QMI Lesson 2: Functions, Their Graphs, & Their Algebra

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Functions

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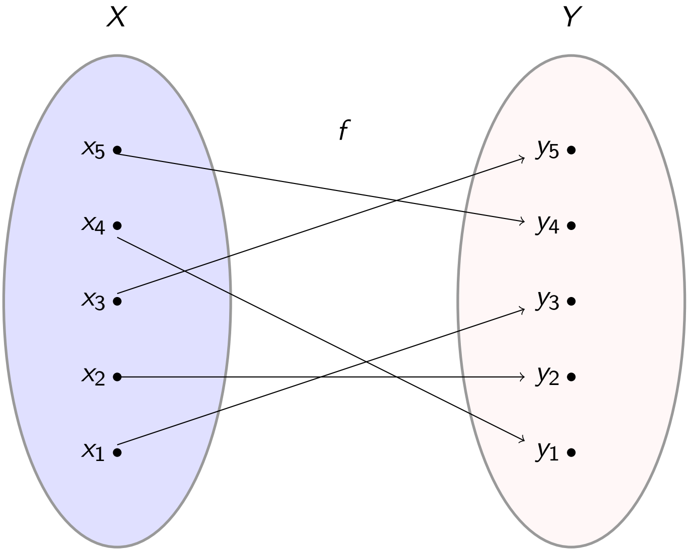
Example (Function)

Let p represent the price assignment at a car lot. Because every car is associated with exactly one dollar value, p is a function from C to P, where C is the set of cars in the lot and P is the set of all possible prices.

Abstract Examples & Non-examples of Functions



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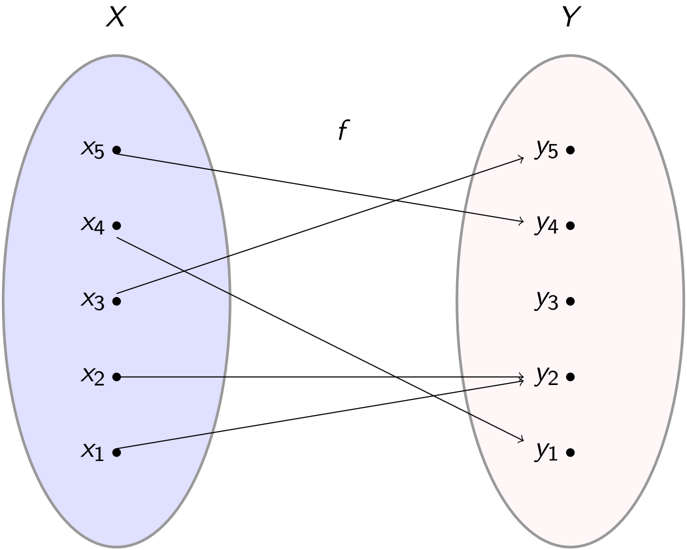

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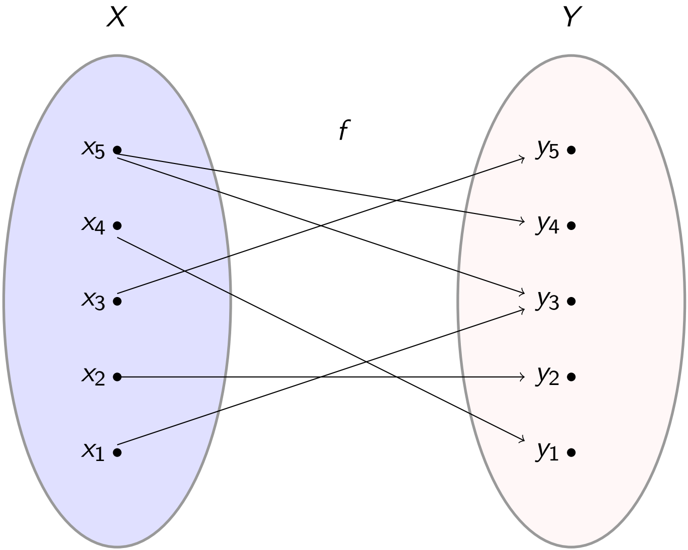

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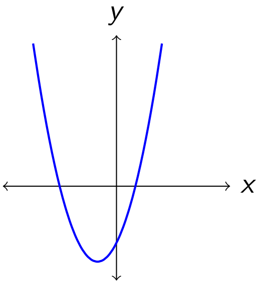

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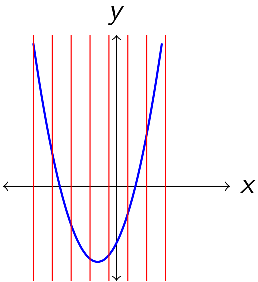

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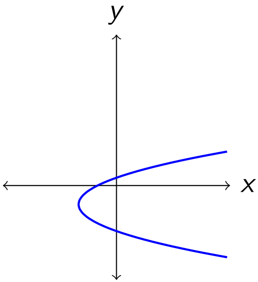

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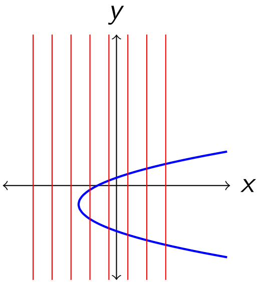

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Evaluating a Function

When a function is determined by an equation, you may evaluate the function simply by plugging in.

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$$f(-1) = 2 \cdot (-1)^3 + 2(-1) - \frac{1}{(-1)} = -2 - 2 + 1 = -3$$

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You can use these rules to infer domains.

Domains of Functions: Example 1

A piece of cardboard (4-by-5 feet) has squares cut out of its corners. The cardboard is then folded into a box. Write an equation for the volume of the box when squares of side x are cut out of the corners. Determine its domain.

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Thus, $V(x) = (4 - 2x)(5 - 2x)(x) = 20x - 18x^2 + 4x^3$. Now, the dimensions must all be non-negative, so $x \geq 0$, $4 - 2x \geq 0$, $5 - 2x \geq 0 \implies 0 \leq x$, $x \leq 2$, $x \leq 2.5$. Therefore, the domain must me $0 \leq x \leq 2$.

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The domain is $(-2, 2)$.

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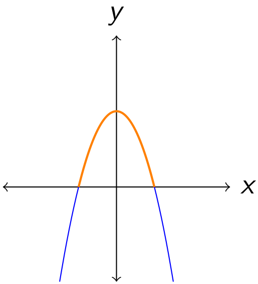
When determining where $(2 - x)(2 + x) > 0$, you can look at the graph $g(x) = (2 - x)(2 + x)$.



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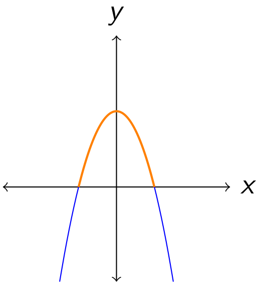

What's the range of $g(x)$? $(-\infty, 4)$.

Piecewise Functions

Definition (Piecewise-Defined Functions)

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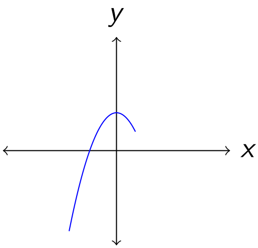
$$f(x) = \begin{cases} x & x > 1 \\ -x^2 + 2 & x \leq 1 \end{cases}$$



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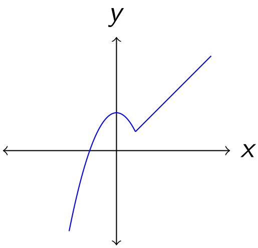

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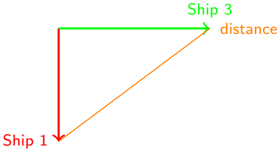
A ship leaves port headed south at 15mph. Another ship leaves headed east at 20mph. Create a formula for the distance between these two ships after t hours. How far are they apart after 4 hours?



The distance in t of Ship 1 from Ship 2, then must be this distance between the two points, which are $(0, -15t)$ and $(20t, 0)$, so $d(t) = \sqrt{(-20t)^2 + (15t)^2} = 25\sqrt{t^2} = 25t$.

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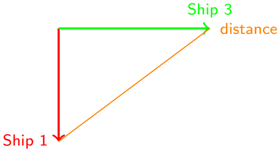

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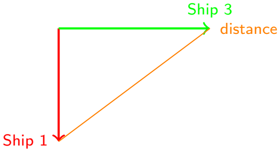

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So, we have then that $d(4) = 100$ miles.

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What are the domains of these functions?

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Definition (Total Profit)

The total profit is the difference between the total revenue and the total cost.

Example

Purifi creates water filters. The firm runs a total monthly fixed cost of \$12000, and its cost per unit (x) is given by $-0.002x^2 + 12x$. Its revenue per unit is given by $0.0001x^2 + 20x$. Give a formula for total profit. What profit is realized when 5000 units are sold?

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$P(x) = 0.0021x^2 + 8x - 12000$. And $P(5000)$=80500.

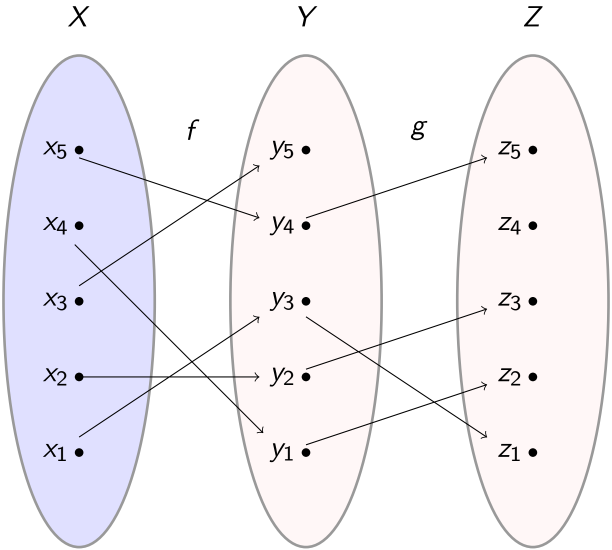
Composition of Functions

Definition (Composition of Functions)

If f and g are two functions, then the composition of f and g is the function $g \circ f$ defined by $(g \circ f)(x) = g(f(x))$.

The domain of $g \circ f$ is the set of all x in the domain of f such that $f(x)$ is in the domain of g.

Composition of Functions: Abstract Example



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And, $(g \circ f)(x) = \sqrt{(2x^2-1)-1} = \sqrt{2x^2-2}$.
The domain of $g \circ f$ is $(-\infty, -1] \cup [1, \infty)$.

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Assignment

Read 2.3. Do problems 14, 16, 20, 22, 36, 74 in 2.1 and 6, 18, 30, 34, 40, 66 in 2.2.