QMI Lesson 16:
Derivative of Logarithmic and Exponential Functions

C C Moxley

Samford University Brock School of Business

The Derivative of Exponential Fucntions

Exponential functions appear so widely in mathematical models that understanding their rates of change is immensely important. We have

The Derivative of Exponential Fucntions

Exponential functions appear so widely in mathematical models that understanding their rates of change is immensely important. We have

Theorem (Derivative of $f(x) = e^x$)

If $f(x) = e^x$, then $f'(x) = e^x$.

The Derivative of Exponential Fucntions

Exponential functions appear so widely in mathematical models that understanding their rates of change is immensely important. We have

Theorem (Derivative of $f(x) = e^x$)

If $f(x) = e^x$, then $f'(x) = e^x$.

Note: The exponential function is the only function which is its own derivative!

The Derivative of Exponential Fucntions

Exponential functions appear so widely in mathematical models that understanding their rates of change is immensely important. We have

Theorem (Derivative of $f(x) = e^x$)

If $f(x) = e^x$, then $f'(x) = e^x$.

Note: The exponential function is the only function which is its own derivative! We also have

The Derivative of Exponential Fucntions

Exponential functions appear so widely in mathematical models that understanding their rates of change is immensely important. We have

Theorem (Derivative of $f(x) = e^x$)

If $f(x) = e^x$, then $f'(x) = e^x$.

Note: The exponential function is the only function which is its own derivative! We also have

Theorem (Derivative of $f(x) = a^x$)

If $f(x) = a^x$, then $f'(x) = a^x \ln(a)$.

Verifying the Derivative of Exponential Fucntions

A rigorous proof that $\frac{d}{dx}\left(e^x\right) = e^x$ is beyond the scope of this course, but we will show that the derivative makes sense.

Verifying the Derivative of Exponential Fucntions

A rigorous proof that $\frac{d}{dx}(e^x) = e^x$ is beyond the scope of this course, but we will show that the derivative makes sense.

$$\lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h}.$$

Verifying the Derivative of Exponential Fucntions

A rigorous proof that $\frac{d}{dx}(e^x) = e^x$ is beyond the scope of this course, but we will show that the derivative makes sense.

$$\lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h-1)}{h} = e^x\lim_{h\to 0}\frac{e^h-1}{h}.$$

Using the table below, we see that $\lim_{h\to 0}\frac{e^h-1}{h} = 1$

Verifying the Derivative of Exponential Fucntions

A rigorous proof that $\dfrac{d}{dx}(e^x) = e^x$ is beyond the scope of this course, but we will show that the derivative makes sense.

$$\lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h-1)}{h} = e^x\lim_{h\to 0}\frac{e^h-1}{h}.$$

Using the table below, we see that $\displaystyle\lim_{h\to 0}\frac{e^h-1}{h} = 1$

h	0.1	0.01	0.001	-0.1	-0.01	-0.001
$\frac{e^h-1}{h}$	1.0517	1.0050	1.0005	0.9516	0.9950	0.9995

So, $\displaystyle\lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = e^x\lim_{h\to 0}\frac{e^h-1}{h} = e^x\cdot 1 = e^x$

The Derivative of $f(x) = a^x$.

We can now prove rigorously that $\frac{d}{dx}(a^x) = a^x \ln(a)$.

The Derivative of $f(x) = a^x$.

We can now prove rigorously that $\frac{d}{dx}(a^x) = a^x \ln(a)$.

Recall: $e^{\ln(a)} = a$. Therefore,

The Derivative of $f(x) = a^x$.

We can now prove rigorously that $\frac{d}{dx}(a^x) = a^x \ln(a)$.

Recall: $e^{\ln(a)} = a$. Therefore,

$$\frac{d}{dx} a^x =$$

The Derivative of $f(x) = a^x$.

We can now prove rigorously that $\frac{d}{dx}(a^x) = a^x \ln(a)$.

Recall: $e^{\ln(a)} = a$. Therefore,

$$\frac{d}{dx}a^x = \frac{d}{dx}\left(e^{\ln(a)}\right)^x =$$

The Derivative of $f(x) = a^x$.

We can now prove rigorously that $\frac{d}{dx}(a^x) = a^x \ln(a)$.

Recall: $e^{\ln(a)} = a$. Therefore,

$$\frac{d}{dx} a^x = \frac{d}{dx}\left(e^{\ln(a)}\right)^x = \frac{d}{dx} e^{x \ln(a)} \overset{\text{chain rule}}{=}$$

The Derivative of $f(x) = a^x$.

We can now prove rigorously that $\frac{d}{dx}(a^x) = a^x \ln(a)$.

Recall: $e^{\ln(a)} = a$. Therefore,

$$\frac{d}{dx}a^x = \frac{d}{dx}\left(e^{\ln(a)}\right)^x = \frac{d}{dx}e^{x\ln(a)} \overset{\textit{chain rule}}{=}$$

$$\tfrac{d}{dx}(x\ln(a)) \cdot \tfrac{d}{dx}\left(e^{\ln(a)}\right)^x =$$

The Derivative of $f(x) = a^x$.

We can now prove rigorously that $\dfrac{d}{dx}(a^x) = a^x \ln(a)$.

Recall: $e^{\ln(a)} = a$. Therefore,

$$\frac{d}{dx}a^x = \frac{d}{dx}\left(e^{\ln(a)}\right)^x = \frac{d}{dx}e^{x\ln(a)} \overset{\textit{chain rule}}{=}$$

$$\tfrac{d}{dx}(x\ln(a)) \cdot \tfrac{d}{dx}\left(e^{\ln(a)}\right)^x = \ln(a) \cdot \left(e^{\ln(a)}\right)^x =$$

The Derivative of $f(x) = a^x$.

We can now prove rigorously that $\frac{d}{dx}(a^x) = a^x \ln(a)$.

Recall: $e^{\ln(a)} = a$. Therefore,

$$\frac{d}{dx}a^x = \frac{d}{dx}\left(e^{\ln(a)}\right)^x = \frac{d}{dx}e^{x\ln(a)} \overset{\text{chain rule}}{=}$$

$$\frac{d}{dx}(x\ln(a)) \cdot \frac{d}{dx}\left(e^{\ln(a)}\right)^x = \ln(a) \cdot \left(e^{\ln(a)}\right)^x = a^x \ln(a).$$

The Derivative of $g(x) = e^{f(x)}$

In the same manner as in the previous slide, we can calculate

$$\frac{d}{dx} e^{f(x)} =$$

The Derivative of $g(x) = e^{f(x)}$

In the same manner as in the previous slide, we can calculate

$$\frac{d}{dx}e^{f(x)} = e^{f(x)} \cdot \frac{d}{dx}f(x) = e^{f(x)} \cdot f'(x).$$

The Derivative of $g(x) = e^{f(x)}$

In the same manner as in the previous slide, we can calculate

$$\frac{d}{dx}e^{f(x)} = e^{f(x)} \cdot \frac{d}{dx}f(x) = e^{f(x)} \cdot f'(x).$$

Theorem (The Chain Rule for Exponential Functions)

If f is differentiable at x, then $\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$.

The Derivative of $g(x) = e^{f(x)}$

In the same manner as in the previous slide, we can calculate

$$\frac{d}{dx}e^{f(x)} = e^{f(x)} \cdot \frac{d}{dx}f(x) = e^{f(x)} \cdot f'(x).$$

Theorem (The Chain Rule for Exponential Functions)

If f is differentiable at x, then $\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x).$

The proof of this theorem is just a direct application of the chain rule!

Examples

Find the derivative of the following:

Examples

Find the derivative of the following:

- $\frac{d}{dx}[x^2 e^x]$

Examples

Find the derivative of the following:

- $\frac{d}{dx}[x^2 e^x] = \frac{d}{dx}[x^2]e^x + x^2 \frac{d}{dx}[e^x]$

Examples

Find the derivative of the following:

- $\frac{d}{dx}[x^2e^x] = \frac{d}{dx}[x^2]e^x + x^2\frac{d}{dx}[e^x] = 2xe^x + x^2e^x$

Examples

Find the derivative of the following:

- $\frac{d}{dx}[x^2e^x] = \frac{d}{dx}[x^2]e^x + x^2\frac{d}{dx}[e^x] = 2xe^x + x^2e^x = xe^x(2+x).$
- $\frac{d}{dx}[(e^x + 2)^{\frac{3}{2}}]$

Examples

Find the derivative of the following:

- $\frac{d}{dx}[x^2e^x] = \frac{d}{dx}[x^2]e^x + x^2\frac{d}{dx}[e^x] = 2xe^x + x^2e^x = xe^x(2+x).$
- $\frac{d}{dx}[(e^x + 2)^{\frac{3}{2}}] = \frac{3}{2}(e^t + 2)^{\frac{1}{2}}\frac{d}{dx}[e^t + 2]$

Examples

Find the derivative of the following:

- $\frac{d}{dx}[x^2e^x] = \frac{d}{dx}[x^2]e^x + x^2\frac{d}{dx}[e^x] = 2xe^x + x^2e^x = xe^x(2+x)$.
- $\frac{d}{dx}[(e^x+2)^{\frac{3}{2}}] = \frac{3}{2}(e^t+2)^{\frac{1}{2}}\frac{d}{dx}[e^t+2] = \frac{3}{2}(e^t+2)^{\frac{1}{2}}e^t$

Examples

Find the derivative of the following:

- $\frac{d}{dx}[x^2e^x] = \frac{d}{dx}[x^2]e^x + x^2\frac{d}{dx}[e^x] = 2xe^x + x^2e^x = xe^x(2+x).$
- $\frac{d}{dx}[(e^x+2)^{\frac{3}{2}}] = \frac{3}{2}(e^t+2)^{\frac{1}{2}}\frac{d}{dx}[e^t+2] = \frac{3}{2}(e^t+2)^{\frac{1}{2}}e^t = \frac{3}{2}e^t(e^t+2)^{\frac{1}{2}}.$
- $\frac{d}{dx}\left[\frac{x}{e^x}\right]$

Examples

Find the derivative of the following:

- $\frac{d}{dx}[x^2 e^x] = \frac{d}{dx}[x^2]e^x + x^2\frac{d}{dx}[e^x] = 2xe^x + x^2e^x = xe^x(2+x)$.
- $\frac{d}{dx}[(e^x+2)^{\frac{3}{2}}] = \frac{3}{2}(e^t+2)^{\frac{1}{2}}\frac{d}{dx}[e^t+2] = \frac{3}{2}(e^t+2)^{\frac{1}{2}}e^t = \frac{3}{2}e^t(e^t+2)^{\frac{1}{2}}$.
- $\frac{d}{dx}\left[\frac{x}{e^x}\right] = \frac{e^x(x)' - x(e^x)'}{(e^x)^2} = \frac{e^x - xe^x}{e^{2x}}$

Examples

Find the derivative of the following:

- $\dfrac{d}{dx}[x^2 e^x] = \dfrac{d}{dx}[x^2]e^x + x^2 \dfrac{d}{dx}[e^x] = 2xe^x + x^2 e^x = xe^x(2+x).$
- $\dfrac{d}{dx}[(e^x+2)^{\frac{3}{2}}] = \dfrac{3}{2}(e^t+2)^{\frac{1}{2}} \dfrac{d}{dx}[e^t+2] = \dfrac{3}{2}(e^t+2)^{\frac{1}{2}} e^t =$ $\dfrac{3}{2}e^t(e^t+2)^{\frac{1}{2}}.$
- $\dfrac{d}{dx}\left[\dfrac{x}{e^x}\right] = \dfrac{e^x(x)' - x(e^x)'}{(e^x)^2} = \dfrac{e^x - xe^x}{e^{2x}} = \dfrac{e^x(1-x)}{e^{2x}}$

Examples

Find the derivative of the following:

- $\frac{d}{dx}[x^2e^x] = \frac{d}{dx}[x^2]e^x + x^2\frac{d}{dx}[e^x] = 2xe^x + x^2e^x = xe^x(2+x).$
- $\frac{d}{dx}[(e^x+2)^{\frac{3}{2}}] = \frac{3}{2}(e^t+2)^{\frac{1}{2}}\frac{d}{dx}[e^t+2] = \frac{3}{2}(e^t+2)^{\frac{1}{2}}e^t =$ $\frac{3}{2}e^t(e^t+2)^{\frac{1}{2}}.$
- $\frac{d}{dx}\left[\frac{x}{e^x}\right] = \frac{e^x(x)' - x(e^x)'}{(e^x)^2} = \frac{e^x - xe^x}{e^{2x}} = \frac{e^x(1-x)}{e^{2x}} =$ $\frac{1-x}{e^x}.$

Examples

Find the derivative of the following:

Examples

Find the derivative of the following:

- $\frac{d}{dx}[e^{2x}]$

Examples

Find the derivative of the following:

- $\frac{d}{dx}[e^{2x}] = (2x)'e^{2x}$

Examples

Find the derivative of the following:

- $\dfrac{d}{dx}[e^{2x}] = (2x)'e^{2x} = 2e^{2x}.$
- $\dfrac{d}{dx}[e^{-3x}]$

Examples

Find the derivative of the following:

- $\dfrac{d}{dx}[e^{2x}] = (2x)'e^{2x} = 2e^{2x}$.
- $\dfrac{d}{dx}[e^{-3x}] = (-3x)'e^{-3x}$

Examples

Find the derivative of the following:

- $\dfrac{d}{dx}[e^{2x}] = (2x)'e^{2x} = 2e^{2x}$.
- $\dfrac{d}{dx}[e^{-3x}] = (-3x)'e^{-3x} = -3e^{-3x}$.
- $\dfrac{d}{dx}[e^{-2x^2-x}]$

Examples

Find the derivative of the following:

- $\frac{d}{dx}[e^{2x}] = (2x)'e^{2x} = 2e^{2x}.$
- $\frac{d}{dx}[e^{-3x}] = (-3x)'e^{-3x} = -3e^{-3x}.$
- $\frac{d}{dx}[e^{-2x^2-x}] = (-2x^2 - x)'e^{-2x^2-x}$

Examples

Find the derivative of the following:

- $\frac{d}{dx}[e^{2x}] = (2x)'e^{2x} = 2e^{2x}$.
- $\frac{d}{dx}[e^{-3x}] = (-3x)'e^{-3x} = -3e^{-3x}$.
- $\frac{d}{dx}[e^{-2x^2-x}] = (-2x^2 - x)'e^{-2x^2-x} = (-4x-1)e^{-3x}$.

Examples

Find the derivative of the following:

Examples

Find the derivative of the following:

- $\frac{d}{dx}[xe^{-2x}]$

Examples

Find the derivative of the following:

- $\frac{d}{dx}[xe^{-2x}] = \frac{d}{dx}[x]e^{-2x} + x\frac{d}{dx}[e^{-2x}]$

Examples

Find the derivative of the following:

- $\dfrac{d}{dx}[xe^{-2x}] = \dfrac{d}{dx}[x]e^{-2x} + x\dfrac{d}{dx}[e^{-2x}] = e^{-2x} + x(-2)e^{-2x}$

Examples

Find the derivative of the following:

- $\frac{d}{dx}[xe^{-2x}] = \frac{d}{dx}[x]e^{-2x} + x\frac{d}{dx}[e^{-2x}] = e^{-2x} + x(-2)e^{-2x} = (1-2x)e^{-2x}$
- $\frac{d}{dx}\left[\frac{e^t}{e^t + e^{-t}}\right]$

Examples

Find the derivative of the following:

- $\frac{d}{dx}[xe^{-2x}] = \frac{d}{dx}[x]e^{-2x} + x\frac{d}{dx}[e^{-2x}] = e^{-2x} + x(-2)e^{-2x} = (1-2x)e^{-2x}$
- $\frac{d}{dx}\left[\frac{e^t}{e^t + e^{-t}}\right] = \frac{(e^t + e^{-t})(e^t)' - (e^t + e^{-t})'e^t}{(e^t + e^{-t})^2}$

Examples

Find the derivative of the following:

- $\dfrac{d}{dx}[xe^{-2x}] = \dfrac{d}{dx}[x]e^{-2x} + x\dfrac{d}{dx}[e^{-2x}] = e^{-2x} + x(-2)e^{-2x} = (1-2x)e^{-2x}$
- $\dfrac{d}{dx}\left[\dfrac{e^t}{e^t + e^{-t}}\right] = \dfrac{(e^t + e^{-t})(e^t)' - (e^t + e^{-t})'e^t}{(e^t + e^{-t})^2} = \dfrac{(e^t + e^{-t})(e^t) - (e^t - e^{-t})e^t}{(e^t + e^{-t})^2}$

Examples

Find the derivative of the following:

- $\frac{d}{dx}[xe^{-2x}] = \frac{d}{dx}[x]e^{-2x} + x\frac{d}{dx}[e^{-2x}] = e^{-2x} + x(-2)e^{-2x} = (1-2x)e^{-2x}$
- $\frac{d}{dx}\left[\frac{e^t}{e^t + e^{-t}}\right] = \frac{(e^t + e^{-t})(e^t)' - (e^t + e^{-t})'e^t}{(e^t + e^{-t})^2} = \frac{(e^t + e^{-t})(e^t) - (e^t - e^{-t})e^t}{(e^t + e^{-t})^2} = \frac{e^{2t} + 1 - e^{2t} + 1}{(e^t + e^{-t})^2}$

Examples

Find the derivative of the following:

- $\frac{d}{dx}[xe^{-2x}] = \frac{d}{dx}[x]e^{-2x} + x\frac{d}{dx}[e^{-2x}] = e^{-2x} + x(-2)e^{-2x} = (1-2x)e^{-2x}$
- $\frac{d}{dx}\left[\frac{e^t}{e^t+e^{-t}}\right] = \frac{(e^t+e^{-t})(e^t)' - (e^t+e^{-t})'e^t}{(e^t+e^{-t})^2} = \frac{(e^t+e^{-t})(e^t) - (e^t-e^{-t})e^t}{(e^t+e^{-t})^2} = \frac{e^{2t}+1-e^{2t}+1}{(e^t+e^{-t})^2} = \frac{2}{(e^t+e^{-t})^2}.$

Example

Find the inflection points of $f(x) = e^{-x^2}$.

Example

Find the inflection points of $f(x) = e^{-x^2}$.

Well, $f'(x) = -2xe^{-x^2}$ and

Example

Find the inflection points of $f(x) = e^{-x^2}$.

Well, $f'(x) = -2xe^{-x^2}$ and $f''(x) = (-2)(e^{-x^2}) - 2x(-2x)e^{-x^2}$

Example

Find the inflection points of $f(x) = e^{-x^2}$.

Well, $f'(x) = -2xe^{-x^2}$ and $f''(x) = (-2)(e^{-x^2}) - 2x(-2x)e^{-x^2}$
$= -2e^{-x^2}(1 - 2x^2)$.

Example

Find the inflection points of $f(x) = e^{-x^2}$.

Well, $f'(x) = -2xe^{-x^2}$ and $f''(x) = (-2)(e^{-x^2}) - 2x(-2x)e^{-x^2}$
$= -2e^{-x^2}(1 - 2x^2)$. So, the second derivative is zero when

$$1 - 2x^2 = 0,$$

i.e. when $x = \pm\frac{1}{\sqrt{2}}$,

Example

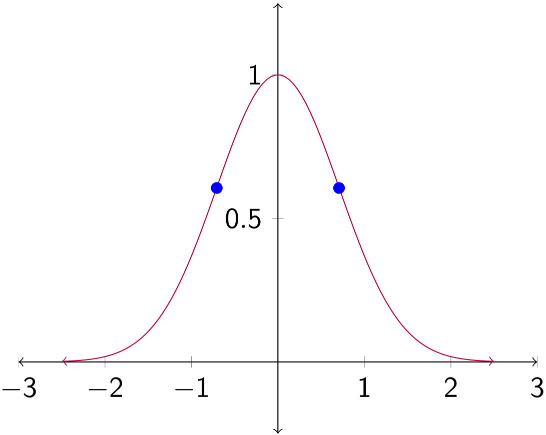
Find the inflection points of $f(x) = e^{-x^2}$.

Well, $f'(x) = -2xe^{-x^2}$ and $f''(x) = (-2)(e^{-x^2}) - 2x(-2x)e^{-x^2}$ $= -2e^{-x^2}(1 - 2x^2)$. So, the second derivative is zero when

$$1 - 2x^2 = 0,$$

i.e. when $x = \pm\frac{1}{\sqrt{2}}$, and the inflection points are $\left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$

Graph of $f(x) = 2^x$



Exponential Growth

A quantity $Q(t) = Q_0 e^{kt}$ experiences exponential growth as t grows (and is positive).

Exponential Growth

A quantity $Q(t) = Q_0 e^{kt}$ experiences exponential growth as t grows (and is positive). Show that a quantity experiencing exponential growth has a growth rate that is directly proportional to the amount of the quantity present.

Exponential Growth

A quantity $Q(t) = Q_0e^{kt}$ experiences exponential growth as t grows (and is positive). Show that a quantity experiencing exponential growth has a growth rate that is directly proportional to the amount of the quantity present. Recall: Two things a and b are directly proportional if their ratio is a constant, i.e. if $\frac{a}{b} = c$ with c a constant.

Exponential Growth

A quantity $Q(t) = Q_0 e^{kt}$ experiences exponential growth as t grows (and is positive). Show that a quantity experiencing exponential growth has a growth rate that is directly proportional to the amount of the quantity present. Recall: Two things a and b are directly proportional if their ratio is a constant, i.e. if $\frac{a}{b} = c$ with c a constant.

Well, $Q'(t) = kQ_0 e^{kt}$, so

$$\frac{Q'(t)}{Q(t)} = k,$$

as desired.

Example from Section 5.3

Blakely Properties owns a building in the commercial district of Birmingham. The present value of the market price of the property is given by $P(t) = 300000e^{-0.09t+0.5\sqrt{t}}$, where P is in dollars and t in years from present (for the next ten years). Find the optimal present value of the building's market price.

Example from Section 5.3

Blakely Properties owns a building in the commercial district of Birmingham. The present value of the market price of the property is given by $P(t) = 300000e^{-0.09t+0.5\sqrt{t}}$, where P is in dollars and t in years from present (for the next ten years). Find the optimal present value of the building's market price.

Well, $P'(t) =$

Example from Section 5.3

Blakely Properties owns a building in the commercial district of Birmingham. The present value of the market price of the property is given by $P(t) = 300000e^{-0.09t+0.5\sqrt{t}}$, where P is in dollars and t in years from present (for the next ten years). Find the optimal present value of the building's market price.

Well, $P'(t) = \left(-0.09 + \frac{1}{4\sqrt{x}}\right) 300000e^{-0.09t+0.5\sqrt{t}}$. So, setting $P'(t) = 0$, we get

Example from Section 5.3

Blakely Properties owns a building in the commercial district of Birmingham. The present value of the market price of the property is given by $P(t) = 300000e^{-0.09t+0.5\sqrt{t}}$, where P is in dollars and t in years from present (for the next ten years). Find the optimal present value of the building's market price.

Well, $P'(t) = \left(-0.09 + \frac{1}{4\sqrt{x}}\right) 300000e^{-0.09t+0.5\sqrt{t}}$. So, setting $P'(t) = 0$, we get

$$\begin{aligned} 0 &= -0.09 + \frac{1}{4\sqrt{t}} \\ 0.09(4) &= \frac{1}{\sqrt{t}} \\ \frac{1}{0.09(4)} &= \sqrt{t} \\ 7.72 &\approx t. \end{aligned}$$

Example from Section 5.3

Blakely Properties owns a building in the commercial district of Birmingham. The present value of the market price of the property is given by $P(t) = 300000e^{-0.09t+0.5\sqrt{t}}$, where P is in dollars and t in years from present (for the next ten years). Find the optimal present value of the building's market price.

Example from Section 5.3

Blakely Properties owns a building in the commercial district of Birmingham. The present value of the market price of the property is given by $P(t) = 300000e^{-0.09t+0.5\sqrt{t}}$, where P is in dollars and t in years from present (for the next ten years). Find the optimal present value of the building's market price.

We check the critical points $x = 0, 7.72, 10$ and see that the maximum present value of market price is \$600,779.

The Derivative of the Logarithmic Function

Now we turn to the logarithmic function.

The Derivative of the Logarithmic Function

Now we turn to the logarithmic function.

Theorem (The Derivative of the Natural Logarithm)

If $x \neq 0$*, then* $\dfrac{d}{dx} \ln |x| = \dfrac{1}{x}$.

The Derivative of the Logarithmic Function

Now we turn to the logarithmic function.

Theorem (The Derivative of the Natural Logarithm)

If $x \neq 0$, *then* $\dfrac{d}{dx} \ln |x| = \dfrac{1}{x}$.

What's more, we have the following theorem for logarithms of any base.

Theorem (The Derivative of Logarithmic Functions)

If $f(x) = \log_b(|x|)$, *then when* $x \neq 0$ *we have* $f'(x) = \frac{1}{x \ln(b)}$.

The Derivative of the Logarithmic Function

Now we turn to the logarithmic function.

Theorem (The Derivative of the Natural Logarithm)

If $x \neq 0$*, then* $\frac{d}{dx} \ln |x| = \frac{1}{x}$.

What's more, we have the following theorem for logarithms of any base.

Theorem (The Derivative of Logarithmic Functions)

If $f(x) = \log_b(|x|)$*, then when* $x \neq 0$ *we have* $f'(x) = \frac{1}{x \ln(b)}$.

As an exercise, you might try to show why these derivatives work!

Examples

Find the following derivatives.

- $\dfrac{d}{dx}\dfrac{\ln x}{x}$

Examples

Find the following derivatives.

- $\frac{d}{dx}\frac{\ln x}{x} = \frac{x(\ln x)' - \ln x(x)'}{x^2}$

Examples

Find the following derivatives.

- $\frac{d}{dx}\frac{\ln x}{x} = \frac{x(\ln x)' - \ln x(x)'}{x^2} = \frac{xx^{-1} - \ln x}{x^2}$

Examples

Find the following derivatives.

- $\dfrac{d}{dx}\dfrac{\ln x}{x} = \dfrac{x(\ln x)' - \ln x(x)'}{x^2} = \dfrac{xx^{-1} - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}.$
- $\dfrac{d}{dx}(x \ln x)$

Examples

Find the following derivatives.

- $\dfrac{d}{dx}\dfrac{\ln x}{x} = \dfrac{x(\ln x)' - \ln x(x)'}{x^2} = \dfrac{xx^{-1} - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}.$
- $\dfrac{d}{dx}(x \ln x) = (x)'(\ln x) + x(\ln x)'$

Examples

Find the following derivatives.

- $\frac{d}{dx}\frac{\ln x}{x} = \frac{x(\ln x)' - \ln x(x)'}{x^2} = \frac{xx^{-1} - \ln x}{x^2} = \frac{1-\ln x}{x^2}$.
- $\frac{d}{dx}(x\ln x) = (x)'(\ln x) + x(\ln x)' = \ln x + xx^{-1}$

Examples

Find the following derivatives.

- $\frac{d}{dx}\frac{\ln x}{x} = \frac{x(\ln x)' - \ln x(x)'}{x^2} = \frac{xx^{-1} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$.
- $\frac{d}{dx}(x \ln x) = (x)'(\ln x) + x(\ln x)' = \ln x + xx^{-1} = \ln x + 1$.
- $\frac{d}{dx}[(\log_3 x)(\log_2 x]$

Examples

Find the following derivatives.

- $\dfrac{d}{dx}\dfrac{\ln x}{x} = \dfrac{x(\ln x)' - \ln x(x)'}{x^2} = \dfrac{xx^{-1} - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}.$
- $\dfrac{d}{dx}(x\ln x) = (x)'(\ln x) + x(\ln x)' = \ln x + xx^{-1} = \ln x + 1.$
- $\dfrac{d}{dx}[(\log_3 x)(\log_2 x] = (\log_3 x)'(\log_2 x) + (\log_3 x)(\log_2 x)'$

Examples

Find the following derivatives.

- $\dfrac{d}{dx}\dfrac{\ln x}{x} = \dfrac{x(\ln x)' - \ln x(x)'}{x^2} = \dfrac{xx^{-1} - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$.
- $\dfrac{d}{dx}(x \ln x) = (x)'(\ln x) + x(\ln x)' = \ln x + xx^{-1} = \ln x + 1$.
- $\dfrac{d}{dx}[(\log_3 x)(\log_2 x] = (\log_3 x)'(\log_2 x) + (\log_3 x)(\log_2 x)' =$
  $\dfrac{1}{x \ln 3}(\log_2 x) + (\log_3 x)\dfrac{1}{x \ln 2}$.

The Chain Rule for Logarithmic Fuctions

We can combine the chain rule with the derivative of the natural logarithmic function to get the following rule.

The Chain Rule for Logarithmic Fuctions

We can combine the chain rule with the derivative of the natural logarithmic function to get the following rule.

Theorem (The Chain Rule for the Natural Logarithm)

If $f(x) > 0$, *then we have* $\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$.

The Chain Rule for Logarithmic Fuctions

We can combine the chain rule with the derivative of the natural logarithmic function to get the following rule.

Theorem (The Chain Rule for the Natural Logarithm)

If $f(x) > 0$, then we have $\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$.

The proof of this theorem is just a direct application of the Chain Rule. You might consider proving this theorem as an exercise.

The Chain Rule for Logarithmic Fuctions

We can combine the chain rule with the derivative of the natural logarithmic function to get the following rule.

Theorem (The Chain Rule for the Natural Logarithm)

If $f(x) > 0$*, then we have* $\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$.

The proof of this theorem is just a direct application of the Chain Rule. You might consider proving this theorem as an exercise. We also have

Theorem (The Chain Rule for Logarithmic Functions)

If $f(x) > 0$*, then we have* $\frac{d}{dx}[\log_b f(x)] = \frac{f'(x)}{\log(b)f(x)}$.

Examples

Find the following derivatives.

- $\dfrac{d}{dx} \ln(x^2 + 1)$

Examples

Find the following derivatives.

- $\dfrac{d}{dx}\ln(x^2+1) = \dfrac{(x^2+1)'}{x^2+1}$

Examples

Find the following derivatives.

- $\dfrac{d}{dx}\ln(x^2+1) = \dfrac{(x^2+1)'}{x^2+1} = \dfrac{2x}{x^2+1}.$
- $\dfrac{d}{dx}\ln((x^2+1)(x^3+2)^6)$

Examples

Find the following derivatives.

- $\dfrac{d}{dx}\ln(x^2+1) = \dfrac{(x^2+1)'}{x^2+1} = \dfrac{2x}{x^2+1}.$
- $\dfrac{d}{dx}\ln((x^2+1)(x^3+2)^6) = \dfrac{d}{dx}[\ln(x^2+1) + 6\ln(x^3+2)]$

Examples

Find the following derivatives.

- $\dfrac{d}{dx}\ln(x^2+1) = \dfrac{(x^2+1)'}{x^2+1} = \dfrac{2x}{x^2+1}.$
- $\dfrac{d}{dx}\ln((x^2+1)(x^3+2)^6) = \dfrac{d}{dx}[\ln(x^2+1) + 6\ln(x^3+2)] =$

  $\dfrac{d}{dx}\ln(x^2+1) + 6\dfrac{d}{dx}\ln(x^3+2)$

Examples

Find the following derivatives.

- $\dfrac{d}{dx}\ln(x^2+1) = \dfrac{(x^2+1)'}{x^2+1} = \dfrac{2x}{x^2+1}$.
- $\dfrac{d}{dx}\ln((x^2+1)(x^3+2)^6) = \dfrac{d}{dx}[\ln(x^2+1)+6\ln(x^3+2)] =$

  $\dfrac{d}{dx}\ln(x^2+1)+6\dfrac{d}{dx}\ln(x^3+2) = \dfrac{2x}{x^2+1}+\dfrac{6(3x^2)}{x^3+2}$

Examples

Find the following derivatives.

- $\frac{d}{dx}\ln(x^2+1) = \frac{(x^2+1)'}{x^2+1} = \frac{2x}{x^2+1}$.
- $\frac{d}{dx}\ln((x^2+1)(x^3+2)^6) = \frac{d}{dx}[\ln(x^2+1) + 6\ln(x^3+2)] =$

  $\frac{d}{dx}\ln(x^2+1) + 6\frac{d}{dx}\ln(x^3+2) = \frac{2x}{x^2+1} + \frac{6(3x^2)}{x^3+2} =$

  $\frac{2x}{x^2+1} + \frac{18x^2}{x^3+2}$

Examples

Find the following derivative.

- $\dfrac{d}{dt}\ln(t^2 e^{-t^2})$

Examples

Find the following derivative.

- $\dfrac{d}{dt}\ln(t^2 e^{-t^2}) = \dfrac{d}{dx}(2\ln t + t^2 \ln e)$

Examples

Find the following derivative.

- $\dfrac{d}{dt}\ln(t^2 e^{-t^2}) = \dfrac{d}{dx}(2\ln t + t^2 \ln e) =$

  $\frac{d}{dx}(2\ln t) + \frac{d}{dx}(t^2)$

Examples

Find the following derivative.

- $\frac{d}{dt}\ln(t^2e^{-t^2}) = \frac{d}{dx}(2\ln t + t^2\ln e) =$
  $\frac{d}{dx}(2\ln t) + \frac{d}{dx}(t^2) = \frac{2}{t} - 2t$

Examples

Find the following derivative.

- $\dfrac{d}{dt}\ln(t^2e^{-t^2}) = \dfrac{d}{dx}(2\ln t + t^2\ln e) =$

  $\frac{d}{dx}(2\ln t) + \frac{d}{dx}(t^2) = \frac{2}{t} - 2t = \frac{2(1-t^2)}{t}$.

Examples

Find the following derivative.

- $\dfrac{d}{dt}\ln(t^2e^{-t^2}) = \dfrac{d}{dx}(2\ln t + t^2\ln e) =$
  $\frac{d}{dx}(2\ln t) + \frac{d}{dx}(t^2) = \frac{2}{t} - 2t = \frac{2(1-t^2)}{t}$.

Clearly, it is easier to differentiate logarithmic fucntions when they are written in an expanded form. We can exploit the ease of this process to simplify the differentiation of other functions.

Examples

Find the following derivative.

- $\dfrac{d}{dt}\ln(t^2e^{-t^2}) = \dfrac{d}{dx}(2\ln t + t^2\ln e) =$
  $\frac{d}{dx}(2\ln t) + \frac{d}{dx}(t^2) = \frac{2}{t} - 2t = \frac{2(1-t^2)}{t}$.

Clearly, it is easier to differentiate logarithmic fucntions when they are written in an expanded form. We can exploit the ease of this process to simplify the differentiation of other functions. In particular, **logarithmic differentiation** can make functions that are products of other functions easier to differentiate.

Logarithmic Differentiation

You an take the following steps to differentiate functions involving many products.

Logarithmic Differentiation

You an take the following steps to differentiate functions involving many products.

1. Take the natural logarithm of both sides of the expression and rewrite any complicated expression using the rules of logarithms.

Logarithmic Differentiation

You an take the following steps to differentiate functions involving many products.

1. Take the natural logarithm of both sides of the expression and rewrite any complicated expression using the rules of logarithms.
2. Differentiate both sides of the expression with respect to x.

Logarithmic Differentiation

You an take the following steps to differentiate functions involving many products.

1. Take the natural logarithm of both sides of the expression and rewrite any complicated expression using the rules of logarithms.
2. Differentiate both sides of the expression with respect to x.
3. Solve for $\frac{dy}{dx}$, i.e. for y'.

Example

Find the derivatives of the given function

- $y = (x)(x+1)(x^2+1) \implies$

Example

Find the derivatives of the given function

- $y = (x)(x+1)(x^2+1) \implies \ln y = \ln[(x)(x+1)(x^2+1)] =$

Example

Find the derivatives of the given function

- $y = (x)(x+1)(x^2+1) \implies \ln y = \ln[(x)(x+1)(x^2+1)] = \ln x + \ln(x+1) + \ln(x^2+1)$

Example

Find the derivatives of the given function

- $y = (x)(x+1)(x^2+1) \implies \ln y = \ln[(x)(x+1)(x^2+1)] = \ln x + \ln(x+1) + \ln(x^2+1) \implies$

Example

Find the derivatives of the given function

- $y = (x)(x+1)(x^2+1) \implies \ln y = \ln[(x)(x+1)(x^2+1)] = \ln x + \ln(x+1) + \ln(x^2+1) \implies$

  $(\ln y)' =$

Example

Find the derivatives of the given function

- $y = (x)(x+1)(x^2+1) \implies \ln y = \ln[(x)(x+1)(x^2+1)] = \ln x + \ln(x+1) + \ln(x^2+1) \implies$

  $(\ln y)' = (\ln x + \ln(x+1) + \ln(x^2+1))'$

Example

Find the derivatives of the given function

- $y = (x)(x+1)(x^2+1) \implies \ln y = \ln[(x)(x+1)(x^2+1)] = \ln x + \ln(x+1) + \ln(x^2+1) \implies$

  $(\ln y)' = (\ln x + \ln(x+1) + \ln(x^2+1))' \implies$

Example

Find the derivatives of the given function

- $y = (x)(x+1)(x^2+1) \implies \ln y = \ln[(x)(x+1)(x^2+1)] = \ln x + \ln(x+1) + \ln(x^2+1) \implies$

  $(\ln y)' = (\ln x + \ln(x+1) + \ln(x^2+1))' \implies$

  $\frac{y'}{y} = \frac{1}{x} + \frac{1}{x+1} + \frac{2x}{x^2+1}$

Example

Find the derivatives of the given function

- $y = (x)(x+1)(x^2+1) \implies \ln y = \ln[(x)(x+1)(x^2+1)] = \ln x + \ln(x+1) + \ln(x^2+1) \implies$

  $(\ln y)' = (\ln x + \ln(x+1) + \ln(x^2+1))' \implies$

  $\frac{y'}{y} = \frac{1}{x} + \frac{1}{x+1} + \frac{2x}{x^2+1} \implies y' = y(\frac{1}{x} + \frac{1}{x+1} + \frac{2x}{x^2+1})$

Example

Find the derivatives of the given function

- $y = (x)(x+1)(x^2+1) \implies \ln y = \ln[(x)(x+1)(x^2+1)] = \ln x + \ln(x+1) + \ln(x^2+1) \implies$

  $(\ln y)' = (\ln x + \ln(x+1) + \ln(x^2+1))' \implies$

  $\frac{y'}{y} = \frac{1}{x} + \frac{1}{x+1} + \frac{2x}{x^2+1} \implies y' = y(\frac{1}{x} + \frac{1}{x+1} + \frac{2x}{x^2+1}) = (x)(x+1)(x^2+1)\left(\frac{1}{x} + \frac{1}{x+1} + \frac{2x}{x^2+1}\right)$

Example

Find the derivatives of the given function

- $y = (x^2)(x-1)(x^2+4)^3 \implies$

Example

Find the derivatives of the given function

- $y = (x^2)(x-1)(x^2+4)^3 \implies \ln y = \ln[(x^2)(x-1)(x^2+4)^3] =$

Example

Find the derivatives of the given function

- $y = (x^2)(x-1)(x^2+4)^3 \implies \ln y = \ln[(x^2)(x-1)(x^2+4)^3] = 2\ln x + \ln(x-1) + 3\ln(x^2+4)$

Example

Find the derivatives of the given function

- $y = (x^2)(x-1)(x^2+4)^3 \implies \ln y = \ln[(x^2)(x-1)(x^2+4)^3] = 2\ln x + \ln(x-1) + 3\ln(x^2+4) \implies$

Example

Find the derivatives of the given function

- $y = (x^2)(x-1)(x^2+4)^3 \implies \ln y = \ln[(x^2)(x-1)(x^2+4)^3] = 2\ln x + \ln(x-1) + 3\ln(x^2+4) \implies$

  $(\ln y)' =$

Example

Find the derivatives of the given function

- $y = (x^2)(x-1)(x^2+4)^3 \implies \ln y = \ln[(x^2)(x-1)(x^2+4)^3] = 2\ln x + \ln(x-1) + 3\ln(x^2+4) \implies$
  $(\ln y)' = (2\ln x + \ln(x-1) + 3\ln(x^2+4))'$

Example

Find the derivatives of the given function

- $y = (x^2)(x-1)(x^2+4)^3 \implies \ln y = \ln[(x^2)(x-1)(x^2+4)^3] = 2\ln x + \ln(x-1) + 3\ln(x^2+4) \implies$

  $(\ln y)' = (2\ln x + \ln(x-1) + 3\ln(x^2+4))' \implies$

Example

Find the derivatives of the given function

- $y = (x^2)(x-1)(x^2+4)^3 \implies \ln y = \ln[(x^2)(x-1)(x^2+4)^3] = 2\ln x + \ln(x-1) + 3\ln(x^2+4) \implies$

  $(\ln y)' = (2\ln x + \ln(x-1) + 3\ln(x^2+4))' \implies$

  $\frac{y'}{y} = \frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+4}$

Example

Find the derivatives of the given function

- $y = (x^2)(x-1)(x^2+4)^3 \implies \ln y = \ln[(x^2)(x-1)(x^2+4)^3] = 2\ln x + \ln(x-1) + 3\ln(x^2+4) \implies$
  $(\ln y)' = (2\ln x + \ln(x-1) + 3\ln(x^2+4))' \implies$
  $\frac{y'}{y} = \frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+4} \implies y' = y(\frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+4})$

Example

Find the derivatives of the given function

- $y = (x^2)(x-1)(x^2+4)^3 \implies \ln y = \ln[(x^2)(x-1)(x^2+4)^3] = 2\ln x + \ln(x-1) + 3\ln(x^2+4) \implies$
  $(\ln y)' = (2\ln x + \ln(x-1) + 3\ln(x^2+4))' \implies$
  $\frac{y'}{y} = \frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+4} \implies y' = y(\frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+4}) =$
  $(x^2)(x-1)(x^2+4)^3\left(\frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+4}\right)$

Example

The population of a town t months after the opening of an auto assembly plant in the surrounding area is given by the function $P(t) = 18000e^{-(\ln 9)e^{-0.1t}}$. What is the relative rate of growth of the population 6 months after the opening of the auto assembly plant?

Example

The population of a town t months after the opening of an auto assembly plant in the surrounding area is given by the function $P(t) = 18000e^{-(\ln 9)e^{-0.1t}}$. What is the relative rate of growth of the population 6 months after the opening of the auto assembly plant?

We need to calculate $P'(t)/P(t)$,

Example

The population of a town t months after the opening of an auto assembly plant in the surrounding area is given by the function $P(t) = 18000e^{-(\ln 9)e^{-0.1t}}$. What is the relative rate of growth of the population 6 months after the opening of the auto assembly plant?

We need to calculate $P'(t)/P(t)$, but we'll use logarithmic differentiation.

$$\ln(P) = \ln 18000 - (\ln 9)e^{-0.1t}$$

Example

The population of a town t months after the opening of an auto assembly plant in the surrounding area is given by the function $P(t) = 18000e^{-(\ln 9)e^{-0.1t}}$. What is the relative rate of growth of the population 6 months after the opening of the auto assembly plant?

We need to calculate $P'(t)/P(t)$, but we'll use logarithmic differentiation.

$$\ln(P) = \ln 18000 - (\ln 9)e^{-0.1t}$$

So, $\frac{P'}{P} = (\ln 9)(0.1)e^{-0.1t}$

Example

The population of a town t months after the opening of an auto assembly plant in the surrounding area is given by the function $P(t) = 18000e^{-(\ln 9)e^{-0.1t}}$. What is the relative rate of growth of the population 6 months after the opening of the auto assembly plant?

We need to calculate $P'(t)/P(t)$, but we'll use logarithmic differentiation.

$$\ln(P) = \ln 18000 - (\ln 9)e^{-0.1t}$$

So, $\frac{P'}{P} = (\ln 9)(0.1)e^{-0.1t}$ and so $\frac{P'(6)}{P(6)} = 0.1(\ln 9)e^{-0.1(6)} \approx 0.121$.

Example

The population of a town t months after the opening of an auto assembly plant in the surrounding area is given by the function $P(t) = 18000e^{-(\ln 9)e^{-0.1t}}$. What is the relative rate of growth of the population 6 months after the opening of the auto assembly plant?

We need to calculate $P'(t)/P(t)$, but we'll use logarithmic differentiation.

$$\ln(P) = \ln 18000 - (\ln 9)e^{-0.1t}$$

So, $\frac{P'}{P} = (\ln 9)(0.1)e^{-0.1t}$ and so $\frac{P'(6)}{P(6)} = 0.1(\ln 9)e^{-0.1(6)} \approx 0.121$.

So, 6 months after the auto assembly plant opens, the population will be growing at a relative rate of approximately 12.1%.

Assignment

Read 5.6. Do problems 8, 16, 28, 32, 46, 76 in 5.4 and 16, 30, 40, 48, 58, 78, 84 in 5.5.