QMI Lesson 4:
Limits

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Motivating Limits

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- the area under a curve
- the probability of an event

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As we approach 2 on the right-hand of our interval, the average velocity appears to approach 16.

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The Limit

Definition (Limit of a Function)

The function f has the limit L as x approaches a, written

$$\lim_{x \to a} f(x) = L,$$

if the value of $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to (but not equal to) a.

Evaluating a Limit

There are some general approaches to evaluating limits you may find useful.

1. At a point of continuity, just plug in.
2. At a point of discontinuity, if the discontinuity is removable, you can evaluate using the continuous part.
3. Limits do not exist at singularities and non-removable discontinuities.

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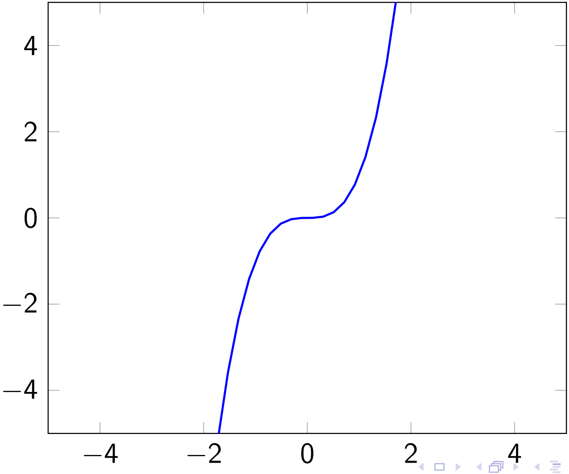
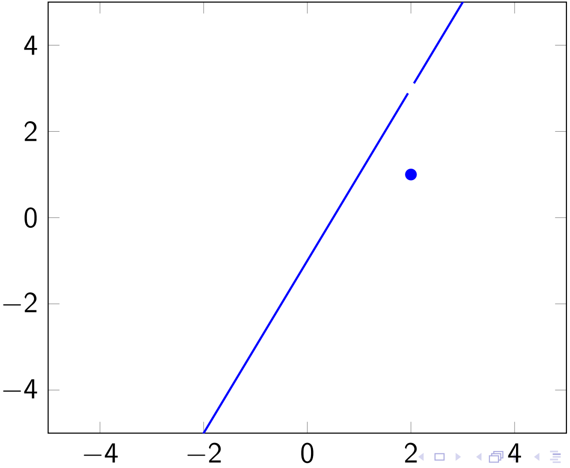


Example: Removable Discontinuity

$$f(x) = \begin{cases} 1 & x = 2 \\ 2x - 1 & x \neq 1 \end{cases}$$



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In this case, we can just look at the continuous part and plug in.

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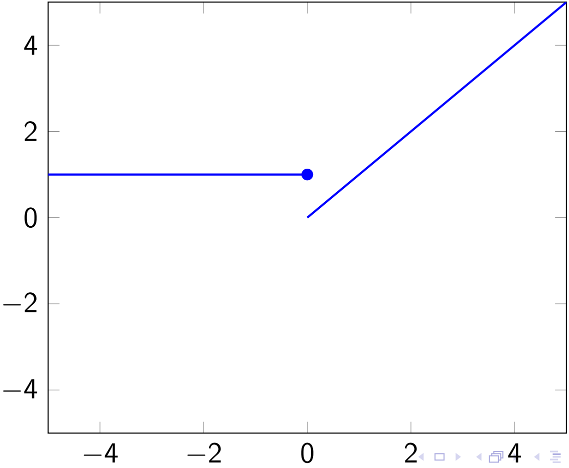

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Evaluate $\lim_{x \to 0} f(x)$. As we approach 0 from the left, $f(x)$ approaches 1. As we approach 0 from the right, $f(x)$ approaches 0. So, $\lim_{x \to 0} f(x)$ **does not exist**.

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In this case, we also say the limit does not exist. In fact, if the function approaches infinity on either side of the x-value, the limit cannot exist.

Graph of “Infinite” Limit

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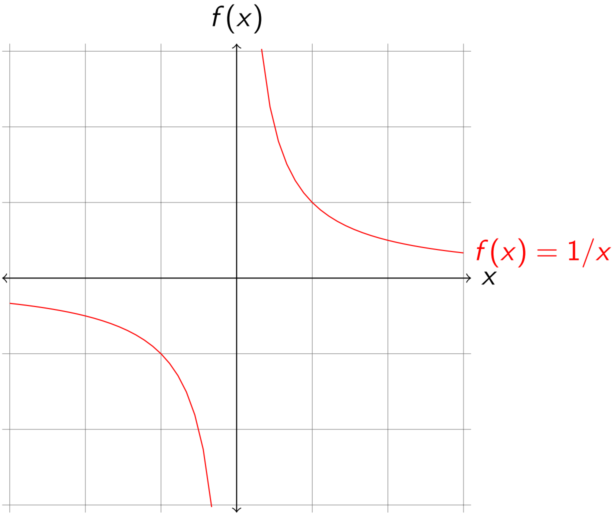

Other Limit Issues

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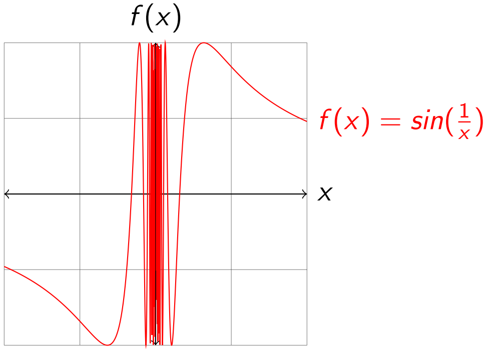

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3. $$\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M$$

Properties of Limits Continued

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5. $$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}.$$

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$$\underset{1\&2\&3}{=} \frac{2[\lim_{x\to 2} x]^2 - 3\lim_{x\to 2} x}{(\lim_{x\to 2} x + \lim_{x\to 2} 2)(\lim_{x\to 2} x - \lim_{x\to 2} 4)}$$

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Notice, the truth of this statement arises from the last computation. (Read the theorem carefully.)

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then it's easy to see that $\lim_{t \to 2} v(t) = 16$.

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The first step usually involves factorization/cancellation (like in our previous example) or multiplication by conjugates in the numerator and denominator.

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$$\frac{1}{\sqrt{x+1}+1}.$$

Thus, $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{2}$.

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Definition (Limit of a Function at Infinity)

A function f has a limit L as x increases without bound, written $\lim_{x \to \infty} f(x) = L$, if $f(x)$ can be made arbitrarily close to L by taking x large enough.

A function f has a limit L as x decreases without bound, written $\lim_{x \to -\infty} f(x) = L$, if $f(x)$ can be made arbitrarily close to L by taking x negative and large (in absolute value) enough.

Theorems for Limits at Infinity

Theorem

For all $n > 0$, $\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$, *so long as* $\frac{1}{x^n}$ *is defined.*

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Specifically, for all polynomials $p(x)$ *and* $q(x)$, $\lim_{x \to \pm\infty} \frac{p(x)}{q(x)} = 0$ *if the degree of* $q(x)$ *is greater than the degree of* $p(x)$.

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Moreover, if the degree of $p(x)$ and $q(x)$ are the same, then $\lim_{x \to \pm\infty} \frac{p(x)}{q(x)} = \frac{\tilde{p}}{\tilde{q}}$, where $\tilde{p}$ and $\tilde{q}$ are the leading coefficients of p and q respectively.

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Finally, if the degree of $p(x)$ is greater than the degree of $q(x)$, then $\lim_{x \to \pm\infty} \frac{p(x)}{q(x)}$ **does not exist**.

Examples

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Other Limit Issues

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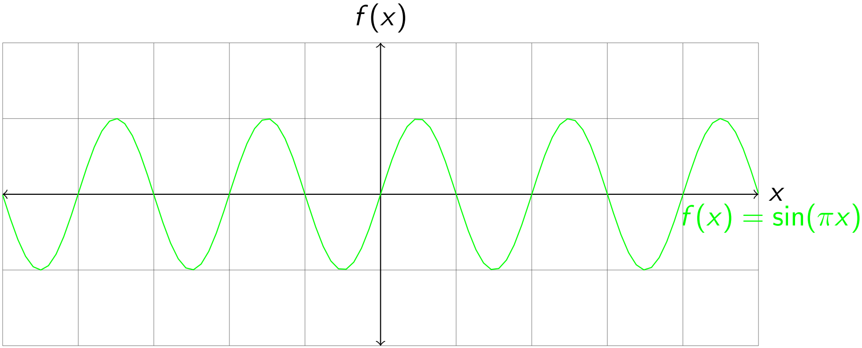

Assignment

Read 2.5. Do problems 6, 12, 16, 34, 46, 60, 62, 68, 76, 96 in 2.4.