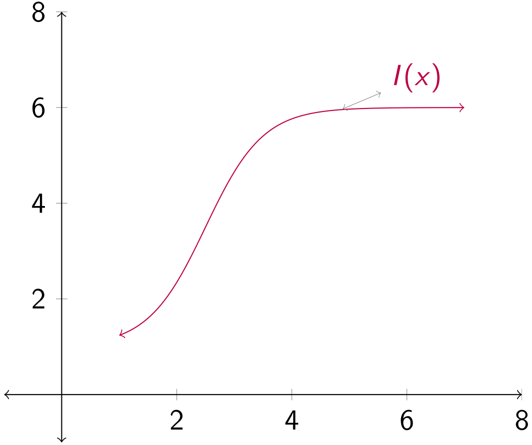
Example: The CPI and Inflation

The Consumer Price Index (CPI) is a measure of prices of good across the economy. If $I(t)$ is a function giving the CPI at a certain time t, then how could you describe $\frac{I'(t)}{I(t)}$?

This is the *relative rate of change in the consumer price index*, i.e. it is the **inflation rate**. Because the CPI is always positive, the inflation rate takes the same sign as the rate of change of the CPI, i.e. of $I'(t)$.

The second derivative $I''(t)$ describes the acceleration of the CPI. You can think of this as the rate of change of inflation. (But is it?) What would it mean for $I'(t)$ to be positive with $I''(t)$ negative? This would mean that CPI was **increasing at a decreasing rate**. This would be the case if you paid \$400 more for goods this year than you did last yearbut will pay only \$300 more for goods next year than you did this year.

Graph: The CPI and Inflation



Implicit Differentiation

Often, a function is not given in its explicit form, i.e. with one variable isolated on one side of the equation.

Implicit Differentiation

Often, a function is not given in its explicit form, i.e. with one variable isolated on one side of the equation.

Implicit form: $yx + x = 1$.

Implicit Differentiation

Often, a function is not given in its explicit form, i.e. with one variable isolated on one side of the equation.

Implicit form: $yx + x = 1$.
Explicit form: $y = \frac{1}{x} - 1$.

Implicit Differentiation

Often, a function is not given in its explicit form, i.e. with one variable isolated on one side of the equation.

Implicit form: $yx + x = 1$.
Explicit form: $y = \frac{1}{x} - 1$.

Sometimes, it is difficult or impossible to write a function in its explicit form!

Implicit Differentiation

Often, a function is not given in its explicit form, i.e. with one variable isolated on one side of the equation.

Implicit form: $yx + x = 1$.
Explicit form: $y = \frac{1}{x} - 1$.

Sometimes, it is difficult or impossible to write a function in its explicit form! How can we find the derivative of these sorts of functions?

Implicit Differentiation

Often, a function is not given in its explicit form, i.e. with one variable isolated on one side of the equation.

Implicit form: $yx + x = 1$.
Explicit form: $y = \frac{1}{x} - 1$.

Sometimes, it is difficult or impossible to write a function in its explicit form! How can we find the derivative of these sorts of functions?

Simply use the Chain Rule!

Implicit Differentiation: Steps

If you wanted to find $\frac{dy}{dx}$ with $y^3x^2 + 6x^2 = y + 12$, you must

1. Differentiate both sides of the equation with respect to x, using the Chain Rule where necessary,

Implicit Differentiation: Steps

If you wanted to find $\frac{dy}{dx}$ with $y^3x^2 + 6x^2 = y + 12$, you must

1. Differentiate both sides of the equation with respect to x, using the Chain Rule where necessary, remembering that y is really $y(x)$, a function and not a variable!
2. Solve for y', i.e. for $\frac{dy}{dx}$.

Implicit Differentiation: Steps

With $\frac{dy}{dx}$ with $y^3x^2 + 6x^2 = y + 12$, we have (differentiating both sides with respect to x)

Implicit Differentiation: Steps

With $\frac{dy}{dx}$ with $y^3x^2 + 6x^2 = y + 12$, we have (differentiating both sides with respect to x)

$$\begin{aligned} y^3(2x) + 3y^2\frac{dy}{dx}x^2 + 12x = \frac{dy}{dx} &\implies \\ 2y^3x + 12x = \frac{dy}{dx} - 3y^2\frac{dy}{dx}x^2 &\implies \\ 2y^3x + 12x = \frac{dy}{dx}(1 - 3y^2x^2) &\implies \\ \frac{2y^3x + 12x}{1 - 3y^2x^2} = \frac{dy}{dx} \end{aligned}$$

Implicit Differentiation: Steps

With $\frac{dy}{dx}$ with $y^3x^2 + 6x^2 = y + 12$, we have (differentiating both sides with respect to x)

$$
\begin{aligned}
y^3(2x) + 3y^2\frac{dy}{dx}x^2 + 12x = \frac{dy}{dx} &\implies \\
2y^3x + 12x = \frac{dy}{dx} - 3y^2\frac{dy}{dx}x^2 &\implies \\
2y^3x + 12x = \frac{dy}{dx}(1 - 3y^2x^2) &\implies \\
\frac{2y^3x + 12x}{1 - 3y^2x^2} = \frac{dy}{dx}
\end{aligned}
$$

So, at (1,2), the slope of the tangent line is $\frac{2 \cdot 2^3(1) + 12(1)}{1 - 3(2)^2(1)^2}$

Implicit Differentiation: Steps

With $\frac{dy}{dx}$ with $y^3x^2 + 6x^2 = y + 12$, we have (differentiating both sides with respect to x)

$$y^3(2x) + 3y^2\frac{dy}{dx}x^2 + 12x = \frac{dy}{dx} \implies$$
$$2y^3x + 12x = \frac{dy}{dx} - 3y^2\frac{dy}{dx}x^2 \implies$$
$$2y^3x + 12x = \frac{dy}{dx}(1 - 3y^2x^2) \implies$$
$$\frac{2y^3x + 12x}{1 - 3y^2x^2} = \frac{dy}{dx}$$

So, at (1,2), the slope of the tangent line is $\frac{2\cdot2^3(1)+12(1)}{1-3(2)^2(1)^2} = -\frac{28}{11}$.

Implicit Differentiation: Steps

With $\frac{dy}{dx}$ with $y^3x^2 + 6x^2 = y + 12$, we have (differentiating both sides with respect to x)

$$y^3(2x) + 3y^2\frac{dy}{dx}x^2 + 12x = \frac{dy}{dx} \implies$$
$$2y^3x + 12x = \frac{dy}{dx} - 3y^2\frac{dy}{dx}x^2 \implies$$
$$2y^3x + 12x = \frac{dy}{dx}(1 - 3y^2x^2) \implies$$
$$\frac{2y^3x + 12x}{1 - 3y^2x^2} = \frac{dy}{dx}$$

So, at (1,2), the slope of the tangent line is $\frac{2\cdot 2^3(1)+12(1)}{1-3(2)^2(1)^2} = -\frac{28}{11}$.

Note: It's not always necessary to find an explicit expression for $\frac{dy}{dx}$.

Marginal Rate of Technical Substitution

The chief economist of a nation estimates the output of the country as $Q(x, y) = 10x^{\frac{3}{4}}y^{\frac{1}{4}}$, where x is the amount of money spent on labor and y is the amount spent on capital, all measured in billions of dollars.

Marginal Rate of Technical Substitution

The chief economist of a nation estimates the output of the country as $Q(x, y) = 10x^{\frac{3}{4}}y^{\frac{1}{4}}$, where x is the amount of money spent on labor and y is the amount spent on capital, all measured in billions of dollars. What's the output when $y = 81$ and $x = 625$?

Marginal Rate of Technical Substitution

The chief economist of a nation estimates the output of the country as $Q(x, y) = 10x^{\frac{3}{4}}y^{\frac{1}{4}}$, where x is the amount of money spent on labor and y is the amount spent on capital, all measured in billions of dollars. What's the output when $y = 81$ and $x = 625$? Suppose the economist wants the nation's output to remain constant at that level. By how much should the amount spent on capital be changed if the amount spent on labor is to increased by \$1 billion?

Marginal Rate of Technical Substitution

The chief economist of a nation estimates the output of the country as $Q(x, y) = 10x^{\frac{3}{4}}y^{\frac{1}{4}}$, where x is the amount of money spent on labor and y is the amount spent on capital, all measured in billions of dollars. What's the output when $y = 81$ and $x = 625$? Suppose the economist wants the nation's output to remain constant at that level. By how much should the amount spent on capital be changed if the amount spent on labor is to increased by $1 billion?

Well, $Q(625, 81) = 3750$. So, we need to solve $Q(626, 81 + \delta y) = 3750$.

Marginal Rate of Technical Substitution

The chief economist of a nation estimates the output of the country as $Q(x, y) = 10x^{\frac{3}{4}}y^{\frac{1}{4}}$, where x is the amount of money spent on labor and y is the amount spent on capital, all measured in billions of dollars. What's the output when $y = 81$ and $x = 625$? Suppose the economist wants the nation's output to remain constant at that level. By how much should the amount spent on capital be changed if the amount spent on labor is to increased by \$1 billion?

Well, $Q(625, 81) = 3750$. So, we need to solve $Q(626, 81 + \delta y) = 3750$. This yields that $\delta y = -0.38756$.

Marginal Rate of Technical Substitution

The chief economist of a nation estimates the output of the country as $Q(x, y) = 10x^{\frac{3}{4}}y^{\frac{1}{4}}$, where x is the amount of money spent on labor and y is the amount spent on capital, all measured in billions of dollars. What's the output when $y = 81$ and $x = 625$? Suppose the economist wants the nation's output to remain constant at that level. By how much should the amount spent on capital be changed if the amount spent on labor is to increased by \$1 billion?

Well, $Q(625, 81) = 3750$. So, we need to solve $Q(626, 81 + \delta y) = 3750$. This yields that $\delta y = -0.38756$. But we can also approximate this change using the derivative.

Marginal Rate of Technical Substitution

We get the implicit form

$$3750 = 10x^{\frac{3}{4}}y^{\frac{1}{4}} \implies$$
$$x^{\frac{3}{4}}y^{\frac{1}{4}} = 375 \implies$$
$$\frac{dy}{dx}\frac{1}{4}y^{-\frac{3}{4}}x^{\frac{3}{4}} + \frac{3}{4}x^{-\frac{1}{4}}y^{\frac{1}{4}} = 0 \implies$$
$$\frac{dy}{dx} = (4x^{-\frac{3}{4}}y^{\frac{3}{4}})(-\frac{3}{4}x^{-\frac{1}{4}}y^{\frac{1}{4}}) = -3\frac{y}{x}$$

So, at (625,81), we have $\frac{dy}{dx} = -0.3888$, which approximates the exact answer.

Marginal Rate of Technical Substitution

We get the implicit form

$$
\begin{aligned}
3750 = 10x^{\frac{3}{4}}y^{\frac{1}{4}} &\implies \\
x^{\frac{3}{4}}y^{\frac{1}{4}} = 375 &\implies \\
\frac{dy}{dx}\frac{1}{4}y^{-\frac{3}{4}}x^{\frac{3}{4}} + \frac{3}{4}x^{-\frac{1}{4}}y^{\frac{1}{4}} = 0 &\implies \\
\frac{dy}{dx} = (4x^{-\frac{3}{4}}y^{\frac{3}{4}})(-\frac{3}{4}x^{-\frac{1}{4}}y^{\frac{1}{4}}) = -3\frac{y}{x}&
\end{aligned}
$$

So, at (625,81), we have $\frac{dy}{dx} = -0.3888$, which approximates the exact answer. The negative of this quantity is called the marginal rate of technical substitution.

Marginal Rate of Technical Substitution

We get the implicit form

$$\begin{aligned} 3750 = 10x^{\frac{3}{4}}y^{\frac{1}{4}} \implies \\ x^{\frac{3}{4}}y^{\frac{1}{4}} = 375 \implies \\ \frac{dy}{dx}\frac{1}{4}y^{-\frac{3}{4}}x^{\frac{3}{4}} + \frac{3}{4}x^{-\frac{1}{4}}y^{\frac{1}{4}} = 0 \implies \\ \frac{dy}{dx} = (4x^{-\frac{3}{4}}y^{\frac{3}{4}})(-\frac{3}{4}x^{-\frac{1}{4}}y^{\frac{1}{4}}) = -3\frac{y}{x} \end{aligned}$$

So, at (625,81), we have $\frac{dy}{dx} = -0.3888$, which approximates the exact answer. The negative of this quantity is called the marginal rate of technical substitution. Generally speaking, it measures the rate at which a producer is technically capable of reducing one input (capital) in favor of another (labor) while maintaining the same output.

Related Rates

The framework of a related rates problem follows this description:

Related Rates

The framework of a related rates problem follows this description: Suppose we know that x and y are two quantities depending on t and that there is some expression relating x to y. Can we then find a relationship between $\frac{dy}{dt}$ and $\frac{dx}{dt}$.

Related Rates

The framework of a related rates problem follows this description: Suppose we know that x and y are two quantities depending on t and that there is some expression relating x to y. Can we then find a relationship between $\frac{dy}{dt}$ and $\frac{dx}{dt}$. We do so using these steps:

1. Assign a variable to each quantity and draw a diagram if needed.

Related Rates

The framework of a related rates problem follows this description: Suppose we know that x and y are two quantities depending on t and that there is some expression relating x to y. Can we then find a relationship between $\frac{dy}{dt}$ and $\frac{dx}{dt}$. We do so using these steps:

1. Assign a variable to each quantity and draw a diagram if needed.
2. Write the *given* values of the variables and their rates of of change with respect to t.

Related Rates

The framework of a related rates problem follows this description: Suppose we know that x and y are two quantities depending on t and that there is some expression relating x to y. Can we then find a relationship between $\frac{dy}{dt}$ and $\frac{dx}{dt}$. We do so using these steps:

1. Assign a variable to each quantity and draw a diagram if needed.
2. Write the *given* values of the variables and their rates of of change with respect to t.
3. Find an equation relating x and y.

Related Rates

The framework of a related rates problem follows this description: Suppose we know that x and y are two quantities depending on t and that there is some expression relating x to y. Can we then find a relationship between $\frac{dy}{dt}$ and $\frac{dx}{dt}$. We do so using these steps:

1. Assign a variable to each quantity and draw a diagram if needed.
2. Write the *given* values of the variables and their rates of of change with respect to t.
3. Find an equation relating x and y.
4. Differentiate the equation with respect to t.

Related Rates

The framework of a related rates problem follows this description: Suppose we know that x and y are two quantities depending on t and that there is some expression relating x to y. Can we then find a relationship between $\frac{dy}{dt}$ and $\frac{dx}{dt}$. We do so using these steps:

1. Assign a variable to each quantity and draw a diagram if needed.
2. Write the *given* values of the variables and their rates of of change with respect to t.
3. Find an equation relating x and y.
4. Differentiate the equation with respect to t.
5. Replace the variables and their derivatives by the numerical data from Step 2 and solve the resulting equation for the required rate of change.

Related Rates

A supplier is willing to make a x thousand solid-state drives for \$p given the demand equation $x^2 - 3xp + p^2 = 5$. How fast is the supply of drives changing when the price per drive is \$11, the quantity supplied is 4000 drives, and the price of the drives is increasing at the rate of \$0.10 per drive each week?

Related Rates

A supplier is willing to make a x thousand solid-state drives for \$p given the demand equation $x^2 - 3xp + p^2 = 5$. How fast is the supply of drives changing when the price per drive is \$11, the quantity supplied is 4000 drives, and the price of the drives is increasing at the rate of \$0.10 per drive each week?

Steps 1-3 have been taken care of for us. So,

Related Rates

A supplier is willing to make a x thousand solid-state drives for \$p given the demand equation $x^2 - 3xp + p^2 = 5$. How fast is the supply of drives changing when the price per drive is \$11, the quantity supplied is 4000 drives, and the price of the drives is increasing at the rate of \$0.10 per drive each week?

Steps 1-3 have been taken care of for us. So,

$$x^2 - 3xp + p^2 = 5 \implies 2xx'(t) - 3xp'(t) - 3px'(t) + 2pp'(t) = 0$$

Related Rates

A supplier is willing to make a x thousand solid-state drives for \$p given the demand equation $x^2 - 3xp + p^2 = 5$. How fast is the supply of drives changing when the price per drive is \$11, the quantity supplied is 4000 drives, and the price of the drives is increasing at the rate of \$0.10 per drive each week?

Steps 1-3 have been taken care of for us. So,

$$x^2 - 3xp + p^2 = 5 \implies 2xx'(t) - 3xp'(t) - 3px'(t) + 2pp'(t) = 0$$

Thus, plugging in, we get
$2(4)x'(t) - 3(4)(0.1) - 3(11)x'(t) + 2(11)(0.1) = 0.$

Related Rates

A supplier is willing to make a x thousand solid-state drives for \$p given the demand equation $x^2 - 3xp + p^2 = 5$. How fast is the supply of drives changing when the price per drive is \$11, the quantity supplied is 4000 drives, and the price of the drives is increasing at the rate of \$0.10 per drive each week?

Steps 1-3 have been taken care of for us. So,

$$x^2 - 3xp + p^2 = 5 \implies 2xx'(t) - 3xp'(t) - 3px'(t) + 2pp'(t) = 0$$

Thus, plugging in, we get $2(4)x'(t) - 3(4)(0.1) - 3(11)x'(t) + 2(11)(0.1) = 0$. Solving for $x'(t)$, we get $x'(t) = 0.04$,

Related Rates

A supplier is willing to make a x thousand solid-state drives for \$p given the demand equation $x^2 - 3xp + p^2 = 5$. How fast is the supply of drives changing when the price per drive is \$11, the quantity supplied is 4000 drives, and the price of the drives is increasing at the rate of \$0.10 per drive each week?

Steps 1-3 have been taken care of for us. So,

$$x^2 - 3xp + p^2 = 5 \implies 2xx'(t) - 3xp'(t) - 3px'(t) + 2pp'(t) = 0$$

Thus, plugging in, we get $2(4)x'(t) - 3(4)(0.1) - 3(11)x'(t) + 2(11)(0.1) = 0$. Solving for $x'(t)$, we get $x'(t) = 0.04$, meaning that the supply is increasing at the rate of 40 drives per week.

The Differential

We have actually been using the (unit) differential (where $dx = 1$) without naming it. We've been using it to estimate things like marginal cost, etc. We will now formalize it.

The Differential

We have actually been using the (unit) differential (where $dx = 1$) without naming it. We've been using it to estimate things like marginal cost, etc. We will now formalize it.

First, though, note that the increment between two points is simply the change between them. So if we have two x values $x_1 = 2.01$ and $x_2 = 2.02$, then the increment (Δx) is just $2.02 - 2.01 = 0.01 = \Delta x$.

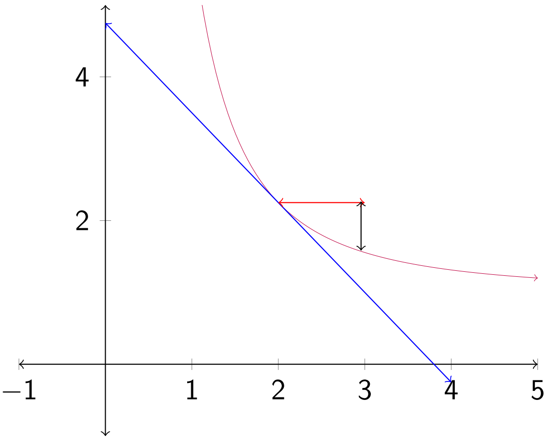
The Differential

We have actually been using the (unit) differential (where $dx = 1$) without naming it. We've been using it to estimate things like marginal cost, etc. We will now formalize it.

First, though, note that the increment between two points is simply the change between them. So if we have two x values $x_1 = 2.01$ and $x_2 = 2.02$, then the increment (Δx) is just $2.02 - 2.01 = 0.01 = \Delta x$.

The corresponding y increment (Δy) if $y = f(x)$ is given by $f(x_1 + \Delta x) - f(x_1) = \Delta y$.

Increments: Graph



Here, Δx is given in red and Δy is given in black.

Definition: The Differential

In this definition, we assume that f is differentiable at x.

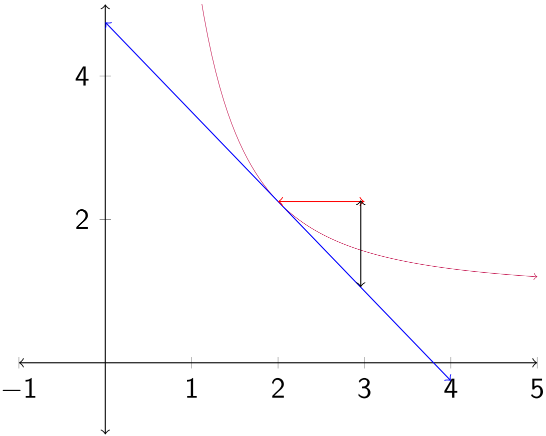
Definition: The Differential

In this definition, we assume that f is differentiable at x.

Definition (Differential of f at x)

The differential dx of the independent variable x is given by $dx = \Delta x$.
The differential dy of the dependent variable y is given by $dy = f'(x)\Delta x = f'(x)dx$.

Increments: Graph



Here, $dx = \Delta x$ is given in red and $dy = f'(x)dx$ is given in black.

Example

Suppose the side of a cube is measured with maximum percentage error of 2%. Use differentials to estimate the maximum **percentage** error in the calculated volume of the cube.

Example

Suppose the side of a cube is measured with maximum percentage error of 2%. Use differentials to estimate the maximum **percentage** error in the calculated volume of the cube.

Well, the volume of a cube is given by $s^3 = V$, where s is the length of a side and V is the volume.

Example

Suppose the side of a cube is measured with maximum percentage error of 2%. Use differentials to estimate the maximum **percentage** error in the calculated volume of the cube.

Well, the volume of a cube is given by $s^3 = V$, where s is the length of a side and V is the volume. Thus, $3s^2 ds = dV$, so

$$\frac{3s^2 ds}{s^3} = \frac{dV}{V}.$$

Example

Suppose the side of a cube is measured with maximum percentage error of 2%. Use differentials to estimate the maximum **percentage** error in the calculated volume of the cube.

Well, the volume of a cube is given by $s^3 = V$, where s is the length of a side and V is the volume. Thus, $3s^2ds = dV$, so

$$\frac{3s^2ds}{s^3} = \frac{dV}{V}.$$

Now, the right-hand side of this equation is the percentage differential of the volume.

Example

Suppose the side of a cube is measured with maximum percentage error of 2%. Use differentials to estimate the maximum **percentage** error in the calculated volume of the cube.

Well, the volume of a cube is given by $s^3 = V$, where s is the length of a side and V is the volume. Thus, $3s^2 ds = dV$, so

$$\frac{3s^2 ds}{s^3} = \frac{dV}{V}.$$

Now, the right-hand side of this equation is the percentage differential of the volume. And on the left-hand side, we have $\frac{3s^2 ds}{s^3} = \frac{3ds}{s}$,

Example

Suppose the side of a cube is measured with maximum percentage error of 2%. Use differentials to estimate the maximum **percentage** error in the calculated volume of the cube.

Well, the volume of a cube is given by $s^3 = V$, where s is the length of a side and V is the volume. Thus, $3s^2 ds = dV$, so

$$\frac{3s^2 ds}{s^3} = \frac{dV}{V}.$$

Now, the right-hand side of this equation is the percentage differential of the volume. And on the left-hand side, we have $\frac{3s^2 ds}{s^3} = \frac{3ds}{s}$, which is three times the percentage differential of the length of one side.

Example

Suppose the side of a cube is measured with maximum percentage error of 2%. Use differentials to estimate the maximum **percentage** error in the calculated volume of the cube.

Well, the volume of a cube is given by $s^3 = V$, where s is the length of a side and V is the volume. Thus, $3s^2 ds = dV$, so

$$\frac{3s^2 ds}{s^3} = \frac{dV}{V}.$$

Now, the right-hand side of this equation is the percentage differential of the volume. And on the left-hand side, we have $\frac{3s^2 ds}{s^3} = \frac{3ds}{s}$, which is three times the percentage differential of the length of one side. Thus,

$$\left|\frac{dV}{V}\right| = \left|3 \cdot \frac{ds}{s}\right| \leq 3(0.02) = 0.06.$$

Example

Approximate $\sqrt[3]{28.5}$ using differentials.

Example

Approximate $\sqrt[3]{28.5}$ using differentials.

Consider the function $f(x) = \sqrt[3]{x}$. Then the differential dy is just

Example

Approximate $\sqrt[3]{28.5}$ using differentials.

Consider the function $f(x) = \sqrt[3]{x}$. Then the differential dy is just

$$dy = f'(x)dx = \frac{1}{3x^{\frac{2}{3}}}dx.$$

Example

Approximate $\sqrt[3]{28.5}$ using differentials.

Consider the function $f(x) = \sqrt[3]{x}$. Then the differential dy is just

$$dy = f'(x)dx = \frac{1}{3x^{\frac{2}{3}}}dx.$$

And since $\sqrt[3]{28.5} - \sqrt[3]{27} = \Delta y \approx dy$,

Example

Approximate $\sqrt[3]{28.5}$ using differentials.

Consider the function $f(x) = \sqrt[3]{x}$. Then the differential dy is just

$$dy = f'(x)dx = \frac{1}{3x^{\frac{2}{3}}}dx.$$

And since $\sqrt[3]{28.5} - \sqrt[3]{27} = \Delta y \approx dy$, we have $\sqrt[3]{28.5} \approx \sqrt[3]{27} + dy$, where dy is evaluated at 27 with $dx = 1.5$. We get

Example

Approximate $\sqrt[3]{28.5}$ using differentials.

Consider the function $f(x) = \sqrt[3]{x}$. Then the differential dy is just

$$dy = f'(x)dx = \frac{1}{3x^{\frac{2}{3}}}dx.$$

And since $\sqrt[3]{28.5} - \sqrt[3]{27} = \Delta y \approx dy$, we have $\sqrt[3]{28.5} \approx \sqrt[3]{27} + dy$, where dy is evaluated at 27 with $dx = 1.5$. We get

$$dy = \frac{1}{3 \cdot 27^{\frac{2}{3}}}(1.5) = \frac{1}{27}(1.5) \approx 0.056.$$

Example

Approximate $\sqrt[3]{28.5}$ using differentials.

Consider the function $f(x) = \sqrt[3]{x}$. Then the differential dy is just

$$dy = f'(x)dx = \frac{1}{3x^{\frac{2}{3}}}dx.$$

And since $\sqrt[3]{28.5} - \sqrt[3]{27} = \Delta y \approx dy$, we have $\sqrt[3]{28.5} \approx \sqrt[3]{27} + dy$, where dy is evaluated at 27 with $dx = 1.5$. We get

$$dy = \frac{1}{3 \cdot 27^{\frac{2}{3}}}(1.5) = \frac{1}{27}(1.5) \approx 0.056.$$

So $\sqrt[3]{28.5} \approx \sqrt[3]{27} + 0.056 \approx 3.056$.

Example

Approximate $\sqrt[3]{28.5}$ using differentials.

Consider the function $f(x) = \sqrt[3]{x}$. Then the differential dy is just

$$dy = f'(x)dx = \frac{1}{3x^{\frac{2}{3}}}dx.$$

And since $\sqrt[3]{28.5} - \sqrt[3]{27} = \Delta y \approx dy$, we have $\sqrt[3]{28.5} \approx \sqrt[3]{27} + dy$, where dy is evaluated at 27 with $dx = 1.5$. We get

$$dy = \frac{1}{3 \cdot 27^{\frac{2}{3}}}(1.5) = \frac{1}{27}(1.5) \approx 0.056.$$

So $\sqrt[3]{28.5} \approx \sqrt[3]{27} + 0.056 \approx 3.056$. The actual value is 3.0546.

Assignment

Read 4.1. Do problems 18, 26, 30, 42 in 3.5; 6, 26, 32, 52 in 3.6; and 6, 18, 24, 40 in 3.7.