QMI Lesson 6:
The Derivative

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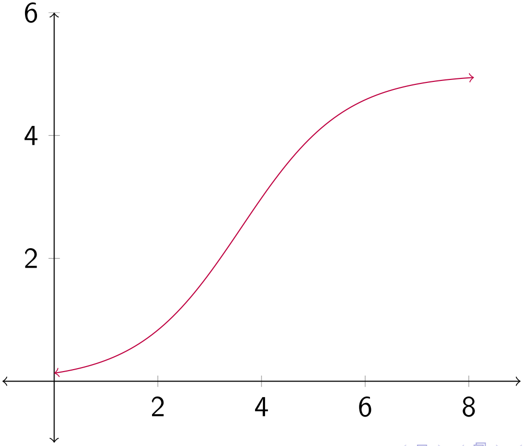
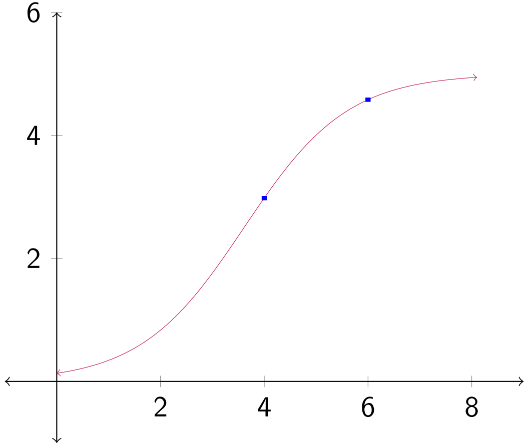
Motivating Example

The graph gives the number of pensions a company pays (y in thousands) at a certain point in time (x=0 corresponds to 2005, x in years).



Motivating Example

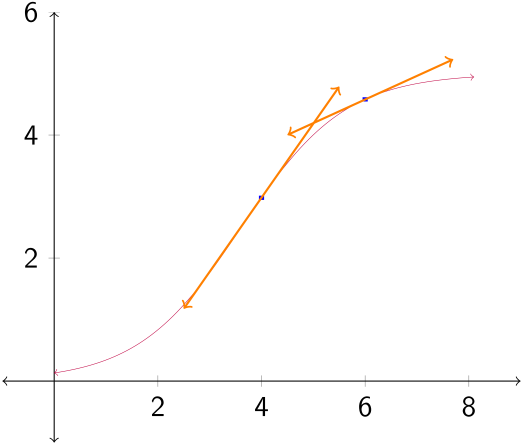
Consider the points on the graph $(6, 4.58)$ and $(4, 2.98)$. What can we say about these points in general?



What can we say about the **rates of change** at these two points?

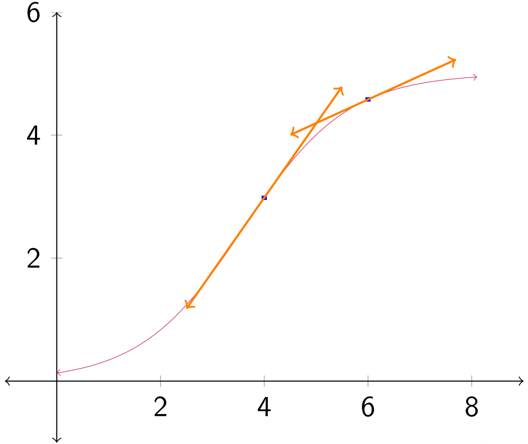
Motivating Example

Well, to do this, we look at the slope of the graph at these points, i.e. the **tangent line**.



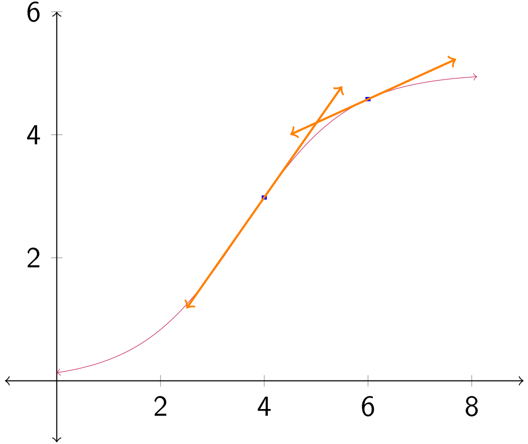
Motivating Example

So we can tell now that, although the number of pensions has increased from 2009 to 2011, the rate has decreased. How does this translate to budget planning?



Motivating Example

This means that the business would need to find more money in 2010 to cover the increase in number of pensions than it would need to find in 2012.



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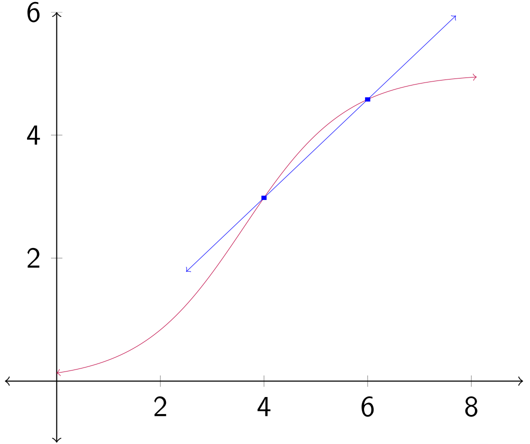
Tangent as Limit of Secants

We are trying to calculate the tangent line at $x = 4$. We do this by drawing secant lines between $(4, f(4))$ and $(4 + h, f(4 + h))$, letting $h \to 0$.



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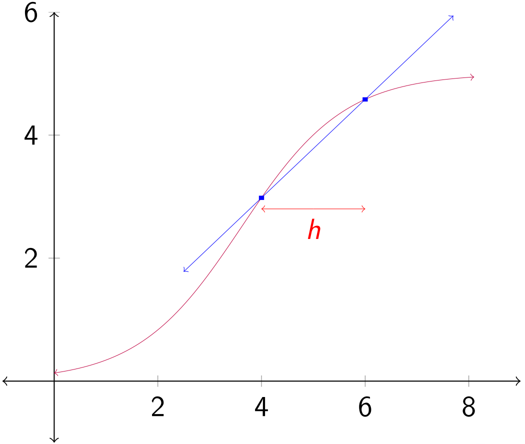

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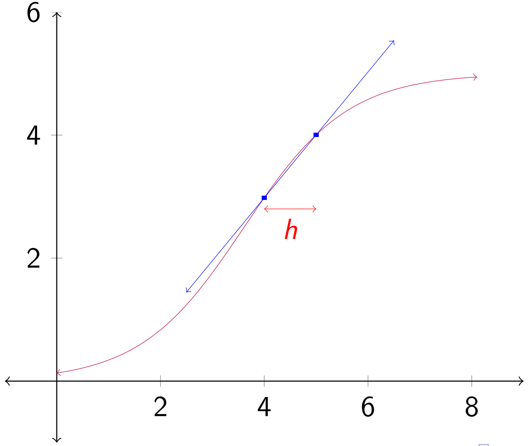

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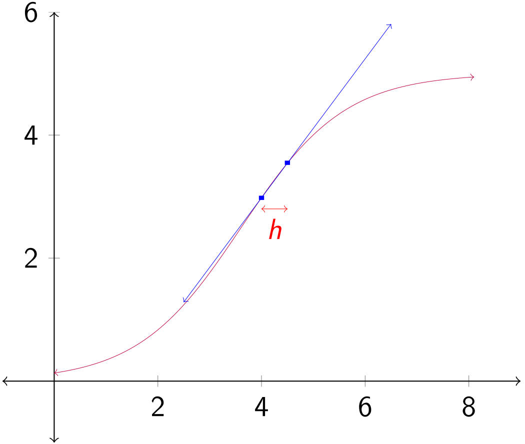

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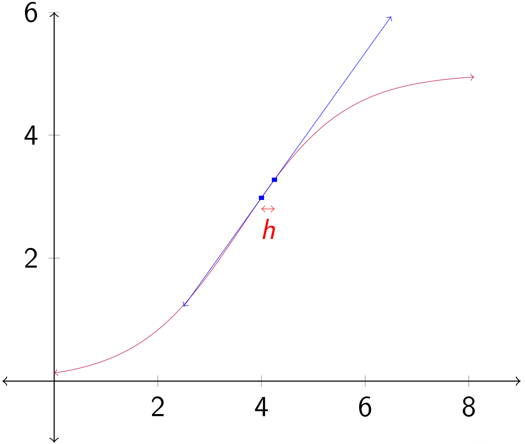

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$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The Derivative and the Slope of a Tangent Line

Definition

The derivative of a function f with respect to the variable x is the function f' where

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The domain of f' is simply all points x at which the limit exists.

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The derivative also gives the slope of the tangent line at $(x, f(x))$.

The Derivative and Average Rate of Change

Definition

The average rate of change m of a function f between $(x, f(x))$ and $(x+h, f(x+h))$ is just the slope of the secant line connecting those points.

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The derivative relates the average rate of change to the **instantaneous** rate of change through the limit as $h \to 0$.

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Computing a Derivative

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1. Compute $f(x+h)$.
2. Calculate the difference $f(x+h) - f(x)$.
3. Form the quotient
$$\frac{f(x+h) - f(x)}{h}.$$
4. Take the limit
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

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$$y - \left(-\frac{1}{4}\right) = \frac{1}{4}(x - (2)),$$

or, equivalently, (slope-intercept)

$$y = \frac{1}{4}x - \frac{3}{4}.$$

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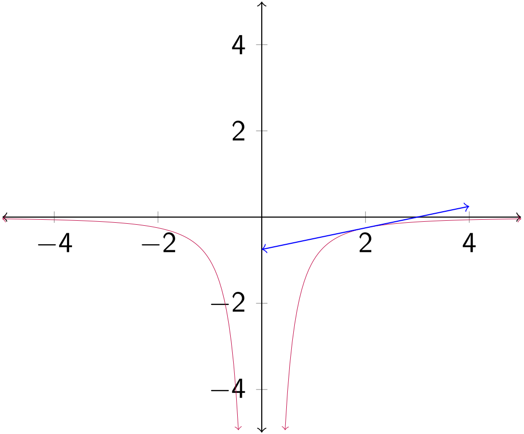

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So, the tangent line is has slope zero at (-2,-8)

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What does it mean for the tangent line to be horizontal?

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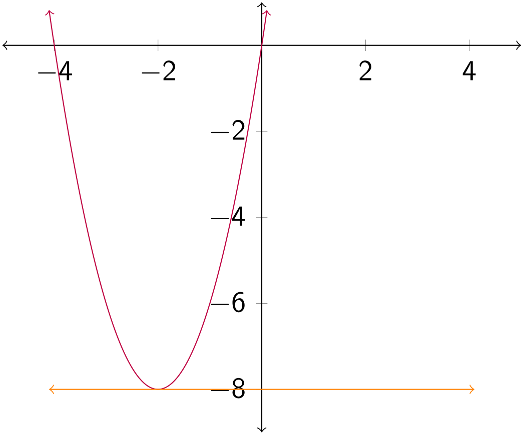
What does it mean for the tangent line to be horizontal? It means that *the instantaneous rate of change is zero*!



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Non-Differentiability: Corner

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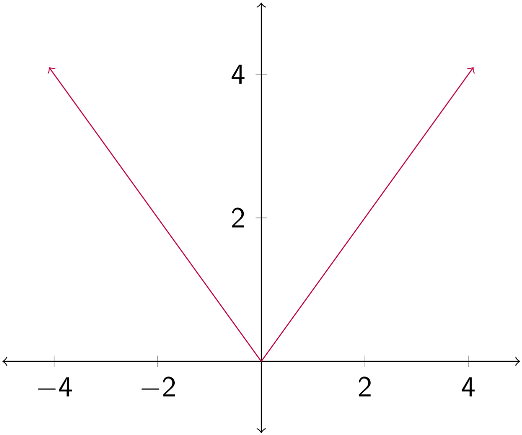
Non-Differentiability: Corner

The graph of $f(x) = |x|$. Observe the corner at the origin.



Non-Differentiability: Vertical Tangent Line

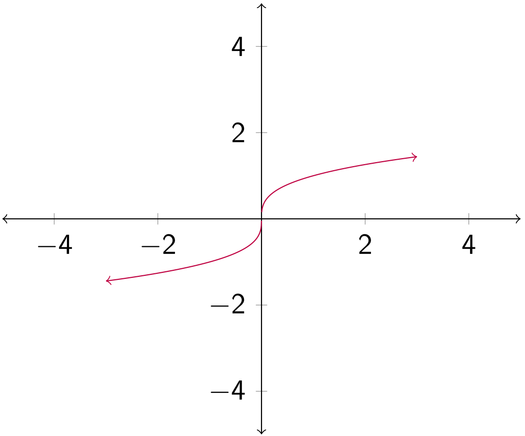
Consider $f(x) = x^{\frac{1}{3}}$. Then,

$$f'(0) = \lim_{h \to 0} \frac{h^{\frac{1}{3}}}{h} = \lim_{h \to 0} \frac{1}{h^{\frac{2}{3}}}$$

which does not exist because as $h \to 0^+$, $\frac{1}{h^{\frac{2}{3}}}$ increases without bound.

Non-Differentiability: Vertical Tangent Line

The graph of $f(x) = x^{\frac{1}{3}}$. Observe the vertical tangent line that the y-axis makes with the graph.



Applied Example

The weekly demand for car washes at GooGoo's Car Wash can be modeled by $d(p) = 15625 - p^2$, where the quantity demanded d depends on the price p. Find the average rate of change in the unit demand for car washes if the price changes from \$30 to \$50, from \$30 to \$35, and from \$30 to \$31.

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And, $\frac{d(35)-d(30)}{5} = -65$ car washes per dollar increase.

And, $\frac{d(31)-d(30)}{1} = -61$ car washes per dollar increase.

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Well, $f'(30) = \lim_{h \to 0} \frac{(15625 - (30+h)^2) - (15625 - 30^2)}{h} =$

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Well, $f'(30) = \lim_{h \to 0} \frac{(15625-(30+h)^2)-(15625-30^2)}{h} =$
$\lim_{h \to 0} \frac{-900-60h-h^2+900}{h} =$

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Assignment

Read 3.1-3.2. Do problems 6, 14, 22, 26, 34, 38, 44, 62 in 2.6.