QMI Lesson 15: Compound Interest

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$$\underbrace{1000\left(\frac{0.06}{2}\right)}_{\textit{interest1}} + \underbrace{\left(1000\left(\frac{0.06}{2}\right) + 1000\right)\left(\frac{0.06}{2}\right)}_{\textit{interest2}} + \underbrace{1000}_{\textit{principal}} = 1060.$$

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where $e \approx 2.718$ is the natural constant. This is **continuous compounding**!

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$$
\begin{aligned}
A_1 &= P(1+i) \\
A_2 &= [A_1](1+i) = P(1+i)(1+i) = P(1+i)^2 \\
A_3 &= [A_2](1+i) = P(1+i)^2)(1+i) = P(1+i)^3 \\
&\vdots \\
A_m &= [A_{m-1}](1+i) = P(1+i)^{m-1} = P(1+i)^m
\end{aligned}
$$

Compound Interest

The calculations from the previous slide yield the following formula.

Definition (Amount from Compound Interest)

$A = P\left(1 + \frac{r}{n}\right)^{nt}$, where

$A =$ the total value amount of the investment after t,
$r =$ the nominal rate of interest per time t,
$P =$ the principal,
$n =$ number of conversions periods in time t,
$t =$ the term (often given in years).

Example

Find the accumulated amount if \$1000 is invested at 8% annually for a term of three years with (a) annual, (b) semiannual, (c) quarterly, (d) monthly, and (e) daily compounding.

(a) Here, $P = 1000$, $r = 0.08$, $n = 1$, and $t = 3$, so we get

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Definition (Effective Rate of Interest)

$r_{eff} = \left(1 + \frac{r}{n}\right)^n - 1$ is the effective rate of interest, where
$r =$ the nominal interest rate per year and
$n =$ the number of conversion periods (compoundings) per year.

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$$r_{eff} = \left(1 + \frac{0.051}{2}\right)^{2} - 1 \approx 0.05165.$$

So the 5.1% annual rate compounded semiannually is preferable.

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Why would such a formula be useful? If you know you needed the amount A at a future time, you could calculate how much P would need to be invested now. For this reason, we call A the **future value** and P the **present value** of an investment.

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$$P = 49158.6\left(1+\frac{0.1}{4}\right)^{-4(5)}$$

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$$P = 49158.6\left(1+\frac{0.1}{4}\right)^{-4(5)} \approx 30000.$$

Continuous Compounding: Recall $e = \lim_{u \to \infty} \left(1 + \frac{1}{u}\right)^u$

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$$\begin{aligned}
\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} &= P\left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}\right]^{t} \\
&= P\left[\lim_{n\to\infty}\left(1+\frac{1}{u}\right)^{ur}\right]^{t} \\
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Blakely Properties owns a building in the commercial district of Birmingham. Because of the city's successful urban renewal program, there is a miniboom in urban property. The market value of the property is given by $V(t) = 300000e^{0.5\sqrt{t}}$, where V is in dollars and t in years from present. If the expected rate of appreciation is 9% compounded continously for the next 10 years, find an expression for the present value $P(t)$ of the market price valid for the next 10 years. Compute $P(7)$, $P(8)$, and $P(9)$. Interpret your results.

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So the present value of the property seems to decrease after a certain period of growth, meaning there is an optimal time for the owners to sell. Later, we'll show this time is exactly $t = 7.72$ years with valued \$600779.

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$$2000 = 1000\left(1 + \frac{r}{12}\right)^{12(6)} \implies 2 = \left(1 + \frac{r}{12}\right)^{12(6)}$$
$$\implies \ln 2 = \ln\left(1 + \frac{r}{12}\right)^{6(12)} = 72 \ln\left(1 + \frac{r}{12}\right)$$

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Thus, we need a rate of about 11.61% to double an investment in 6 years with interest compounded monthly.

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So it will take approximately 3.43 years.

Assignment

Read 5.4-5.5. Do problems 4, 8, 14, 22, 28, 30, 38, 40, 48, 50, 52 in 5.3.