QMI Lesson 10:
Applications of the First Derivative

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Increasing and Decreasing Functions

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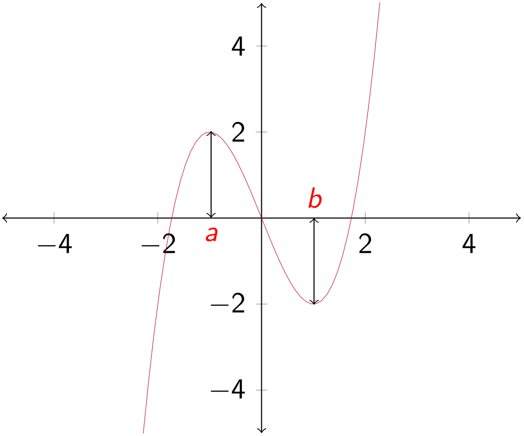
Increasing and Decreasing Functions: Graph



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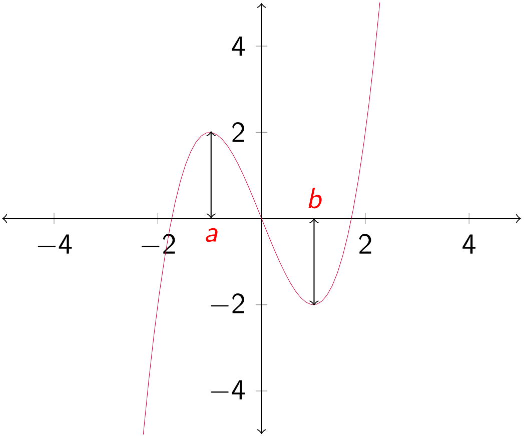

This function is decreasing on the interval (a, b).

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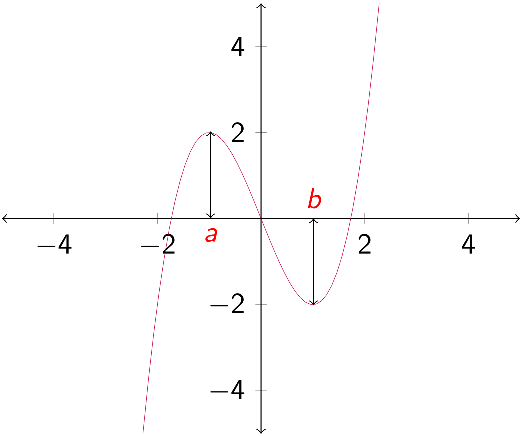

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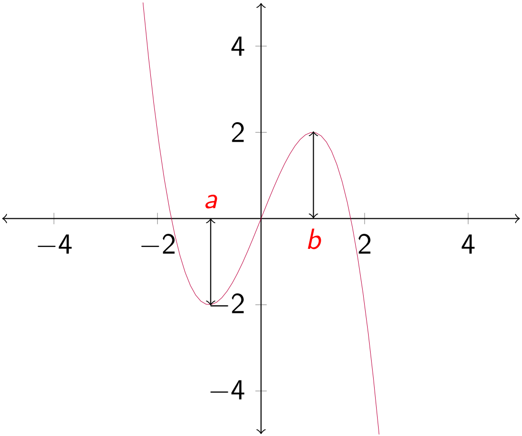

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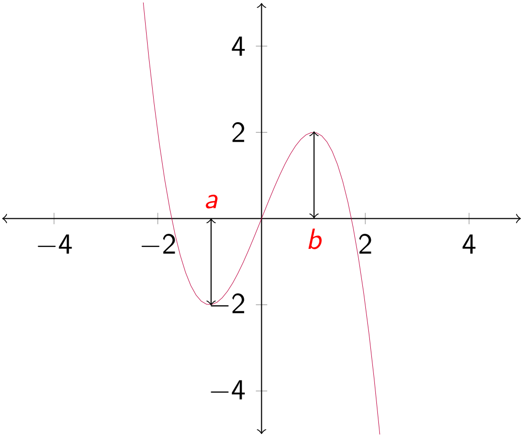

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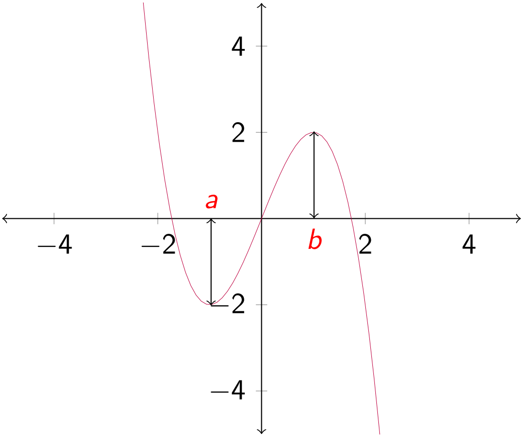

This function is increasing on the interval (a, b). What can you say about the tangent line at the points on this interval?

The Derivative and Inceasing/Decreasing Functions

Theorem

1. *A function is increasing on an interval* (a, b) *if for every* x *in* (a, b)*, we have that* $f'(x) > 0$.

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3. *A function is constant on an interval* (a, b) *if for every* x *in* (a, b), *we have that* $f'(x) = 0$.

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1. Find everywhere that f' is zero or undefined, and break up the number line into intervals with these points (or intervals) as endpoints.
2. Find a test value in each interval. If the derivative is positive at that test value, then the function is increasing on the corresponding interval. Similarly, if the derivative is negative at the test value, then the fucntion is decreasing on the corresponding interval.

Example

Find the intervals on which $f(x) = x + \frac{1}{x}$ is increasing and decreasing.

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Interval	**Test Value** x	$f'(x)$	**Sign of** $f'(x)$
$(-\infty, -1)$	-2	$\frac{3}{4}$	$+$
$(-1, 0)$	$-\frac{1}{2}$	$-\frac{3}{8}$	$-$
$(0, 1)$	$\frac{1}{2}$	$-\frac{3}{8}$	$-$
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So, the function is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 0)$ and $(0, 1)$.

Mean Value Theorem

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A nice feature of a smooth, i.e. differenitable, function is the Mean Value Theorem. Essentially, it says that if you have a secant line connecting two points between which the function is smooth, then there is at least one corresponding point at which the tangent line is parrallel to the this secant line.

Theorem (The Mean Value Theorem)

If f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one c in (a, b) for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

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A farmer's production of apples is governed by $y(t) = 3t^2 + 10$, where y is tons of apples and t is weeks. Show that, at some point, the farmer is producing exactly 30 tons of apples per week.

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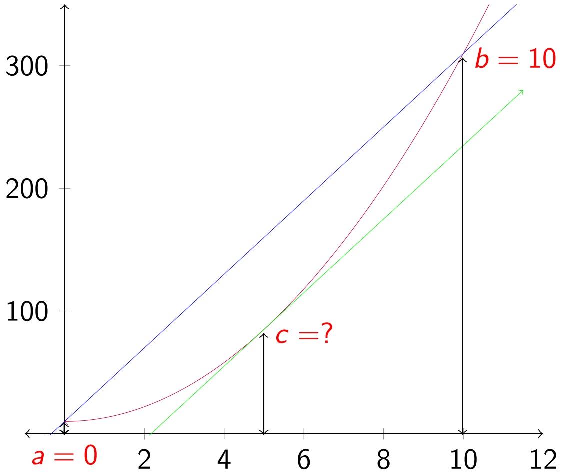
$$\frac{y(10) - y(0)}{10 - 0} = \frac{300 - 10}{10} = 30.$$

Thus, by the MVT, there is some c in $(0, 30)$ at which the rate of production is 30 tons of apples per week.

Graph



Relative Extrema: Definitions

Definition (Relative Maximum)

A relative maximum of a function f occurs at a point c if $f(c) \geq f(x)$ for all points x in some open interval containing c.

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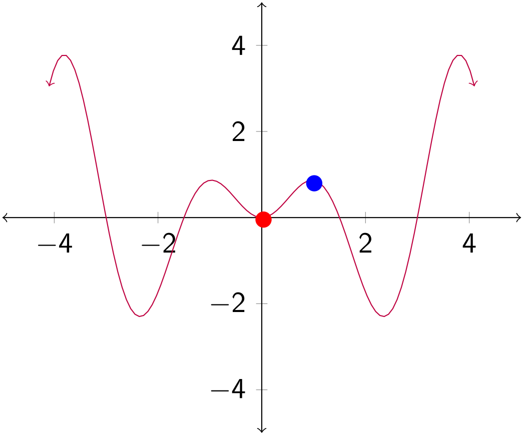

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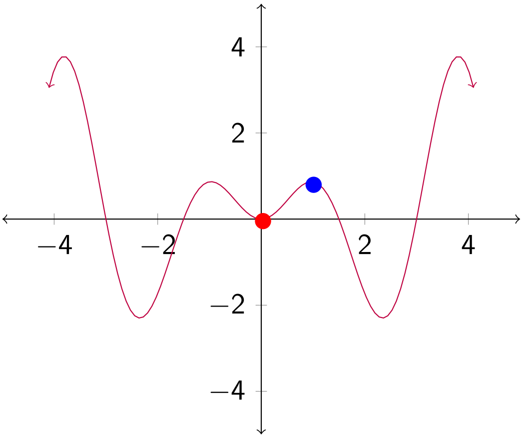

A relative max is in blue and a relative minimum is in red. Where are the other relative maxima/minima?

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Obviously, if a function is increasing and then decreasing, then its derivative must move from positive to negative. So, if the derivative is continuous, then it must be zero at some point. This means that to the right of the point where the derivative is zero, the function must take values less than the value at the point. And to the left of the point, the function must take values less than the value at the point.

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A relative extrema may exist even if the function does not have a continuous derivative. (In which case, we cannot use the above steps.) For example, consider $f(x) = |x|$.

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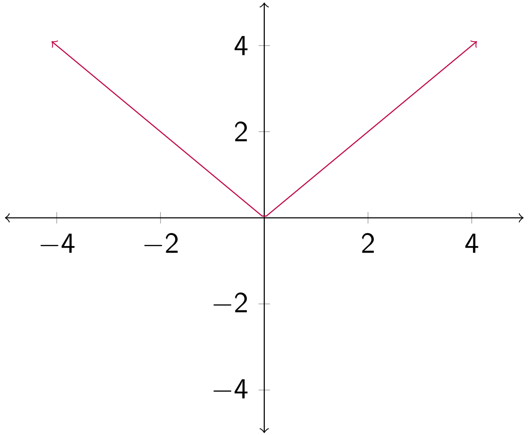

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With this extension, we can find the relative minima and maxima of any continuous function, not just those with continuous derivatives.

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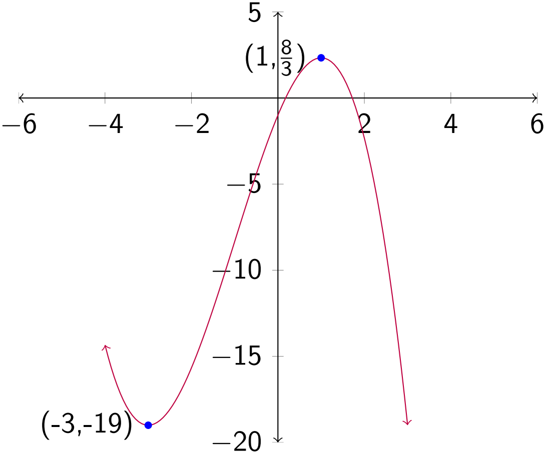

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Thus, a relative maximum ($f(-1) = -2$) occurs at $x = -1$, and a relative minimum ($f(1) = 2$) occurs at $x = 1$.

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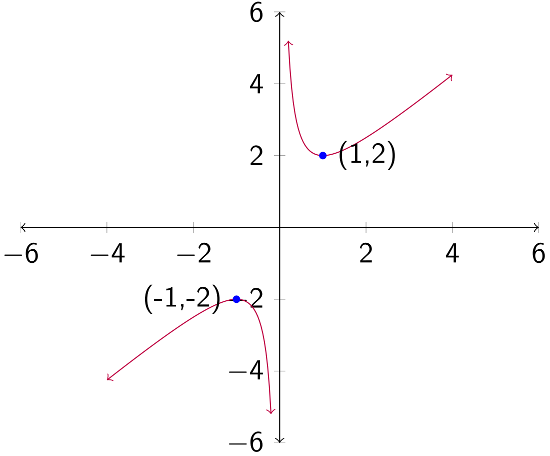

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The profit function for AirSoft is given by $P(x) = -0.02x^2 + 300x - 200000$ in dollars where x is the number of AirSoft guns produced. Find where P is increasing and decreasing and find any relative extrema.

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Example

The profit function for AirSoft is given by $P(x) = -0.02x^2 + 300x - 200000$ in dollars where x is the number of AirSoft guns produced. Find where P is increasing and decreasing and find any relative extrema.

Well, $P'(x) = -0.04x + 300$ which equals zero when $x = 7500$. So we break the number line over this point:

Interval	Test Value	Sign
$(-\infty, 7500)$	0	$f'(0) > 0$
$(7500, \infty)$	10000	$f'(10000) < 0$

Thus, P is decreasing on $(7500, \infty)$, increasing on $(0, 7500)$,and has a relative maximum at $x = 7500$ which is $f(7500) = 925000$.

Assignment

Read 4.2. Do problems 8, 10, 18, 34, 42, 46, 48, 62, 70, 86, 92 in 4.1.