QMI Lesson 19:
Integration by Substitution, Definite Integral, and Area Under Curve

C C Moxley

Samford University Brock School of Business

Substitution Rule

The following rules arise from the chain rule of differentiation.

Substitution Rule

The following rules arise from the chain rule of differentiation.

Theorem (Substitution Rule)

$$\int F'(g(x))g'(x)\ dx = F(g(x)) + C$$

Substitution Rule

The following rules arise from the chain rule of differentiation.

Theorem (Substitution Rule)

$$\int F'(g(x))g'(x) \; dx = F(g(x)) + C$$

Theorem (Generalized Power Rule)

$$\int [g(x)]^n g'(x) \; dx = \frac{1}{n+1} g(x)^{n+1} + C, \qquad (n \neq -1)$$

Substitution Rule

The following rules arise from the chain rule of differentiation.

Theorem (Substitution Rule)

$$\int F'(g(x))g'(x)\ dx = F(g(x)) + C$$

Theorem (Generalized Power Rule)

$$\int [g(x)]^n g'(x)\ dx = \frac{1}{n+1} g(x)^{n+1} + C, \qquad (n \neq -1)$$

Method of Substitution

You can exploit the previous rules to integrate certain functions by following the steps below.

Method of Substitution

You can exploit the previous rules to integrate certain functions by following the steps below.

1. Let $u = g(x)$, where $g(x)$ is part of the integrand, usually the "inside function" of a composite function.

Method of Substitution

You can exploit the previous rules to integrate certain functions by following the steps below.

1. Let $u = g(x)$, where $g(x)$ is part of the integrand, usually the "inside function" of a composite function.
2. Find $du = g'(x)dx$.

Method of Substitution

You can exploit the previous rules to integrate certain functions by following the steps below.

1. Let $u = g(x)$, where $g(x)$ is part of the integrand, usually the "inside function" of a composite function.
2. Find $du = g'(x)dx$.
3. Use the substitution $u = g(x)$ and $du = g'(x)dx$ to convert the entire integral into one involving only u.

Method of Substitution

You can exploit the previous rules to integrate certain functions by following the steps below.

1. Let $u = g(x)$, where $g(x)$ is part of the integrand, usually the "inside function" of a composite function.
2. Find $du = g'(x)dx$.
3. Use the substitution $u = g(x)$ and $du = g'(x)dx$ to convert the entire integral into one involving only u.
4. Evaluate the resulting integral.

Method of Substitution

You can exploit the previous rules to integrate certain functions by following the steps below.

1. Let $u = g(x)$, where $g(x)$ is part of the integrand, usually the "inside function" of a composite function.
2. Find $du = g'(x)dx$.
3. Use the substitution $u = g(x)$ and $du = g'(x)dx$ to convert the entire integral into one involving only u.
4. Evaluate the resulting integral.
5. Replace u by $g(x)$ to obtain the final solution as a function of x.

Example

Find $\int 2x(x^2+3)^4 \; dx$.

Example

Find $\int 2x(x^2+3)^4 \, dx$.

1. Let $u = x^2 + 3$.

Example

Find $\int 2x(x^2+3)^4 \ dx$.

1. Let $u = x^2 + 3$.
2. Then $du = 2x \ dx$.

Example

Find $\int 2x(x^2+3)^4 \; dx$.

1. Let $u = x^2 + 3$.
2. Then $du = 2x \; dx$.
3. Thus, $\int 2x(x^2+3)^4 \; dx = \int \underbrace{(x^2+3)^4}_{u^4} \; \underbrace{(2xdx)}_{du} = \int u^4 \; du$.

Example

Find $\int 2x(x^2+3)^4 \; dx$.

1. Let $u = x^2 + 3$.
2. Then $du = 2x \; dx$.
3. Thus, $\int 2x(x^2+3)^4 \; dx = \int \underbrace{(x^2+3)^4}_{u^4} \; \underbrace{(2xdx)}_{du} = \int u^4 \; du$.
4. $\int u^4 \; du = \frac{u^5}{5} + C$.

Example

Find $\int 2x(x^2+3)^4 \ dx$.

1. Let $u = x^2 + 3$.
2. Then $du = 2x \ dx$.
3. Thus, $\int 2x(x^2+3)^4 \ dx = \int \underbrace{(x^2+3)^4}_{u^4} \ \underbrace{(2xdx)}_{du} = \int u^4 \ du$.
4. $\int u^4 \ du = \frac{u^5}{5} + C$.
5. So, $\int 2x(x^2+3)^4 \ dx = \frac{(x^2+3)^5}{5} + C$.

Example

Find $\int \frac{x}{3x^2+1}\ dx$.

Example

Find $\int \frac{x}{3x^2+1}\ dx$.

1. Let $u = 3x^2 + 1$.

Example

Find $\int \frac{x}{3x^2+1}\ dx$.

1. Let $u = 3x^2 + 1$.
2. Then $du = 6x\ dx$ which means $\frac{du}{6} = x\ dx$.

Example

Find $\int \frac{x}{3x^2+1}\ dx$.

1. Let $u = 3x^2 + 1$.
2. Then $du = 6x\ dx$ which means $\frac{du}{6} = x\ dx$.
3. Thus, $\int \frac{x}{3x^2+1}\ dx = \int \underbrace{\frac{1}{3x^2+1}}_{\frac{1}{u}} \underbrace{(xdx)}_{\frac{du}{6}} = \int \frac{du}{6u}$.

Example

Find $\int \frac{x}{3x^2+1}\ dx$.

1. Let $u = 3x^2 + 1$.
2. Then $du = 6x\ dx$ which means $\frac{du}{6} = x\ dx$.
3. Thus, $\int \frac{x}{3x^2+1}\ dx = \int \underbrace{\frac{1}{3x^2+1}}_{\frac{1}{u}} \underbrace{(xdx)}_{\frac{du}{6}} = \int \frac{du}{6u}$.
4. $\frac{1}{6}\int \frac{du}{u} = \frac{1}{6}\ln u + C$.

Example

Find $\int \frac{x}{3x^2+1}\ dx$.

1. Let $u = 3x^2 + 1$.
2. Then $du = 6x\ dx$ which means $\frac{du}{6} = x\ dx$.
3. Thus, $\int \frac{x}{3x^2+1}\ dx = \int \underbrace{\frac{1}{3x^2+1}}_{\frac{1}{u}} \underbrace{(xdx)}_{\frac{du}{6}} = \int \frac{du}{6u}$.
4. $\frac{1}{6} \int \frac{du}{u} = \frac{1}{6} \ln u + C$.
5. So, $\int \frac{x}{3x^2+1}\ dx = \frac{1}{6} \ln(3x^2 + 1) + C$.

Example

A study forecasts that a new line of computers will have sales of $2000 - 1500e^{-0.05t}$ units per month after t months. Find an expression for the number of computers sold in the first t months.

Example

A study forecasts that a new line of computers will have sales of $2000 - 1500e^{-0.05t}$ units per month after t months. Find an expression for the number of computers sold in the first t months.

We need to calculate $\int 2000 - 1500e^{-0.05t}\ dt$, letting $u = -0.05t \implies du = -0.05dt$.

Example

A study forecasts that a new line of computers will have sales of $2000 - 1500e^{-0.05t}$ units per month after t months. Find an expression for the number of computers sold in the first t months.

We need to calculate $\int 2000 - 1500e^{-0.05t}\ dt$, letting $u = -0.05t \implies du = -0.05dt$. So we get

$$\int 2000 - 1500e^{-0.05t}\ dt = \int 2000 - 1500e^{u}\ (-20)du =$$

Example

A study forecasts that a new line of computers will have sales of $2000 - 1500e^{-0.05t}$ units per month after t months. Find an expression for the number of computers sold in the first t months.

We need to calculate $\int 2000 - 1500e^{-0.05t}\ dt$, letting $u = -0.05t \implies du = -0.05dt$. So we get

$$\int 2000 - 1500e^{-0.05t}\ dt = \int 2000 - 1500e^{u}\ (-20)du =$$

$$\int -40000 + 30000e^{u}\ du = -40000u + 30000e^{u} + C =$$

Example

A study forecasts that a new line of computers will have sales of $2000 - 1500e^{-0.05t}$ units per month after t months. Find an expression for the number of computers sold in the first t months.

We need to calculate $\int 2000 - 1500e^{-0.05t}\ dt$, letting $u = -0.05t \implies du = -0.05dt$. So we get

$$\int 2000 - 1500e^{-0.05t}\ dt = \int 2000 - 1500e^{u}\ (-20)du =$$

$$\int -40000 + 30000e^{u}\ du = -40000u + 30000e^{u} + C =$$

$$2000t + 30000e^{-0.05t} + C.$$

Example

Naturally, no computers were sold at time $t = 0$, so to solve for C, we simply notice

$$2000(0) + 30000e^{-0.05(0)} + C = 0 \implies C = -30000.$$

So we get our expression for the number of computers sold after month t to be

$$2000t + 30000e^{-0.05t} - 30000.$$

Example

Find $\int 2x^3(x^2+1)^{\frac{1}{2}}\ dx$.

Example

Find $\int 2x^3(x^2+1)^{\frac{1}{2}}\ dx$.

1. Let $u = x^2 + 1 \implies x^2 = u - 1$.

Example

Find $\int 2x^3(x^2+1)^{\frac{1}{2}}\ dx$.

1. Let $u = x^2 + 1 \implies x^2 = u - 1$.
2. Then $du = 2x\ dx$.

Example

Find $\int 2x^3(x^2+1)^{\frac{1}{2}}\ dx$.

1. Let $u = x^2 + 1 \implies x^2 = u - 1$.
2. Then $du = 2x\ dx$.
3. Thus, $\int 2x^3(x^2+1)^{\frac{1}{2}}\ dx = \int \underbrace{(x^2)}_{u-1}\underbrace{(x^2+1)^{\frac{1}{2}}}_{u^{\frac{1}{2}}}\ \underbrace{(2xdx)}_{du} =$

   $\int (u-1)u^{\frac{1}{2}}\ du = \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\ du$.

Example

Find $\int 2x^3(x^2+1)^{\frac{1}{2}}\ dx$.

1. Let $u = x^2 + 1 \implies x^2 = u - 1$.
2. Then $du = 2x\ dx$.
3. Thus, $\int 2x^3(x^2+1)^{\frac{1}{2}}\ dx = \int \underbrace{(x^2)}_{u-1} \underbrace{(x^2+1)^{\frac{1}{2}}}_{u^{\frac{1}{2}}} \underbrace{(2xdx)}_{du} =$

   $\int (u-1)u^{\frac{1}{2}}\ du = \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\ du$.
4. $\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\ du = \frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3} + C$.

Example

Find $\int 2x^3(x^2+1)^{\frac{1}{2}}\ dx$.

1. Let $u = x^2 + 1 \implies x^2 = u - 1$.
2. Then $du = 2x\ dx$.
3. Thus, $\int 2x^3(x^2+1)^{\frac{1}{2}}\ dx = \int \underbrace{(x^2)}_{u-1}\underbrace{(x^2+1)^{\frac{1}{2}}}_{u^{\frac{1}{2}}}\ \underbrace{(2xdx)}_{du} =$

   $\int (u-1)u^{\frac{1}{2}}\ du = \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\ du$.
4. $\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\ du = \frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3} + C$.
5. So, $\int 2x(x^2+3)^4\ dx = \frac{2(x^2+1)^{\frac{5}{2}}}{5} - \frac{2(x^2+1)^{\frac{3}{2}}}{3} + C$.

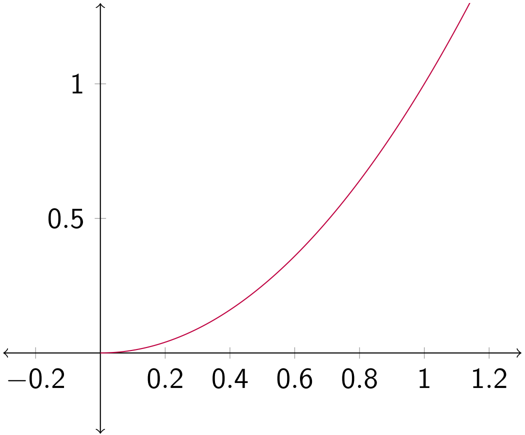
The Definite Integral

Recall when we talked about the **differential** that we interpreted $dy = f'(x)dx$ as an appoximation for Δy.

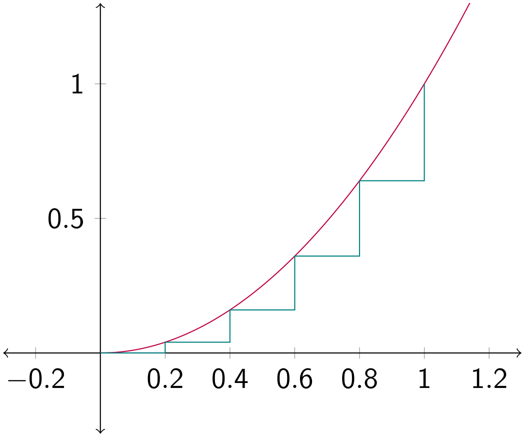
The Definite Integral

Recall when we talked about the **differential** that we interpreted $dy = f'(x)dx$ as an appoximation for Δy. On the graph of a function, we can see Δy and $\Delta x = dx$ forming an area related to the graph of a function. See the next slide for details.

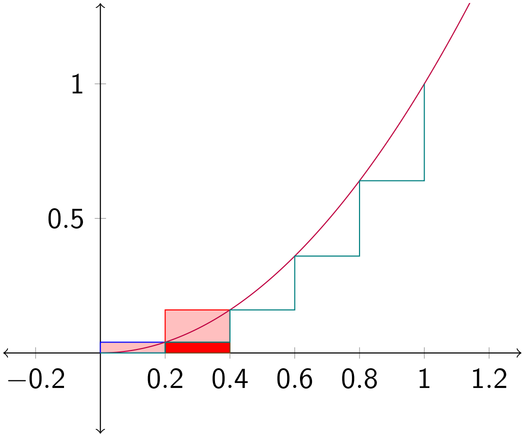
Area Under the Curve of $f(x) = x^2$ on [0,1]



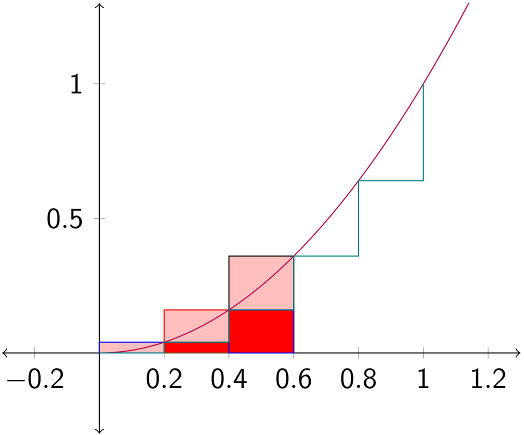
Area Under the Curve of $f(x) = x^2$ on [0,1]



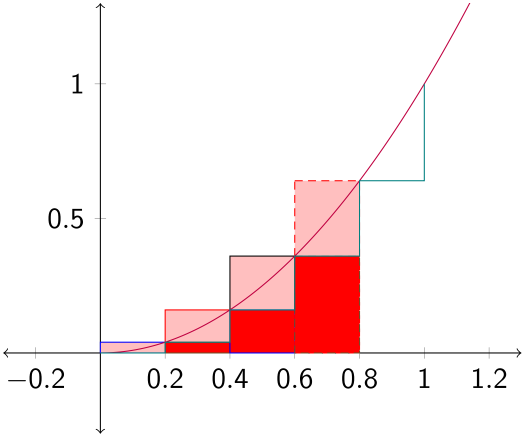
Area Under the Curve of $f(x) = x^2$ on [0,1]



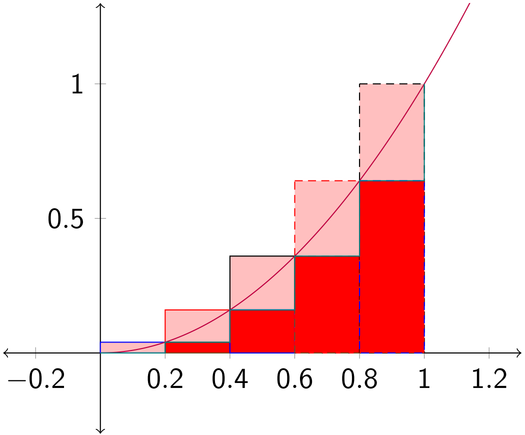
Area Under the Curve of $f(x) = x^2$ on [0,1]



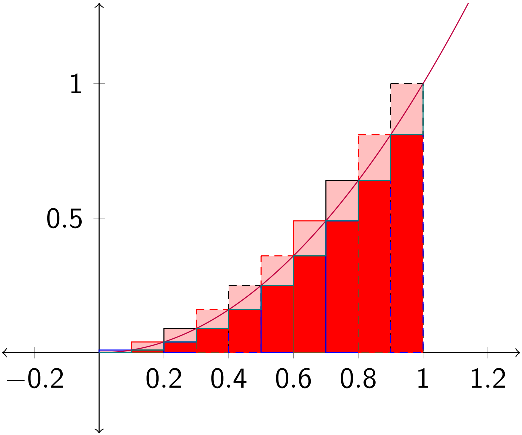
Area Under the Curve of $f(x) = x^2$ on [0,1]



Area Under the Curve of $f(x) = x^2$ on [0,1]



Improving Area Under the Curve of $f(x) = x^2$ on [0,1]



Numerically Approximating Area Under the Curve of $f(x) = x^2$ on [0,1]

Our first division of the area under $f(x)$ on [0,1] into five intervals gave the approximation of the area as

Numerically Approximating Area Under the Curve of $f(x) = x^2$ on [0,1]

Our first division of the area under $f(x)$ on [0,1] into five intervals gave the approximation of the area as

$$\frac{1}{5}f\left(\frac{1}{5}\right)+\frac{1}{5}f\left(\frac{2}{5}\right)+\frac{1}{5}f\left(\frac{3}{5}\right)+\frac{1}{5}f\left(\frac{4}{5}\right)+\frac{1}{5}f\left(1\right)=0.44,$$

Numerically Approximating Area Under the Curve of $f(x) = x^2$ on [0,1]

Our first division of the area under $f(x)$ on $[0,1]$ into five intervals gave the approximation of the area as

$$\frac{1}{5}f\left(\frac{1}{5}\right) + \frac{1}{5}f\left(\frac{2}{5}\right) + \frac{1}{5}f\left(\frac{3}{5}\right) + \frac{1}{5}f\left(\frac{4}{5}\right) + \frac{1}{5}f\left(1\right) = 0.44,$$

And our second division of the area under $f(x)$ on $[0,1]$ into ten intervals gave the approximation

Numerically Approximating Area Under the Curve of $f(x) = x^2$ on [0,1]

Our first division of the area under $f(x)$ on [0,1] into five intervals gave the approximation of the area as

$$\frac{1}{5}f\left(\frac{1}{5}\right) + \frac{1}{5}f\left(\frac{2}{5}\right) + \frac{1}{5}f\left(\frac{3}{5}\right) + \frac{1}{5}f\left(\frac{4}{5}\right) + \frac{1}{5}f\left(1\right) = 0.44,$$

And our second division of the area under $f(x)$ on [0,1] into ten intervals gave the approximation

$$\frac{1}{10}\left(f(0.1) + f(0.2) + \cdots + f(0.9) + f(1)\right) = 0.385.$$

Example

Let R be the region under the graph of $f(x) = 16 - x^2$ on the interval [1,3]. Find an approximation of the area of R by using four subintervals of equal length and picking the midpoint of each subinterval to evaluate the height of the approximating rectagle.

Well, our four subintervals are [1,1.5], [1.5,2], [2,2.5], and [2.5,3].

Example

Let R be the region under the graph of $f(x) = 16 - x^2$ on the interval [1,3]. Find an approximation of the area of R by using four subintervals of equal length and picking the midpoint of each subinterval to evaluate the height of the approximating rectagle.

Well, our four subintervals are [1,1.5], [1.5,2], [2,2.5], and [2.5,3].The midpoints of these intervals are, respectively, 1.25, 1.75, 2.25, and 2.75.

Example

Let R be the region under the graph of $f(x) = 16 - x^2$ on the interval [1,3]. Find an approximation of the area of R by using four subintervals of equal length and picking the midpoint of each subinterval to evaluate the height of the approximating rectagle.

Well, our four subintervals are [1,1.5], [1.5,2], [2,2.5], and [2.5,3].The midpoints of these intervals are, respectively, 1.25, 1.75, 2.25, and 2.75. So our approximation is:

$$\frac{1}{2}(f(1.25) + f(1.75) + f(2.25) + f(2.75)) = 23.375$$

Riemann Sums and the Area Under a Curve

We can generalize this process of approximation via Riemann sums and pass through the limit to calculate the actual area under the graph of a function.

Definition (The Area Under the Graph of a Function)

Riemann Sums and the Area Under a Curve

We can generalize this process of approximation via Riemann sums and pass through the limit to calculate the actual area under the graph of a function.

Definition (The Area Under the Graph of a Function)

Let f be a nonnegative continuous function on $[a,b]$. Then the area A of the region under the graph of f is given by

$$A = \lim_{n\to\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$

where x_1, x_2, ..., x_n are arbitrary points in the n subintervals of $[a,b]$ of equal width $\Delta x = \frac{b-a}{n}$.

Riemann Sums and the Area Under a Curve

We can generalize this process of approximation via Riemann sums and pass through the limit to calculate the actual area under the graph of a function.

Definition (The Area Under the Graph of a Function)

Let f be a nonnegative continuous function on $[a,b]$. Then the area A of the region under the graph of f is given by

$$A = \lim_{n\to\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$

where x_1, x_2, ..., x_n are arbitrary points in the n subintervals of $[a,b]$ of equal width $\Delta x = \frac{b-a}{n}$.

If f is continuous on $[a, b]$, then this limit always exists, to the definition is not degenerate.

The Definite Integral

Definition (The Definite Integral)

Let f be a function on $[a,b]$. If the limit

$$\lim_{n\to\infty}[f(x_1)+f(x_2)+\cdots+f(x_n)]\Delta x$$

exists and is the same for all choices of x_1, x_2, ..., x_n in the n subintervals of $[a,b]$ of equal width $\Delta x = \frac{b-a}{n}$, then this limit is called the definite integral of f from a to b and is denoted

$$\int_a^b f(x)dx = \lim_{n\to\infty}[f(x_1)+f(x_2)+\cdots+f(x_n)]\Delta x$$

The Definite Integral

Definition (The Definite Integral)

Let f be a function on $[a,b]$. If the limit

$$\lim_{n\to\infty}[f(x_1)+f(x_2)+\cdots+f(x_n)]\Delta x$$

exists and is the same for all choices of x_1, x_2, ..., x_n in the n subintervals of $[a,b]$ of equal width $\Delta x = \frac{b-a}{n}$, then this limit is called the definite integral of f from a to b and is denoted

$$\int_a^b f(x)dx = \lim_{n\to\infty}[f(x_1)+f(x_2)+\cdots+f(x_n)]\Delta x$$

We call a the lower limit of integration and b the upper limit of integration.

Notes on the Definite Integral

It is useful to note that

- a function is called integrable on an interval $[a, b]$ if its definite integral exists on that interval.

Notes on the Definite Integral

It is useful to note that

- a function is called integrable on an interval $[a,b]$ if its definite integral exists on that interval.
- a function which is continuous on a closed interval is automatically integrable, but a function need not be continuous ot be integrable.

Notes on the Definite Integral

It is useful to note that

- a function is called integrable on an interval $[a, b]$ if its definite integral exists on that interval.
- a function which is continuous on a closed interval is automatically integrable, but a function need not be continuous ot be integrable.
- if a function is non-negative, then its definite integral on an open interval is equal to the area under its curve.

Notes on the Definite Integral

It is useful to note that

- a function is called integrable on an interval $[a, b]$ if its definite integral exists on that interval.
- a function which is continuous on a closed interval is automatically integrable, but a function need not be continuous ot be integrable.
- if a function is non-negative, then its definite integral on an open interval is equal to the area under its curve.
- a definite integral is a number whereas an indefinite integral is a family of functions.

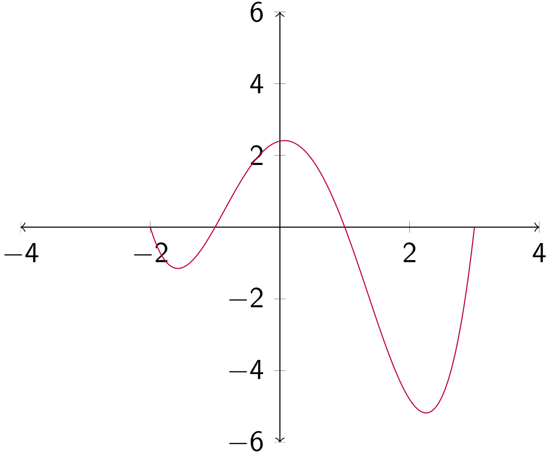
Geometric Interpretations of the Definite Integral

If f is non-negative on $[a, b]$, as we mentioned, the definite integral

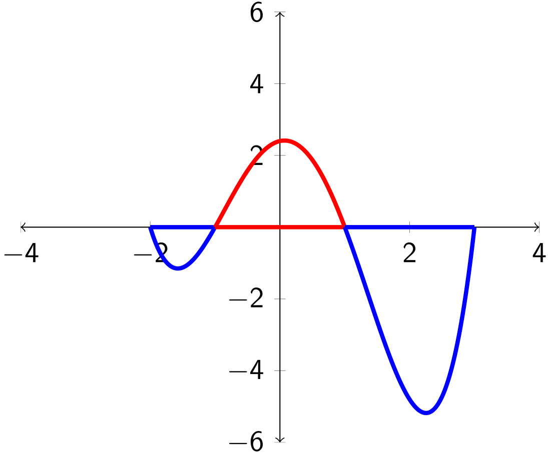
$$\int_a^b f(x)\ dx$$

is the area between the curve and the x-axis. But if f takes both positive and negative values, then the definite intergral is equal to the area of the region below the graph but above the x-axis minus the area of the region blow the x-axis but above f. We can see this graphically.

The Definite Integral and Area Generalized



The Definite Integral and Area Generalized



Assignment

Read 6.4. Do problems 14, 26, 36, 50, 54, 66 in 6.2 and 2, 10, 14, 16, 18 in 6.3.